On the Calculation of the Stability Radius of an Optimal or an Approximate Schedule

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Abstract: The main objective of this paper is to stimulate interest in stability analysis for scheduling problems. In spite of impressive theoretical results in sequencing and scheduling, up to now the implementation of scheduling algorithms with a rather deep mathematical background in production planning, scheduling and control, and in other real-life problems with sequencing aspects is limited. In classical scheduling theory, mainly deterministic systems are considered and the processing times of all operations are supposed to be given in advance. Such problems do not often arise in practice: Even if the processing times are known before applying a scheduling procedure, OR workers are forced to take into account the precision of equipment, which is used to calculate the processing times, round-off errors in the calculation of a schedule, errors within the practical realization of a schedule, machine breakdowns, additional jobs and so on. This paper is devoted to the calculation of the stability radius of an optimal or an approximate schedule. We survey some recent results in this field and derive new results in order to make this approach more suitable for practical use. Computational results on the calculation of the stability radius for randomly generated job shop scheduling problems are presented. The extreme values of the stability radius are considered in more detail. The new results are amply illustrated with examples.

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1 Introduction

This paper is devoted to a stability analysis of an optimal or approximate solution of discrete optimization problems, mainly from the field of scheduling theory. It basically consists of two parts, which are devoted to two different settings of scheduling problems and which are completed by appropriate calculation methods. The main idea is to combine in one paper different types of scheduling settings and to show that the same stability analysis may be suitable for optimization problems of different complexity. Due to this, results derived for one problem type may also be used for other types (directly or as a possible subject for additional research).

The first part (Sections 2 - 6) deals with the general shop scheduling problem, where a set of jobs has to be processed on a set of machines. Each maximal non-preemptive processing of a job on a machine is called an operation and there are given precedence constraints between certain operations. In the first part of the paper, we consider as objective the minimization of the makespan. A suitable model for representing this scheduling problem is the disjunctive or mixed graph (see Section 2). We investigate the stability ball of an optimal digraph representing a semi-active schedule s for a general shop (scheduling) problem, i.e., a ball in the space of the numerical input data such that within this ball schedule s remains optimal. In Section 3 we present a formal definition of the stability radius (the maximal value of the radius of such a stability ball) and survey some previous results. An illustrative example of the job shop problem with the makespan criterion (as a special case of the general shop problem) is given in Section 4. In Section 5 we improve a known algorithm for calculating the stability radius for the general shop problem. In Section 6 we discuss computational results on the calculation of the stability radii for randomly generated job shop problems.

Stability results for simpler (but still NP-hard) scheduling problems, which may be formulated as a linear binary program, are surveyed and improved in the second part of the paper (Sections 7 – 14). This part is devoted to a stability analysis of an approximate solution of such types of optimization problems. All results presented in the second part are not only valid for scheduling problems of this type, but for the general class of linear binary programming problems as well. Some basic results on the stability analysis of an approximate solution are given in Section 7. Section 8 deals with the stability region of an approximate solution. The derived properties are illustrated on an example in Section 9. An upper bound on the stability radius is derived in Section 10. Section 11 presents necessary and sufficient conditions for a zero stability radius of an approximate solution. In Section 12 lower bounds for the stability radius are derived. The calculation of the stability radius of an approximate solution is treated in Section 13. In Section 14 we demonstrate the calculation of the stability radius on the example from Section 9. Finally, some open questions in stability analysis are formulated in Section 15.

2 The General Shop Problem

Most scheduling problems may be represented as extremal problems on disjunctive (mixed) graphs [9, 16, 22, 23]. The only requirement for this representation is the prohibition of preemptions of operations. In the first part of the paper we use the disjunctive graph model to represent the input data of the so-called general shop problem usually denoted by $G//\Phi$ and defined as follows.

There is a set $Q = \{1, 2, \dots, q\}$ of operations that have to be processed on the machines of a set $M = \{M_1, M_2, \dots, M_m\}$. Let Q_k denote the set of operations that have to be processed on machine $M_k \in M$: $Q = \cup_{k=1}^{m} Q_k$, $Q_k \neq \emptyset$, $Q_k \cap Q_l = \emptyset$, $k = 1, 2, \dots, m$, $l = 1, 2, \dots, m$, $k \neq l$. At any time each machine can process at most one operation and the processing time $p_i \geq 0$ of operation $i \in Q_k$ on machine $M_k \in M$ is given before scheduling. Preemptions of operations are not allowed and this implies that a schedule of the operations Q on the machines M may be defined by the completion times c_i or by the starting times $c_i - p_i$ of all operations $i \in Q$. We assume in the following that $c_i \geq p_i$ holds for each $i \in Q$. The set of operations Q is supposed to be partially ordered by the given precedence constraints $\rightarrow$: if $i \rightarrow j$ is given, then

$$c_i \leq c_j - p_j \tag{2.1}$$

must hold for any feasible schedule. Since at any time a machine can process at most one operation, the conditions $i \in Q_k$ and $j \in Q_k$ imply one of the following inequalities:

$$c_i \leq c_j - p_j \quad \text{or} \quad c_j \leq c_i - p_i. \tag{2.2}$$

The general shop problem $G//\Phi$ is to find a feasible schedule $(c_1, c_2, \dots, c_q)$ in order to minimize the value of a given non-decreasing objective function $\Phi(c_1, c_2, \dots, c_q)$. The problem data are represented by means of a disjunctive (or mixed) graph $G = (Q, C, D)$ as follows. The set Q of operations is the set of vertices, a non-negative weight p_i being assigned to each vertex $i \in Q$. C is the set of directed (conjunctive) arcs, representing conditions (2.1): $C = \{(i, j) \mid i \rightarrow j,\ i \in Q,\ j \in Q\}$. D is the set of pairs of directed (disjunctive) arcs, representing conditions (2.2): $D = \{(i, j), (j, i) \mid i \in Q_k;\ j \in Q_k;\ i \not\rightarrow j;\ j \not\rightarrow i;\ k = 1, 2, \dots, m\}$.

An analogous model may be given in terms of a mixed graph $G^0 = (Q, C, D^0)$ using an undirected edge $[i, j]$ instead of a pair of disjunctive arcs $\{(i, j), (j, i)\}$: $D^0 = \{[i, j] \mid i \in Q_k;\ j \in Q_k;\ i \not\rightarrow j;\ j \not\rightarrow i;\ k = 1, 2, \dots, m\}$.

One can note that conditions (2.2) may be implied not only by the same machine M_k, which has to process operations i and j, but also by the same job in the case of a non-fixed technological route which includes both operations i and j (like in the open shop problem). We should mention that all known and new results presented in the first part of the paper (i.e. in Sections 2 - 6) remain valid also for such type of the conditions (2.2). However, for the sake of simplicity we shall restrict our presentation to the case that machine M_k is the only reason for the occurrence of conditions (2.2).

While solving problem $G//\Phi$, each pair of disjunctive arcs $\{(i, j), (j, i)\}$ must be settled, i.e., one of these arcs must be added to a subset $D_s \subset D$ of chosen arcs and the other one must be rejected [9, 16, 22]. The choice of arc (i, j) ((j, i), respectively) defines a precedence of operation i (operation j) over operation j (operation i) on their common machine $M_k \in M$. A feasible schedule is defined by a subset $D_s \subset D$ such that

(*) $(i, j) \in D_s$ if and only if $(j, i) \in D \setminus D_s$, and

(**) the digraph $G_s = (Q, C \cup D_s, \emptyset)$ has no circuits.

Since the objective function is non-decreasing in the completion times, we may consider only semi-active schedules [9, 23]. Let $P(G) = \{G_1, G_2, \dots, G_\lambda\}$ be the set of all digraphs

G_s that satisfy both conditions (*) and (**). On the one hand, each digraph $G_s \in P(G)$ defines a unique semi-active schedule $s = (c_1(s), c_2(s), \ldots, c_q(s))$, where $c_i(s)$ is the earliest completion time of operation $i \in Q$ with respect to the digraph G_s. On the other hand, each semi-active schedule defines a unique digraph $G_s \in P(G)$. In the following we call the digraph $G_s \in P(G)$ optimal if s is an optimal schedule.

The general shop problem (and many of its special cases) is NP-hard in the strong sense for all criteria considered in classical scheduling theory [9, 23], but one can find an optimal schedule $s = (c_1(s), c_2(s), \ldots, c_q(s))$ in $O(q^2)$ time after having constructed an optimal digraph G_s. It follows that the main difficulty of problem $G//\Phi$ consists in constructing an optimal digraph $G_s = (Q, C \cup D_s, \emptyset)$, in other words, in constructing the best set D_s of chosen arcs. Because of its importance, set D_s is called the signature of a schedule s [18, 20, 22].

3 The Stability Radius of an Optimal Digraph

One of the main questions under consideration is as follows. How can one vary the processing times $p_i, i \in Q$, in the problem $G//\Phi$ such that an optimal schedule remains optimal? Note that any variation of the processing times changes an optimal schedule s, however, the optimal digraph $G_s = (Q, C \cup D_s, \emptyset)$ may remain the same and the signature D_s of an optimal schedule s is more stable. Also, in practice it is often not so important to know exactly an optimal solution (i.e., the times when the operations have to be started and have to be completed), but rather the optimal sequences in which the operations have to be processed on the machines $M_k \in M$ (this is again due to the fact that optimal sequences are more stable than an optimal schedule). Therefore, following [8, 17, 18] we investigate the stability of an optimal digraph G_s, which represents a solution of problem $G//\Phi$ in a compact form. We concretize the above question: Under which largest independent changes in the components of the vector of the processing times $p = (p_1, p_2, \ldots, p_q)$ does the digraph G_s remain optimal? Next, we introduce these notions in a formal way.

Let R^q be the set of all non-negative real vectors p with the maximum metric. The distance $r(p, p')$ between the vectors $p \in R^q$ and $p' \in R^q$ is equal to $\max\{|p_i - p'_i| \mid i \in Q\}$, where $|p_i - p'_i|$ denotes the absolute value of the difference $p_i - p'_i$.

Definition 1 *The closed ball $O_\varrho(p)$ with the radius ϱ and the centre p in the space of all q-dimensional real vectors is called a stability ball of an optimal digraph G_s, if for any vector $p' \in O_\varrho(p) \cap R^q$ of processing times the schedule s remains optimal.*

Note that a stability ball may include also q-dimensional vectors with negative real components.

Definition 2 *The radius ϱ of the largest stability ball $O_\varrho(p)$ of the optimal digraph G_s is called the stability radius of G_s and is denoted by $\varrho_s(p)$.*

In the remainder of this section we survey recent results on stability analysis and in Section 5 we derive some results for problem $G//C_{max}$ with the makespan criterion: $\Phi(c_1, c_2, \ldots, c_q) = \max\{c_i \mid i \in Q\} = C_{max}$. Let $< \mu >$ denote the set of vertices which form a path μ in the digraph G_s and let $l^p(\mu)$ be the length of this path: $l^p(\mu) = \sum_{i \in <\mu>} p_i$. Obviously, the value of $\max\{c_i(s) \mid i \in Q\}$ of a schedule s is equal to the length of a critical (longest) path in G_s and, hence, in the case of the makespan criterion we have to determine a feasible schedule s such that the length of a critical path in G_s is minimized:

$$\max_{\mu \in H'_s} l^p(\mu) = \min_{k=1,\dots,\lambda} \max_{\nu \in H'_k} l^p(\nu), \tag{3.1}$$

where H'_k denotes the set of all paths in the digraph G_k. Since the processing times are non-negative, we can consider in (3.1) only dominant paths. The path $\mu \in H'_k$ is called dominant [8, 18, 20], if there is no other path $\nu \in H'_k$ such that $< \mu > \subseteq < \nu >$. Otherwise, we write that path μ is dominated by path ν. Let H_k and H denote the sets of all dominant paths in the digraphs $G_k = (Q, C \cup D_k, \emptyset)$ and $(Q, C, \emptyset)$, respectively.

It has been shown in [17, 18] that, if s is an optimal schedule of problem $G//C_{max}$, the value $\varrho_s(p)$ either satisfies the inequalities

$$0 \leq \varrho_s(p) \leq \max\{p_i \mid i = 1, 2, \dots, q\} = p_* \tag{3.2}$$

or it is infinitely large: $\varrho_s(p) = \infty$.

Let H^p_k denote the set of all dominant critical paths in the digraph $G_k \in P(G)$ with the vector $p \in R^q$ of weights. Obviously, $H^p_k \subseteq H_k$ holds for each $k = 1, 2, \dots, \lambda$. We denote the set of all optimal schedules by $\phi(p)$. The following theorems (see [17, 18]) characterize the extreme values of the stability radius.

Theorem 1 *For an optimal schedule $s \in \phi(p)$ of problem $G//C_{max}$, the strict inequality $\varrho_s(p) > 0$ holds if and only if for any path $\mu \in H^p_s \setminus H$ and any other optimal schedule $k \in \phi(p)$ (provided that $|\phi(p)| > 1$) there exists a path $\nu \in H^p_k$ such that $< \mu > \subseteq < \nu >$.*

Theorem 2 *For problem $G//C_{max}$, we have $\varrho_s(p) = \infty$ if and only if for any path $\mu \in H_s \setminus H$ and any digraph $G_k \in P(G)$ there exists a path $\nu \in H_k$ such that $< \mu > \subseteq < \nu >$.*

Unfortunately, for problem $G//C_{max}$ it is difficult to verify the conditions of Theorems 1 and 2. In [8], simpler (in the computational sense) necessary and sufficient conditions have been derived for a given job shop problem $J//C_{max}$ to have at least one optimal digraph with an infinitely large stability radius. It has been shown that the latter conditions can be verified in $O(q^2)$ time and this is also the complexity of actually constructing an optimal makespan schedule s with $\varrho_s(p) = \infty$. Similar results have been obtained for problem $J//L_{max}$ of minimizing maximum lateness. It has also been proven in [8] that for a problem $J//\Phi$ with any other classical criterion $\Phi(c_1, c_2, \dots, c_q)$, presented e.g. in [9], there does not exist an optimal schedule s with an infinitely large stability radius. Note also that for the flow shop problem $F//C_{max}$ and the open shop problem $O//C_{max}$, there exists an optimal schedule s with $\varrho_s(p) = \infty$ only for very small examples. More precisely, if $|J^*| > 1$ and $m > 1$, we have $\varrho_s(p) \leq p_*$ for problem $O//C_{max}$, and if $|J^*| \geq 2$ and $m \geq 2$, we have $\varrho_s(p) \leq p_*$ for problem $F//C_{max}$ ($|J^*|$ denotes the cardinality of the set of jobs $J^* = (J_1, J_2, \dots, J_{|J^*|})$).

A general formula for calculating $\varrho_s(p)$ for problem $G//C_{max}$ has been given in [17, 18]. In Section 5 we slightly simplify this formula in order to reduce the required time for calculating the stability radius and to include the case $\varrho_s(p) = \infty$. To illustrate the above notions and Theorems 1 and 2, we consider in the following section an example of problem $J//C_{max}$ with two jobs and two machines.

4 Example 1

Let a job shop problem be specified by the mixed graph $G^0 = (Q, C, D^0)$ given in Fig. 1. The first job consists of operations 1 and 2, and the second job consists of operations 3, 4, and 5. So we have the precedence constraints $1 \to 2$, $3 \to 4$ and $4 \to 5$. The assignment of the operations to the machines is as follows: $Q_1 = \{1, 4\}$, $Q_2 = \{2, 3, 5\}$. The vector $p = (10, 20, 30, 40, 20)$ defines the processing times of the operations $Q = \{1, 2, 3, 4, 5\}$. Hereafter we shall refer to this example as Example 1.

F i g u r e 1

For Example 1 we get $P(G) = \{G_1, G_2, G_3, G_4, G_5\}$ with the following signatures of all semi-active schedules: $D_1 = \{(1,4), (3,2), (2,5)\}$, $D_2 = \{(1,4), (3,2), (5,2)\}$, $D_3 = \{(1,4), (2,3), (2,5)\}$, $D_4 = \{(4,1), (3,2), (2,5)\}$ and $D_5 = \{(4,1), (3,2), (5,2)\}$. The corresponding sets of dominant paths are the following: $H_1 = \{(1,2,5), (3,2,5), (1,4,5), (3,4,5)\}$, $H_2 = \{(1,4,5,2), (3,4,5,2)\}$, $H_3 = \{(1,2,3,4,5)\}$, $H_4 = \{(3,4,1,2,5)\}$, $H_5 = \{(3,4,1,2), (3,4,5,2)\}$. The (makespan) optimal digraph $G_1 = (Q, C \cup D_1, \emptyset)$ is shown in Fig. 2 and it defines the unique optimal semi-active schedule $(10, 50, 30, 70, 90)$.

F i g u r e 2

Since there exists only one optimal digraph G_1, we conclude that $\varrho_1(p) > 0$ due to Theorem 1. On the other hand, the value $\varrho_1(p)$ cannot be infinitely large, since there exist the path $\mu = (1, 2, 5)$ in the set $H_1 \backslash H$ and the digraph $G_5 \in P(G)$ such that for any path $\nu \in H_5$ the inclusion $< \mu > \subseteq < \nu >$ does not hold. Indeed $\{1, 2, 5\} \not\subseteq \{1, 2, 3, 4\}$ and $\{1, 2, 5\} \not\subseteq \{2, 3, 4, 5\}$. Therefore due to Theorem 2, we have $\varrho_1(p) < \infty$ and, as a result, the inequalities $0 < \varrho_1(p) \leq 40 = p_*$ hold (see (3.2)).

5 The Calculation of $\varrho_s(p)$ for Problem $G//C_{max}$

Next, we derive a formula for calculating the stability radius which, similar to that proven in [18], is based on the enumeration and comparison of the dominant paths of an optimal and other feasible digraphs. However, using the dominance relation between the set of paths of an optimal and that of a feasible digraph, we shall reduce the set of paths which have to be compared (see set H_{sk} in the following proof) while calculating the stability radius. Due to this, the new formula often leads to a smaller running time for some scheduling problems (see calculation for Example 1 at the end of this section). Moreover, while the formula from [18] is valid only for finite values $\varrho_s(p)$ and it does not identify the case when the stability radius is infinitely large, the new formula holds for the general case. More precisely, the calculation of $\varrho_s(p)$ in accordance with the new formula indicates the case $\varrho_s(p) = \infty$ (if it occurs).

First, assume that $\varrho_s(p) < \infty$ holds for the given optimal schedule $s \in \phi(p)$ of problem $G//C_{max}$. Using Definition 2 and equality (3.1), we can conclude that

$$\varrho_s(p) = \inf\{r(p, p') \mid p' \in R^q, \max_{\mu \in H_s} l^{p'}(\mu) > \min_{k=1,\dots,\lambda} \max_{\nu \in H_k} l^{p'}(\nu)\}.$$

Therefore, to find the stability radius $\varrho_s(p)$ it is sufficient to construct a vector $p' \in R^q$ which satisfies the following three conditions:
1) there exists a digraph $G_k \in P(G)$ such that

$$\max_{\mu \in H_s} l^{p'}(\mu) = \max_{\nu \in H_k} l^{p'}(\nu); \tag{5.1}$$

2) for any given real $\epsilon > 0$ which is close to zero, there exists a vector p^ϵ such that $r(p', p^\epsilon) = \epsilon$ and the inequality

$$\max_{\mu \in H_s} l^{p^\epsilon}(\mu) > \max_{\nu \in H_k} l^{p^\epsilon}(\nu) \tag{5.2}$$

is satisfied for at least one digraph $G_k \in P(G)$;
3) the distance $r(p, p')$ achieves its minimal value among the distances between vector p and the other vectors in the space R^q which satisfy both conditions 1 and 2 above.

After having constructed such a vector $p' \in R^q$, one can define the stability radius of the digraph G_s: $\varrho_s(p) = r(p, p')$, since the critical path in the digraph G_s becomes larger than that of the digraph G_k for any $p^\epsilon \in R^q$ with positive real ϵ, which may be as small as possible (see condition 2), and so the digraph G_s is no longer optimal, while in the ball $O_{r(p,p')}(p)$ digraph G_s remains optimal (see condition 3).

Thus, the calculation of the stability radius is reduced to an extremal problem on a given set of weighted digraphs $P(G) = \{G_1, G_2, \dots, G_\lambda\}$ with a variable vector p of weights assigned to the vertices of each digraph $G_i \in P(G)$. As it follows from equality (5.1) and inequality (5.2), the main objects for such a calculation are the sets of dominant paths $H_k, k = 1, 2, \dots, \lambda$. Similarly to [18], we look next for a vector $p' = p(r) = (p_1(r), p_2(r), \dots, p_q(r)) \in R^q$ with the components $p_i(r) \in \{p_i, p_i + r, p_i - r\}$ on the basis of a direct comparison of the paths from the set H_s and the paths from the sets H_k, where $k = 1, 2, \dots, \lambda$ and $k \neq s$.

Let the value $l^p(\nu)$ be greater than the length of a critical path in an optimal digraph G_s. To satisfy equality (5.1), the length of a path $\nu \in H_k$ may not be greater than that of at least one path $\mu \in H_s$ and there is a path $\nu \in H_k$ with a length equal to the length of a critical path in G_s. Thus, if we have calculated

$$r_\nu = \min_{\mu \in H_s} \frac{l^p(\nu) - l^p(\mu)}{|<\mu> \cup <\nu>| - |<\mu> \cap <\nu>|}, \tag{5.3}$$

we obtain the equality $\max_{\mu \in H_s} l^{p(r)}(\mu) = l^{p(r)}(\nu)$ for the vector $p(r) = p(r_\nu)$ with the components

$$p_i(r) = \begin{cases} p_i + r_\nu, & \text{if } i \in <\mu>, \\ p_i - r_\nu, & \text{if } i \in <\nu> \setminus <\mu>, \\ p_i, & \text{if } i \notin <\mu> \cup <\nu>. \end{cases} \tag{5.4}$$

On the other hand, to reach equality (5.1) for the whole digraph G_k, we have to repeat the calculation (5.3) for each path $\nu \in H_k$ with $l^p(\nu) > l^p_s$, where l^p_s denotes the length of a critical path in G_s. Thus, instead of the vector $p(r_\nu)$ we have to consider the vector $p(r) = p(r_{G_k})$ calculated according to formula (5.4), where

$$r_{G_k} = \min_{\mu \in H_s} \max_{\nu \in H_k;\, l^p(\nu) > l^p_s} \frac{l^p(\nu) - l^p(\mu)}{|<\mu> \cup <\nu>| - |<\mu> \cap <\nu>|}. \tag{5.5}$$

Let us now consider inequality (5.2). Since the processing times are non-negative, this inequality may not be valid for a vector $p^\epsilon \in R^q$ if path μ is dominated by path ν: $<\mu> \subseteq$

$< \nu >$. Thus we can restrict our consideration to the subset H_{sk} of the set H_s of all paths, which are not dominated by paths from H_k:

$$H_{sk} = \Big\{ \mu \in H_s \mid \text{ there is no path } \nu \in H_k \text{ such that } < \mu > \subseteq < \nu > \Big\}.$$

Thus, we can replace H_s in equality (5.5) by H_{sk}. To obtain the desired vector $p' \in R^q$, we have to use equality (5.5) for each digraph $G_k \in P(G), k \neq s$. Let r denote the minimum of such a value r_{G_k}: $r = r_{G_k^*} = \min\{r_{G_k} \mid G_k \in P(G), k \neq s\}$, and let $v^* \in H_{k^*}$ and $\mu^* \in H_{sk^*}$ be paths at which the value r_{G^*} has been reached:

$$r_{G_{k^*}} = r_{\nu^*} = \frac{l^p(\nu^*) - l^p(\mu^*)}{\mid < \mu^* > \cup < \nu^* > \mid - \mid < \mu^* > \cap < \nu^* > \mid}.$$

Taking into account (5.4), we note that, if $r_{\nu^*} \leq p_i$ for each $i \in < \nu^* > \setminus < \mu^* >$, the vector $p(r) = p(r_{\nu^*})$ does not contain negative components, i.e., $p(r) \in R^q$. For the general case we have obtained only a lower bound for the stability radius:

$$\varrho_s(p) \geq r = \min_{k=1,\ldots,\lambda;\, k \neq s} \ \min_{\mu \in H_{sk}} \ \max_{\nu \in H_k;\, l^p(\nu) > l^p_s} \frac{l^p(\nu) - l^p(\mu)}{\mid < \mu > \cup < \nu > \mid - \mid < \mu > \cap < \nu > \mid}. \quad (5.6)$$

This bound is tight. Indeed, if $\varrho_s(p) \leq p_i$ for each $i \in < \nu^* > \setminus < \mu^* >$, then $\varrho_s(p) = r$ due to the above remark. For practical use, we note that $\varrho_s(p) = r$ in (5.6) if $\varrho_s(p) \leq \min\{p_i \mid i \in Q\}$.

To obtain the exact value of $\varrho_s(p)$ in the general case, we follow [18]: Let $p^0_{\nu\mu}$ be equal to zero and let $(p^1_{\nu\mu}, p^2_{\nu\mu}, \ldots, p^{w_{\nu\mu}}_{\nu\mu})$ denote a non-decreasing sequence of the processing times of the operations from the set $< \nu > \setminus < \mu >$, where $w_{\nu\mu} = \mid < \nu > \setminus < \mu > \mid$. We obtain the following assertion.

Theorem 3 *If $s \in \phi(p)$ holds for problem $G//C_{max}$, then*

$$\varrho_s(p) = \min_{k=1,\ldots,\lambda; k \neq s} \ \min_{\mu \in H_{sk}} \ \max_{\nu \in H_k; l^p(\nu) > l^p_s} \ \max_{\beta=0,\ldots,w_{\nu\mu}} \frac{l^p(\nu) - l^p(\mu) - \sum_{\alpha=0}^{\beta} p^{\alpha}_{\nu\mu}}{\mid < \mu > \cup < \nu > \mid - \mid < \mu > \cap < \nu > \mid - \beta}. \quad (5.7)$$

Now we can reject the above assumption that $\varrho_s(p) < \infty$: When coding formula (5.7), we start with setting $\varrho_s(p) = \infty$. If $H_{sk} = \emptyset$ for any $k = 1, 2, \ldots, \lambda,\ k \neq s$ (see Theorem 2), we do not change the initial value of $\varrho_s(p)$ which indicates that the stability radius is infinitely large. Thus formula (5.7) gives the exact value of the stability radius for any optimal digraph $G_s \in P(G)$, including the extreme values 0 and ∞ of $\varrho_s(p)$. Note also that, if only a subset of the processing times (say, $P \subseteq \{p_1, p_2, \ldots, p_q\}$) can be changed, but the other ones cannot, formulas (5.6) and (5.7) remain valid provided that the difference $\mid < \mu > \cup < \nu > \mid - \mid < \mu > \cap < \nu > \mid$ is replaced by the difference $|\{< \mu > \cup < \nu >\} \cap P| - | < \mu > \cap < \nu > \cap P|$.

On the basis of Example 1 (see Section 4), we show that the calculation of $\varrho_s(p)$ may be simplified considerably due to the use of H_{sk} instead of H_s. First we compare the sets H_1 and H_2 (see Section 3). Obviously, the paths $(1,2,5) \in H_1$ and $(1,4,5) \in H_1$ are dominated by the path $(1,4,5,2) \in H_2$. The paths $(3,2,5) \in H_1$ and $(3,4,5) \in H_1$ are dominated by the path $(3,4,5,2) \in H_2$. Thus, we have $H_{1,2} = \emptyset$. Similarly, one can verify that $H_{1,3} = H_{1,4} = \emptyset$.

So for Example 1 only the set $H_{1,5}$ is nonempty: $H_{1,5} = \{(1,2,5),(1,4,5)\}$, and we have to compare the lengths of four pairs of paths. For path $\nu_1 = (3,4,1,2) \in H_5$ and for the two paths from $H_{1,5}$, we have

$$r_{\nu_1} = \min\left\{\frac{100-50}{3}, \frac{100-70}{3}\right\} = 10.$$

For path $\nu_2 = (3,4,5,2) \in H_5$ and for the two paths from $H_{1,5}$, we have

$$r_{\nu_2} = \min\left\{\frac{110-50}{3}, \frac{110-70}{3}\right\} = 13\frac{1}{3}.$$

Thus, we can calculate $r = r_{G_5} = \max\{10, 13\frac{1}{3}\} = 13\frac{1}{3}$, $\nu^* = \nu_2 = (3,4,5,2)$ and $\mu^* = (1,4,5)$. Since $r \leq p_i$ holds for each $i \in \{2,3\} = < \nu^* > \setminus < \mu^* >$, we conclude that $\varrho_1(p) = r = 13\frac{1}{3}$. While the calculation of $\varrho_1(p)$ on the basis of formula (5.7) requires to compare the lengths of four pairs of paths, the calculation of $\varrho_1(p)$ on the basis of formula (13) in [18] requires to consider the lengths of 24 pairs of paths. Moreover, in fact, we use here the simpler formula (5.6) as equality $\varrho_s(p) = r$ on the basis of our earlier remark about the tightness of (5.6). According to (5.4), we can calculate vector $p' = p(r)$:

$$p' = (10 + 13\frac{1}{3}, 20 - 13\frac{1}{3}, 30 - 13\frac{1}{3}, 40 + 13\frac{1}{3}, 20 + 13\frac{1}{3}) = (23\frac{1}{3}, 6\frac{2}{3}, 16\frac{2}{3}, 53\frac{1}{3}, 33\frac{1}{3}),$$

for which we have $l_1^{p'} = l_5^{p'} = 110$, $l_2^{p'} = 116\frac{2}{3}$, $l_3^{p'} = l_4^{p'} = 133\frac{1}{3}$. Thus, $\phi(p') = \{1,5\}$ holds (see condition 1) and for any given small $\epsilon > 0$ we can construct the vector $p^\epsilon = (23\frac{1}{3} + \epsilon, 6\frac{2}{3}, 16\frac{2}{3}, 53\frac{1}{3}, 33\frac{1}{3})$ for which $\phi(p^\epsilon) = \{5\}$. Due to $H_{1,2} = H_{1,3} = H_{1,4} = \emptyset$, it is easy to see that condition 3 is satisfied, too.

6 Computational Results

In this section we give some computational results of the program for calculating the stability radii for all optimal schedules $\phi(p)$ of a problem $G//C_{max}$. The input and output data of the program are as follows.

Input data: The mixed graph $G^0 = (Q, C, D^0)$ with the weights $p = (p_1, p_2, \ldots, p_q)$.

Output data: The number $|P(G)|$ of feasible semi-active schedules; the number $|\phi(p)|$ of semi-active optimal schedules. For each optimal schedule $s \in \phi(p)$, the output data include the stability radius $\varrho_s(p)$, its signature D_s and all signatures D_k, paths $\mu^* \in H_{sk}$ and $\nu^* \in H_k$ at which the stability radius $\varrho_s(p)$ has been reached (see Section 5). If there was more than one optimal schedule, we calculated the minimal, average and maximal differences of their stability radii.

We coded an algorithm based on the formulas (5.6) and (5.7) in FORTRAN. To restrict the number of digraphs $G_k \in P(G)$, with which an optimal digraph G_s has to be compared, we use the simple bounds from [21]. To this end, we compare digraph G_s consecutively with the digraphs G_k from $P(G)$ in non-decreasing order of the objective function values (the makespans). The bound from [21] is used as stopping rule, since, due to this rule, the digraphs with large makespan value need not be considered.

When randomly generating the test instances, we distributed the operations evenly over the machines and then the operations assigned to the same machine have been evenly distributed over the jobs. We considered the following 15 types of job shop problems $J//C_{max}$.

1: $|J^*| = 3, n_1 = n_2 = n_3 = 3, m = 3.$
2: $|J^*| = 3, n_1 = 3, n_2 = 2, n_3 = 4, m = 3.$
3: $|J^*| = 3, n_1 = n_2 = n_3 = 4, m = 4.$
4: $|J^*| = 4, n_1 = n_2 = n_3 = n_4 = 3, m = 6.$
5: $|J^*| = 4, n_1 = n_2 = 3, n_3 = n_4 = 2, m = 4.$
6: $|J^*| = 4, n_1 = 4, n_2 = 3, n_3 = n_4 = 2, m = 6.$
7: $|J^*| = 5, n_1 = n_2 = 4, n_3 = n_4 = 3, n_5 = 2, m = 7.$
8: $|J^*| = 5, n_1 = n_2 = 3, n_3 = n_4 = n_5 = 2, m = 6.$
9: $|J^*| = 6, n_1 = n_2 = n_3 = n_4 = 3, n_5 = n_6 = 2, m = 7.$
10: $|J^*| = 6, n_1 = n_2 = 4, n_3 = n_4 = 3, n_5 = n_6 = 2, m = 8.$
11: $|J^*| = 5, n_1 = 5, n_2 = n_3 = 4, n_4 = n_5 = 3, m = 9.$
12: $|J^*| = 6, n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = 3, m = 8.$
13: $|J^*| = 6, n_1 = n_2 = 5, n_3 = n_4 = 3, n_5 = n_6 = 2, m = 10.$
14: $|J^*| = 7, n_1 = n_2 = n_3 = n_4 = 3, n_5 = n_6 = n_7 = 2, m = 9.$
15: $|J^*| = 7, n_1 = n_2 = n_3 = n_4 = n_5 = 3, n_6 = n_7 = 2, m = 9.$

Here n_i denotes the number of operations per job $J_i \in J^* = \{J_1, J_2, \ldots, J_{|J^*|}\}$. First we generated 100 (pseudo)random instances of each of the types 1 - 10, where the processing times are uniformly distributed real numbers in the segment $[p_{min}, p_{max}] = [10, 1000]$. The results are given in Table 1. For each type of problems (column 1 in Table 1) we calculated the stability radii of all makespan optimal semi-active schedules of all 100 randomly generated instances. The minimal, average and maximal values of $\varrho_s(p)$ among all 100 instances are presented in columns 2, 3 and 4, respectively. Moreover, each calculated stability radius has been divided by $p_{AVE} = (\sum_{i \in Q} p_i)/q$, and the obtained minimal, average and maximal values for the whole series of instances are presented in columns 5, 6 and 7, respectively. If there was more than one makespan optimal schedule for the same instance, we calculated the minimal, average and maximal differences between them. The average and maximal differences among all 100 instances of a series are presented in columns 8 and 9. Note that the minimum of these values was equal to zero for each type of problems 1 - 10. The minimal, average and maximal numbers of feasible semi-active schedules among 100 instances of a series are presented in columns 10, 11 and 12. The average and maximal numbers of optimal schedules are presented in columns 13 and 14. For each of these 10 types, there was at least one instance with only one makespan optimal schedule. Column 15 contains the numbers NPO of instances (among 100 considered in each series) with two or more makespan optimal schedules. For each instance the CPU time, which was necessary for constructing the whole set of feasible semi-active schedules and calculating the stability radii for all optimal schedules, has been measured. The average value of this CPU time (in seconds on a PC 486, 33 MHz) for each type of problems is given in column 16.

T a b l e 1

We can observe that (at least for the considered types of problems) the optimal digraph is often not unique. Only for the problems of types 1 and 3 we obtained a value of NPO less than 50 %. The largest numbers of optimal schedules were obtained for the problems of type 10: On average, each of these instances has more than 16 makespan optimal schedules, and there was an instance even with 66 makespan optimal schedules. So, one can conclude that from a practical point of view, it makes sense to look for a makespan optimal schedule with the largest value of the stability radius. The difference of the stability radii of the optimal schedules of the same problem type may be rather large (see columns 8 and 9). This difference reached the value 301 for an instance of type 8, and on average, for all problem types 1 – 10, this difference was between 30.09 and 56.73.

From columns 2 – 7 it follows that makespan optimal schedules are stable (i.e. they have a strictly positive stability radius), and therefore $\varrho_s(p)$ may be used as a good measure of stability of makespan optimal schedules. It is worth to note that in this simulation study, we never obtained an unstable makespan optimal schedule, i.e., one with zero stability radius. However, in series of types 4, 5 and 9, there were optimal schedules with very small relative values of $\varrho_s(p) : \varrho_s(p)/p_{AVE} = 0.01$.

T a b l e 2

On the basis of problems of types 11 – 15, we investigated the influence of different ranges of the variations of the processing times: $[10, 100], [10, 1000]$ and $[100, 1000]$. To this end we generated 10 problems of each of the types 11 – 15. The results are given in Table 2. Of course, larger processing times imply larger values of the stability radii (see columns 2, 3 and 4 in Table 2), but the relative values of the stability radii do not differ very much for different ranges of the processing times (see columns 5, 6 and 7). Indeed, we have obtained the following segments for the average values of $\varrho_{(p)}/p_{AVE}$: [4.39, 10.05], [4.96, 13.28] and [3.76, 8.16].

F i g u r e 3

In conclusion of the first part of the paper we present in Fig. 3 one randomly generated instance of type 10 (see Table 1), where the processing times p_i are given near the vertices $i \in \{1, 2, \dots, 18\}$ and operations 0 (start) and $*$ (finish) are fictitious. To give an impression on the above simulation study, we present the following additional information for the mixed graph which is drawn in Fig. 3. This instance has 1728 feasible semi-active schedules, 12 of them are makespan optimal: 4 optimal schedules have the smallest stability radius 9.99, 4 other optimal schedules have the stability radius 14.86, and the other optimal schedules have the stability radii 25.67, 31.08, 58.16 and 63.57, respectively. The optimal value of the objective function (makespan) for this instance is equal to 2886.47. The largest value of the objective function among all feasible semi-active schedules is equal to 6750.76. While calculating all 12 stability radii for this instance, we considered about 29 % of all feasible semi-active schedules due to the mentioned bounds from [21]. However, it is worth to note that to find the exact values of the stability radii (without guarantee that these are indeed the exact values of the stability radii), it was sufficient to consider only 0.52 % of all feasible semi-active schedules. The construction of the whole set of feasible semi-active schedules and the calculation of the stability radii for all optimal schedules for this instance took 9.78 seconds.

We can note that the time needed to calculate the stability radii for all optimal schedules for an instance increases exponentially with the number of edges in the corresponding mixed graph. In our experiments, for different types of problems the average CPU time on a PC 486 (33 MHz) for such a calculation varied from 0.11 seconds for instances of type 6 up to 46.92 seconds for instances of type 11. However, the bottleneck for considering problem types of larger size (using only internal memory) is connected with the required number of dominant paths to be considered, i.e., with the required internal memory of the computer. A possible way to overcome this memory restriction is to generate the dominant paths of the feasible digraphs systematically and to consider them one by one, without storing them.

7 Stability Analysis of an ϵ-Approximate Solution

As follows from the above theoretical and computational results (see also [21]), it is possible to find the exact value of $\varrho_s(p)$ for general shop problems only with very small dimensions since the used formulas (5.6) and (5.7) are based on a direct comparison of dominant paths in the set (or in the subset) of feasible digraphs $P(G)$. To avoid such an enormous enumeration, we shall now consider rather simple scheduling problems (essentially simpler than $G//C_{max}$, but still NP-hard in the general case), for which the set of dominant paths H_s of each solution s consists of a unique path. Moreover, instead of the 'min-max' criterion, considered in Sections 1 – 6, we shall consider the 'min-sum' criterion.

More precisely, in the remainder of the paper we restrict our attention to those scheduling problems, which may be represented in terms of linear binary programming [3, 6, 11] (or, similarly, in terms of a linear trajectory problem [4, 5, 19]). A concept of stability analysis for the latter problem has been developed in [4, 5, 10, 11, 12, 24] and in some other papers (see [20] for the extensive survey). It should be noted that most results have been obtained for the stability radius of the whole set of optimal trajectories, i.e., for the largest radius of an open ball in the space of the numerical input data such that a new optimal trajectory does not arise. Unfortunately, the set of all optimal trajectories is often unknown since its cardinality may be large. Even if the optimal trajectory is unique for the problem, this information is usually inaccessible for OR workers. On the other hand, the investigation of the stability radius of one optimal trajectory of such a problem has the following drawback: The stability radius of an optimal solution of a linear trajectory problem is equal to zero, if at least one alternative optimal solution exists. Therefore, in [3, 6, 19] the stability of an ϵ-approximate solution has been investigated.

In Sections 7 - 14 we survey known stability results for ϵ-approximate solutions, and prove some new ones. For simplicity, we use here the notations from [3, 6, 19, 20] which are more suitable for linear binary programming. However, we try to keep most notations (for the criterion, the stability radius and the variable data) close to those used in the first part of the paper. We hope that this will not imply any ambiguity for the reader since we do not use cross-references between the first and second parts (with the exception of Section 15, which contains some concluding remarks).

Let $N = \{1, 2, \ldots, n\}$ and $X \subseteq \{0, 1\}^n$ be the set of all feasible vectors (feasible solutions). For a given vector $p \in R^n$ and a feasible solution $x \in X$, let $\Phi(p, x) = \sum_{i \in N} p_i x_i$ be the objective function. We assume that the set X of feasible solutions is finite and does not depend on the vector $p = (p_1, p_2, \ldots, p_n)$ of objective coefficients. For brevity, we shall call p the objective vector.

The linear binary programming problem under consideration is to find an optimal vector (optimal solution) $x^p = (x_1^p, x_2^p, \dots, x_n^p) \in X$ with

$$\Phi(p, x^p) = \min\{\Phi(p, x) \mid x \in X\}. \tag{7.1}$$

We investigate problem (7.1) under the assumption that all or a subset of the objective coefficients $p_1, p_2, \dots, p_n$ can change their values in comparison with the initial ones. We first consider the stability region and the stability ball of an ϵ-approximate solution of problem (7.1), where $\epsilon \geq 0$. Let $x \in X$ be an ϵ-approximate solution of the problem (7.1), i.e., the condition

$$\Phi(p, x) \leq (1 + \epsilon) \cdot \Phi(p, x^p) \tag{7.2}$$

holds. We investigate the situation when w given components of the objective vector p, $1 \leq w \leq n$, can be changed after solving problem (7.1) but the remaining $n - w$ components of p cannot be changed. Without loss of generality we assume that the first w components $p_1, p_2, \dots, p_w$ of the objective vector p can be changed (unstable components). Hence, the values of the objective coefficients $p_{w+1}, p_{w+2}, \dots, p_n$ are reliable, but the values of the objective coefficients $p_1, p_2, \dots, p_w$ can change after solving problem (7.1).

From now on, we shall consider only objective vectors $p \in R^n$, where the last $n - w$ components are equal to given values, say $p_{w+1} = \bar{p}_{w+1},\ p_{w+2} = \bar{p}_{w+2}, \dots,\ p_n = \bar{p}_n$, and for such objective vectors we define $\tilde{p} = (p_1, p_2, \dots, p_w) \in R^w$ as the vector of its first w components.

Definition 3 *The set of all objective vectors $\tilde{p} \in R^w$, for which $x \in X$ is an ϵ-approximate solution, is called the stability region of x and it is denoted by $K_\epsilon^w(x)$:*

$$K_\epsilon^w(x) = \{\tilde{p} \in R^w \mid \Phi(p, x) \leq (1 + \epsilon) \cdot \Phi(p, x^p)\}. \tag{7.3}$$

Definition 4 *The closed ball $O_\varrho^w(p)$ with radius ϱ and vector $\tilde{p}$ as centre is called a stability ball of ϵ-approximate solution $x \in X$ if $O_\varrho^w(p) \cap R^w \subseteq K_\epsilon^w(x)$.*

Definition 5 *Let $x \in X$ be an ϵ-approximate solution with respect to the objective vector p. The radius ϱ of the largest stability ball $O_\varrho^w(p)$ of x is called the stability radius of x and it is denoted by $\varrho_\epsilon^w(x, p)$.*

In the next section we prove some simple properties of the set $K_\epsilon^w(x)$.

8 The Stability Region

For every $x \in X$, define $\tilde{x} = (x_1, x_2, \dots, x_w)$ as the vector of its first w components. Furthermore, define $\tilde{X} = \{\tilde{x} \in X^w \mid x \in X\}$, and let U be a reflexive binary relation on the set $\tilde{X}$ with maximal cardinality such that $(\tilde{x}, \tilde{x}') \in U$ if and only if for all $i \in \{1, 2, \dots, w\}$ from the equality $x_i = 1$ the equality $x_i' = 1$ follows. In other words, we have $(x, x') \in U$ if and only if the set of indices of the variables having the value 1 in x is a subset of those in x'. Due to the maximal cardinality, the relation U is uniquely determined for each $\tilde{X} \subseteq \{0, 1\}^w$. In the following, we use the vectors $\tilde{x}' = (x_1', x_2', \dots, x_w')$ and $x' = (x_1', x_2', \dots, x_w', x_{w+1}', \dots, x_n')$.

Define $K_\epsilon^w = \bigcap_{x \in X} K_\epsilon^w(x)$. Obviously, K_ϵ^w contains all objective vectors p that are contained in the stability region of every feasible solution $x \in X$. Using the above notations, we can formulate and then prove the following assertions about the stability region $K_\epsilon^w(x)$ with $x \in X, w \leq n$ and $\epsilon \geq 0$.

Property 1: The stability region $K_\epsilon^w(x)$ and the set K_ϵ^w are polyhedra.

Property 2: If $\epsilon_1 < \epsilon_2$, then $K_{\epsilon_1}^w(x) \subseteq K_{\epsilon_2}^w(x)$.

Property 3: If the inclusion $(\tilde{x}, \tilde{x}') \in U$ and the inequality

$$\sum_{i=w+1}^{n} t_i x_i \leq \sum_{i=w+1}^{n} t_i x_i' \tag{8.1}$$

hold, then $K_\epsilon^w(x') \subseteq K_\epsilon^w(x)$.

Property 4: We have $\bigcup_{x \in X} K_\epsilon^w(x) = R^w$.

Property 5: At least one of the regions $K_\epsilon^w(x), x \in X$, is unbounded.

Proof: First we show that $K_\epsilon^w(x)$ can be represented as the set of solutions of a system of linear inequalities. From (7.1) and (7.3) it follows that the set $K_\epsilon^w(x)$ contains all objective vectors p which satisfy the inequality

$$\sum_{i \in N} p_i x_i \leq (1+\epsilon) \cdot \min \left\{ \sum_{i \in N} p_i x_i' \mid x' \in X \right\}. \tag{8.2}$$

Consequently, inequality (8.2) can be written as

$$\begin{aligned} \sum_{i=1}^{w} p_i[x_i - (1+\epsilon)x_i'] \quad &\leq \quad b', \qquad x' \in X, \\ p_i \quad &\geq \quad 0, \qquad i = 1, 2, \ldots, w, \end{aligned} \tag{8.3}$$

where

$$b' = (1+\epsilon) \cdot \sum_{i=w+1}^{n} \bar{p}_i x_i' - \sum_{i=w+1}^{n} \bar{p}_i x_i.$$

Hence, the stability region $K_\epsilon^w(x)$ is given as the set of solutions of system (8.3) of linear inequalities with the variable objective coefficients $p_1, p_2, \ldots, p_w$. Thus, $K_\epsilon^w(x)$ is a polyhedron (note that $K_\epsilon^w(x) = \emptyset$ is possible). K_ϵ^w is also a polyhedron because it is the intersection of polyhedra. Thus, Property 1 holds.

The validity of Property 2 immediately follows from the definition of $K_\epsilon^w(x)$. Indeed, if the inequality $\Phi(p, x) \leq (1+\epsilon) \cdot \Phi(p, x^p)$ holds for $\epsilon = \epsilon_1$, then it is also satisfied for each $\epsilon = \epsilon_2$ with $\epsilon_1 < \epsilon_2$.

Let the conditions of Property 3 be satisfied, objective vector $\tilde{p}^0 \in K_\epsilon^w(x')$ be arbitrary and p^0 be equal to $(p_1^0, p_2^0, \ldots, p_w^0, \bar{p}_{w+1}, \ldots, \bar{p}_n)$. We consider the value

$$\Phi(p^0, x) = \sum_{i=1}^{w} p_i^0 x_i + \sum_{i=w+1}^{n} \bar{p}_i x_i.$$

Because of $(\tilde{x}, \tilde{x}') \in U$, the inequality

$$\sum_{i=1}^{w} p_i^0 x_i \leq \sum_{i=1}^{w} p_i^0 x_i'$$

holds. Considering (8.1), we obtain $\Phi(p^0, x) \leq \Phi(p^0, x') \leq (1+\epsilon) \cdot \Phi(p^0, x^{p^0})$, i.e., the vector x is an ϵ-approximate solution of problem (7.1) for any objective vector $\tilde{p}^0$ from the region $K_\epsilon^w(x')$. Hence, $K_\epsilon^w(x') \subseteq K_\epsilon^w(x)$ holds which proves Property 3.

For the proof of Property 4 we note that, since the set X is finite, there exists an ϵ-approximate solution of problem (7.1) for any given $p \in R^w$ and any given $\epsilon \geq 0$. Especially, we can take the vector x^p, automatically satisfying inequality (7.2), as ϵ-approximate solution of problem (7.1).

Property 4 means that for any $\epsilon \geq 0$, there exists a finite covering of the space R^w by stability regions $K_\epsilon^w(x)$. Because we have a finite number of regions in the set $\{K_\epsilon^w(x) \mid x \in X\}$ but R^w is unbounded, at least one stability region must be unbounded, i.e., Property 5 holds. ■

Property 1 can be strengthened for the important special case when $w = n$ (considered in [6]).

Property 6: The set $K_\epsilon^n(x)$ is a closed convex cone with the origin in $(p_1 = 0, p_2 = 0, \ldots, p_n = 0) \in R^n$.

Proof: Indeed, for any $\lambda \geq 0$ we obtain from (7.2), $\Phi(\lambda p, x) = \lambda \Phi(p, x) \leq (1+\epsilon) \cdot \lambda \Phi(p, x^p) \leq (1+\epsilon) \cdot \Phi(\lambda p, x^{\lambda p})$. Hence, we have $\lambda p \in K_\epsilon^n(x)$ for $\lambda \geq 0$ which proves Property 6. ■

As we shall illustrate in the following example, the set $K_\epsilon^w(x)$ (and, consequently, in general the set K_ϵ^w) can be empty. However, in the case $w = n$, due to Property 6 any set $K_\epsilon^w(x)$ is non-empty: In fact, the set K_ϵ^n is not $\emptyset$ because the n-dimensional vector $(p_1 = 0, p_2 = 0, \ldots, p_n = 0)$ belongs to K_ϵ^n for any $\epsilon \geq 0$.

9 Example 2

A broad class of discrete optimization problems can be formulated as a linear binary programming problem, e.g., a one-machine scheduling problem with sequence dependent setup times between the processing of the jobs which is equivalent to the traveling salesman problem that was, by the way, the first linear trajectory problem for which a formula for calculating the stability radius of the whole set of optimal trajectories has been derived [10].

Here we consider as an illustrative example the scheduling problem of minimizing the sum of processing times of the jobs on parallel (but not identical) machines. Assume that we have m machines that have to process l jobs, where the processing times do not depend on the jobs but on the machine and on the processing order. The matrix P gives the processing times $p_{ij} \geq 0$ if a job is the j-th job in the job order of machine i. The problem is to assign all l jobs to the machines such that the sum of processing times becomes minimal.

Let $m = 2,\ l = 3$ and the matrix $P = [p_{ij}]_{(m,l)}$ of the initial data be as follows:

$$P = \begin{bmatrix} 10 & 5 & 5 \\ 7 & 7 & 7 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \end{bmatrix}.$$

We set $p = (p_1, p_2, \ldots, p_6)$ and $n = m \cdot l$. Let $x_{ij} = 1$ if at least j jobs are processed on machine i and $x_{ij} = 0$ otherwise. Analogously, we denote

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}.$$

The set X of feasible vectors for this problem can be described as follows:

$$\begin{aligned} X &= \Big\{ x \in X^n \mid \sum_{i=1}^{n} x_i = l;\ [x_i = 1,\ lu < i \leq l(u+1),\ u \geq 0,\ u \text{ is integer}] \\ &\quad \Rightarrow [x_{lu+1} = x_{lu+2} = \ldots = x_{i-1} = 1] \Big\}. \end{aligned}$$

For the given values of m and l, we have four feasible vectors:

$$\begin{aligned} X &= \{x^1 = (1,1,1,0,0,0), x^2 = (1,1,0,1,0,0), \\ &\quad x^3 = (1,0,0,1,1,0), x^4 = (0,0,0,1,1,1)\}. \end{aligned}$$

First, we determine the stability regions $K_\epsilon^2(x^i)$ for $i \in \{1,2,3,4\}$ with $p = (p_1, p_2, 5, 7, 7, 7)$ and $\epsilon = 0$. For $x = x^1$, system (7.3) is as follows:

$$\begin{aligned} p_1 + p_2 + 5 &\leq p_1 + p_2 + 7 \\ p_1 + p_2 + 5 &\leq p_1 + 14 \\ p_1 + p_2 + 5 &\leq 21 \\ p_1 &\geq 0 \\ p_2 &\geq 0 \end{aligned}$$

The above system is equivalent to

$$\begin{aligned} p_2 &\leq 9 \\ p_1 + p_2 &\leq 16 \\ p_1 &\geq 0 \\ p_2 &\geq 0 \end{aligned}$$

For $x = x^2$, system (7.3) is inconsistent:

$$\begin{aligned} p_1 + p_2 + 7 &\leq p_1 + p_2 + 5 \\ p_1 + p_2 + 7 &\leq p_1 + 14 \\ p_1 + p_2 + 7 &\leq 21 \\ p_1 &\geq 0 \\ p_2 &\geq 0 \end{aligned}$$

For $x = x^3$, system (7.3) is equivalent to:

$$\begin{aligned} p_2 &\geq 9 \\ p_1 &\leq 7 \\ p_1 &\geq 0 \end{aligned}$$

For $x = x^4$, system (7.3) is equivalent to:

$$\begin{aligned} p_1 + p_2 &\geq 16 \\ p_1 &\geq 7 \\ p_2 &\geq 0 \end{aligned}$$

The above stability regions are given in Fig. 4. They form a covering of the space R^w with $w = 2$. Note that $K_0^2(x^2) = \emptyset$ and, consequently, $K_0^2 = \emptyset$. Both regions $K_0^2(x^3)$ and $K_0^2(x^4)$ are unbounded.

Figure 4

Analogously, we determine the stability regions $K_\epsilon^2(x^i)$ for $i \in \{1, 2, 3, 4\}$ with $p = (p_1, p_2, 5, 7, 7, 7)$ and $\epsilon = 0.5$. It is easy to see that the regions $K_{0.5}^2(x^1), K_{0.5}^2(x^2), K_{0.5}^2(x^3)$ and $K_{0.5}^2(x^4)$, respectively, are polyhedra given by the solutions of the following systems of linear inequalities:

$$\begin{aligned} 1)\quad p_1 - 0.5p_2 &\leq 16 \\ p_1 + p_2 &\leq 26.5 \\ p_1 &\geq 0 \\ p_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} 2)\quad -0.5p_1 + p_2 &\leq 14 \\ p_1 + p_2 &\leq 24.5 \\ p_1 &\geq 0 \\ p_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} 3)\quad p_1 + 3p_2 &\geq 13 \\ p_1 &\leq 17.5 \\ p_1 &\geq 0 \\ p_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} 4)\quad p_1 + p_2 &\geq 9 \\ p_1 &\geq 0 \\ p_2 &\geq 0 \end{aligned}$$

These stability regions are shown in Fig. 5. The bounds of these regions are shaded in the same way as in Fig. 4, only the stability region of x^2 additionally occurs in a new form of shading. For the regions $K_{0.5}^2(x') = K_{0.5}^2(x^2)$ and $K_{0.5}^2(x) = K_{0.5}^2(x^1)$ the conditions of Property 3 are satisfied. Hence, $K_{0.5}^2(x') \subseteq K_{0.5}^2(x)$ holds.

Figure 5

We note that the polyhedron $K^2_{0.5}$ (where each feasible solution is an ϵ-approximate solution with $\epsilon = 0.5$) is given by the set of solutions of the following system:

$$\begin{aligned} -0.5p_1 + p_2 &\leq 14 \\ p_1 + p_2 &\leq 24.5 \\ p_1 + p_2 &\geq 9 \\ p_1 + 3p_2 &\geq 13 \\ p_1 &\leq 17.5 \\ p_1 &\geq 0 \\ p_2 &\geq 0 \end{aligned}$$

Fig. 6 illustrates Property 2 if $p = (p_1, p_2, 5, 7, 7, 7), w = 2, x = x^1$ and $\epsilon \in \{0,\ 0.5,\ 1\}$.

F i g u r e 6

Obviously, the determination of the stability regions is possible only for special problems of the type (7.1) with a very small number of variables. Moreover, for $w > 3$ we need a special form even to draw the stability region in the plane.

10 An Upper Bound on $\varrho^w_\epsilon(x,p)$

Similarly to Section 3, if the ball $O^w_\varrho(p)$ is a stability ball of an ϵ-approximate solution x for any positive ϱ, then we write $\varrho^w_\epsilon(x,p) = \infty$. In the following we strengthen an upper bound for the stability radius given in [19] by using the following result about the existence of an infinitely large stability radius of an ϵ-approximate solution.

Theorem 4 *The stability radius $\varrho^w_\epsilon(x,p)$ equals ∞ if and only if for any vector $x' \in X$ the condition $(\tilde{x}, \tilde{x}') \in U$ and the inequality*

$$\sum_{i=w+1}^{n} p_i x_i \leq (1+\epsilon) \cdot \sum_{i=w+1}^{n} p_i x'_i \qquad (10.1)$$

hold.

The proof of Theorem 4 can be found in [3] and [19]. Applying this theorem, we obtain the following bound on $\varrho^w_\epsilon(x,p)$.

Theorem 5 *If $\varrho^w_\epsilon(x,p) < \infty$, then we have $\varrho^w_\epsilon(x,p) \leq p_\epsilon$, where*

$$p_\epsilon = \max\left\{ \max\{p_i \mid 1 \leq i \leq w\},\ (1+\epsilon) \cdot \sum_{i=w+1}^{n} p_i - \sum_{i=w+1}^{n} p_i x_i \right\}. \qquad (10.2)$$

Proof: Due to Theorem 4, one of the following conditions must be satisfied because of $\varrho_\epsilon^w(x,p) < \infty$:

(i) there exists a vector $x' \in X$ such that $(\tilde{x}, \tilde{x}') \notin U$ or

(ii) inequality (10.1) does not hold.

First we consider the case (i), i.e., there exists an index j with $1 \le j \le w$ such that $x_j = 1$ and $x'_j = 0$. We consider the following vector $p' = (p'_1, p'_2, \ldots, p'_n) \in R^n$. Here we set $p'_j = p'_\epsilon > p_\epsilon$, where p_ϵ is given by (10.2). For each i with $1 \le i \le w$ and $i \ne j$, we set $p'_i = 0$. For the stable components of the vector p' we have, of course, $p'_i = p_i$ $(w+1 \le i \le n)$.

Now we evaluate $\Phi(p', x)$ for the vector p':

$$
\begin{aligned}
\Phi(p', x) &= \sum_{i \in N} p'_i x_i \\
&= \sum_{i=1}^{m} p'_i x_i + \sum_{i=w+1}^{n} p'_i x_i \\
&= p'_\epsilon + \sum_{i=w+1}^{n} p'_i x_i \\
&= p'_\epsilon + \sum_{i=w+1}^{n} p_i x_i \\
&> (1+\epsilon) \cdot \sum_{i=w+1}^{n} p_i \\
&= (1+\epsilon) \cdot \sum_{i=w+1}^{n} p'_i \\
&\ge (1+\epsilon) \cdot \sum_{i=1}^{w} p'_i x'_i + (1+\epsilon) \cdot \sum_{i=w+1}^{n} p'_i x'_i \\
&= (1+\epsilon) \cdot \sum_{i=1}^{n} p'_i x'_i \\
&\ge (1+\epsilon) \cdot \Phi(p', x^{p'}).
\end{aligned}
$$

The first of the above inequalities holds because of $p'_\epsilon > p_\epsilon$ and the second inequality follows from

$$\sum_{i=1}^{w} p'_i x'_i = 0 \quad \text{and} \quad \sum_{i=w+1}^{n} p_i \ge \sum_{i=w+1}^{n} p_i x'_i.$$

Consequently, the vector x is not an ϵ-approximate solution of problem (7.1) for objective vector $p = p'$. Therefore, we have $\varrho_\epsilon^w(x,p) < r(p,p') \le p'_\epsilon$. Obviously, the value of p'_ϵ can be arbitrarily close to p_ϵ with $p_\epsilon < p'_\epsilon$ and the inequality $\varrho_\epsilon^w(x,p) < r(p,p')$ is satisfied. Thus, if condition (i) holds, we have the inequality $\varrho_\epsilon^w(x,p) \le p_\epsilon$.
Let condition (ii) be satisfied, i.e., there exists a vector $x' \in X$ for which the inequality

$$\sum_{i=w+1}^{n} p_i x_i > (1+\epsilon) \cdot \sum_{i=w+1}^{n} p_i x'_i$$

holds. Then we consider the objective vector $p'' = (p''_1, p''_2, \ldots, p''_n)$ with the components

$$p''_i = \begin{cases} 0 & \text{for } i = 1, 2, \ldots, w, \\ p_i & \text{for } i = w+1, w+2, \ldots, n. \end{cases}$$

Now we evaluate $\Phi(p'', x)$:

$$\begin{aligned} \Phi(p'', x) &= \sum_{i \in N} p''_i x_i \\ &= \sum_{i=w+1}^{n} p_i x_i \\ &> (1+\epsilon) \cdot \sum_{i=w+1}^{n} p_i x'_i \\ &= (1+\epsilon) \cdot \sum_{i=1}^{n} p''_i x'_i \\ &\geq (1+\epsilon) \cdot \min \left\{ \sum_{i=1}^{n} p''_i x''_i \mid (x''_1, x''_2, \ldots, x''_w, x''_{w+1}, \ldots, x''_n) \in X \right\} \\ &= (1+\epsilon) \cdot \Phi(p'', x^{p''}). \end{aligned}$$

Hence, the vector $x \in X$ is not an ϵ-approximate solution of the problem (7.1) for objective vector $p = p''$ and $\varrho^w_\epsilon(x, p) \leq r(p, p'') \leq \max\{p_i \mid 1 \leq i \leq w\}$. Thus, the theorem has been proved. ■

From Theorem 5 we immediately obtain the following corollary.

Corollary 1 *If $\varrho^n_\epsilon(x, p) < \infty$, then $\varrho^n_\epsilon(x, p) \leq p_* = \max\{p_i \mid 1 \leq i \leq n\}$.*

It is easy to see that the upper bound $\varrho^n_\epsilon(x, p) \leq p_*$ is tight and that the bound

$$\varrho^n_\epsilon(x, p) \leq (1+\epsilon) \cdot \sum_{i=w+1}^{n} p_i - \sum_{i=w+1}^{n} p_i x_i$$

is tight under the condition that the vector $(x_1, x_2, \ldots, x_w, 1, 1, \ldots, 1)$ is feasible.

While the upper bound of $\varrho^w_\epsilon(x, p)$ is implied from Theorem 4 about an infinite stability radius, a lower bound will be implied from the claims about the zero stability radius considered in the next section.

11 A Zero Value of $\varrho^w_\epsilon(x, p)$

Necessary and sufficient conditions for $\varrho^w_\epsilon(x, p) = 0$ have been given in [19], which are valid only for $\tilde{x} \neq \tilde{0}$, where $\tilde{0}$ is the zero vector in R^w. Here we prove separately criteria for the case $\tilde{x} \neq \tilde{0}$ as well as $\tilde{x} = \tilde{0}$.

Theorem 6 *Let $\tilde{p} \in K_\epsilon^w(x)$, $\tilde{x} \neq \tilde{0}$, $w \leq n$, $\epsilon > 0$ and $p_i > 0$ for each $i = 1, 2, \ldots, w$. Then we have $\varrho_\epsilon^w(x,p) = 0$ if and only if condition (7.2) is satisfied as equality.*

Proof: a) Sufficiency: Let the condition (7.2) be satisfied as equality, i.e.,

$$\Phi(p,x) = (1+\epsilon) \cdot \Phi(p, x^p). \tag{11.1}$$

Since $\tilde{x} \neq \tilde{0}$ and $p_i > 0$ for each $i = 1, 2, \ldots, w$, the value $\Phi(p,x)$ is not equal to zero. Because we have $\epsilon > 0$ in (7.1), we conclude that vector x is not a solution of the problem (7.1). Then we take any nonzero component of $\tilde{x}$, say $\tilde{x}_i = 1$, $i \in \{1, 2, \ldots, w\}$, and consider the following two possible cases:
Case (i): $x_i^p = 0$. Let the positive real number $\gamma > 0$ be arbitrarily small. Then the objective vector $p^\gamma = (p_1, p_2, \ldots, p_{i-1}, p_i + \gamma, p_{i+1}, \ldots, p_w, p_{w+1}, \ldots, p_n) \in R^n$ satisfies the following conditions:

$$\begin{aligned}
\Phi(p^\gamma, x) &= \gamma + \Phi(p,x) \\
&= \gamma + (1+\epsilon) \cdot \Phi(p, x^p) \\
&= \gamma + (1+\epsilon) \cdot \Phi(p^\gamma, x^p) \\
&\geq \gamma + (1+\epsilon) \cdot \Phi(p^\gamma, x^{p^\gamma}) \\
&> (1+\epsilon) \cdot \Phi(p^\gamma, x^{p^\gamma}).
\end{aligned}$$

Thus, we have $\tilde{p}^\gamma \notin K_\epsilon^w(x)$ and, because of $r(\tilde{p}, \tilde{p}^\gamma) = \gamma$, the set $O_\gamma^w(p) \cap R^w$ is not a subset of $K_\epsilon^w(x)$. Consequently, the ball $O_\gamma^w(p)$ is not a stability ball of the ϵ-approximate solution x. Since this holds for any arbitrarily small $\gamma > 0$, we have $\varrho_\epsilon^w(x,p) = 0$.
Case (ii): $x_i^p = 1$. Let us consider the objective vector $p^{(\gamma)} = (p_1, \ldots, p_{i-1}, p_i - \gamma, p_{i+1}, \ldots, p_w, p_{w+1}, \ldots, p_n)$. If $0 < \gamma \leq p_i$, then $p^{(\gamma)} \in R^w$ due to $p_i > 0$. Since $\epsilon > 0$, the objective vector $p^{(\gamma)}$ satisfies the following relations:

$$\begin{aligned}
\Phi(p^{(\gamma)}, x) &= \Phi(p,x) - \gamma \\
&= (1+\epsilon) \cdot \Phi(p, x^p) - \gamma \\
&> (1+\epsilon) \cdot \Phi(p, x^p) - (1+\epsilon) \cdot \gamma \\
&= (1+\epsilon) \cdot \Phi(p^{(\gamma)}, x^p) \\
&\geq (1+\epsilon) \cdot \Phi(p^{(\gamma)}, x^{p^{(\gamma)}}).
\end{aligned}$$

Thus, we have $p^{(\gamma)} \notin K_\epsilon^w(x)$ and the ball $O_\gamma^w(p)$ is not a stability ball of the ϵ-approximate solution x. Because this holds for any arbitrarily small $\gamma > 0$, we conclude again that $\varrho_\epsilon^w(x,p) = 0$.

b) Necessity will be proved by contradiction: Assume that $\varrho_\epsilon^w(x,p) = 0$ and that condition (7.2) is satisfied as a strict inequality, i.e., we have

$$(1+\epsilon) \cdot \Phi(p, x^p) - \Phi(p,x) = \Delta_p > 0. \tag{11.2}$$

We show that there exists a real number $\varrho > 0$ such that inequality (11.2) remains valid for any objective vector $p' \in R^n$ with $\tilde{p}' \in O_\varrho^w(p) \cap R^w$. We set $\varrho = \beta = \Delta_p/(3+\epsilon)w > 0$. Because of $r(\tilde{p}, \tilde{p}') \leq \beta$, we obtain

$$\Phi(p', x) - \Phi(p,x) \leq w \cdot \beta = \Delta_p/(3+\epsilon) \tag{11.3}$$

and

$$(1+\epsilon)\cdot\Phi(p,x^p)-(1+\epsilon)\cdot\Phi(p',x^{p'}) \;\leq\; (1+\epsilon)\cdot\Phi(p,x^{p'})-(1+\epsilon)\cdot\Phi(p',x^{p'}) \\ \leq\; w\cdot\beta\cdot(1+\epsilon)=\Delta_p\cdot(1+\epsilon)/(3+\epsilon). \tag{11.4}$$

We summarize the left and the right terms of the inequalities (11.3) and (11.4):

$$\begin{aligned} & \Phi(p',x)-\Phi(p,x)+(1+\epsilon)\cdot\Phi(p,x^p)-(1+\epsilon)\cdot\Phi(p',x^{p'}) \\ \leq\; & \Delta_p/(3+\epsilon)+\Delta_p\cdot(1+\epsilon)/(3+\epsilon) \\ =\; & \Delta_p\cdot(2+\epsilon)/(3+\epsilon). \end{aligned}$$

From this we get

$$\begin{aligned} (1+\epsilon)\cdot\Phi(p',x^{p'})-\Phi(p',x) \;&\geq\; (1+\epsilon)\cdot\Phi(p,x^p)-\Phi(p,x)-\Delta_p\cdot(2+\epsilon)/(3+\epsilon) \\ &=\; \Delta_p-\Delta_p\cdot(2+\epsilon)/(3+\epsilon) \\ &=\; \Delta_p/(3+\epsilon)>0. \end{aligned}$$

Thus, $\tilde{p}'\in O^w_\beta(p)\cap R^w$ implies $\tilde{p}'\in K^w_\epsilon(x)$, i.e. the ball $O^w_\beta(p)$ is a stability ball of the ϵ-approximate solution x. We obtain a contradiction to the assumption: $\varrho^w_\epsilon(x,p)\geq\beta=\Delta_p/(3+\epsilon)w>0$. ■

Next we prove the analogy to Theorem 6 for the case $\tilde{x}=\tilde{0}$.

Theorem 7 *Let $\tilde{p}\in K^w_\epsilon(x)$, $\tilde{x}=\tilde{0}$, $w\leq n$, $\epsilon>0$ and $\tilde{p}_i>0$ for each $i=1,2,\ldots,w$. Then we have $\varrho^w_\epsilon(x,p)=0$ if and only if condition (7.2) is satisfied as an equality and there exists an optimal solution x^p of problem (7.1) for which $\tilde{x}^p\neq 0$.*

Proof: a) Sufficiency: Since $\tilde{x}^p\neq 0$, there exists a nonzero component $x^p_i=1$, $i\in\{1,2,\ldots,w\}$. As in case (ii) in the proof of Theorem 6, we consider the objective vector $p^{(\gamma)}$ and by using arguments quite similar as before we obtain $\varrho^w_\epsilon(x,p)=0$.

b) Necessity: Let $\varrho^w_\epsilon(x,p)=0$. The case when condition (7.2) is satisfied as a strict inequality can be considered similarly as in the proof of the necessity of Theorem 6. To complete the proof, we still have to consider the situation when $\Delta_p=0$ holds (where Δ_p is as defined in (11.2)) but for any solution x^p of problem (7.1) we have $\tilde{x}^p=0$. Since $x_i=x^p_i=0$ for any $i\in\{1,2,\ldots,w\}$, due to the equality

$$\Phi(p,x)=(1+\epsilon)\cdot\Phi(p,x^p), \tag{11.5}$$

we have

$$\Phi(p,x)=\sum_{i=w+1}^{n}p_ix_i=(1+\epsilon)\sum_{i=w+1}^{n}p_ix^p_i<(1+\epsilon)\cdot\sum_{i=1}^{n}p_ix'_i=(1+\epsilon)\cdot\Phi(p,x').$$

for any vector $x'\in X$ which is not a solution of problem (7.1). Now we calculate the value

$$\Delta_0=\min\{(1+\epsilon)\cdot\Phi(p,x')-\Phi(p,x)\mid x'\in X\backslash\{x\},\ \Phi(p,x')>\Phi(p,x^p)\}>0.$$

Setting $\beta = \beta_0 = \Delta_0/(3+\epsilon)w > 0$ and repeating the steps performed for the case $\Delta_p > 0$ in the proof of the necessity of Theorem 6, we obtain the inequality

$$(1+\epsilon)\cdot\Phi(p',x') - \Phi(p',x) > 0 \tag{11.6}$$

which holds for any objective vector $\tilde{p}' \in O^w_\beta(p) \cap R^w$.

Due to $\tilde{x} = \tilde{0}$ and $x^p = 0$ for any solution x^p of problem (7.1), any variation of the objective vector $\tilde{p}$ does not affect the validity of equation (11.5). Thus, for any objective vector $\tilde{p}'$ both inequality (11.6) for any vector $x' \in X$ which is not a solution of problem (7.1) and inequality (11.5) (with p replaced by p') for any solution x^p of problem (7.1) are valid. We can conclude that $O^w_\beta(p)$ is a stability ball of the ϵ-approximate solution x which contradicts to the assumption that the stability radius of x is equal to zero.

■

In the above proofs we have actually derived a lower bound on the stability radius of an ϵ-approximate solution. It is presented in the next section, along with a lower bound on the stability radius of an optimal solution.

12 Lower Bounds on $\varrho^w_\epsilon(x,p)$

While proving the necessity of Theorem 6 and the necessity of Theorem 7 for the case $\Delta_p > 0$, we obtained the following lower bound for the value $\varrho^w_\epsilon(x,p)$.

Corollary 2 *Let $\tilde{p} \in K^w_\epsilon(x)$, $w \le n$, $\epsilon \ge 0$. If condition (7.2) is satisfied as a strict inequality, then we have $\varrho^w_\epsilon(x,p) \ge \Delta_p/(3+\epsilon)w > 0$.*

Note that any strictly positive lower bound can be used as a conservative estimate of the stability radius, i.e., $O^w_\varrho(p)$ is a stability ball if $\varrho > 0$ is equal to a lower bound on $\varrho^w_\epsilon(x,p)$. In particular, one may be interested in lower bounds on the stability radius of an optimal solution x^p. For optimal solutions we have the following lower bound, which is better than the one given in Corollary 2.

Theorem 8 *Let x^p be an optimal solution of problem (7.1) and let $v(x^p)$ denote the number of 1's in vector $\tilde{x}^p$, then*

$$\varrho^w_\epsilon(x^p,p) \ge \min\left\{\frac{\Delta_p}{(1+\epsilon)\cdot w - \epsilon\cdot v(x^p)}, \frac{\Delta_p}{(1+\epsilon)\cdot w - v(x^p)}\right\}.$$

Proof: For any pair of feasible solutions $y,z \in X$, let $D(y,z)$ denote the set of components which are equal to 1 in $\tilde{y}$ and 0 in $\tilde{z}$, and let $E(y,z)$ denote the set of components which are equal to 1 in both vectors $\tilde{y}$ and $\tilde{z}$.

Let L be equal to $\min\left\{\frac{\Delta_p}{(1+\epsilon)\cdot w-\epsilon v\cdot(x^p)}, \frac{\Delta_p}{(1+\epsilon)\cdot w-v(x^p)}\right\}$ and suppose $r(p,p') \le L$. For an arbitrary feasible solution x' we have

$$\sum_{i=1}^{n} p'_i x^p_i \ - (1+\epsilon) \cdot \sum_{i=1}^{n} p'_i x'_i =$$

$$\sum_{i=1}^{n} p_i x^p_i \ + \sum_{i=1}^{w} (p'_i - p_i) x^p_i \ - (1+\epsilon) \cdot \sum_{i=1}^{n} p_i x'_i \ - (1+\epsilon) \cdot \sum_{i=1}^{w} (p'_i - p_i) x'_i =$$

$$\sum_{i=1}^{n} p_i x^p_i \ - (1+\epsilon) \cdot \sum_{i=1}^{n} p_i x'_i \ +$$
$$\sum_{i \in D(x^p, x')} (p'_i - p_i) \ - \epsilon \cdot \sum_{i \in E(x^p, x')} (p'_i - p_i) \ - (1+\epsilon) \cdot \sum_{i \in D(x', x^p)} (p'_i - p_i) \leq$$

$$\sum_{i=1}^{n} p_i x^p_i \ - (1+\epsilon) \cdot \sum_{i=1}^{n} p_i x^p_i \ +$$
$$\sum_{i \in D(x^p, x')} L \ - \epsilon \cdot \sum_{i \in E(x^p, x')} (-L) \ - (1+\epsilon) \cdot \sum_{i \in D(x', x^p)} (-L) =$$

$$-\Delta_p \ + \sum_{i \in D(x^p, x')} L \ + \epsilon \cdot \sum_{i \in E(x^p, x')} L \ + (1+\epsilon) \cdot \sum_{i \in D(x', x^p)} L$$

Now if $\epsilon \leq 1$, then this is at most

$$-\Delta_p \ + \sum_{i \in D(x^p, x')} L \ + \sum_{i \in E(x^p, x')} L \ + (1+\epsilon) \cdot \sum_{i \in D(x', x^p)} L \leq$$

$$-\Delta_p \ + v(x^p) \cdot L \ + (1+\epsilon) \cdot (w - v(x^p)) \cdot L =$$

$$-\Delta_p \ + [(1+\epsilon) \cdot w - \epsilon \cdot v(x^p)] L = 0.$$

Hence, x^p is an ϵ-approximate solution for p'.

For the case $\epsilon > 1$ the proof is similar.

■

Next, we present a new algorithm for calculating the stability radius.

13 The Calculation of the Stability Radius

In [19] the following theorem has been proved.

Theorem 9 *The closed ball $O^w_\varrho(p)$ with $\varrho > 0$ is a stability ball of an ϵ-approximate solution x of problem (7.1) if and only if the condition $(\tilde{p} + \delta) \in K^w_\epsilon(x)$ is satisfied for all 2^w vectors $\delta = (\delta_1, \delta_2, \ldots, \delta_w) \in R^w$ with the components $\delta_i \in \{\varrho, \max\{-p_i, -\varrho\}\}$ for $i = 1, 2, \ldots, w$.*

We first show that the above characterization of a stability ball may be simplified.

Theorem 10 *The closed ball $O^w_\varrho(p)$ with $\varrho > 0$ is a stability ball of an ϵ-approximate solution x of problem (7.1) if and only if the condition $(\tilde{p}+\delta) \in K^w_\epsilon(x)$ is satisfied for all vectors $\delta = (\delta_1, \delta_2, \ldots, \delta_w) \in R^w$ which have $\delta_i \in \{\varrho, \max\{-p_i, -\varrho\}\}$ if $x_i = 1$, and $\delta_i = \max\{-p_i, -\varrho\}$ if $x_i = 0$, $i = 1, 2, \ldots, w$.*

Proof: It suffices to show that the condition of Theorem 10 implies the seemingly stronger condition of Theorem 9. Suppose the condition of Theorem 10 is satisfied and consider any vector δ' with components $\delta'_i \in \{\varrho, \max\{-p_i, -\varrho\}\}$, $i = 1, 2, \ldots, w$. We define the vector δ'' by $\delta''_i = \delta'_i$ if $x_i = 1$ and $\delta''_i = \max\{-p_i, -\varrho\}$ if $x_i = 0$, $i = 1, 2, \ldots, w$. We have

$$\sum_{i=1}^{n}(p_i + \delta'_i)x_i = \sum_{i=1}^{n}(p_i + \delta''_i)x_i \leq (1+\epsilon)\cdot\sum_{i=1}^{n}(p_i + \delta''_i)x'_i \leq (1+\epsilon)\cdot\sum_{i=1}^{n}(p_i + \delta'_i)x'_i,$$

where the equality holds because δ'' and δ' differ only in components for which the corresponding component of x is equal to 0, the first inequality is true because the condition of Theorem 10 is satisfied, and the last inequality follows from the fact that $\delta'' \leq \delta$ and all components of x' are non-negative. Since $x' \in X$ is arbitrary, it now follows that $(\tilde{p}+\delta') \in K^w_\epsilon(x)$, which completes the proof.

■

Let us return to Example 2 (see Section 9). We shall show that for objective vector $p = (10, 5, 5, 7, 7, 7)$ the ball $O^2_3(p)$ with the centre $\tilde{p} = (10, 5)$ is not a stability ball for the optimal solution x^1 (i.e., if $\epsilon = 0$). Indeed, the point $(10 + 3, 5 + 3) = (13, 8)$ does not belong to the set $K^2_0(x^1)$ since inequality (7.2) is not satisfied with $\epsilon = 0$: $13 + 8 + 5 > (1 + 0)\cdot\min\{13 + 8 + 7, 13 + 7 + 7, 7 + 7 + 7\}$. On the other hand, it is not difficult to show that this ball is a stability ball of the ϵ-approximate solution x^1 with $p = (10, 5, 5, 7, 7, 7)$, $\tilde{p} = (10,\ 5)$ and $\epsilon = 0.5$. Indeed, checking inequality (7.2), we obtain that the vectors $(13, 8), (13, 2), (7, 8)$ and $(7, 2)$ belong to the set $K^2_{0.5}(x^1)$. It follows from Theorem 10 that the ball $O^2_3(p)$ is a stability ball of the ϵ-approximate solution x^1 with $\epsilon = 0.5$. In Fig. 6 this ball is shaded.

From Fig. 6 it follows that the radius of this ball is not the largest possible one for $\epsilon = 0.5$: We have the strict inequality $\varrho^2_{0.5}(x^1, p) > 3$. The exact value of the above stability radius will be calculated in Section 14.

Now let us consider x^4. None of the components of $\tilde{x}^4$ is equal to 1. Therefore, to check whether $O^2_3(p)$ is a stability ball of the ϵ-approximate solution x^4 with $\epsilon = 0.5$, it suffices to check only whether $(7, 2)$ belongs to the set $K^2_{0.5}(x^4)$. It is left to the reader to verify that this is indeed the case, and that $\varrho^2_{0.5}(x^4, p) = 3$.

Theorem 10 allows us to calculate the stability radius, as is shown in the proof of the next theorem.

Theorem 11 *Let x be an ϵ-approximate solution and let $v(x)$ denote the number of 1's in $\tilde{x}$. If the calculation of $\Phi(p', x_{p'})$ can be done in $O(g(n))$ time for any objective vector $p' \in R^n$, then the stability radius $\varrho_\epsilon^w(x,p)$ can be calculated in $O(2^{v(x)} \cdot w \cdot g(n))$ time.*

Proof: The proof is constructive. We denote the set of indices of 1's in the vector $\tilde{x}$ by V : $V = \{i \mid x_i = 1;\ i = 1, 2, \ldots, w\}$. For every subset $I \subseteq V$ and for every $\varrho \geq 0$, we define the vector $\delta[I, \varrho] \in R^w$ by $\delta_i = \varrho$ if $i \in I$, and $\delta_i = \max\{-p_i, -\varrho\}$ if $i \in \{1, 2, \ldots, w\} \backslash I$. Let ϱ_I denote the largest value of ϱ for which $\tilde{p} + \delta[I, \varrho] \in K_\epsilon^w(x)$. We propose to calculate ϱ_I for every $I \subseteq V$. It follows from Theorem 10 that the stability radius is equal to the minimum of these $2^{v(x)}$ values.

Let us consider a fixed subset $I \subseteq V$. For any $\varrho \geq 0$, we define $p[I, \varrho] \in R^n$ by $\tilde{p}[I, \varrho] = \tilde{p} + \delta[I, \varrho]$ and $p[I, \varrho]_i = p_i$ for $i = w+1, w+2, \ldots, n$. Now we consider $\Phi(p[I, \varrho], x)$, the value of solution x, as a function of $\varrho \geq 0$. If ϱ increases from 0, then initially this function is linear with slope equal to $|I| - |V \backslash I|$. When ϱ becomes equal to $\min\{p_i \mid \ i \in V \backslash I\}$, the slope of the function changes into $|I| - |\{i \in V \backslash I \mid \varrho < p_i\}|$, and so on. It follows that $\Phi(p[I, \varrho], x)$ is a continuous and piecewise linear function of ϱ, with breakpoints occuring exactly at the values $\varrho = p_i$, $i \in V \backslash I$.

From the observations above, it also follows that for every $x' \in X$, the function $\Phi(p[I, \varrho], x')$ is continuous and piecewise linear, with breakpoints occuring at some subset of the values $\varrho = p_i$, $i \in \{1, 2, \ldots, w\} \backslash I$. Let us therefore refer to the values $p_i, i \in \{1, 2, \ldots, w\} \backslash I$, as critical points. For convenience, we also define 0 and ∞ to be critical points. Hence, between two consecutive critical points, the functions $\Phi(p[I, \varrho], x')$, $x' \in X$, are all linear. Moreover, the slope of each of these functions is an integer in the range from $|I| - w$ to $|I|$, with the extreme values occuring when $x'_i = 1$ if and only if $i \notin I$, and when $x'_i = 1$ if and only if $i \in I$, respectively.

Furthermore, we define for $\varrho \geq 0$ the function $H_I(\varrho)$ as $H_I(\varrho) = (1+\epsilon) \cdot \min_{x' \in X}\{\Phi(p[I, \varrho], x')\}$. Since between two consecutive critical points, the functions $(1+\epsilon) \cdot \Phi(p[I, \varrho], x')$, $x' \in X$, are all linear, it follows that on such an interval the function $H_I(\varrho)$ is the minimum of a finite number of linear functions. It is well-known that this implies that $H_I(\varrho)$ is continuous, piecewise linear and concave on these intervals (see [2]). Since between consecutive critical points the functions $\Phi(p[I, \varrho], x')$, $x' \in X$, each have an integer slope in the range from $|I| - w$ to $|I|$, it follows that the slope of $H_I(\varrho)$ is always in the set $\{(1+\epsilon)(|I| - w), (1+\epsilon)(|I| - w + 1), \ldots, (1+\epsilon)(|I|)\}$. Because of concavity, the slope of $H_I(\varrho)$ is non-increasing, which implies that $H_I(\varrho)$ has at most w breakpoints on any interval between two consecutive critical points. Also note that the continuity of the functions $(1+\epsilon) \cdot \Phi(p[I, \varrho], x')$, $x \in X$, implies that $H_I(\varrho)$ is continuous for $\varrho \geq 0$.

The shapes of the functions $\Phi(p[I, \varrho], x)$ and $H_I(\varrho)$ between two consecutive critical points ϱ_1 and ϱ_2, imply that if $\Phi(p[I, \varrho], x) \leq H_I(\varrho)$ for both $\varrho = \varrho_1$ and $\varrho = \varrho_2$, then $\Phi(p[I, \varrho], x) \leq H_I(\varrho)$ for all $\varrho \in [\varrho_1, \varrho_2]$, i.e., x is an ϵ-approximate solution on the complete interval. By evaluating the functions $\Phi(p[I, \varrho], x)$ and $H_I(\varrho)$ at the critical points, we can find the largest finite critical point, say ϱ', for which x is an ϵ-approximate solution. This takes $O(w \cdot g(n))$ time, because there are at most $w+1$ finite critical points and calculating $H_I(\varrho)$ at each of them boils down to calculating $\Phi(p', x_{p'})$ for a given $p' \in R^n$, which requires $O(g(n))$ time.

We have already observed that on the interval between ϱ' and its next critical point ϱ'', $\Phi(p[I, \varrho], x)$ is linear, while $H_I(\varrho)$ is concave and piecewise linear with at most w breakpoints.

There exists a method (see [2]) which determines all the linear pieces of $H_I(\varrho)$ on this interval in $O(B \cdot g(n))$ time, where B is the number of breakpoints of $H_I(\varrho)$ on the interval. Hence, it takes $O(w \cdot g(n))$ time to determine the linear pieces. Once this has been done, the value ϱ_I is found by calculating the largest intersection point of $\Phi(p[I, \varrho], x)$ and $H_I(\varrho)$ on the interval. Note that there always exists an intersection point, if ϱ'' is finite. Hence, if the functions do not intersect, $\varrho'' = \infty$ and also $\varrho_I = \infty$. If there is more than one intersection point, the set of intersection points is a complete linear piece of $H_I(\varrho)$. In any case, the largest intersection point can be found in time bounded by the number of linear pieces of $H_I(\varrho)$.

Thus, for each subset $I \subseteq V$, it is possible to calculate ϱ_I in $O(w \cdot g(n))$ time. Since we need to consider $2^{v(x)}$ subsets, the result now follows. ■

The value $v(x)$ may be significantly less than w. In particular, if the number of unstable components increases when n grows, the problem structure may prevent $v(x)$ from growing as fast as w. For instance, in the traveling salesman problem, if the unstable components are the distances with respect to one specific city, then w is of the same order as the number of cities, but $v(x) = 2$ for every feasible solution x.

On the one hand, note that even for fixed w, the asymptotic bound $O(2^{v(x)} \cdot w \cdot g(x))$ in Theorem 11 is exponential if $g(n)$ is an exponential function. This is no surprise, since it has been shown in [15] and [25] that, even if $w = 1$, computing the stability radius exactly for any $\epsilon \geq 0$, is NP-hard if the original optimization problem is NP-hard. On the other hand, we obtain the following 'positive result'.

Corollary 3 *Let problem (7.1) be polynomially solvable. We consider the maximum number of decision variables corresponding to unstable components of objective vector p which can be chosen equal to 1 in any feasible solution. If, when n increases, this number grows as $O(\log\ n)$, then the stability radius $\varrho_\epsilon^w(x, p)$ of any ϵ-approximate solution x can be calculated in polynomial time.*

Proof: The function $g(n)$ in Theorem 11 is now polynomial in n, and the growth of $v(x)$ is $O(\log\ n)$. Hence, the running time $O(2^{v(x)} \cdot w \cdot g(n))$ of the above algorithm becomes $O(n \cdot w \cdot g(n))$, which is clearly polynomial in n. ■

The next section illustrates the algorithm for calculating $\varrho_\epsilon^w(x, p)$, presented implicitly in the proof of Theorem 11.

14 The Calculation of $\varrho_\epsilon^w(x, p)$ for Example 2

For Example 2 (see Section 9), we calculate $\varrho_{0.5}^2(x^1, p)$ with $p = (10, 5, 5, 7, 7, 7)$. Since $\tilde{x}^1 = (1, 1)$, we have $V = \{1, 2\}$, which means that the subsets I are $\emptyset$, $\{1\}$, $\{2\}$ and $\{1, 2\}$. For each of these subsets we have to calculate ϱ_I.

Let us consider the calculation of $\varrho_{\{2\}}$ in detail. The vector $p[\{2\}, \varrho]$ is equal to $(\max\{0, 10 - \varrho\}, 5+\varrho, 5, 7, 7, 7)$. Therefore, the values of $\Phi(p[\{2\}, \varrho], x^i) = \sum_{j=1}^{6} p[\{2\}, \varrho]_j x_j^i$, $i \in \{1, 2, 3, 4\}$, are as shown in Fig. 7.

Figure 7

Note that some of the functions have a breakpoint at $\varrho = 10$, which corresponds to p_1. There are no other breakpoints, since 1 is the only element of $V \backslash I$. Hence, the critical points are 0, 10 and ∞. From Fig. 7 we see that $H_{\{2\}}(\varrho)$, which is defined as $1.5 \cdot \min\{\Phi(p[\{2\}, \varrho], x^i) \mid i \in \{1, 2, 3, 4\}\}$, is given by the following piecewise linear function:

$$H_{\{2\}}(\varrho) = \begin{cases} 1.5 \cdot (20), & \text{if } 0 \leq \varrho < 4, \\ 1.5 \cdot (24 - \varrho), & \text{if } 4 \leq \varrho < 10, \\ 1.5 \cdot (14), & \text{if } 10 \leq \varrho. \end{cases}$$

Note that this function is indeed concave between consecutive critical points.

Figure 8

In Fig. 8, the function is drawn together with $\Phi(p[\{2\}, \varrho], x^1)$. The largest value of ϱ for which $\Phi(p[\{2\}, \varrho], x^1) \leq H_{\{2\}}(\varrho)$ is equal to 11. Hence, this is the value of $\varrho_{\{2\}}$. Our algorithm, however, does not construct the two functions completely, before it determines $\varrho_{\{2\}}$. What the algorithm does, is the following. It first considers the smallest positive critical point, $\varrho = 10$. The values $\Phi(p[\{2\}, 10], x^1)$ and $H_{\{2\}}(10)$ are calculated. The first calculation is trivial, since x^1 is a fixed solution. For the second calculation we need to solve an instance of the scheduling problem (see the proof of Theorem 11, where this takes $g(n)$ time). Since $\Phi(p[\{2\}, 10], x^1) \leq H_{\{2\}}(10)$, the algorithm proceeds with the next critical point. This point is ∞, and therefore the functions $\Phi(p[\{2\}, \varrho], x^1)$ and $H_{\{2\}}(\varrho)$ are considered for arbitrarily large values of ϱ. Because $\Phi(p[\{2\}, \varrho], x^1) > H_{\{2\}}(\varrho)$ for such values, we conclude that $\varrho_{\{2\}} \in [10, \infty)$. To find the value of $\varrho_{\{2\}}$, we first construct $H_{\{2\}}(\varrho)$ on the interval, which is easy in this case since the function is linear on the interval. (How the construction is done in general will be described below.) Subsequently $\varrho_{\{2\}}$ is calculated as the (largest) intersection point of $H_{\{2\}}(\varrho)$ and $\Phi(p[\{2\}, \varrho], x^1)$.

Figure 9

In Fig. 9, we have illustrated the calculation of $\varrho_{\{1,2\}}$. (It is left to the reader to verify that the functions have indeed the shown shape.) In this case the only critical points are 0 and ∞. Note again that $H_{\{1,2\}}(\varrho)$ is indeed concave between these two values. The method to construct this function can briefly be described as follows. First we determine optimal solutions corresponding to the endpoints of the interval. These solutions define linear functions, which are possible pieces of the function to be constructed. Next the intersection point of the two linear functions is calculated. In our case, this is $\varrho = 0.5$, and therefore $H_{\{1,2\}}(0.5)$ is calculated. Since this does not yield a new solution (linear function), it can be concluded that all the linear parts of $H_{\{1,2\}}(\varrho)$ have been found. In general, we continue this procedure by calculating the intersection points of the newly found linear function with the concave lower envelope of the linear functions found earlier. The procedure stops if no new intersection points are generated. We refer to [2] for a proof of the correctness of this procedure, which requires solving $O(B)$ instances of the scheduling problem, where B is the number of breakpoints of the function to be constructed.

From Fig. 9, we see that $\varrho_{\{1,2\}} = 5.75$. It is now left to the reader to verify that $\varrho_{\emptyset} = \infty$ and $\varrho_{\{1\}} = 16.5$. Hence, we calculate the stability radius: $\varrho^2_{0.5}(x^1, p) = \min\{11,\ 5.75,\ 16.5,\ \infty\} = 5.75$. The correctness of this result can be checked in Fig. 6, where the ball with this radius is drawn.

15 Concluding Remarks

In deterministic scheduling theory the processing times are supposed to be given in advance, i.e., before applying a scheduling procedure. More general cases have been considered in stochastic scheduling (see [14]), where p_i is a random variable with a known distribution of probabilities. However, in practice such functions may also be unknown. The results surveyed and developed in this paper may be considered as an attempt to initialize further investigations of scheduling problems under conditions of uncertainty.

We have applied the same stability analysis for a large class of scheduling problems: Those which may be represented as linear binary programming problems and more general scheduling problems which may be represented as extremal problems on a disjunctive (mixed) graph. Of course, the complexity of the problems has to be taken into account: The stability results which seems to be appropriate for the general shop problems (see Sections 2 – 6) are rather rough for the linear binary programming problems which allow the deriviation of deeper mathematical results and more efficient algorithms.

In turn, stability properties of an optimization problem may be used to characterize its complexity. We can illustrate this on the job shop problem from the first part, and on the traveling salesman and assignment problem from the second part. (The latter corresponds to an optimal distribution of n jobs to m parallel non-identical machines in a single-stage system). The stability radius of an optimal schedule for problem $J//C_{max}$ is usually strictly positive, even if the optimal schedule is not unique (see Theorem 1 and the computational results in Section 6). On the other hand, it is easy to show that $\varrho^{\omega}_0(x, p) = 0$ if there exist at least two optimal solutions for a traveling salesman problem (or for an assignment problem). For this reason, the main focus in the second part of the paper was on a stability analysis of ϵ-approximate solutions (see Theorems 6 and 7). Such a difference of the complexity of the considered problems is not implied by the different type of objective functions ('min-max' for problem $G//C_{max}$ and 'min-sum' for problem (7.1)): As follows from [1], a general shop problem with mean flow time criterion (i.e., of type 'min-sum') becomes even more difficult for a stability analysis than problem $G//C_{max}$.

Possible trends for future research may be the investigation of connections between the complexity of scheduling problems and the complexity of calculating the stability radius of an optimal schedule (see [15, 20, 25]). Recall that in [15, 25] it was shown that the existence of a polynomial algorithm for calculating $\varrho^1_0(x, p)$ implies a polynomial algorithm for problem (7.1). In [25] a similar implication was also proven for the case $\epsilon > 0$. Moreover, in [15] it was shown that if problem (7.1) is polynomially solvable, then $\varrho^1_0(x, p)$ may be calculated in polynomial time. Thus the value $\varrho^1_0(x, p)$ may be calculated in polynomial time for a given optimal solution x of the assignment problem, while a similar calculation for the traveling salesman problem requires exponential time, unless $P = NP$. An interesting subject for research may be connected with a generalization of the result from [15]: Is it possible to find the stability radius $\varrho_s(p)$ of an optimal schedule s in polynomial time, if s may be constructed in polynomial time?

The setting of scheduling problems in the first part of the paper is so general that it is unlikely to find simple answers to those questions, which are usual for deterministic scheduling problems. So, future research may also focus on determining classes of rather simple scheduling problems for which it is possible to find the stability radius of an optimal or an approximate solution in a reasonable time, e.g., if the stability region for such a class will be convex, an algorithm similar to that developed in the proof of Theorem 11 may be applied. Therefore, another interesting topic of research is to establish that for some types of scheduling problems the stability region of an optimal schedule is a convex set.

An important question is connected with the determination of simple conditions (preferably conditions which can be verified in polynomial time) for the validity of $\varrho_s(p) = 0$ similar to those derived in [8] for $\varrho_s(p) = \infty$ for the problems $J//C_{max}$ and $J//L_{max}$. It is also of interest to develop simple bounds for $\varrho_s(p)$ and $\varrho^w_\epsilon(x,p)$ (see e.g. (2.2), Theorem 5, Corollary 2 and Theorem 8). An interesting question is how a branch-and-bound algorithm, which is often used for NP-hard scheduling problems, can be combined with calculating (bounds on) the stability radius of an optimal or ϵ-approximate schedule (see [13, 21]).

Finally, we note that the above approach to stability analysis is not the only possible one (see survey [20]). For example, a completely different approach to stability analysis is discussed in [7], where the sensitivity of a heuristic algorithm with respect to the variation of the processing time of one job is investigated. Note also that stability analysis is a well-studied topic in linear programming. For instance, in [26] a tolerance approach is presented. A similar concept could be applied to some scheduling problems.

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