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Companion Based Matrix Functions: Description and Minimal Factorization

H. Bart and L. G. Kroon
Erasmus University Rotterdam
P.O. Box 1738
3000 DR Rotterdam, The Netherlands

ABSTRACT

Companion based matrix functions are rational matrix functions admitting a minimal realization involving state space matrices that are first companions. Necessary and sufficient conditions are given for a rational matrix function to be companion based. Minimal factorization of such functions is discussed in detail. It is shown that the property of being companion based is hereditary with respect to minimal factorization. Also, the issue of minimal factorization is reduced to a division problem for pairs of monic polynomials of the same degree. In this context, a connection with the Euclidean algorithm is made. The results apply to canonical Wiener-Hopf factorization as well as to complete factorization. The analysis of the latter leads to a combinatorial problem involving the eigenvalues of the state space matrices. The algorithmic aspects of this problem are intimately related to the two machine flow shop problem and Johnson's rule from job scheduling theory.

1. INTRODUCTION

The material in this paper is concerned with rational $n \times n$ matrix functions W that are analytic at ∞ with $W(\infty) = I_n$, the $n \times n$ identity matrix. From systems theory it is known that if W is such a matrix function, then it can be written in the form

$$W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B, \tag{1.1}$$

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where A is an $m \times m$ matrix, B is an $m \times n$ matrix, and C is an $n \times m$ matrix. An expression (1.1) is called a *realization* of W. The realization (1.1) is called a *minimal* realization if m is the smallest possible integer such that W admits a realization (1.1). We say that W is *companion based* if it admits a minimal realization (1.1) where A and $A^{\times} = A - BC$ are *first companion matrices*.

As will be explained in a forthcoming paper [10], both minimal and complete factorization of a companion based matrix function W are issues that are closely related to the two machine flow shop problem (2MFSP) and Johnson's rule from job scheduling theory. In the current paper, however, the emphasis is on the study of companion based matrix functions per se.

In Section 2 we collect together some material on companion matrices and rational matrix functions that will be needed later. In particular, we discuss the subjects of minimal and complete factorization of rational matrix functions. We also consider simultaneous similarity of two companion matrices with their transposes. In this context an interesting connection with the Bezout matrix appears.

Section 3 is devoted to the description of companion based matrix functions. Here outer product representations play an important role. These are representations of a rational matrix function of the form

$$W(\lambda) = I_n + \frac{1}{p(\lambda)} \begin{bmatrix} w_1(\lambda) \\ w_2(\lambda) \\ \vdots \\ w_{n-1}(\lambda) \\ w_n(\lambda) \end{bmatrix} [\mu_1 \ \ \mu_2 \ \ \cdots \ \ \mu_{n-1} \ \ \mu_n],$$

where the polynomials $p, w_1, \dots, w_n$ and the complex numbers $\mu_1, \dots, \mu_n$ satisfy certain conditions. Among other things, it will be shown that all companion based 2×2 matrix functions can be obtained in a simple way from the functions

$$\begin{bmatrix} 1 & \dfrac{r(\lambda)}{p(\lambda)} \\ 0 & \dfrac{p^{\times}(\lambda)}{p(\lambda)} \end{bmatrix} = I_2 + \frac{1}{p(\lambda)} \begin{bmatrix} r(\lambda) \\ p^{\times}(\lambda) - p(\lambda) \end{bmatrix} [0 \ \ 1].$$

Here p and $p^{\times}$ are monic polynomials of the same positive degree m, r is a polynomial of degree less than m, and p, $p^{\times}$ and r do not have any common zero.

Section 4 is concerned with minimal factorization of companion based rational matrix functions. We prove that the property of being companion based is hereditary with respect to minimal factorization. That is, if W is companion based and $W = UV$ is a minimal factorization of W, then U and V are companion based as well. Furthermore, we present necessary and sufficient conditions guaranteeing that W is companion based if $W = UV$ is a minimal product of companion based matrix functions U and V. It is also shown that for a companion based matrix function W there exists a one-to-one correspondence between its minimal factorizations and specific factorizations of two monic polynomials associated with W, namely its pole and zero polynomial. The results apply to canonical Wiener-Hopf factorization as well as to complete factorization.

In Section 5 we discuss some algorithmic aspects related to the construction of minimal and, in particular, complete factorizations of companion based matrix functions. We first describe how Johnson's rule from job scheduling theory can be used to determine whether a given companion based matrix function admits complete factorization. Thereafter we make the results of Section 4 explicit for the companion based 2×2 matrix functions. That is, we show how all minimal factorizations of a companion based 2×2 matrix function can be obtained. Here a connection with the Euclidean algorithm is made.

As a final part of this introductory section we give an overview of some notation and conventions that are use in this paper. The notation $\mathscr{C}^n$ is used for the set of complex (column) n-vectors. The $n \times n$ identity matrix is denoted by I_n. The superscript ^{T} signals the operation of taking transposes. The characteristic polynomial of an $n \times n$ matrix A is denoted by p_A. So $p_A(\lambda) = \det(\lambda I_n - A)$. If W is a matrix function, then the matrix functions W^T and W^{-1} are defined by $W^T(\lambda) = W(\lambda)^T$ and $W^{-1}(\lambda) = W(\lambda)^{-1}$. For polynomials $p_1, \dots, p_k$, the notation $\gcd(p_1, \dots, p_k) = 1$ means that $p_1, \dots, p_k$ do not have any common zero. Finally, we follow the convention that nonzero constant functions are viewed as polynomials of degree zero. The zero function is viewed as a polynomial of degree -1.

2. PRELIMINARIES

In this section we collect together some properties of first companion matrices and some preliminaries on systems theory that are used in this paper.

2.1. *Similarity of First and Second Companion Matrices*

A matrix A is called a *first companion $m \times m$ matrix* if it has the form

$$A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{m-1} \end{bmatrix}, \tag{2.1}$$

where $a_0, \dots, a_{m-1}$ are complex numbers. More specifically, we sometimes call the matrix A in (2.1) the first companion matrix associated with the monic polynomial $p(\lambda) = \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_0$. Recall that this polynomial is precisely the characteristic polynomial p_A of A. *Second companion matrices* are the transposes of first companion matrices. For basic material on companion matrices, see Lancaster and Tismenetsky [22].

It is well known that a square matrix and its transpose are always similar. For companion matrices this statement can be made more explicit. Indeed, if the matrix A is given by (2.1), then the so-called *symmetrizer* H of A is the matrix

$$H = \begin{bmatrix} a_1 & a_2 & \cdots & a_{m-1} & 1 \\ a_2 & & \cdot & \cdot & 0 \\ \vdots & \cdot & \cdot & \cdot & \vdots \\ a_{m-1} & \cdot & \cdot & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{2.2}$$

and it is well known that $HA = A^T H$. Since it is obvious that H is invertible, this proves that A and A^T are similar.

We shall now discuss this matter for pairs of different companion matrices. The following proposition is a special case of a result of Bart and Thijsse [11, 12]. We present a new proof exhibiting a connection with the Bezout matrix. For information on the Bezout matrix, see [22, Section 13.3].

PROPOSITION 2.1. *Let A and Z be different first companion $m \times m$ matrices. Then there exists an invertible $m \times m$ matrix S such that $SAS^{-1} = A^T$ and $SZS^{-1} = Z^T$ if and only if A and Z do not have a common eigenvalue.*

Proof. Let S be an invertible $m \times m$ matrix such that $SAS^{-1} = A^T$ and $SZS^{-1} = Z^T$, and assume that α is a common eigenvalue of A and Z. Then the vector $v = [1 \ \alpha \ \cdots \ \alpha^{m-1}]^T$ is a common eigenvector of A and Z corresponding to the eigenvalue α. Hence Sv is a common eigenvector of A^T and Z^T corresponding to the eigenvalue α. Now choose $x, y \in \mathscr{C}^m$ such that

$$[1 \ \lambda \ \cdots \ \lambda^{m-1}]x = \frac{p_A(\lambda)}{\lambda - \alpha}, \qquad [1 \ \lambda \ \cdots \ \lambda^{m-1}]y = \frac{p_Z(\lambda)}{\lambda - \alpha}, \quad (2.3)$$

where $p_A(\lambda) = \det(\lambda I_m - A)$ and $p_Z(\lambda) = \det(\lambda I_m - Z)$. Then $x \in \mathrm{Ker}(\alpha I_m - A^T)$ and $y \in \mathrm{Ker}(\alpha I_m - Z^T)$, as can be verified by writing out the corresponding equations and taking into account that $x_{m-1} = 1$. Since A^T and Z^T are nonderogatory, the eigenspaces $\mathrm{Ker}(\alpha I_m - A^T)$ and $\mathrm{Ker}(\alpha I_m - Z^T)$ are one-dimensional (cf. Lancaster and Tismenetsky [22]). Therefore both x and y are scalar multiples of Sv. Taking into account that the right hand sides in (2.3) are monic polynomials, it follows that $p_A = p_Z$. But this contradicts the assumption $A \neq Z$. Thus the "only if" part of the proposition has been proved.

Next consider the Bezout matrix $\mathscr{B}$ associated with p_A and p_B. That is, $\mathscr{B} = (b_{ij})$ is the $m \times m$ matrix defined by

$$\frac{p_A(\lambda)p_Z(\mu) - p_A(\mu)p_Z(\lambda)}{\lambda - \mu} = \sum_{i=1}^{m}\sum_{j=1}^{m} b_{ij}\lambda^{i-1}\mu^{j-1}.$$

By the Barnett factorization theorem, $\mathscr{B} = Hp_Z(A)$, where H is the symmetrizer of A given by (2.2). Using that $HA = A^TH$, we now obtain $\mathscr{B}A = Hp_Z(A)A = HAp_Z(A) = A^THp_Z(A) = A^T\mathscr{B}$. By interchanging the roles of A and Z, we also get $\mathscr{B}Z = Z^T\mathscr{B}$. Indeed, the Bezout matrix associated with p_Z and p_A is $-\mathscr{B}$. The "if part" of the proposition is now an immediate consequence of the well-known fact that $\mathscr{B}$ is invertible if and only if p_A and p_Z do not have a common zero. ∎

The matrix S appearing in Proposition 2.1 is essentially unique. Indeed, if S is any invertible $m \times m$ matrix such that $SAS^{-1} = A^T$ and $SZS^{-1} = Z^T$, then S is a scalar multiple of the Bezout matrix $\mathscr{B}$ associated with p_A and p_Z.

To see this, we argue as follows. Write $A - Z = bc^T$ where b and c are nonzero vectors in $\mathscr{C}^m$ such that the $m \times m$ matrix $C = [b \ Ab \ \cdots \ A^{m-1}b]$ is invertible. This is possible, since A and Z are different first companion matrices. For b one can take the mth unit vector in $\mathscr{C}^m$. Now $cb^T = A^T - Z^T = S(A - Z)S^{-1} = (Sb)(c^TS^{-1})$. As a rank factorization is essentially unique, there exists $\sigma \neq 0$ such that $Sb = \sigma c$. Analogously, $cb^T = A^T -$

$Z^T = \mathscr{B}(A - Z)\mathscr{B}^{-1} = (\mathscr{B}b)(c^T\mathscr{B}^{-1})$, and there exists $\tau \neq 0$ such that $\mathscr{B}b = \tau c$. It follows that

$$SC = [Sb \;\; SAb \;\; \cdots \;\; SA^{m-1}b] = \sigma\left[c \;\; A^Tc \;\; \cdots \;\; (A^T)^{m-1}c\right].$$

In a similar way we have

$$\mathscr{B}C = [\mathscr{B}b \;\; \mathscr{B}Ab \;\; \cdots \;\; \mathscr{B}A^{m-1}b] = \tau\left[c \;\; A^Tc \;\; \cdots \;\; (A^T)^{m-1}c\right].$$

Hence $\tau SC = \sigma\mathscr{B}C$. Since the matrix C is invertible, this identity implies that $S = (\sigma/\tau)\mathscr{B}$ and the desired result has been obtained.

2.2. *Review of Rational Matrix Functions*

In this subsection we review some material from systems theory. The material is concerned with rational $n \times n$ matrix functions. *It will be assumed in this paper that these functions are analytic at* ∞ *with value* I_n, *the* $n \times n$ *identity matrix*. The relevant references are Bart et al. [4], Bart et al. [5], DeWilde and Vandewalle [14], Gohberg et al. [17], Kailath [19], Kalman [20], Kalman et al. [21], and Sahnovic [25].

Let W be a rational $n \times n$ matrix function. By a *realization* of W we mean a representation of the form

$$W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B, \tag{2.4}$$

where A is an $m \times m$ matrix, B is an $m \times n$ matrix, and C is an $n \times m$ matrix. If W satisfies the standing assumption formulated above, then it is always possible to find such a representation.

The standing assumption implies that W^T and W^{-1} are also well-defined rational $n \times n$ matrix functions. A realization (2.4) of W implies a realization of W^T, namely

$$W^T(\lambda) = I_n + B^T(\lambda I_m - A^T)^{-1}C^T. \tag{2.5}$$

It is customary to write $A^\times$ for the matrix $A - BC$. With this notation, (2.4) implies

$$W^{-1}(\lambda) = I_n - C(\lambda I_m - A^\times)^{-1}B. \tag{2.6}$$

The smallest possible m for which a rational matrix function W admits a realization (2.4) is called the *McMillan degree* of W and is denoted by $\delta(W)$. It equals the total number of poles of W counted according to pole multiplicity. A discussion of this notion is given after the next paragraph. Note that $\delta(W) = 0$ if and only if $W(\lambda) = I_n$ for all λ.

The realization (2.4) is called *minimal* if $m = \delta(W)$. The minimality of (2.4) implies that of (2.5) and (2.6). In particular, the McMillan degrees of W, W^T and W^{-1} are the same. Minimal realizations are essentially unique: if (2.4) is a minimal realization of W, then all minimal realizations of W can be obtained by replacing A, B, and C with SAS^{-1}, SB, and CS^{-1}, respectively where S is an invertible $m \times m$ matrix. This result is known as the state space isomorphism theorem.

At this point we shall explain the notion of pole multiplicity already referred to above. Let W be a rational $n \times n$ matrix function and let α be a complex number. In a deleted neighborhood of α we have the Laurent expansion

$$W(\lambda) = \sum_{k=-r}^{\infty} (\lambda - \alpha)^k W_k, \tag{2.7}$$

where r is a positive integer not smaller than the order of α as a pole of W. Write

$$\delta(W, \alpha) = \operatorname{rank} \begin{bmatrix} W_{-r} & W_{-r+1} & \cdots & W_{-2} & W_{-1} \\ 0 & W_{-r} & \ddots & & W_{-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & W_{-r} & W_{-r+1} \\ 0 & 0 & \cdots & 0 & W_{-r} \end{bmatrix}. \tag{2.8}$$

Then $\delta(W, \alpha)$ does not depend on the choice of r, and $\delta(W, \alpha)$ is not smaller than the pole order of W at α. Also, $\delta(W, \alpha) = 0$ if and only if α is pole of W of order zero, i.e., W is analytic at α. The number $\delta(W, \alpha)$ is called the *local degree* or the *pole multiplicity* of W at α. As was mentioned already, the McMillan degree of W equals the number of poles of W counted according to pole multiplicity. In other words,

$$\delta(W) = \sum_{\mu \in \mathscr{C}} \delta(W, \mu).$$

Here the summation can be restricted to those μ that are genuine poles of W.

Note that α is a pole of W if and only if α is a pole of W^T. A complex number α is called a *zero* of W if it is a pole of W^{-1}. The *zero multiplicity* of α as a zero of W then equals the pole multiplicity $\delta(W^{-1}, \alpha)$ of α as a pole of W^{-1}.

To facilitate later discussions, we now associate two scalar polynomials with W, namely its *pole polynomial* p_W and its *zero polynomial* $p_W^{\times}$. They are defined by

$$p_W(\lambda) = (\lambda - \alpha_1)\cdots(\lambda - \alpha_m), \qquad p_W^{\times}(\lambda) = (\lambda - \alpha_1^{\times})\cdots(\lambda - \alpha_m^{\times}),$$

where $\alpha_1, \dots, \alpha_m$ are the poles of W counted according to pole multiplicity and $\alpha_1^{\times}, \dots, \alpha_m^{\times}$ are the zeros of W counted according to zero multiplicity. Both p_W and $p_W^{\times}$ are monic and have degree $\delta(W)$.

Again let (2.4) be a minimal realization of W. Then α is a pole of W with pole order r if and only if α is a pole of $(\lambda I_m - A)^{-1}$ with pole order r. Furthermore, α is a pole of W with pole multiplicity k if and only if α is an eigenvalue of A with algebraic multiplicity k. Thus the poles of W coincide with the eigenvalues of A. In particular, $p_W = p_A$, and similarly, $p_W^{\times} = p_{A^{\times}}$. Also,

$$\det W(\lambda) = \frac{p_W^{\times}(\lambda)}{p_W(\lambda)}.$$

To see this, let (2.4) be a minimal realization of W, and note that

$$\begin{aligned}
\det W(\lambda) &= \det\left[I_n + C(\lambda I_m - A)^{-1}B\right] \\
&= \det\left[I_m + (\lambda I_m - A)^{-1}BC\right] \\
&= \det\left[I_m + (\lambda I_m - A)^{-1}(\{\lambda I_m - A^{\times}\} - \{\lambda I_m - A\})\right] \\
&= \det\left[(\lambda I_m - A)^{-1}(\lambda I_m - A^{\times})\right] \\
&= \frac{\det(\lambda I_m - A^{\times})}{\det(\lambda I_m - A)} = \frac{p_W^{\times}(\lambda)}{p_W(\lambda)}.
\end{aligned}$$

Here we used the well-known identity $\det(I + PQ) = \det(I + QP)$ valid for matrices P and Q of appropriate sizes.

A pole α of W is called *geometrically simple* if its pole multiplicity equals its pole order. So a pole α is geometrically simple if and only if α is an eigenvalue of A of geometric multiplicity 1. Hence all poles of W are geometrically simple if and only if A is nonderogatory.

The McMillan degree $\delta(W)$ is sublogarithmic in the following sense. If $W = W_1 \cdots W_k$ is a factorization of W, then

$$\delta(W) \leqslant \delta(W_1) + \cdots + \delta(W_k). \tag{2.9}$$

Of special interest are factorizations with equality in (2.9). These are called *minimal factorizations*. In such factorizations pole-zero cancellation does not occur (cf. Bart et al. [4]). Note that the following holds true. Let $W = W_1 \cdots W_k$ be a factorization of W. Then this factorization is minimal if and only if $p_W = p_{W_1} \cdots p_{W_k}$ or, equivalently, $p_W^\times = p_W^\times \cdots p_{W_k}^\times$. A minimal factorization $W = W_1 \cdots W_k$ induces a minimal factorization of W^T and W^{-1}, namely $W^T = W_k^T \cdots W_1^T$ and $W^{-1} = W_k^{-1} \cdots W_1^{-1}$. There exist nontrivial rational matrix functions without any nontrivial minimal factorization.

A rational matrix function is called *elementary* if it has McMillan degree one. A *complete factorization* is a minimal factorization involving elementary factors only. Thus a complete factorization is a factorization of the form

$$W(\lambda) = \left(I_n + \frac{1}{\lambda - \alpha_1} R_1\right) \cdots \left(I_n + \frac{1}{\lambda - \alpha_m} R_m\right), \tag{2.10}$$

where m is the McMillan degree W, $\alpha_1, \ldots, \alpha_m$ are the poles of W counted according to pole multiplicity, and $R_1, \ldots, R_m$ are $n \times n$ matrices of rank 1. With the complete factorization (2.10) of W we can associate complete factorizations of W^T and W^{-1}. Indeed,

$$W^T(\lambda) = \left(I_n + \frac{1}{\lambda - \alpha_m} R_m^T\right) \cdots \left(I_n + \frac{1}{\lambda - \alpha_1} R_1^T\right). \tag{2.11}$$

Also, each R_i can be written as $R_i = c_i b_i^T$ for some nonzero vectors b_i and c_i. Hence if we define $\alpha_i^\times = \alpha_i - b_i^T c_i = \alpha_i - \operatorname{trace} R_i$ for $i = 1, \ldots, m$, then (2.10) implies

$$W^{-1}(\lambda) = \left(I_n - \frac{1}{\lambda - \alpha_m^\times} R_m\right) \cdots \left(I_n - \frac{1}{\lambda - \alpha_1^\times} R_1\right). \tag{2.12}$$

3. COMPANION BASED MATRIX FUNCTIONS

Let W be a rational $n \times n$ matrix function that is analytic at ∞ with $W(\infty) = I_n$ (standing assumption). We say that W is *companion based* if it admits a minimal realization

$$W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B, \tag{3.1}$$

where A and $A^\times$ are first companion matrices. If W is companion based, then the poles of W, W^T, and W^{-1} are geometrically simple. This is clear from the material presented in Section 2.2.

In this section we study companion based matrix functions with prescribed pole and zero polynomial. In Section 3.1 we give a general description of such matrix functions. In Section 3.2 we specialize to companion based matrix functions admitting an outer product representation. The minimally sized companion based matrix functions with prescribed pole and zero polynomial are studied in Section 3.3. Finally, in Section 3.4 we give another representation of companion based matrix functions that is closely related to the outer product representation and that is useful in the study of minimal factorizations.

PROPOSITION 3.1. *Let W be a companion based $n \times n$ matrix function. Then W^{-1} is companion based as well. Furthermore, W^T is companion based if and only if either $p_W = p_W^\times$ or* $\gcd(p_W; p_W^\times) = 1$.

Recall that p_W and $p_W^\times$ are the pole polynomial and the zero polynomial of W, respectively. These polynomials were defined in Section 2.2.

Proof. Let (3.1) be a minimal realization of W such that A and $A^\times$ are first companions. Then (2.6) is a minimal realization of W^{-1}. Now $A^\times$ is a first companion, and the same is true for $(A^\times)^\times = A^\times + BC = A$. This proves the first part of the proposition.

Next, suppose W is companion based and $p_W = p_W^\times$ or $\gcd(p_W; p_W^\times) = 1$. If W is given by (3.1), then (2.5) is a minimal realization of W^T. Recall that $p_W = p_A$, $p_W^\times = p_{A^\times}$, and that A and $A^\times$ are first companion matrices. Thus $A = A^\times$, or A and $A^\times$ do not have any common eigenvalue. So, according to the material presented in Section 2.1, there exists an invertible

$m \times m$ matrix S such that $SAS^{-1} = A^T$ and $SA^{\times}S^{-1} = (A^{\times})^T$. By substituting SAS^{-1} for A^T in (2.5), we find that

$$W^T(\lambda) = I_n + B^T S(\lambda I_m - A)^{-1} S^{-1} C^T$$

is a minimal realization of W^T as well. Since $A - S^{-1}C^T B^T S = S^{-1}(A^T - C^T B^T)S = S^{-1}(A^{\times})^T S = A^{\times}$, we may conclude that W^T is companion based.

Finally, assume that W^T is companion based. Then W^T admits a minimal realization

$$W^T(\lambda) = I_n + C_1(\lambda I_m - A_1)^{-1} B_1, \tag{3.2}$$

where A_1 and $A_1 - B_1 C_1$ are first companions. Clearly, W and W^T have the same pole polynomial. So A and A_1 have the same characteristic polynomial, which implies $A = A_1$. Similarly, we find $A^{\times} = A_1^{\times}$ as well. Now (2.5) and (3.2) are two minimal realizations of W^T. So the state space isomorphism theorem guarantees the existence of an invertible $m \times m$ matrix S such that $SAS^{-1} = A^T$ and $SA^{\times}S^{-1} = (A^{\times})^T$. By Proposition 2.1, we find $A = A^{\times}$, or A and $A^{\times}$ do not have any common eigenvalue. Thus $p_W = p_W^{\times}$ or $\gcd(p_W; p_W^{\times}) = 1$. ■

3.1. *General Description*

Throughout this section p is a monic polynomial of positive degree m given by $p(\lambda) = \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0$. With the polynomial p we associate the $m \times m$ matrix function $R_p(\lambda) = p(\lambda)(\lambda I_m - A)^{-1}$, where A is the $m \times m$ companion matrix associated with p. If we write $R_p(\lambda) = [r_{ij}(\lambda)]_{i,j=1}^m$, then it is easy to verify that

$$r_{ij}(\lambda) = \begin{cases} a_{j-1}\lambda^{i+j-2} + \cdots + a_0\lambda^{i-2}, & i > j, \\ \lambda^{m+i-j-1} + a_{m-1}\lambda^{m+i-j-2} + \cdots + a_j\lambda^{i-1}, & i \leq j. \end{cases}$$

Note that R_p is an $m \times m$ matrix polynomial of degree $m - 1$. The last column of R_p equals $[1 \ \lambda \ \cdots \ \lambda^{m-1}]^T$.

Let h be a k-vector polynomial (i.e. a $k \times 1$ matrix polynomial) of degree not exceeding $m - 1$. Then there exists a unique $k \times m$ matrix H such that

$$h(\lambda) = H[1 \ \lambda \ \cdots \ \lambda^{m-1}]^T = HR_p(\lambda)e_m.$$

Here e_m is the mth unit vector $[0 \ \cdots \ 0 \ 1]^T$ in $\mathscr{C}^m$. The matrix H will be called the *coefficient matrix* of h.

THEOREM 3.2. *Let W be a rational $n \times n$ matrix function with $\delta(W) = m$, and let p and $p^\times$ be different monic polynomials of the same positive degree m. Then the following statements are equivalent:*

(i) *W is companion based, $p_W = p$ and $p_W^\times = p^\times$.*
(ii) *There exists an invertible $n \times n$ matrix T such that W has the form*

$$W(\lambda) = I_n + \frac{1}{p(\lambda)} T \begin{bmatrix} 0 & h(\lambda) & HR_p(\lambda)G \\ 0 & 0 & 0 \end{bmatrix} T^{-1},$$

where h is a k-vector polynomial of degree not exceeding $m-1$, the bottom entry of h is $p^\times - p$, H is the coefficient matrix of h, and G is an $m \times (n-k)$ matrix. Here k is a positive integer not exceeding n.

Proof. Suppose (i) is satisfied, so W is companion based, $p_W = p$, and $p_W^\times = p^\times$. Then W can be written in the form (3.1) where A and $A^\times$ are the first companion matrices associated with p and $p^\times$, respectively. Now $BC = A - A^\times$ is a rank 1 matrix which can be written as $BC = e_m v^T$ where $p^\times(\lambda) - p(\lambda) = v^T[1 \ \lambda \ \cdots \ \lambda^{m-1}]^T = v^T R_p(\lambda) e_m$. A routine argument (involving column and row operations) shows that there exists an invertible $n \times n$ matrix T such that BT and $T^{-1}C$ have the form

$$BT = [0 \ \ e_m \ \ B_0], \qquad T^{-1}C = \begin{bmatrix} C_0 \\ v^T \\ 0 \end{bmatrix},$$

where B_0 is an $m \times (n-k)$ matrix and C_0 is a $(k-1) \times m$ matrix. Here k is a positive integer not exceeding n. So we get

$$\begin{aligned} T^{-1}W(\lambda)T &= I_n + T^{-1}C(\lambda I_m - A)^{-1}BT \\ &= I_n + \frac{1}{p(\lambda)} T^{-1}CR_p(\lambda)BT \\ &= I_n + \frac{1}{p(\lambda)} \begin{bmatrix} 0 & C_0 R_p(\lambda) e_m & C_0 R_p(\lambda) B_0 \\ 0 & v^T R_p(\lambda) e_m & v^T R_p(\lambda) B_0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Now we introduce

$$h(\lambda) = \begin{bmatrix} C_0 R_p(\lambda) e_m \\ v^T R_p(\lambda) e_m \end{bmatrix}.$$

Then h is a k-vector polynomial of degree not exceeding $m-1$, and the bottom entry $v^T R_p(\lambda) e_m$ of h equals $p^\times - p$. Also,

$$H = \begin{bmatrix} C_0 \\ v^T \end{bmatrix}$$

is the coefficient matrix of h. Putting $G = B_0$ we see that (ii) is satisfied.

Conversely, suppose there exists a positive integer k and an invertible $n \times n$ matrix T with the properties described in (ii). Write

$$H = \begin{bmatrix} C_0 \\ v^T \end{bmatrix},$$

where $v \in \mathscr{C}^m$ and C_0 is a $(k-1) \times m$ matrix. Then $v^T R_p(\lambda) e_m = p^\times(\lambda) - p(\lambda)$ and

$$T^{-1} W(\lambda) T = I_n + \frac{1}{p(\lambda)} \begin{bmatrix} C_0 \\ v^T \\ 0 \end{bmatrix} R_p(\lambda) [0 \ \ e_m \ \ G]$$

$$= I_n + \begin{bmatrix} C_0 \\ v^T \\ 0 \end{bmatrix} (\lambda I_m - A)^{-1} [0 \ \ e_m \ \ G],$$

where A is the first companion associated with p. Put

$$BT = [0 \ \ e_m \ \ G], \qquad T^{-1} C = \begin{bmatrix} C_0 \\ v^T \\ 0 \end{bmatrix}.$$

Then W admits the realization (3.1). Observe that $A^\times = A - BC = A - e_m v^T$. This implies that $A^\times$ is the first companion matrix associated with $p^\times$.

Since $m = \delta(W)$, the realization (3.1) is minimal. Hence W is companion based. Clearly, $p_W = p$ and $p_W^\times = p^\times$. So (i) is satisfied. ∎

Theorem 3.2 deals with the case where $p \neq p^\times$. The case where $p = p^\times$ can be treated in a similar way. For this case the following observation is useful. If $p = p^\times$, then $A = A^\times$, which implies $BC = A - A^\times = 0$. Further, if B is an $m \times n$ matrix, C is an $n \times m$ matrix, and $BC = 0$, then there exists an invertible $n \times n$ matrix T such that

$$B = [0 \;\; B_0]T^{-1}, \qquad C = T\begin{bmatrix} C_0 \\ 0 \end{bmatrix},$$

where B_0 is an $m \times k$ matrix, C_0 is a $(m-k) \times m$ matrix, and k is a positive integer not exceeding m.

The description of companion based matrix functions in Theorem 3.2 is not completely satisfactory, because it is based on the condition $\delta(W) = m$. This condition does not play a role in the implication (i) ⇒ (ii), but it does appear in the proof of (ii) ⇒ (i). However, in this proof it can be avoided under the assumption $\gcd(p; h_1, \dots, h_k) = 1$, which is sufficient to guarantee $\delta(W) = m$. This idea is worked out in more detail for a special case in the next section.

3.2. *Outer Product Representations*

Let p and $p^\times$ be monic polynomials of the same positive degree m, and let W be a rational $n \times n$ matrix function. By an *outer product representation of* W with respect to p and $p^\times$ we mean a representation of the form

$$W(\lambda) = I_n + \frac{1}{p(\lambda)}\begin{bmatrix} w_1(\lambda) \\ w_2(\lambda) \\ \vdots \\ w_{n-1}(\lambda) \\ w_n(\lambda) \end{bmatrix}[\mu_1 \;\; \mu_2 \;\; \cdots \;\; \mu_{n-1} \;\; \mu_n], \tag{3.3}$$

where $\mu_1, \dots, \mu_n$ are complex numbers, not all equal to zero, $w_1, \dots, w_n$ are polynomials of degree less than m, and $\mu_1 w_1 + \cdots + \mu_n w_n = p^\times - p$.

A few comments are in order. First it should be noted that if W admits an outer product representation with respect to p and $p^\times$, then rank $[W(\lambda) -$

$I_n] \leqslant 1$ for all but a finite number of λ. Further, if W admits an outer product representation with respect to p and $p^\times$, then

$$W^{-1}(\lambda) = I_n - \frac{1}{p^\times(\lambda)} \begin{bmatrix} w_1(\lambda) \\ w_2(\lambda) \\ \vdots \\ w_{n-1}(\lambda) \\ w_n(\lambda) \end{bmatrix} [\mu_1 \;\; \mu_2 \;\; \cdots \;\; \mu_{n-1} \;\; \mu_n],$$

where $-(\mu_1 w_1 + \cdots + \mu_n w_n) = -(p^\times - p) = p - p^\times$. Thus W admits an outer product representation with respect to p and $p^\times$ if and only if W^{-1} admits an outer product representation with respect to $p^\times$ and p. Finally, a straightforward computation, based on the identity $\det(I + PQ) = \det(I + QP)$, shows that, if W admits an outer product representation with respect to p and $p^\times$, then

$$\det W(\lambda) = \frac{p^\times(\lambda)}{p(\lambda)}.$$

In particular, the unique rational scalar matrix function W admitting an outer product representation with respect to p and $p^\times$ is $W(\lambda) = p^\times(\lambda)/p(\lambda)$.

PROPOSITION 3.3. *Let W be a rational $n \times n$ matrix function, and let p and $p^\times$ be monic polynomials of the same positive degree m. Then the following statements are equivalent:*

(i) *W admits an outer product representation with respect to p and $p^\times$;*

(ii) *W admits a realization $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$, where A is the first companion matrix associated with p, $A^\times$ is the first companion matrix associated with $p^\times$, and* rank $B = 1$;

(iii) *W admits a realization $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$, where the characteristic polynomials of A and $A^\times$ are p and $p^\times$, respectively, and where* rank $B = 1$.

For a fourth equivalent statement, see Proposition 3.9 below.

Proof. Suppose (3.3) is an outer product representation of W with respect to p and $p^{\times}$. Let A be the first companion matrix associated with p. Then it is well known that

$$(\lambda I_m - A)^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \frac{1}{p(\lambda)}\begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-2} \\ \lambda^{m-1} \end{bmatrix}.$$

Since $w_1, \dots, w_n$ are polynomials of degree less than m, there exists a unique $n \times m$ matrix C such that

$$\begin{bmatrix} w_1(\lambda) \\ w_2(\lambda) \\ \vdots \\ w_{n-1}(\lambda) \\ w_n(\lambda) \end{bmatrix} = C\begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-2} \\ \lambda^{m-1} \end{bmatrix}.$$

Furthermore, we define the $m \times n$ matrix B as follows.

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}[\mu_1 \ \ \mu_2 \ \ \cdots \ \ \mu_{n-1} \ \ \mu_n].$$

Combining these definitions, we find

$$I_n + C(\lambda I_m - A)^{-1}B$$

$$= I_n + \frac{1}{p(\lambda)}\begin{bmatrix} w_1(\lambda) \\ w_2(\lambda) \\ \vdots \\ w_{n-1}(\lambda) \\ w_n(\lambda) \end{bmatrix}[\mu_1 \ \ \mu_2 \ \ \cdots \ \ \mu_{n-1} \ \ \mu_n] = W(\lambda).$$

Recall that A is the first companion matrix associated with p. The structure of the matrix B implies that $A^\times$ is a first companion matrix as well. In fact, $A^\times$ is the first companion matrix associated with $p^\times$, because $p(\lambda) = \det(\lambda I_m - A)$ and

$$\frac{p^\times(\lambda)}{p(\lambda)} = \det W(\lambda) = \frac{\det(\lambda I_m - A^\times)}{\det(\lambda I_m - A)}.$$

Since rank $B = 1$ we may conclude that (i) implies (ii). It is evident that (ii) implies (iii). So it remains to be shown that (iii) implies (i).

For this we argue as follows. Assume $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$, where the characteristic polynomials of A and $A^\times$ are p and $p^\times$, respectively, and where rank $B = 1$. Then the $m \times n$ matrix B can be factorized as

$$B = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_{m-1} \\ \nu_m \end{bmatrix} [\mu_1 \ \ \mu_2 \ \ \cdots \ \ \mu_{n-1} \ \ \mu_n],$$

where at least one product $\mu_i \nu_j$ is nonzero. Furthermore, by Cramer's rule, the matrix function $C(\lambda I_m - A)^{-1}$ can be written in the form

$$C(\lambda I_m - A)^{-1} = \frac{1}{p(\lambda)} D(\lambda),$$

where $D(\lambda)$ is an $n \times m$ matrix polynomial of degree less than m. Now put

$$\begin{bmatrix} w_1(\lambda) \\ w_2(\lambda) \\ \vdots \\ w_{n-1}(\lambda) \\ w_n(\lambda) \end{bmatrix} = D(\lambda) \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_{m-1} \\ \nu_m \end{bmatrix}.$$

Then $w_1, \dots, w_n$ are polynomials of degree less than m, and (3.3) is satisfied. Finally, the relation $\mu_1 w_1 + \cdots + \mu_n w_n = p^{\times} - p$ can be deduced as follows.

$$\frac{p^{\times}(\lambda)}{p(\lambda)} = \det W(\lambda) = 1 + \frac{\mu_1 w_1(\lambda) + \cdots + \mu_n w_n(\lambda)}{p(\lambda)}.$$

■

PROPOSITION 3.4. *Let W be a rational $n \times n$ matrix function, and let p and $p^{\times}$ be monic polynomials of the same positive degree m. Suppose W admits an outer product representation* (3.3) *with respect to p and $p^{\times}$. Then the following statements are equivalent:*

(i) $\delta(W) = m$;

(ii) $\gcd(p; w_1, \dots, w_n; p^{\times}) = 1$;

(iii) *W is companion based, $p_W = p$, and $p_W^{\times} = p^{\times}$.*

Condition (ii) can be replaced by $\gcd(p; w_1; \dots; w_n) = 1$ *or* $\gcd(w_1; \dots; w_n; p^{\times}) = 1$. *This is clear from the identity* $\mu_1 w_1 + \cdots + \mu_n w_n = p^{\times} - p$.

Proof. Since p has degree m, it is clear that (iii) implies (i). To prove the reverse implication, we write W in the form (3.1) where A and $A^{\times}$ are the first companion matrices associated with p and $p^{\times}$, respectively. This is possible by Proposition 3.3. If $\delta(W) = m$, then the realization is minimal, and it follows that (iii) is satisfied.

It remains to be shown that (i) and (ii) are equivalent. For this we argue as follows. Let α be a pole of W and write $r = r(\alpha)$ for the order of α as a pole of W. Let the Laurent expansion of W at α be given by (2.7). Then the pole multiplicity $\delta(W, \alpha)$ of W at α is given by (2.8). For $j = 1, \dots, r$, the $n \times n$ matrix W_{-j} in the matrix (2.8) has the form $W_{-j} = x_j[\mu_1 \ \mu_2 \ \cdots \ \mu_{n-1} \ \mu_n]$ where $x_j \in \mathscr{C}^m$. This is clear from (3.3). Hence $\delta(W, \alpha)$ does not exceed the rank of the following $nr \times r$ matrix

$$\begin{bmatrix} x_r & x_{r-1} & \cdots & \cdots & x_2 & x_1 \\ 0 & x_r & x_{r-1} & & & x_2 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & x_r & x_{r-1} \\ 0 & 0 & \cdots & \cdots & 0 & x_r \end{bmatrix}.$$

Since $x_r \neq 0$, the rank of this matrix equals $r = r(\alpha)$. So $\delta(W, \alpha) \leqslant r(\alpha)$. On the other hand, we always have $\delta(W, \alpha) \geqslant r(\alpha)$. Thus $\delta(W, \alpha) = r(\alpha)$.

It follows that $\delta(W) = m$ if and only if $\Sigma_\alpha \delta(W, \alpha) = m$, where the summation is taken over all poles of W. From (3.3) we see that $r(\alpha)$ does not exceed the multiplicity $m_p(\alpha)$ of α as a zero of p. Also, the multiplicities of the zeros of p add up to m. Hence $\delta(W) = m$ if and only if the following holds: a complex number α is a pole of W if and only if α is a zero of p, and in that case $r(\alpha) = m_p(\alpha)$. But this is, in turn, equivalent to the requirement that $\gcd(p; w_1; \dots; w_n) = 1$. ■

Although Propositions 3.3 and 3.4 give some information on the class of companion based matrix functions, they do not provide an exhaustive description of this class.

EXAMPLE 3.5. Let the rational matrix function W be defined by

$$W(\lambda) = I_3 + \frac{1}{\lambda^3 - 1}\begin{bmatrix} 0 & \lambda^2 & 1 \\ 0 & \lambda & \lambda^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then W admits a minimal realization $W(\lambda) = I_3 + C(\lambda I_3 - A)^{-1}B$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, A is a first companion matrix, and it can be verified easily that $A^\times$ is a first companion matrix as well. It follows that W is companion based. On the other hand, W does not admit an outer product representation, since $\operatorname{rank}[W(\lambda) - I_3] = 2$ for all but a finite number of λ.

3.3. *Companion Based Matrix Functions of Minimal Size*

In this subsection we describe the minimally sized companion based matrix functions with prescribed pole and zero polynomial. Let p and $p^\times$ be two monic polynomials of the same positive degree m. By $n_{\min}(p, p^\times)$ we denote the smallest possible n for which there exists a companion based $n \times n$ matrix function W with $p_W = p$ and $p_W^\times = p^\times$. First we consider the case where p and $p^\times$ are different and have at least one common zero.

THEOREM 3.6. *If p and $p^\times$ are different monic polynomials of the same positive degree m with at least one common zero, then $n_{\min}(p, p^\times) = 2$.*

Moreover, for a rational 2×2 *matrix function* W, *the following statements are equivalent*:

(i) W *is companion based*, $p_W = p$, *and* $p_W^\times = p^\times$,

(ii) $\delta(W) = m$, *and* W *admits an outer product representation with respect to* p *and* $p^\times$.

Proof. Since p and $p^\times$ have at least one common zero, a scalar rational function with pole polynomial p and zero polynomial $p^\times$ does not exist. So $n_{\min}(p, p^\times) \geqslant 2$. On the other hand, it is clear from Proposition 3.3 that $n_{\min}(p, p^\times) \leqslant 2$. To see this, consider the matrix function

$$\begin{bmatrix} 1 & \dfrac{1}{p(\lambda)} \\ 0 & \dfrac{p^\times(\lambda)}{p(\lambda)} \end{bmatrix} = I_2 + \frac{1}{p(\lambda)}\begin{bmatrix} 1 \\ p^\times(\lambda) - p(\lambda) \end{bmatrix}[0 \ \ 1]. \tag{3.4}$$

According to Proposition 3.4, this matrix function is companion based and has pole polynomial p_W and zero polynomial $p_W^\times$. It follows that $n_{\min}(p, p^\times) = 2$.

We now turn to the second part of the theorem. From Proposition 3.4 we know that (ii) implies (i). So we have to show that (i) implies (ii). Here the fact that W is a 2×2 matrix function plays a crucial role. Suppose W is as in (i), and let

$$W(\lambda) = I_2 + C(\lambda I_m - A)^{-1}B$$

be a minimal realization of W such that A and $A^\times$ are the first companion matrices corresponding to p and $p^\times$, respectively. Then $BC = A - A^\times$ has rank 1. Now B is an $m \times 2$ matrix and C is a $2 \times m$ matrix. Hence if rank $B = 2$ and rank $C = 2$, then rank $BC = 2$ as well, contradicting the fact that rank $BC = 1$. Thus rank $B = 1$ or rank $C = 1$. As we shall see in the next paragraph, the latter is impossible, so rank $B = 1$. But then W admits an outer product representation with respect to p and $p^\times$, according to Proposition 3.3. The desired result is now clear from Proposition 3.4.

To see that, indeed, rank $C \neq 1$, we argue as follows. Recall that W^T admits a minimal realization

$$W^T(\lambda) = I_n + B^T(\lambda I_m - A^T)^{-1}C^T.$$

The characteristic polynomial of A^T is p, and that of $A^T - C^T B^T = (A^\times)^T$ is $p^\times$. Assume now that rank $C = 1$. Applying Proposition 3.3 to W^T, we see that W^T admits an outer product representation with respect to p and $p^\times$. The McMillan degree of W^T is the same as that of W, namely m. Therefore Proposition 3.4 guarantees that W^T is a companion based 2×2 matrix function. It follows from Proposition 3.4 guarantees that W^T is a companion based 2×2 matrix function. It follows from Proposition 2.1 that $p_W = p_W^\times$ or $\gcd(p_W; p_W^\times) = 1$. However, both conclusions contradict our assumptions. It follows that rank $C \neq 1$. ∎

Theorem 3.6 remains true when the condition $p \neq p^\times$ is dropped. When $p = p^\times$ the identity $\mu_1 w_1 + \mu_2 w_2 = p^\times - p$ reduces to $w_1 = -(\mu_2/\mu_1)w_2$ or $w_2 = -(\mu_1/\mu_2)w_1$. So in this case the result takes the following form.

THEOREM 3.7. *If p is a monic polynomial of positive degree m, then $n_{\min}(p, p) = 2$. Moreover, for a rational 2×2 matrix function W, the following statements are equivalent:*

(i) *W is companion based, $p_W = p$ and $p_W^\times = p$;*
(ii) *W has the form*

$$W(\lambda) = I_2 + \frac{w(\lambda)}{p(\lambda)} \begin{bmatrix} -\mu_2 \\ \mu_1 \end{bmatrix} [\,\mu_1 \ \ \mu_2\,],$$

where μ_1 and μ_2 are complex numbers, not both equal to zero, w is a polynomial of degree less than m, and $\gcd(p; w) = 1$.

This result reflects the fact that, if $p = p^\times$, then W is companion based if and only if W^T is companion based. Finally, we still have to consider the case $\gcd(p; p^\times) = 1$.

THEOREM 3.8. *If p and $p^\times$ are different monic polynomials of the same positive degree m with $\gcd(p; p^\times) = 1$, then $n_{\min}(p, p^\times) = 1$. In this case $p^\times(\lambda)/p(\lambda)$ is the unique companion based scalar function with pole polynomial p and zero polynomial $p^\times$. Moreover, for a rational 2×2 matrix function W, the following statements are equivalent:*

(i) *W is companion based, $p_W = p$, and $p_W^\times = p^\times$;*
(ii) *$\delta(W) = m$, and W or W^T admits an outer product representation with respect to p and $p^\times$.*

Proof. Obviously, if W is a companion based scalar matrix function with $p_W = p$ and $p_W^{\times} = p^{\times}$, then $W(\lambda) = p^{\times}(\lambda)/p(\lambda)$. Conversely,

$$\frac{p^{\times}(\lambda)}{p(\lambda)} = 1 + \frac{1}{p(\lambda)} w_1(\lambda)\mu_1,$$

where $w_1 = p^{\times} - p$ and $\mu_1 = 1$. Now Proposition 3.4 guarantees that $p^{\times}(\lambda)/p(\lambda)$ is companion based with pole polynomial p and zero polynomial $p^{\times}$.

The second part of Theorem 3.8 can be proved in a similar way to the second part of Theorem 3.6. According to Proposition 2.1, we have in this case again that W is companion based if and only if W^T is companion based. Hence both cases rank $B = 1$ and rank $C = 1$, appearing in the proof of Theorem 3.6, can occur now. The latter explains the symmetry with respect to W and W^T in the theorem. ∎

Note that the assumption $\gcd(p; p^{\times}) = 1$ allows for some special choices of the "parameters" $\mu_1, \dots, \mu_n$, and $w_1, \dots, w_n$ featuring in (3.3). Indeed, by choosing $\mu_1 = 0$, $\mu_2 = 1$, $w_1 = 0$, and $w_2 = p^{\times} - p$, one sees that the 2×2 matrix function W given by

$$W(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{p^{\times}(\lambda)}{p(\lambda)} \end{bmatrix}$$

is companion based with $p_W = p$ and $p_W^{\times} = p^{\times}$. This could already have been guessed from the first part of Theorem 3.8.

3.4. *Another Representation*

In Section 3.2 we concentrated on outer product representations of companion based matrix functions. We shall now discuss another (related) representation, which is particularly useful in studying minimal factorization of companion based 2×2 matrix functions in Sections 4 and 5. Our first observation in this context holds for matrix functions of arbitrary size. It can be viewed as a supplement to Theorem 3.2 and is strongly related to Proposition 3.3.

PROPOSITION 3.9. *Let W be a rational $n \times n$ matrix function, and let p and $p^{\times}$ be monic polynomials of the same positive degree m. Then W admits* *an outer product representation with respect to p and $p^{\times}$ if and only if W can be written in the form*

$$W(\lambda) = T^{-1}\begin{bmatrix} 1 & 0 & \cdots & 0 & \dfrac{r_1(\lambda)}{p(\lambda)} \\ 0 & 1 & \cdots & 0 & \dfrac{r_2(\lambda)}{p(\lambda)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \dfrac{r_{n-1}(\lambda)}{p(\lambda)} \\ 0 & 0 & \cdots & 0 & \dfrac{p^{\times}(\lambda)}{p(\lambda)} \end{bmatrix} T, \tag{3.5}$$

where T is an invertible $n \times n$ matrix and $r_1, \dots, r_{n-1}$ are $n-1$ polynomials of degree less than m.

Proof. Suppose W has the form (3.5). Then W can be written as

$$W(\lambda) = I_n + \frac{1}{p(\lambda)} T^{-1} \begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \\ \vdots \\ r_{n-1}(\lambda) \\ r_n(\lambda) \end{bmatrix} [0 \; 0 \; \cdots \; 0 \; 1] T,$$

where $r_n = p^{\times} - p$. Thus if we define $w_1, \dots, w_n$ and $\mu_1, \dots, \mu_n$ by

$$\begin{bmatrix} w_1(\lambda) \\ w_2(\lambda) \\ \vdots \\ w_{n-1}(\lambda) \\ w_n(\lambda) \end{bmatrix} = T^{-1} \begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \\ \vdots \\ r_{n-1}(\lambda) \\ r_n(\lambda) \end{bmatrix},$$

$$[\mu_1 \;\; \mu_2 \;\; \cdots \;\; \mu_{n-1} \;\; \mu_n] = [0 \; 0 \; \cdots \; 0 \; 1] T,$$

then W has the form (3.3). This is an outer product representation of W with respect to p and $p^\times$. Indeed,

$$\mu_1 w_1(\lambda) + \cdots + \mu_n w_n(\lambda) = [\,\mu_1 \;\; \mu_2 \;\; \cdots \;\; \mu_{n-1} \;\; \mu_n\,]\begin{bmatrix} w_1(\lambda) \\ w_2(\lambda) \\ \vdots \\ w_{n-1}(\lambda) \\ w_n(\lambda) \end{bmatrix}$$

$$= [0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1]\begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \\ \vdots \\ r_{n-1}(\lambda) \\ r_n(\lambda) \end{bmatrix}$$

$$= r_n(\lambda) = p^\times(\lambda) - p(\lambda).$$

Conversely, assume (3.3) is an outer product representation of W with respect to p and $p^\times$. Choose an invertible $n \times n$ matrix T such that the last row of T equals the vector $[\,\mu_1 \;\; \mu_2 \;\; \cdots \;\; \mu_{n-1} \;\; \mu_n]$, and write

$$\begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \\ \vdots \\ r_{n-1}(\lambda) \\ r_n(\lambda) \end{bmatrix} = T\begin{bmatrix} w_1(\lambda) \\ w_2(\lambda) \\ \vdots \\ w_{n-1}(\lambda) \\ w_n(\lambda) \end{bmatrix}.$$

Then $r_1, \dots, r_n$ are polynomials of degree less than m. Since $r_n = \mu_1 w_1 + \cdots + \mu_n w_n = p^\times - p$, the rational matrix function W is of the form (3.5). ■

Note that in this proof, $\gcd(p; w_1; \dots; w_n; p^\times) = 1$ if and only if $\gcd(p; r_1; \dots; r_{n-1}; p^\times) = 1$. It is now obvious that there exist counterparts to Proposition 3.4 and to the second parts of Theorem 3.6, 3.7, and 3.8. For example, we have the following result (cf. Theorems 3.6 and 3.7).

THEOREM 3.10. *Let W be a rational 2×2 matrix function, and let p and $p^\times$ be monic polynomials of the same positive degree m with at least one common zero. Then the following statements are equivalent:*

(i) *W is companion based, $p_W = p$, and $p_W^\times = p^\times$;*
(ii) *W is of the form*

$$W(\lambda) = T^{-1}\begin{bmatrix} 1 & \dfrac{r(\lambda)}{p(\lambda)} \\ 0 & \dfrac{p^\times(\lambda)}{p(\lambda)} \end{bmatrix} T \tag{3.6}$$

where T is an invertible 2×2 matrix, r is a polynomial of degree less than m, and $\gcd(p; r; p^\times) = 1$.

Since p and $p^\times$ have at least one common zero, r cannot be the zero polynomial. Hence (3.6) can be written as

$$W(\lambda) = Q^{-1}(\lambda)\begin{bmatrix} 1 & \dfrac{1}{p(\lambda)} \\ 0 & \dfrac{p^\times(\lambda)}{p(\lambda)} \end{bmatrix} Q(\lambda), \qquad Q(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & r(\lambda) \end{bmatrix} T.$$

In view of this result, the matrix

$$M_0(\lambda) = \begin{bmatrix} 1 & \dfrac{1}{p(\lambda)} \\ 0 & \dfrac{p^\times(\lambda)}{p(\lambda)} \end{bmatrix},$$

already appearing in (3.4), can be viewed as a "model" companion based 2×2 matrix function. When $p = p^\times$, the expression (3.6) becomes

$$W(\lambda) = T^{-1}\begin{bmatrix} 1 & \dfrac{r(\lambda)}{p(\lambda)} \\ 0 & 1 \end{bmatrix} T.$$

This again reflects the fact that in this case W is companion based if and only if this is true for W^T (take for T the reversed identity). In the situation when $\gcd(p; p^{\times}) = 1$, one also has the description (ii), with the understanding that one has to allow for taking transposes (cf. Theorem 3.8).

4. MINIMAL FACTORIZATION

In this section we study minimal factorization of companion based matrix functions. First we show that the property of being companion based is hereditary with respect to minimal factorization.

THEOREM 4.1. *Let U, V, and W be rational $n \times n$ matrix functions, and let $W = UV$ be a minimal factorization of W. If W is companion based, then U and V are companion based as well. Furthermore, in that case* $\gcd(p_U; p_V^{\times}) = 1$.

Proof. We start by choosing minimal realizations of U and V:

$$U(\lambda) = I_n + C_1(\lambda I_{m_1} - A_1)^{-1} B_1, \tag{4.1}$$

$$V(\lambda) = I_n + C_2(\lambda I_{m_2} - A_2)^{-1} B_2. \tag{4.2}$$

Here $m_1 = \delta(U)$ and $m_2 = \delta(V)$. Further, we introduce the matrices

$$A = \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C = [C_1 \ \ C_2]. \tag{4.3}$$

Then A is an $m \times m$ matrix, B is an $m \times n$ matrix, and C is an $n \times m$ matrix where $m = m_1 + m_2$. From the product rule for realizations (cf. Bart et al. [4]) we know that

$$W(\lambda) = I_n + C(\lambda I_m - A)^{-1} B \tag{4.4}$$

is a realization of W, which is minimal, as $m = m_1 + m_2 = \delta(U) + \delta(V) = \delta(W)$. Since W is companion based, the state space isomorphism theorem implies the existence of an invertible $m \times m$ matrix S such that SAS^{-1} and $SA^{\times}S^{-1}$ are first companions. Now it follows from the results of Bart and

Thijsse [12] that $BC = A - A^{\times}$ has rank at most 1 and can be represented in the form $BC = bc^T$ where $b, c \in \mathscr{C}^m$ and b is a common cyclic vector for A and $A^{\times}$ (cf. also Bart and Thijsse [11]). The latter means that

$$\operatorname{rank}[b \; Ab \; \cdots \; A^{m-1}b] = \operatorname{rank}\left[b \; A^{\times}b \; \cdots \; (A^{\times})^{m-1}b\right] = m.$$

Further, we write

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

where b_1, b_2, c_1, and c_2 are vectors of appropriate sizes. Following the standard convention, we put $A_1^{\times} = A_1 - B_1C_1 = A_1 - b_1c_1^T$ and $A_2^{\times} = A_2 - B_2C_2 = A_2 - b_2c_2^T$. Since $A^{\times}$ is of the form

$$A^{\times} = \begin{bmatrix} A_1^{\times} & 0 \\ -b_2c_1^T & A_2^{\times} \end{bmatrix}, \tag{4.5}$$

it follows that b_1 is a cyclic vector for $A_1^{\times}$. Analogously, b_2 is a cyclic vector for A_2. Furthermore, $A_1 - A_1^{\times} = b_1c_1^T$ and $A_2 - A_2^{\times} = b_2c_2^T$. But then it follows that A_1 and $A_1^{\times}$ admit simultaneous reduction to first companion matrices, and that the same holds for A_2 and $A_2^{\times}$ (cf. Bart and Thijsse [11, 12]). Thus both U and V are companion based.

Next, let the matrix M be the direct sum of A_1 and $A_2^{\times}$. Then

$$M = \begin{bmatrix} A_1 & 0 \\ 0 & A_2^{\times} \end{bmatrix} = \begin{bmatrix} A_1 & b_1c_2^T \\ 0 & A_2 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\begin{bmatrix} 0 & c_2^T \end{bmatrix}$$
$$= A - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\begin{bmatrix} 0 & c_2^T \end{bmatrix}.$$

Here $[b_1^T \; b_2^T]^T$ is a cyclic vector for A, which implies that it is a cyclic vector for M as well. As a consequence, M is similar to a first companion matrix. In particular, each eigenvalue of M has geometric multiplicity 1. But this implies that A_1 and $A_2^{\times}$ do not have any common eigenvalue. The latter is equivalent to $\gcd(p_U; p_V^{\times}) = 1$. ∎

In view of Theorem 4.1, it is natural to pose the following question. Suppose $W = UV$ is a minimal factorization of W where both U and V are companion based matrix functions with $\gcd(p_U; p_V^{\times}) = 1$. Does this imply that W is also companion based? In general, the answer is negative. For example, if

$$W(\lambda) = \begin{bmatrix} 1 + \frac{1}{\lambda} & 0 \\ 0 & 1 + \frac{1}{\lambda} \end{bmatrix}, \qquad U(\lambda) = \begin{bmatrix} 1 + \frac{1}{\lambda} & 0 \\ 0 & 1 \end{bmatrix},$$

$$V(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \frac{1}{\lambda} \end{bmatrix},$$

then $\delta(W) = 2$, $\delta(U) = \delta(V) = 1$, and $W = UV$ is a minimal factorization of W. Furthermore, U and V are companion based, and W is not (indeed, $\operatorname{rank}[W(\lambda) - I_2] = 2$ for all $\lambda \neq 0$).

Necessary and sufficient conditions in terms of the realizations (4.1) and (4.2) guaranteeing that a minimal product of companion based matrix functions is companion based as well are provided in Proposition 4.3. This proposition is based on the following well-known auxiliary result.

LEMMA 4.2. *Suppose A and Z are square matrices, a is a cyclic vector for A, and z is a cyclic vector for Z. Let the matrix M be the direct sum of A and Z. Then the following statements are equivalent*:

(i) *A and Z do not have any common eigenvalue*;

(ii) *$\begin{bmatrix} a \\ z \end{bmatrix}$ is a cyclic vector for $M = \begin{bmatrix} A & 0 \\ 0 & Z \end{bmatrix}$.*

Note that if a is a cyclic vector for A, then A is similar to a companion matrix. This is equivalent to the condition that each eigenvalue of A has geometric multiplicity 1. Of course, a similar result holds for Z as well.

Proof. If A and Z have a common eigenvalue, then the corresponding eigenspace of M has dimension 2. This implies that a cyclic vector for M does not exist.

Conversely, if A and Z do not have any common eigenvalue, then each eigenvalue of M has geometric multiplicity 1. Now Hautus' test from systems theory [18] can be used to show that $[a^T \; z^T]^T$ is a cyclic vector for M whenever a and z are cyclic vectors for A and Z respectively. The details are left to the reader. ∎

PROPOSITION 4.3. *Let U and V be companion based matrix functions, given by* (4.1) *and* (4.2) *respectively. Suppose further $W = UV$ is a minimal factorization of W, and let A, B and C be given by* (4.3).

(i) *Assume $A_1 \neq A_1^{\times}$ and $A_2 \neq A_2^{\times}$. Then W is companion based if and only if A_1 and $A_2^{\times}$ do not have any common eigenvalue and* rank $BC = 1$.

(ii) *Assume $A_1 = A_1^{\times}$ and $A_2 \neq A_2^{\times}$. Then W is companion based if and only if A_1 and $A_2^{\times}$ do not have any common eigenvalue*, rank $BC = 1$, *$B_1C_2 = b_1c_2^T \neq 0$ where b_1 is a cyclic vector for $A_1 = A_1^{\times}$, and $B_2C_1 = 0$.*

(iii) *Assume $A_1 \neq A_1^{\times}$ and $A_2 = A_2^{\times}$. Then W is companion based if and only if A_1 and $A_2^{\times}$ do not have any common eigenvalue*, rank $BC = 1$, *$B_1C_2 = 0$, and $B_2C_1 = b_2c_1^T \neq 0$ where b_2 is a cyclic vector for $A_2 = A_2^{\times}$.*

(iv) *Assume $A_1 = A_1^{\times}$ and $A_2 = A_2^{\times}$. Then W is companion based if and only if A_1 and $A_2^{\times}$ do not have any common eigenvalue and $BC = 0$.*

Note that the condition that A_1 and $A_2^{\times}$ do not have any common eigenvalue is equivalent to the condition $\gcd(p_U; p_V^{\times}) = 1$. Unfortunately, the conditions in (ii) and (iii) involving B_1C_2 and B_2C_1 (which are not quite elegant) cannot be omitted. Examples of this can be found easily.

Proof. (i): From Theorem 4.1 we know that the condition that A_1 and $A_2^{\times}$ do not have any common eigenvalue [i.e. $\gcd(p_U, p_V^{\times}) = 1$] is necessary. Furthermore, it is obvious that the condition rank $BC = 1$ is necessary as well. Thus we can proceed to prove the sufficiency of the conditions. Write

$$A - A^{\times} = BC = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} c_1^T & c_2^T \end{bmatrix}, \tag{4.6}$$

where b_1, b_2, c_1, and c_2 are vectors of appropriate sizes. Then $A_1 - A_1^{\times} = b_1c_1^T$ and $A_2 - A_2^{\times} = b_2c_2^T$. Since rank factorizations are essentially unique, it follows that b_1 is a cyclic vector for A_1 and b_2 is a cyclic vector for $A_2^{\times}$. Further, Lemma 4.2 implies that

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ is a cyclic vector for the matrix } M = \begin{bmatrix} A_1 & 0 \\ 0 & A_2^{\times} \end{bmatrix}.$$

We also have

$$A = M + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} 0 & c_2^T \end{bmatrix}, \qquad A^{\times} = M - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} c_1^T & 0 \end{bmatrix}.$$

This implies that $[b_1^T \; b_2^T]^T$ is a common cyclic vector for A and $A^\times$. In view of (4.6) and the results of Bart and Thijsse [12], it follows that A and $A^\times$ admit simultaneous reduction to first companion matrices. Thus W is companion based.

(ii): Again, the condition that A_1 and $A_2^\times$ do not have any common eigenvalue and the condition rank $BC = 1$ are necessary. Furthermore, if W is companion based and $A_2 \neq A_2^\times$, then

$$A - A^\times = BC = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} c_1^T & c_2^T \end{bmatrix},$$

where $[b_1^T \; b_2^T]^T$ is a common cyclic vector for A and $A^\times$. Now $b_2 c_2^T \neq 0$ implies $c_2 \neq 0$. Also, if $[b_1^T \; b_2^T]^T$ is a cyclic vector for $A^\times$, then b_1 is a cyclic vector for $A_1 = A_1^\times$, which implies $b_1 \neq 0$. As a consequence, $b_1 c_2^T \neq 0$. Together with $b_1 c_1^T = 0$ and rank $BC = 1$, this implies $b_2 c_1^T = 0$. Thus the necessity of the conditions has been established.

The sufficiency of the conditions can be proved in the same way as in (i). Indeed, using the fact that rank factorizations are essentially unique, one can deduce from the given conditions that

$$A - A^\times = BC = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} 0 & c_2^T \end{bmatrix},$$

where b_1 is the given cyclic vector for $A_1 = A_1^\times$ and b_2 is a common cyclic vector for A_2 and $A_2^\times$. Thus $[b_1^T \; b_2^T]^T$ is a common cyclic vector for A and $A^\times$.

(iii): The proof of (iii) is similar to that of (ii).

(iv): As before, the condition that A_1 and $A_2^\times$ do not have any common eigenvalue is necessary. Furthermore, suppose rank $BC = 1$. Then

$$A - A^\times = BC = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} c_1^T & c_2^T \end{bmatrix},$$

where $[b_1^T \; b_2^T]^T$ is a common cyclic vector for A and $A^\times$. Thus b_1 is a cyclic vector for $A_1 = A_1^\times$, which implies $b_1 \neq 0$. However, $b_1 c_1^T = A_1 - A_1^\times = 0$. As a consequence, $c_1 = 0$. In a similar way it follows that $c_2 = 0$. However, $c_1 = 0$ and $c_2 = 0$ contradicts the assumption rank $BC = 1$. It follows that $BC = 0$. Thus the necessity of the conditions has been established. The sufficiency of the conditions is obvious. ■

In Theorem 4.4 we describe all minimal factorizations of a given companion based matrix function in terms of special factorizations of the associated pole and zero polynomial.

THEOREM 4.4. *Let W be a companion based $n \times n$ matrix function. Let $\mathscr{M}$ be the collection of all ordered pairs (U, V) where U and V are companion based $n \times n$ matrix functions and $W = UV$ is a minimal factorization of W. Furthermore, let $\mathscr{P}$ be the collection of all ordered pairs $(p, p^{\times})$ where p is a monic divisor of the pole polynomial p_W, $p^{\times}$ is a monic divisor of the pole polynomial $p_W^{\times}$,* $\deg p + \deg p^{\times} = \delta(W)$ *and* $\gcd(p; p^{\times}) = 1$. *For $(U, V) \in \mathscr{M}$, define $\Phi(U, V) = (p_U, p_V^{\times})$. Then $\Phi: \mathscr{M} \to \mathscr{P}$ is a well-defined bijection.*

Proof. Let $(U, V) \in \mathscr{M}$. So $W = UV$ is a minimal factorization of W. From Section 2 we know that $p_W = p_U p_V$ and $p_W^{\times} = p_U^{\times} p_V^{\times}$. So p_U is a monic divisor of p_W, and $p_V^{\times}$ is a monic divisor of $p_W^{\times}$. Also $\deg p_U + \deg p_V^{\times} = \delta(U) + \delta(V) = \delta(W)$. Finally, in Theorem 4.1 we have shown that $\gcd(p_U; p_V^{\times}) = 1$. Thus Φ is well defined.

Next we have to prove that Φ is a bijection. To that end, we choose a minimal realization

$$W(\lambda) = I_n + C(\lambda I_m - A)^{-1} B, \tag{4.7}$$

where A and $A^{\times}$ are first companion matrices. Let $W = UV$ be a minimal factorization of W. From Bart et al. [4] we know that there exists a unique direct sum decomposition $\mathscr{C}^m = M \oplus N$ with the following properties: M is A-invariant, n is $A^{\times}$-invariant, and U and V admit the minimal realizations

$$U(\lambda) = I_n + C_M(\lambda I_M - A_M)^{-1} B_M, \tag{4.8}$$

$$V(\lambda) = I_n + C_N(\lambda I_N - A_N)^{-1} B_N. \tag{4.9}$$

Here A_M, A_N, B_M, B_N, C_M, and C_N are determined by

$$A = \begin{bmatrix} A_M & B_M C_N \\ 0 & A_N \end{bmatrix} : M \oplus N \to M \oplus N, \tag{4.10}$$

$$B = \begin{bmatrix} B_M \\ B_N \end{bmatrix} : \mathscr{C}^m \to M \oplus N, \tag{4.11}$$

$$C = \begin{bmatrix} C_M & C_N \end{bmatrix} : M \oplus N \to \mathscr{C}^n, \tag{4.12}$$

and, as a consequence,

$$A^{\times}=\begin{bmatrix} A_M - B_M C_M & 0 \\ -B_N C_M & A_N - B_N C_N \end{bmatrix} : M \oplus N \to M \oplus N. \quad (4.13)$$

Now $p_U(\lambda) = \det(\lambda I_M - A_M)$ and, since A is a first companion, we have $M = \operatorname{Ker} p_U(A)$. Also, $p_V^{\times}(\lambda) = \det(\lambda I_N - A_N + B_N C_N)$ and $N = \operatorname{Ker} p_V^{\times}(A^{\times})$. We conclude that M and N are completely determined by p_U and $p_V^{\times}$. Thus U and V are completely determined by p_U and $p_V^{\times}$. This implies the injectivity of Φ.

Finally, we establish the surjectivity of Φ by constructing $\Phi^{-1}(p, p^{\times})$ for a given pair $(p, p^{\times}) \in \mathscr{P}$. Put $M = \operatorname{Ker} p(A)$ and $N = \operatorname{Ker} p^{\times}(A^{\times})$. Then $A[M] \subset M$ and $A^{\times}[N] \subset N$. To describe M and N we write

$$p(\lambda) = (\lambda - \mu_1)^{n_1} \cdots (\lambda - \mu_s)^{n_s},$$

$$p^{\times}(\lambda) = (\lambda - \mu_{s+1})^{n_{s+1}} \cdots (\lambda - \mu_t)^{n_t},$$

where $\mu_1, \dots, \mu_t$ are t different complex numbers. Recall in this context that $\gcd(p; p^{\times}) = 1$. We also put

$$v_k(z) = \frac{1}{k!} \frac{d^k}{dz^k} [1 \quad z \quad \cdots \quad z^{m-2} z^{m-1}]^T,$$

where z is a complex variable and $k = 0, \dots, m-1$. Thus the matrix F defined by

$$F = \big[v_0(\mu_1), \dots, v_{n_1-1}(\mu_1), v_0(\mu_2), \dots,$$

$$v_{n_2-1}(\mu_2), \dots, v_0(\mu_t), \dots, v_{n_t-1}(\mu_t)\big] \quad (4.14)$$

is a (confluent) Vandermonde matrix (cf. Barnett [3] and Lu [24]). Since A is a first companion matrix, p is a divisor of p_W, and $p_W = \det(\lambda I_m - A)$, it is well known that the space $M = \operatorname{Ker} p(A)$ is spanned by the vectors

$$v_k(\mu_j), \qquad j = 1, \dots, s; \; k = 0, \dots, n_j - 1.$$

Analogously, the space $N = \operatorname{Ker} p^{\times}(A^{\times})$ is spanned by the vectors

$$v_k(\mu_j), \qquad j = s+1, \dots, t;\ k = 0, \dots, n_j - 1.$$

Now $n_1 + \cdots + n_t = \deg p + \deg p^{\times} = \delta(W) = m$. Furthermore, it is well known that for the (confluent) Vandermonte matrix F defined by (4.14) we have

$$\det F = \prod_{j=1}^{t} \prod_{k=j+1}^{t} (\mu_k - \mu_j)^{n_k n_j} \neq 0.$$

Thus $\mathscr{C}^m = M \oplus N$. With respect to this direct sum decomposition, we write A, B, C, and $A^{\times}$ in the form (4.10), (4.11), (4.12), and (4.13) respectively. Further, let $U(\lambda)$ and $V(\lambda)$ be as in (4.8) and (4.9). We claim that $(U, V) = \Phi^{-1}(p, p^{\times})$. Indeed, $W = UV$, and this is a minimal factorization. In other words, $(U, V) \in \mathscr{M}$. Also, $p_U(\lambda) = \det(\lambda I_M - A_M) = p(\lambda)$ and $p_V^{\times}(\lambda) = \det(\lambda I_N - A_N + B_N C_N) = p^{\times}(\lambda)$. So $\Phi(U, V) = (p, p^{\times})$, as desired. ■

Elaborating on the proof of Theorem 4.4, we remark that at first sight Φ^{-1} depends on the choice of the minimal realization (4.7) of W. Since Φ does not, such a dependence cannot exist. The following argument will make this transparent. Suppose

$$W(\lambda) = I_n + C_1(\lambda I_m - A_1)^{-1} B_1 \tag{4.15}$$

is another minimal realization of W such that A_1 and $A_1^{\times}$ are first companions. By the state space isomorphism theorem, there exists an invertible $m \times m$ matrix S such that

$$A_1 = SAS^{-1}, \qquad B_1 = SB, \qquad C_1 = CS^{-1}, \qquad A_1^{\times} = SA^{\times}S^{-1}.$$

These identities imply $\det(\lambda I_m - A) = \det(\lambda I_m - A_1)$ and $\det(\lambda I_m - A^{\times}) = \det(\lambda I_m - A_1^{\times})$. Since the matrices involved are first companion matrices, $A = A_1$ and $A^{\times} = A_1^{\times}$.

From Bart et al. [4] we know that there is a one-to-one correspondence between the minimal factorizations of W on the one hand and the pairs (M, N) of subspaces of $\mathscr{C}^m$ satisfying $A[M] \subset M$, $A^{\times}[N] \subset N$, and $M \oplus N = \mathscr{C}^m$ on the other. Let (M, N) be such a pair, and consider the associated minimal factorization $W = UV$ induced by (4.7) and $W = U_1 V_1$ induced by (4.15). We claim that $U = U_1$ and $V = V_1$. In order to see this, it is sufficient to show that $S[M] \subset M$ and $S[N] \subset N$. Indeed, in that case the matrices S

and S^{-1} can be written as follows with respect to the decomposition $\mathscr{C}^m = M \oplus N$:

$$S = \begin{bmatrix} S_M & 0 \\ 0 & S_N \end{bmatrix}, \qquad S^{-1} = \begin{bmatrix} S_M^{-1} & 0 \\ 0 & S_N^{-1} \end{bmatrix}.$$

Furthermore, the matrices A, B, and C and the matrices A_1, B_1, and C_1 admit the following representation with respect to the decomposition $\mathscr{C}^m = M \oplus N$:

$$A = \begin{bmatrix} A_M & B_M C_N \\ 0 & A_N \end{bmatrix}, \qquad B = [\, B_M \quad B_N \,],$$

$$C = \begin{bmatrix} C_M \\ C_N \end{bmatrix},$$

$$A_1 = \begin{bmatrix} S_M A_M S_M^{-1} & S_M B_M C_N S_N^{-1} \\ 0 & S_N A_N S_N^{-1} \end{bmatrix}, \qquad B_1 = [\, S_M B_M \quad S_N B_N \,],$$

$$C_1 = \begin{bmatrix} C_M S_M^{-1} \\ C_N S_N^{-1} \end{bmatrix}.$$

Using (4.8) and (4.9), one sees that $U = U_1$ and $V = V_1$.

Thus what remains to be shown is that $S[M] \subset M$ and $S[N] \subset N$. Now $A = A_1 = SAS^{-1}$, so $AS = SA$. Since A is a first companion matrix, hence nonderogatory, it follows that S is a polynomial in A. But then the invariant subspace M for A is invariant for S as well. Analogously, N is invariant for S.

EXAMPLE 4.5. In this example we consider the companion based matrix function W defined by

$$W(\lambda) = \begin{bmatrix} 1 & \dfrac{1}{(\lambda^2+1)^2} \\ 0 & \dfrac{\lambda^2+4}{\lambda^2+1} \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{1}{(\lambda-i)^2(\lambda+i)^2} \\ 0 & \dfrac{(\lambda-2i)(\lambda+2i)}{(\lambda-i)(\lambda+i)} \end{bmatrix}.$$

Note that $p_W = (\lambda - i)^2(\lambda + i)^2$ and $p_W^\times = (\lambda - 2i)(\lambda - i)(\lambda + i)(\lambda + 2i)$. Hence if we define $p_1 = (\lambda - i)^2$, $p_2 = (\lambda + i)^2$, $p_1^\times = (\lambda - 2i)(\lambda - i)$, and $p_2^\times = (\lambda + i)(\lambda + 2i)$, then obviously $p_W = p_1 p_2$ and $p_W^\times = p_1^\times p_2^\times$. Furthermore, $\gcd(p_1; p_2^\times) = 1$. By applying techniques that are described further in Section 5, we find the following minimal factorization of W:

$$W(\lambda) = \begin{bmatrix} 1 & \dfrac{r_1(\lambda)}{(\lambda - i)^2} \\ 0 & \dfrac{(\lambda - 2i)(\lambda - i)}{(\lambda - i)^2} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{r_2(\lambda)}{(\lambda + i)^2} \\ 0 & \dfrac{(\lambda + 2i)(\lambda + i)}{(\lambda + i)^2} \end{bmatrix},$$

where $r_1(\lambda) = \frac{1}{36}(-5i\lambda - 11)$ and $r_2(\lambda) = \frac{1}{36}(5i\lambda - 14)$.

A rational matrix function is called *irreducible* if it does not admit any nontrivial minimal factorization. Obviously, each elementary rational matrix function is irreducible. For nonelementary companion based matrix functions we have the following result.

COROLLARY 4.6. *Let W be a companion based matrix function with $\delta(W) \geqslant 2$. Then W is irreducible if and only if there exists a complex number α such that $p_W(\lambda) = p_W^\times(\lambda) = (\lambda - \alpha)^{\delta(W)}$ for all $\lambda \in \mathscr{C}$.*

Theorems 4.1 and 4.4 are concerned with minimal factorizations of companion based matrix functions involving two factors. Analogous results can be obtained for minimal factorizations involving more than two factors. In that context the relevant factorizations of p_W and $p_W^\times$ are of the form

$$p_W = p_1 \cdots p_k, \qquad p_W^\times = p_1^\times \cdots p_k^\times,$$

where $p_1, \dots, p_k$ and $p_1^\times, \dots, p_k^\times$ are monic scalar polynomials with $\deg p_j = \deg p_j^\times$ for $j = 1, \dots, k$ and $\gcd(p_1 \dots p_j; p_{j+1}^\times \dots p_k) = 1$ for $j = 1, \dots, k-1$. The factor W_j of the corresponding minimal factorization $W_1 \cdots W_k$ of W has pole polynomial p_j and zero polynomial $p_j^\times$. The proof of this more general result is basically an induction argument based on the results of Theorems 4.1 and 4.4.

A complete factorization of W is a minimal factorization involving $\delta(W)$ factors. The existence of a complete factorization can be determined by repeated application of the Theorems 4.1 and 4.4. The details are as follows.

COROLLARY 4.7. *If W is a companion based matrix function with $\delta(W) = m$, then W admits complete factorization if and only if there exist orderings $\alpha_1, \dots, \alpha_m$ and $\alpha_1^\times, \dots, \alpha_m^\times$ of the zeros of p_W and $p_W^\times$ such that $\alpha_i \neq \alpha_j^\times$ whenever $i < j$.*

Such orderings exist if and only if there exists an ordering $\beta_1, \dots, \beta_k$ of the (different) common zeros of p_W and $p_W^\times$ such that

$$\sum_{i=1}^{h} m^\times(\beta_i) + \sum_{i=h}^{k} m(\beta_i) \leqslant m+1, \qquad h = 1, \dots, k. \qquad (4.16)$$

Here $m(\beta_i)$ and $m^\times(\beta_i)$ denote the algebraic multiplicity of β_i as a zero of p_W and $p_W^\times$, respectively. In Section 5 we present an algorithm, based on Johnson's rule from job scheduling theory, for verifying whether the combinatorial condition in Corollary 4.7 can be satisfied (see also [11], Remark 2.5 and [12], Proposition 2.2).

If $\alpha_1, \dots, \alpha_m$ is an ordering of the zeros of p_W and $\alpha_1^\times, \dots, \alpha_m^\times$ is an ordering of the zeros of $p_W^\times$ satisfying the condition $\alpha_i \neq \alpha_j^\times$ whenever $i < j$, then W admits complete factorization. In fact, for appropriate rank 1 matrices R_i one has the complete factorizations (2.10), (2.11), and (2.12) of W, W^T, and W^{-1}, respectively. Corollary 4.7 can also be deduced by combining Theorems 3.2 and 6.1 in Bart and Hoogland [6].

In the final part of this section we consider canonical right Wiener-Hopf factorization with respect to the real line of a companion based matrix function W. A canonical right Wiener-Hopf factorization is a special case of a minimal factorization. For information on Wiener-Hopf factorization, see [4], [15], [16], and the references given there. Instead of the real line, one can also consider other closed contours on the Riemann sphere. Sometimes we shall omit the reference to the real line.

COROLLARY 4.8. *Let W be a companion based $n \times n$ matrix function. Then the following statements are equivalent:*

(i) *W admits canonical right Wiener-Hopf factorization with respect to the real line;*

(ii) *the polynomials p_W and $p_W^\times$ have no zeros on the real line, and the number of zeros of p_W in the open upper half plane equals the number of zeros of $p_W^\times$ in the open upper half plane.*

Here the zeros are counted according to multiplicity. The results remains true when *right* Wiener-Hopf factorization is replaced by *left* Wiener-Hopf factorization. In particular, it follows that W admits canonical right Wiener-Hopf factorization if and only if W admits canonical left factorizations (cf. Ball and Ran [2]).

Proof. Let $W = W_- W_+$ be a canonical right Wiener-Hopf factorization. Let the pole and zero polynomials of W_- and W_+ be denoted by p_-, p_+, $p_-^\times$, and $p_+^\times$. Then p_- and $p_-^\times$ have no zeros in the closed lower half plane, and p_+ and $p_+^\times$ have no zeros in the closed upper half plane. Also $p_W = p_- p_+$ and $p_W^\times = p_-^\times p_+^\times$. So p_W and $p_W^\times$ have no zeros on the real line. Also $\deg p_- = \deg p_-^\times$. Thus the number of zeros of p_W in the open upper half plane equals the number of zeros of $p_W^\times$ in the open upper half plane.

Conversely, assume that (ii) is satisfied. Write $p_W = p_- p_+$ and $p_W^\times = p_-^\times p_+^\times$, where p_- and $p_-^\times$ have no zeros in the closed lower half plane and p_+ and $p_+^\times$ have no zeros in the closed upper half plane. By assumption $\deg p_- = \deg p_-^\times$, so $\deg p_- + \deg p_+^\times = \delta(W)$. Clearly, $\gcd(p_-; p_+^\times) = 1$. Apply now Theorem 4.4. This gives a minimal factorization of W, and it should be clear that it is a canonical right Wiener-Hopf factorization. ■

EXAMPLE 4.9. The factorization shown in Example 4.4 is a canonical right Wiener-Hopf factorization with respect to the real line.

Corollary 4.8 is concerned with *canonical* Wiener-Hopf factorization of companion based matrix functions. More general results can be obtained for *noncanonical* Wiener-Hopf factorization of such functions. For instance, it can be shown that for a companion based matrix function either all factorization indices are nonnegative or all factorization indices are nonpositive. The authors intend to return to this topic in a future publication.

5. ALGORITHMIC ASPECTS AND EXAMPLES

Corollary 4.7 shows that a companion based matrix function W admits complete factorization if and only if a combinatorial condition involving the zeros of the polynomials p_W and $p_W^\times$ is satisfied. To check whether or not this condition can be satisfied, a simple algorithm, based on Johnson's rule from job scheduling theory, is available. Assuming that the zeros of p_W and $p_W^\times$ are known, the algorithm is as follows. Write

$$p_W(\lambda) = (\lambda - \beta_1)^{s_1} \cdots (\lambda - \beta_k)^{s_k},$$
$$p_W^\times(\lambda) = (\lambda - \beta_1)^{t_1} \cdots (\lambda - \beta_k)^{t_k}, \qquad (5.1)$$

where $\beta_1, \ldots, \beta_k$ are k different complex numbers. Next, create a list $(d_1, \ldots, d_{2k})$ containing the nonnegative integers $s_1, \ldots, s_k, t_1, \ldots, t_k$ in nondecreasing order. This list is called the D(egrees)-list. Now one proceeds as follows.

1. Start with two empty lists. The first list is called the F(irst)-list and the second list is called the L(ast)-list.
2. Do while the D-list is nonempty:
   a. If the first number in the D-list equals s_b for some b, then put b at the front of the L-list else put b at the rear of the F-list.
   b. Delete s_b and t_b from the D-list.
3. Combine the F-list $(b_1, \ldots, b_q)$ and the L-list $(b_{q+1}, \ldots, b_k)$ into $(b_1, \ldots, b_q, b_{q+1}, \ldots, b_k)$.

Then W admits complete factorization if and only if

$$\sum_{i=1}^{h} t_{b_i} + \sum_{i=h}^{k} s_{b_i} \leqslant m + 1, \qquad h = 1, \ldots, k. \tag{5.2}$$

Note that this condition is similar to (4.16). Furthermore, if the condition (5.2) is satisfied, then $s_{b_k} \leqslant 1$ and $t_{b_1} \leqslant 1$. Thus in that case

$$\underbrace{\beta_{b_1}, \ldots, \beta_{b_1}}_{s_{b_1}\text{ times}}, \underbrace{\beta_{b_2}, \ldots, \beta_{b_2}}_{s_{b_2}\text{ times}}, \ldots, \underbrace{\beta_{b_{k-1}}, \ldots, \beta_{b_{k-1}}}_{s_{b_{k-1}}\text{ times}}, (\beta_{b_k})$$

is an ordering $\alpha_1, \ldots, \alpha_m$ of the zeros of p_W. Here the parentheses around β_{b_k} are used to indicate that β_{b_k} may or may not be present, depending on whether $s_{b_k} = 1$ or $s_{b_k} = 0$. Also,

$$(\beta_{b_1}), \underbrace{\beta_{b_2}, \ldots, \beta_{b_2}}_{t_{b_2}\text{ times}}, \ldots, \underbrace{\beta_{b_{k-1}}, \ldots, \beta_{b_{k-1}}}_{t_{b_{k-1}}\text{ times}}, \underbrace{\beta_{b_k}, \ldots, \beta_{b_k}}_{t_{b_k}\text{ times}}$$

is an ordering $\alpha_1^\times, \ldots, \alpha_m^\times$ of the zeros of $p_W^\times$. The parentheses around β_{b_1} are used as above.

If (5.2) is satisfied, then the above orderings satisfy the desired combinatorial condition $\alpha_i \neq \alpha_j^\times$ whenever $i < j$, which was mentioned in Corollary 4.7. This means that there exist complete factorizations (2.10), (2.11), and (2.12) of W, W^T, and W^{-1}. A justification of the algorithm is given in Bart and Kroon [9, 10].

It is illustrative to provide some background information on the above algorithm. The inequalities (5.2) can be rewritten as

$$\max\left[\{0\} \cup \left\{ \sum_{i=1}^{h} t_{b_i} + \sum_{i=h}^{k} s_{b_i} - (m+1) \,\middle|\, h = 1,\dots,k \right\}\right] = 0. \quad (5.3)$$

The left hand side of (5.3) has an interesting interpretation: it equals the smallest nonnegative integer z for which there exists an ordering $\alpha_1, \dots, \alpha_m$ of the zeros of p_W and an ordering $\alpha_1^\times, \dots, \alpha_m^\times$ of the zeros of $p^\times$ such that

$$\alpha_i \neq \alpha_{j+z}^\times, \qquad 1 \leqslant i < j \leqslant m - z.$$

Actually, this result is a special case of Johnson's rule associated with the two machine flow shop problem (2MFSP) from job scheduling theory (cf. Baker [1]). This connection with 2MFSP will be explained in more detail in the forthcoming paper [10] (see also [9]).

Johnson's rule is a fast algorithm. Initially sorting the degrees of the polynomials p and $p^\times$ can be done in $O(m \log m)$ time. The remaining part of the algorithm takes $O(m)$ time. So the complete procedure takes $O(m \log m)$ time. Note that m is the McMillan degree $\delta(W)$ of W.

From now on, let p and $p^\times$ be monic polynomials of the same positive degree m. In the remainder of this section we specialize to the case of companion based 2×2 matrix functions having p as pole polynomial and $p^\times$ as zero polynomial. In Section 3 we have seen that these functions can be obtained in a simple way from the special function

$$M(\lambda) = \begin{bmatrix} 1 & \dfrac{r(\lambda)}{p(\lambda)} \\ 0 & \dfrac{p^\times(\lambda)}{p(\lambda)} \end{bmatrix}. \quad (5.4)$$

Here r is a polynomial of degree less than m such that $\gcd(p; r; p^\times) = 1$.

We shall describe all minimal factorizations of the companion based matrix function M involving two factors. The basis for our analysis is the following well known lemma (cf. Lang [23]). The proof of this lemma shows that there is a connection with the Sylvester matrix (also called the resultant) associated with two polynomials.

LEMMA 5.1. *Let q_1 and q_2 be two monic scalar polynomials, let $d = \deg q_1 + \deg q_2$, and assume $\gcd(q_1; q_2) = 1$. Then for each polynomial r of degree less than d the polynomial equation*

$$r_1 q_1 + r_2 q_2 = r \tag{5.5}$$

has a unique solution r_1 and r_2 with $\deg r_1 < \deg q_2$ and $\deg r_2 < \deg q_1$.

Proof. Put $s = \deg q_1$ and $t = \deg q_2$. The case $s = 0$ or $t = 0$ is trivial. So we assume that both s and t are positive. Write

$$r_1(\lambda) = x_0 + x_1\lambda + \cdots + x_{t-1}\lambda^{t-1}, \qquad r_2(\lambda) = y_0 + y_1\lambda + \cdots + y_{s-1}\lambda^{s-1}.$$

Then Equation (5.5) can be transformed into a system of d linear equations in the unknowns $x_0, \dots, x_{t-1}$ and $y_0, \dots, y_{s-1}$. The coefficient matrix associated with this system of equations is the resultant (also called the Sylvester matrix) of the polynomials q_1 and q_2. It is well known that the determinant of this matrix equals $\prod_{i=1}^{s}\prod_{j=1}^{t}(\mu_i - \nu_j)$, where $\mu_1, \dots, \mu_s$ are the zeros of q_1 and $\nu_1, \dots, \nu_t$ are the zeros of q_2. Since $\gcd(q_1; q_2) = 1$, the determinant is nonzero. Thus the system of equations is uniquely solvable. ■

In connection with Lemma 5.1, the following observation can be made. As $\gcd(q_1; q_2) = 1$, one can employ the Euclidean algorithm to produce two polynomials a_1 and a_2 such that $a_1 q_1 + a_2 q_2 = 1$. These polynomials can be used next to describe the polynomials r_1 and r_2 featuring in the lemma. Indeed, one can take $r_1 \equiv a_1 r \pmod{q_2}$ and $r_2 \equiv a_2 r \pmod{q_1}$.

Now we will describe all minimal factorizations of the companion based matrix function M given by (5.4). Let p_1 be a monic divisor of p, and let $p_2^{\times}$ be a monic divisor of $p^{\times}$. Assume $\deg p_1 + \deg p_2^{\times} = m$ and $\gcd(p_1; p_2^{\times}) = 1$. Then we have

$$p = p_1 p_2, \qquad p^{\times} = p_1^{\times} p_2^{\times}, \tag{5.6}$$

where $\deg p_1 = \deg p_1^{\times}$ and $\deg p_2 = \deg p_2^{\times}$. From Lemma 5.1 we know that the polynomial equation

$$r_1 p_2^{\times} + r_2 p_1 = r \tag{5.7}$$

has a unique solution r_1, r_2 where r_1 and r_2 are polynomials with $\deg r_1 < \deg p_1$ and $\deg r_2 < \deg p_2$. It is not difficult to see that for these polynomials r_1 and r_2 we have

$$M(\lambda) = \begin{bmatrix} 1 & \dfrac{r(\lambda)}{p(\lambda)} \\ 0 & \dfrac{p^\times(\lambda)}{p(\lambda)} \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{r_1(\lambda)}{p_1(\lambda)} \\ 0 & \dfrac{p_1^\times(\lambda)}{p_1(\lambda)} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{r_2(\lambda)}{p_2(\lambda)} \\ 0 & \dfrac{p_2^\times(\lambda)}{p_2(\lambda)} \end{bmatrix}. \tag{5.8}$$

Combining (5.6) and (5.7) and taking into account $\gcd(p; r, p^\times) = 1$, it is easy to deduce that $\gcd(p_1; r_1, p_1^\times) = \gcd(p_2; r_2, p_2^\times) = 1$. Hence the factors in the right hand side of (5.8) are companion based with McMillan degrees $\deg p_1$ and $\deg p_2$, respectively.

Conversely, let $M = UV$ be any minimal factorization of M. Then (5.6) is satisfied by $p_1 = p_U$, $p_2 = p_V$, $p_1^\times = p_U^\times$, and $p_2^\times = p_V^\times$. We know that $\gcd(p_U; p_V^\times) = 1$, which implies that we can solve Equation (5.7) with $p_1 = p_U$ and $p_2^\times = p_V^\times$. Thus we can produce the minimal factorization (5.8). The first factor in (5.8) has pole polynomial $p_1 = p_U$, and the second factor has zero polynomial $p_2^\times = p_V^\times$. As we have seen in Theorem 4.4, these properties determine the factors uniquely. So the factors in (5.8) are U and V.

Summarizing, we see that the minimal factorization problem for M (involving two factors) has been reduced to finding solutions to the following two subproblems:

(i) Find monic polynomials p_1, p_2, $p_1^\times$, and $p_2^\times$ with $\deg p_1 = \deg p_1^\times$ and $\deg p_2 = \deg p_2^\times$, satisfying $p = p_1 p_2$, $p^\times = p_1^\times p_2^\times$, and $\gcd(p_1; p_2^\times) = 1$.

(ii) Find polynomials r_1 and r_2 with $\deg r_1 < \deg p_1$ and $\deg r_2 < \deg p_2$, satisfying $r_1 p_2^\times = r_2 p_1 = r$.

If subproblem (i) has been solved, then the existence of a (unique) solution to subproblem (ii) is guaranteed by Lemma 5.1.

In accordance with Corollary 4.6, we see again that (5.4) is irreducible if and only if either $m = 1$ or there exists $\alpha \in \mathscr{C}$ such that $p_W(\lambda) = p_W^\times(\lambda) = (\lambda - \alpha)^m$ for all $\lambda \in \mathscr{C}$. In the latter case (5.4) reduces to

$$M(\lambda) = \begin{bmatrix} 1 & \dfrac{r(\lambda)}{(\lambda - \alpha)^m} \\ 0 & 1 \end{bmatrix}.$$

and the condition $\gcd(p; r; p^{\times}) = 1$ amounts to $r(\alpha) \neq 0$. Note that this result shows some similarity with the results of Cohen [13], who describes the monic irreducible 2×2 matrix polynomials.

It is also possible to describe the minimal factorizations of M involving more than two factors. Suppose $p = p_1 \cdots p_h$ and $p^{\times} = p_1^{\times} \cdots p_h^{\times}$, where $p_1, \ldots, p_h$ and $p_1^{\times}, \ldots, p_h^{\times}$ are monic polynomials with $\deg p_j = \deg p_j^{\times}$ for $j = 1, \ldots, h$ and $\gcd(p_1 \ldots p_j; p_{j+1}^{\times} \ldots p_h^{\times}) = 1$ for $j = 1, \ldots, h-1$. An induction argument based on Lemma 5.1 yields that the polynomial equation

$$\sum_{j=1}^{h} r_j p_1 \cdots p_{j-1} p_{j+1}^{\times} \cdots p_h^{\times} = r$$

has a unique solution $r_1, \ldots, r_h$ with $\deg r_j < \deg p_j$ for $j = 1, \ldots, h$. This leads to the minimal factorization

$$M(\lambda) = \begin{bmatrix} 1 & \dfrac{r(\lambda)}{p(\lambda)} \\ 0 & \dfrac{p^{\times}(\lambda)}{p(\lambda)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \dfrac{r_1(\lambda)}{p_1(\lambda)} \\ 0 & \dfrac{p_1^{\times}(\lambda)}{p_1(\lambda)} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{r_2(\lambda)}{p_2(\lambda)} \\ 0 & \dfrac{p_2^{\times}(\lambda)}{p_2(\lambda)} \end{bmatrix} \cdots \begin{bmatrix} 1 & \dfrac{r_h(\lambda)}{p_h(\lambda)} \\ 0 & \dfrac{p_h^{\times}(\lambda)}{p_h(\lambda)} \end{bmatrix}$$

of M such that the jth factor has pole polynomial p_j and zero polynomial $p_j^{\times}$.

As was observed before, M admits complete factorization if and only if there exists an ordering $\alpha_1, \ldots, \alpha_m$ of the zeros of p_M and an ordering $\alpha_1^{\times}, \ldots, \alpha_m^{\times}$ of the zeros of $p_M^{\times}$ such that $\alpha_i \neq \alpha_j^{\times}$ whenever $i < j$. This leads to a complete factorization of M such that α_j and $\alpha_j^{\times}$ are the pole and the zero of the jth factor. To describe this complete factorization, we note that, under the given combinatorial ordering condition, the equation

$$\sum_{j=1}^{m} c_j(\lambda - \alpha_1) \cdots (\lambda - \alpha_{j-1})(\lambda - \alpha_{j+1}^{\times}) \cdots (\lambda - \alpha_m^{\times}) = r(\lambda) \quad (5.9)$$

has a unique solution $c_1, \dots, c_m$ consisting of m complex numbers. The associated complete factorization has the form

$$M(\lambda) = \begin{bmatrix} 1 & \dfrac{r(\lambda)}{p(\lambda)} \\ 0 & \dfrac{p^{\times}(\lambda)}{p(\lambda)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \dfrac{c_1}{\lambda - \alpha_1} \\ 0 & \dfrac{\lambda - \alpha_1^{\times}}{\lambda - \alpha_1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_2}{\lambda - \alpha_2} \\ 0 & \dfrac{\lambda - \alpha_2^{\times}}{\lambda - \alpha_2} \end{bmatrix} \cdots \begin{bmatrix} 1 & \dfrac{c_m}{\lambda - \alpha_m} \\ 0 & \dfrac{\lambda - \alpha_m^{\times}}{\lambda - \alpha_m} \end{bmatrix}.$$

EXAMPLE 5.2. Let the companion based matrix function M be defined by

$$M(\lambda) = \begin{bmatrix} 1 & \dfrac{1}{(\lambda + 1)^3(\lambda - 1)^3} \\ 0 & \dfrac{\lambda^2}{(\lambda - 1)^2} \end{bmatrix}.$$

Then $p_M(\lambda) = (\lambda + 1)^3(\lambda - 1)^3$ and $p_M^{\times}(\lambda) = (\lambda + 1)^3\lambda^2(\lambda - 1)$. Thus, referring to (5.1), we have $k = 3$ and $\beta_1, \beta_2, \beta_3 = -1, 0, 1$. Furthermore, $s_1, s_2, s_3 = 3, 0, 3$ and $t_1, t_2, t_3 = 3, 2, 1$. After applying Johnson's rule, we find that the necessary and sufficient condition (5.2) for the existence of a complete factorization is satisfied. In fact,

$$1, 1, 1, -1, -1, -1, \qquad 1, -1, -1, -1, 0, 0$$

are the unique orderings of the zeros of p_M and $p_M^{\times}$ that satisfy the desired ordering condition ($\alpha_i \neq \alpha_j^\times$ whenever $i < j$). This implies that M admits precisely one complete factorization, namely

$$M(\lambda) = \begin{bmatrix} 1 & \dfrac{c_1}{\lambda - 1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_2}{\lambda - 1} \\ 0 & \dfrac{\lambda + 1}{\lambda - 1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_3}{\lambda - 1} \\ 0 & \dfrac{\lambda + 1}{\lambda - 1} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \dfrac{c_4}{\lambda + 1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_5}{\lambda + 1} \\ 0 & \dfrac{\lambda}{\lambda + 1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_6}{\lambda + 1} \\ 0 & \dfrac{\lambda}{\lambda + 1} \end{bmatrix},$$

where $c_1 = \frac{1}{8}$, $c_2 = -\frac{7}{8}$, $c_3 = \frac{7}{8}$, $c_4 = -\frac{1}{8}$, $c_5 = -1$, and $c_6 = -1$. The equation corresponding to (5.9) that (uniquely) determines these values for c_1 to c_6 is

$$\begin{aligned} &c_1(\lambda + 1)^3\lambda^2 + c_2(\lambda - 1)(\lambda + 1)^2\lambda^2 + c_3(\lambda - 1)^2(\lambda + 1)\lambda^2 \\ &\quad + c_4(\lambda - 1)^3\lambda^2 + c_5(\lambda - 1)^3(\lambda + 1)\lambda + c_6(\lambda - 1)^3(\lambda + 1)^2 = 1. \end{aligned} \tag{5.10}$$

The values of c_1, c_4, and c_6 can be found easily by substituting $\lambda = 1$, $\lambda = -1$, or $\lambda = 0$ into (5.10). Thereafter, c_2, c_3, and c_5 are obtained by taking the derivative of (5.10) and again substituting these values for λ.

EXAMPLE 5.3. Next we consider the companion based matrix function M defined by

$$M(\lambda) = \begin{bmatrix} 1 & \dfrac{1}{(\lambda + 1)^3(\lambda - 1)^3} \\ 0 & \dfrac{(\lambda + 1)\lambda}{(\lambda - 1)^2} \end{bmatrix}.$$

Note that this matrix function is only slightly different from the one studied in Example 5.2. In this case $p_M(\lambda) = (\lambda + 1)^3(\lambda - 1)^3$ and $p_M^\times(\lambda) = (\lambda + 1)^4\lambda(\lambda - 1)$. Thus we have again $k = 3$ and $\beta_1, \beta_2, \beta_3 = -1, 0, 1$.

However, in this case $s_1, s_2, s_3 = 3, 0, 3$ and $t_1, t_2, t_3 = 4, 1, 1$. After applying Johnson's rule, we find that the necessary and sufficient condition (5.2) for the existence of a complete factorization cannot be satisfied. This implies that M does not admit complete factorization.

However, M admits a minimal factorization into five factors: four elementary factors and one factor with McMillan degree 2. This result is obtained by applying techniques that will be described in the forthcoming paper [10] already referred to before.

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