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Factorization and Job Scheduling: A Connection via Companion Based Matrix Functions

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ABSTRACT

A connection is made between two sets of problems. The first set involves factorization problems of specific rational matrix functions, the companion based matrix functions. The second set is concerned with variants of the two machine flow shop problem (2MFSP) from job scheduling theory. In particular, it is shown that with each companion based matrix function one can associate an instance of 2MFSP and vice versa. The latter can be done in such a way that the factorization properties of the companion based matrix function correspond to the combinatorial properties of the instance of 2MFSP.

1. INTRODUCTION

In this paper we study the problems of minimal and complete factorization of companion based $n \times n$ matrix functions in association with variants of the two machine flow shop problem (2MFSP) from job scheduling theory.

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Here a companion based $n \times n$ matrix function W is a rational $n \times n$ matrix function that admits a minimal realization

$$W(\lambda) = I_n + C(\lambda I_m - A)^{-1} B, \tag{1}$$

where the $m \times m$ matrices A and $A^{\times} = A - BC$ are first companion matrices. The class of companion based matrix functions is studied by Bart and Kroon [6]. Among other results, they briefly indicate a connection between the problem of complete factorization of companion based matrix functions and 2MFSP. In the present paper this connection is described in detail. Also, a more general connection between specific minimal factorizations of companion based matrix functions and variants of 2MFSP is presented.

In Section 2 we provide background material on rational matrix functions and companion based matrix functions, and in Section 3 we give a description of the standard version of 2MFSP. In Section 4 we indicate how an instance of 2MFSP can be associated with a companion based matrix function and vice versa. Then, in Section 5, we prove the following result: *If W is a companion based matrix function, J is an instance of 2MFSP, and W and J are associated, then W admits complete factorization if and only if $\mu(J) \leqslant \delta(W) + 1$.* Here $\mu(J)$ denotes the minimum makespan of J, and $\delta(W)$ denotes the McMillan degree of W.

In Section 6 we describe a number of generalizations of this result. These generalizations involve the Max-Degree problem and the Number problem, as well as two variants of 2MFSP. The Max-Degree problem and the Number problem are generalizations of the problem of complete factorization. In fact, the Max-Degree problem is the problem of determining a minimal factorization where the maximum McMillan degree over the factors is minimum; the Number problem is the problem of finding a minimal factorization with a maximum number of nontrivial factors.

In the mentioned variants of 2MFSP a number of jobs, each one consisting of two operations, have to be processed by two machines within a given deadline. Processing the second operation of a job may start already before processing the first operation of the job has been completed. However, in 2MFSP-MR the objective is to minimize the maximum (reduced) infeasibility of the jobs, whereas in 2MFSP-TR the total (reduced) infeasibility of the jobs is to be minimized. The combinatorial properties of these variants of 2MFSP are investigated by Bart and Kroon [7]. The present paper is concluded in Section 7, where some additional results are pointed out.

Finally, it should be noted that all examples in this paper are based on companion based 2×2 matrix functions of the type discussed by Bart and

Kroon [6]. However, the results of this paper are also valid for arbitrary companion based $n \times n$ matrix functions.

2. RATIONAL MATRIX FUNCTIONS

In this section we present some background material on rational $n \times n$ matrix functions that is used in this paper. We also give a brief review of the results of Bart and Kroon [6] on minimal factorization of companion based matrix functions.

Throughout this paper all rational $n \times n$ matrix functions are assumed to be analytic at ∞ with value I_n, the $n \times n$ identity matrix. Relevant references are Bart et al. [2], Bart et al. [3], DeWilde and Vandewalle [10], Gohberg et al. [12], Kailath [14], Kalman [15], Kalman et al. [16], and Sahnovic [19].

Let W be a rational $n \times n$ matrix function which, according to the standing assumption, is analytic at ∞ with $W(\infty) = I_n$. By a *realization* of W we mean a representation of the form

$$W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B, \tag{2}$$

where A is an $m \times m$ matrix, B is an $m \times n$ matrix, and C is an $n \times m$ matrix. It is known that it is always possible to find such a representation (cf. Bart et al. [2] and the references given there).

If (2) is a realization of W, then

$$W^{-1}(\lambda) = I_n - C(\lambda I_m - A + BC)^{-1}B \tag{3}$$

is a realization of the rational matrix function W^{-1} given by $W^{-1}(\lambda) = W(\lambda)^{-1}$. It is customary to write $A^{\times}$ for the matrix $A - BC$. With this notation (3) becomes $W^{-1}(\lambda) = I_n - C(\lambda I_m - A^{\times})^{-1}B$.

The smallest possible m for which a given rational matrix function W admits a realization (2) is called the *McMillan degree* of W and is denoted by $\delta(W)$. It equals the total number of poles of W counted according to pole multiplicity. A discussion of this notion is given after the next paragraph. Note that $\delta(W) = 0$ if and only if $W(\lambda) = I_n$ for all λ.

The realization (2) is called *minimal* if $m = \delta(W)$. Minimal realizations are essentially unique: if (2) is a minimal realization of W, then all minimal realizations of W can be obtained by replacing A, B, and C with SAS^{-1}, SB, and CS^{-1} respectively, where S is an invertible $m \times m$ matrix. This result is known as the *state space isomorphism theorem.*

Now let us come back to the notion of pole multiplicity. To that end, let α be a complex number. In a deleted neighborhood of α we have the Laurent expansion

$$W(\lambda) = \sum_{k=-r}^{\infty} (\lambda - \alpha)^k W_k \tag{4}$$

for W, where r is a positive integer not less than the order of α as a pole of W. Write

$$\delta(W, \alpha) = \operatorname{rank}\begin{bmatrix} W_{-r} & W_{-r+1} & \cdots & W_{-2} & W_{-1} \\ 0 & W_{-r} & & & W_{-2} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & W_{-r} & W_{-r+1} \\ 0 & 0 & \cdots & 0 & W_{-r} \end{bmatrix}. \tag{5}$$

Then $\delta(W, \alpha)$ does not depend on the choice of r, and $\delta(W, \alpha)$ is not less than the pole order of W at α. Also, $\delta(W, \alpha) = 0$ if and only if W is analytic at α. The number $\delta(W, \alpha)$ is called the *local degree* or the *pole multiplicity* of W at α.

As was mentioned already, the McMillan degree of W equals the number of poles of W counted according to pole multiplicity. In other words,

$$\delta(W) = \sum_{\alpha \in \mathbb{C}} \delta(W, \alpha).$$

Here the summation can be restricted to those α that are genuine poles of W.

A complex number α is called a *zero* of W if it is a pole of W^{-1}. The *zero multiplicity* of α as a zero of W then equals the pole multiplicity $\delta(W^{-1}, \alpha)$ of α as a pole of W^{-1}.

To facilitate later discussions, we associate two polynomials with a rational matrix function W. The *pole polynomial* p_W and the *zero polynomial* $p_W^{\times}$ of W are defined by

$$p_W(\lambda) = (\lambda - \alpha_1) \cdots (\lambda - \alpha_m), \quad p_W^{\times}(\lambda) = (\lambda - \alpha_1^{\times}) \cdots (\lambda - \alpha_m^{\times}), \tag{6}$$

where $\alpha_1, \dots, \alpha_m$ are the poles of W counted according to pole multiplicity and $\alpha_1^\times, \dots, \alpha_m^\times$ are the zeros of W counted according to zero multiplicity. Obviously, both p_W and $p_W^\times$ are monic and have degree $m = \delta(W)$. If (2) is a minimal realization of W, then p_W and $p_W^\times$ are the characteristic polynomials of A and $A^\times$ respectively.

The McMillan degree $\delta(W)$ is sublogarithmic in the following sense: If $W = W_1 \cdots W_r$ is a factorization of W, then

$$\delta(W) \leqslant \delta(W_1) + \cdots + \delta(W_r). \tag{7}$$

Of special interest are factorizations with equality in (7). These are factorizations in which pole-zero cancellation does not occur (cf. Bart et al. [2]). They are called *minimal factorizations*. There exist nontrivial rational matrix functions without any nontrivial minimal factorization.

A rational matrix function is called *elementary* if its McMillan degree equals one. A *complete factorization* is a minimal factorization involving elementary factors only. Thus a factorization of a rational matrix function W is complete if it has the form

$$W(\lambda) = \left(I_n + \frac{1}{\lambda - \alpha_1} R_1\right) \cdots \left(I_n + \frac{1}{\lambda - \alpha_m} R_m\right), \tag{8}$$

where m is the McMillan degree of W, where $\alpha_1, \dots, \alpha_m$ are the poles of W counted according to pole multiplicity, and where $R_1, \dots, R_m$ are $n \times n$ matrices of rank 1.

As a final part of this section we give a review of the results of Bart and Kroon [6] on minimal factorization of companion based matrix functions. As already mentioned in the introduction, a rational matrix function W is said to be *companion based* if it admits a minimal realization (2) where both A and $A^\times$ are first companion matrices. For basic material on companion matrices, see Lancaster and Tismenetsky [17].

One main result of Bart and Kroon [6] states that the property of being companion based is hereditary with respect to minimal factorization. The exact formulation of this statement is as follows.

THEOREM 1. *If W is a companion based matrix function and $W = UV$ is a minimal factorization of W, then U and V are companion based as well.*

Bart and Kroon [6] also prove that there exists a one-to-one correspondence between the minimal factorizations of a companion based matrix function W and specific factorizations of the pole polynomial p_W and the zero polynomial $p_W^\times$ of W. The details of this correspondence are expressed in Theorem 2.

THEOREM 2. *Let W be a companion based matrix function with pole polynomial p_W and zero polynomial $p_W^\times$.*

(i) *Suppose $W = W_1 \cdots W_r$ is a minimal factorization of W where W_j has pole polynomial p_j and zero polynomial $p_j^\times$ $(j = 1, \ldots, r)$. Then $p_W = \Pi_{j=1}^r p_j$, $p_W^\times = \Pi_{j=1}^r p_j^\times$ and $\gcd(\Pi_{j=1}^i p_j; \Pi_{j=i+1}^r p_j^\times) = 1$ $(i = 1, \ldots, r-1)$.*

(ii) *Suppose $p_W = \Pi_{j=1}^r p_j$ and $p_W^\times = \Pi_{j=1}^r p_j^\times$ where $p_1, \ldots, p_r$ and $p_1^\times, \ldots, p_r^\times$ are monic polynomials with $\deg p_j = \deg p_j^\times$ and $\gcd(\Pi_{j=1}^i p_j; \Pi_{j=i+1}^r p_j^\times) = 1$ $(i = 1, \ldots, r-1)$. Then there exist unique companion based matrix functions $W_1, \ldots, W_r$ such that $W = W_1 \cdots W_r$ is a minimal factorization of W, and W_j has pole polynomial p_j and zero polynomial $p_j^\times$ $(j = 1, \ldots, r)$.*

The case $r = m$ obviously corresponds to complete factorization. Hence Theorem 2 implies Corollary 3 (cf. Bart and Hoogland [4]).

COROLLARY 3. *Let W be a companion based matrix function with pole polynomial p_W and zero polynomial $p_W^\times$. Then W admits complete factorization if and only if there exist orderings $\alpha_1, \ldots, \alpha_m$ of the zeros of p_W and $\alpha_1^\times, \ldots, \alpha_m^\times$ of the zeros of $p_W^\times$ such that $\alpha_s \neq \alpha_t^\times$ whenever $s < t$.*

Corollary 3 already indicates that the existence of a complete factorization is equivalent to a combinatorial condition involving the zeros of p_W and $p_W^\times$. In Section 5 this equivalence is described in detail. Generalizations of this result are discussed in Section 6.

A rational matrix function is called *irreducible* if it does not admit any nontrivial minimal factorization. Corollary 4 provides a complete description of the irreducible companion based matrix functions.

COROLLARY 4. *Let W be a companion based matrix function with pole polynomial p_W and zero polynomial $p_W^\times$. Then W is irreducible if and only if W is elementary or $p_W(\lambda) = p_W^\times(\lambda) = (\lambda - \alpha)^c$ for some complex number α and some positive integer c.*

If W is elementary and $p_W \neq p_W^\times$, we say that W has *type 1*. If $p_W(\lambda) = p_W^\times(\lambda) = (\lambda - \alpha)^c$ for some complex number α and some positive integer c, we say that W has *type 2 with degree c*. This terminology turns out to be useful in later considerations.

In Section 7 we present a generalization of Corollary 4. In particular, we describe the smallest possible McMillan degree of a nontrivial factor that can appear in a minimal factorization of a companion based matrix function W, either as an arbitrary middle factor, as a left factor, or as a right factor.

3. THE TWO MACHINE FLOW SHOP PROBLEM

In this section we describe the standard version of 2MFSP and some properties of the optimal schedules of instances of 2MFSP. In an instance of 2MFSP there are k jobs that have to be processed by two machines. Each job consists of two operations. The first and the second operation of job j are called O_j^1 and O_j^2 respectively. The first operation O_j^1 must be processed on the first machine, and the second operation O_j^2 must be processed on the second machine. Each machine can be processing at most one operation at the same time. In standard 2MFSP, processing O_j^2 on the second machine cannot start until processing O_j^1 on the first machine has been completed.

The processing times of all operations are given and fixed. The processing time of O_j^1 is denoted by s_j, and the processing time of O_j^2 is denoted by t_j. Hence an instance J of 2MFSP consists of k tuples (s_j, t_j) specifying the processing times of the operations. *Throughout this paper we assume that all processing times are nonnegative integers*. This is not a serious restriction. What it amounts to is that the processing times are rationals and that the time unit is chosen appropriately. Furthermore, in order to avoid trivialities, we also assume that for each job j either s_j or t_j is nonzero.

If we have a feasible schedule (that is, a schedule satisfying the specified rules), then the length of the time interval required to carry out all jobs is called the *makespan* of the schedule. In standard 2MFSP the objective is to find a feasible schedule with *minimum makespan*. The minimum makespan of an instance J is denoted by $\mu(J)$. In the literature the makespan is sometimes also called the *maximum completion time*. In that case the minimally obtained maximum completion time of an instance J is denoted by $C_{\max}(J)$.

It is well known that each instance of 2MFSP has an optimal *nonpreemptive* schedule (cf. Baker [1]). That is, the optimal schedule has the additional property that, once a machine has started processing an operation, it does not start processing another operation until the first operation has been completed. It is also well known that each instance of 2MFSP has an optimal *permutation* schedule. A schedule is a permutation schedule if it is nonpreemptive and for all $i \neq j$ the operations O_i^2 and O_j^2 are processed in the same order as the operations O_i^1 and O_j^1.

These properties of 2MFSP can be proved in a straightforward way by exchange arguments and by using the fact that, given a feasible schedule, an operation on the first machine can be pushed backward in time without violating the predecessor constraints. Similarly, an operation on the second machine can be pushed forward in time without violating the predecessor constraints.

An optimal permutation schedule for an instance of 2MFSP with k jobs can be obtained by the application of *Johnson's rule* (cf. Baker [1] and Johnson [13]). With Johnson's rule an optimal permutation schedule is constructed as follows:

1. Define the sets V_1 and V_2 by $V_1 = \{j \mid s_j \leqslant t_j\}$ and $V_2 = \{j \mid s_j \geqslant t_j\}$.
2. Put the jobs in V_1 in order of increasing s_j, and put the jobs in V_2 in order of decreasing t_j.
3. Process the jobs in V_1 first, and process the jobs in V_2 thereafter.

Sorting the jobs in the sets V_1 and V_2 can be accomplished in $O(k \log k)$ time. Thus the running time of Johnson's rule is $O(k \log k)$. Therefore 2MFSP belongs to the class of easy problems that can be solved in polynomial time (cf. Garey and Johnson [11]).

4. COMPANION BASED MATRIX FUNCTIONS AND 2MFSP

In this section we indicate how a companion based matrix function can be associated with an instance of 2MFSP and vice versa. This association is essential in the description of the connection between minimal and complete factorization of companion based matrix functions and variants of 2MFSP. As we shall see, the association is essentially one-to-one. In subsequent sections it is shown that, if a companion based matrix function W and an instance J of 2MFSP are associated, then the factorization properties of W are reflected in the combinatorial properties of J and vice versa.

Let W be a companion based $n \times n$ matrix function, and let J be an instance of 2MFSP with k jobs (s_j, t_j) where for $j = 1, \dots, k$ either s_j or t_j is positive. We say that W and J are *associated* if the pole polynomial p_W and the zero polynomial $p_W^\times$ of W can be written in the form

$$p_W(\lambda) = (\lambda - \beta_1)^{t_1}(\lambda - \beta_2)^{t_2} \cdots (\lambda - \beta_k)^{t_k}, \tag{9}$$

$$p_W^\times(\lambda) = (\lambda - \beta_1)^{s_1}(\lambda - \beta_2)^{s_2} \cdots (\lambda - \beta_k)^{s_k}, \tag{10}$$

where each β_j is a pole of W, a zero of W (i.e. a pole of W^{-1}), or both $(j = 1, \ldots, k)$. If β_j is a pole and not a zero of W, then $s_j = 0$ and $t_j > 0$. If β_j is a zero and not a pole of W, then $s_j > 0$ and $t_j = 0$. If β_j is both a pole and a zero of W, then $s_j > 0$ and $t_j > 0$. Note that $\sum_{j=1}^k s_j = \sum_{j=1}^k t_j = \delta(W)$.

It is obvious that for a given companion based matrix function W there exists an instance J of 2MFSP such that W and J are associated. This instance of 2MFSP is unique up to the ordering of the jobs.

Conversely, if J is an instance of 2MFSP with k jobs as in the preceding paragraph and satisfying $\sum_{j=1}^k s_j = \sum_{j=1}^k t_j$, then there do exist companion based matrix functions W such that W and J are associated. The latter can be seen as follows. First, choose k different complex numbers $\beta_1, \ldots, \beta_k$ in an arbitrary way. Next, introduce the polynomials $p(\lambda) = (\lambda - \beta_1)^{t_1}(\lambda - \beta_2)^{t_2} \cdots (\lambda - \beta_k)^{t_k}$ and $q(\lambda) = (\lambda - \beta_1)^{s_1}(\lambda - \beta_2)^{s_2} \cdots (\lambda - \beta_k)^{s_k}$. Finally, define the rational matrix function W by

$$W(\lambda) = \begin{bmatrix} 1 & \dfrac{1}{p(\lambda)} \\ 0 & \dfrac{q(\lambda)}{p(\lambda)} \end{bmatrix}.$$

It is not difficult to see that $p_W = p$ and $p_W^\times = q$. Furthermore, W is a companion based matrix function. It follows that W and J are associated. Also, if R is any invertible 2×2 matrix, then $R^{-1}WR$ and J are associated as well. A similar construction can be used to find an $n \times n$ companion based matrix function W such that W and J are associated. For details, see Bart and Kroon [6].

If J is an instance of 2MFSP that does not satisfy the condition $\sum_{j=1}^k s_j = \sum_{j=1}^k t_j$, then this condition can be met by the addition of at least one appropriate dummy job for which only one of the processing times is positive. In this way one obtains an instance J' of 2MFSP that satisfies the desired condition and that is essentially the same as J. In particular, $\mu(J) = \mu(J')$.

Hence, if J is an instance of 2MFSP, then there exist several companion based matrix functions W such that W and J are associated. However, as will become clear in the following sections, all these functions have basically the same factorization properties. So, from a factorization point of view, these functions can be identified with each other. In this sense, we have uniqueness here as well.

5. COMPLETE FACTORIZATION AND 2MFSP

In this section we describe the connection between complete factorization of companion based matrix functions and 2MFSP. Theorem 5 can be viewed as a reformulation of the result described in Section 5 of Bart and Kroon [6]. For the convenience of the reader we give the full proof of this theorem in the language developed here.

THEOREM 5. *Let W be a companion based matrix function, let J be an instance of 2MFSP, and assume W and J are associated. Then W admits complete factorization if and only if $\mu(J) \leqslant \delta(W) + 1$.*

Proof. Let the pole polynomial p_W and the zero polynomial $p_W^\times$ of W be given by (9) and (10) respectively, and write $m = \delta(W)$.

Suppose $\mu(J) \leqslant m + 1$. Then there exists a permutation schedule for J with makespan $m + 1$. As the operations on the first machine can be pushed backward in time and $\sum_{j=1}^{k} s_j = m$, it may be assumed that the first machine is occupied during the time interval $(0, m)$. Similarly, it may be assumed that the second machine is occupied during the time interval $(1, m + 1)$. As a consequence, the start and finish times of all operations are integers. Now we define the sequences $\alpha_1, \ldots, \alpha_m$ and $\alpha_1^\times, \ldots, \alpha_m^\times$ as follows:

$\alpha_t = \beta_j$ if the second machine is processing O_j^2 in the time interval $(t, t + 1)$;
$\alpha_t^\times = \beta_j$ if the first machine is processing O_j^1 in the time interval $(t - 1, t)$.

Then $\alpha_1, \ldots, \alpha_m$ is a well-defined ordering of the zeros of p_W, and $\alpha_1^\times, \ldots, \alpha_m^\times$ is a well-defined ordering of the zeros of $p_W^\times$. Furthermore,

$$\alpha_s \neq \alpha_t^\times, \qquad s < t. \tag{11}$$

The condition (11) is a consequence of the fact that processing the second operation of a job in J cannot start until processing the first operation of the job has been completed. Thus Corollary 3 implies that W admits complete factorization.

Conversely, suppose W admits complete factorization. Then Corollary 3 implies that there exist orderings $\alpha_1, \ldots, \alpha_m$ of the zeros of p_W and $\alpha_1^\times, \ldots, \alpha_m^\times$ of the zeros of $p_W^\times$ satisfying the condition [11]. Now a feasible schedule for J with makespan $m + 1$ is obtained in the following way:

in the time interval $(t - 1, t)$ the first machine is processing operation O_j^1 where j is chosen in such a way that $\alpha_t^x = \beta_j$;

in the time interval $(t, t+1)$ the second machine is processing operation O_j^2 where j is chosen in such a way that $\alpha_t = \beta_j$.

The obtained schedule is feasible, because the condition [11] is satisfied. Furthermore, the schedule may be preemptive. However, in that case it can be transformed into a nonpreemptive schedule with the same makespan, as was pointed out earlier. As the obtained schedule has makespan $m+1$, the minimum makespan of J does not exceed $m+1$. ∎

A more refined analysis, based on the results of Bart and Hoogland [4], reveals that there exists a one-to-one correspondence between the complete factorizations of W and the feasible preemptive schedules for J with makespan $m+1$ where the first machine is occupied in the interval $(0, m)$, the second machine is occupied in the interval $(1, m+1)$, and all preemptions occur at integer time instants.

EXAMPLE 1. Let the companion based matrix function W be defined by

$$W(\lambda) = \begin{bmatrix} 1 & \dfrac{1}{(\lambda+1)^3(\lambda-1)^3} \\ 0 & \dfrac{\lambda^2}{(\lambda-1)^2} \end{bmatrix}.$$

Then $p_W(\lambda) = (\lambda+1)^3(\lambda-1)^3$ and $p_W^{\times}(\lambda) = (\lambda+1)^3\lambda^2(\lambda-1)$. Thus in the notation introduced before, we have $k = 3$ and $\beta_1, \beta_2, \beta_3 = -1, 0, 1$. The associated instance J of 2MFSP consists of the jobs $(s_1, t_1) = (3,3)$, $(s_2, t_2) = (2, 0)$, and $(s_3, t_3) = (1, 3)$. By applying Johnson's rule, we obtain the optimal permutation schedule $(3, 1, 2)$ which is shown in Figure 1. Note that the sets V_1 and V_2 appearing in Johnson's rule (see the end of Section 3) are $V_1 = \{3\}$ and $V_2 = \{1, 2\}$.

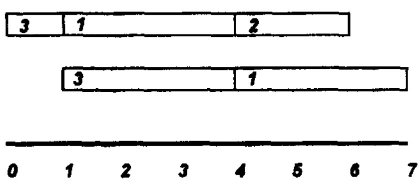

FIG. 1. The optimal permutation schedule $(3, 1, 2)$ for the instance J.

The optimal permutation schedule (3, 1, 2) for J corresponds to orderings

$$1,1,1,-1,-1,-1 \quad \text{and} \quad 1,-1,-1,-1,0,0,$$

of the zeros of p_W and $p_W^{\times}$ respectively, satisfying the desired ordering condition (11). In particular, W admits a complete factorization

$$W(\lambda) = \begin{bmatrix} 1 & \dfrac{c_1}{\lambda - 1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_2}{\lambda - 1} \\ 0 & \dfrac{\lambda + 1}{\lambda - 1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_3}{\lambda - 1} \\ 0 & \dfrac{\lambda + 1}{\lambda - 1} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \dfrac{c_4}{\lambda + 1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_5}{\lambda + 1} \\ 0 & \dfrac{\lambda}{\lambda + 1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_6}{\lambda + 1} \\ 0 & \dfrac{\lambda}{\lambda + 1} \end{bmatrix},$$

where the pole and the zero of the ith factor correspond to the ith zero of p_W and $p_W^{\times}$ respectively. The equation to be satisfied by the complex numbers c_1 to c_6 is

$$c_1(\lambda + 1)^3\lambda^2 + c_2(\lambda - 1)(\lambda + 1)^2\lambda^2 + c_3(\lambda - 1)^2(\lambda + 1)\lambda^2$$

$$+ c_4(\lambda - 1)^3\lambda^2 + c_5(\lambda - 1)^3(\lambda + 1)\lambda + c_6(\lambda - 1)^3(\lambda + 1)^2 = 1. \quad (12)$$

Values for c_1, c_4, and c_6 can be found by substituting $\lambda = 1$, $\lambda = -1$, or $\lambda = 0$ into Equation (12). Thereafter, c_2, c_3, and c_5 are obtained by taking the derivative of (12) and again substituting these values for λ. This leads to $c_1 = \frac{1}{8}$, $c_2 = -\frac{7}{8}$, $c_3 = \frac{7}{8}$, $c_4 = -\frac{1}{8}$, $c_2 = -1$, and $c_6 = -1$.

The general idea behind the above method for computing the constants $c_1, \dots, c_6$ also works for subsequent examples. We omit the details there.

6. GENERALIZATIONS

In this section we present two generalizations of Theorem 5. In particular, we describe connections between the so-called Max-Degree problem and the number problem for companion based matrix functions on one hand, and two variants of 2MFSP on the other.

We first describe the Max-Degree problem and the Number problem in Section 6.1. The involved variants of 2MFSP are described in Section 6.2. The connections between these topics, together with some illustrative examples, are presented in Section 6.3. The proofs of these results are given in Section 6.4.

6.1. *The Max-Degree Problem and the Number Problem*

Since not every rational matrix function admits complete factorization, it is useful to consider minimal factorizations that are optimal in a more general sense. To that end, let W be a rational matrix function. Then the following minimal factorization problems, differing from each other by their objectives, are distinguished:

The Max-Degree problem: Minimize the maximum McMillan degree over the factors that appear in a minimal factorization of W. That is, the objective is to minimize $\max_i\{\delta(W_i) \mid W = W_1 \cdots W_r\}$, where the minimum is taken over all possible minimal factorizations $W = W_1 \cdots W_r$ (and there is no restriction on the number of factors r). The minimum obtainable value is denoted by $\gamma_1(W)$.

The Number problem: Maximize the number of nontrivial factors in a minimal factorization of W. That is, maximize r over all possible minimal factorizations $W = W_1 \cdots W_r$ containing only nontrivial factors (i.e. factors not identically equal to the appropriate identity matrix). The maximum obtainable number of nontrivial factors is denoted by $\nu_1(W)$.

Note that one will not only be interested in the values $\gamma_1(W)$ and $\nu_1(W)$, but also in the corresponding minimal factorizations of W. Note further that both the Max-Degree problem and the Number problem are generalizations of the problem of complete factorization. Indeed, a rational matrix function W admits complete factorization if and only if $\gamma_1(W) = 1$ or, equivalently, $\nu_1(W) = \delta(W)$.

6.2. *2MFSP-MR and 2MFSP-TR*

In this subsection we describe the variants of 2MFSP that are closely connected (and in some sense even equivalent) with the factorization problems introduced in Section 6.1.

To that end, let J be an instance of 2MFSP with k jobs (s_j, t_j), and let the *deadline* $\tau(J)$ be an integer satisfying $\tau(J) \geqslant \max\{\sum_{j=1}^{k} s_j, \sum_{j=1}^{k} t_j\}$. In the variants of 2MFSP to be studied it is required that all jobs be completed within $\tau(J)$ time units. In order to make this meaningful, one has to relax the predecessor constraints. That is, in these variants of 2MFSP it is allowed that processing O_j^2 on the second machine already starts before processing O_j^1 on the first machine has been completed. However, the objective is to minimize such infeasibilities in a prescribed way. For a motivation to study problems of this type we refer to Baker [1].

To make things more precise, let σ be a schedule for the given instance J satisfying the deadline $\tau(J)$. Now we introduce the following notation: if O is an operation with a positive processing time, then $S(O)$ and $F(O)$ denote the start and finish time of this operation in the schedule σ. Furthermore, if $s_j = 0$, then we put $S(O_j^1) = F(O_j^1) = 0$, and if $t_j = 0$, then we put $S(O_j^2) = F(O_j^2) = \tau(J)$. Now the *reduced infeasibility* of job j, denoted by I_j, is defined by

$$I_j = \max\left\{0, F\left(O_j^1\right) - S\left(O_j^2\right) - 1\right\}. \tag{13}$$

Next, the following variants of 2MFSP are distinguished:

2*MFSP-MR*: In 2MFSP-MR the objective is to find a schedule such that $\max_{j=1}^{k} I_j$ is minimum. If J is an instance of 2MFSP-MR, then the optimal value of the objective function is denoted by $\gamma_2(J)$.

2*MFSP-TR*: In 2MFSP-TR the objective is to find a schedule such that $\sum_{j=1}^{k} I_j$ is minimum. If J is an instance of 2MFSP-TR, then the optimal value of the objective function is denoted by $\nu_2(J)$.

If in these problems one works with the ordinary (nonreduced) infeasibilities defined by $\max\{0, F(O_j^1) - S(O_j^2)\}$ instead of the reduced infeasibilities defined by (13), then one obtains the variants 2MFSP-M and 2MFSP-T of 2MFSP. The problems 2MFSP-M and 2MFSP-MR are essentially the same, and a similar statement holds for the problems 2MFSP-T and 2MFSP-TR.

However, if one works with the reduced infeasibilities, then the connection between the Max-Degree problem and the Number problem for companion based matrix functions and the variants of 2MFSP can be expressed more easily. The latter statement is especially true for the connection between the number problem and 2MFSP-TR. For further details on these topics we refer to Bart and Kroon [7, 8].

The combinatorial properties of the above variants of 2MFSP are described by Bart and Kroon [7]. They establish the following result, which will be used later on.

LEMMA 6. *Every instance of* 2*MFSP-MR or* 2*MFSP-TR has an optimal permutation schedule satisfying* $S(O_j^1) \leqslant S(O_j^2)$ *and* $F(O_j^1) \leqslant F(O_j^2)$ *for* $j = 1, \ldots, k$.

An optimal permutation schedule for an instance of 2MFSP-MR can be obtained by first applying Johnson's rule, and next shifting the operations on the second machine $\mu(J) - \tau(J)$ time units backward. It follows that $\gamma_2(J) = \max\{\mu(J) - \tau(J) - 1, 0\}$. Thus the computational complexity of 2MFSP-MR is $O(k \log k)$, where k represents the number of jobs.

Unfortunately, Johnson's rule does not always produce an optimal permutation schedule for an instance of 2MFSP-TR (see section 6.3 for an example). We conjecture the problem 2MFSP-TR to be NP-hard. For more information we refer to Bart and Kroon [7].

Both 2MFSP-MR and 2MFSP-TR bear some analogy to the variant of 2MFSP described by Mitten [18]. In the latter variant a maximum infeasibility of each job is prescribed, and the objective is to find a schedule that minimizes the makespan, whereas in 2MFSP-MR and 2MFSP-TR the makespan is given as a deadline, and the objective is to minimize the infeasibilities of the jobs in some sense. Mitten shows that his variant of 2MFSP can be solved by an extension of Johnson's rule. We briefly come back to Mitten's variant of 2MFSP in Section 7.

6.3. *Connections and Examples*

In this subsection we describe the connections between the Max-Degree problem and the Number problem for companion based matrix functions on one hand, and 2MFSP-MR and 2MFSP-TR on the other. We also present two illustrative examples.

It is convenient to start out with the following definition. Let W be a companion based matrix function, and let J be an instance of 2MFSP-MR or 2MFSP-TR with k jobs (s_j, t_j) and deadline $\tau(J)$. Then we say that W and J are *associated* if p_W and $p_W^{\times}$ are given by (9) and (10), and if $\tau(J) = \delta(W)$ (cf. Section 4).

Now Theorem 7 describes the connection between the Max-Degree problem for companion based matrix functions and 2MFSP-MR.

THEOREM 7. *Let W be a companion based matrix function, let J be an instance of 2MFSP-MR, and assume W and J are associated. Then $\gamma_1(W) = \gamma_2(J) + 1$.*

The conclusion of Theorem 7 can also be written as $\gamma_1(W) = \max\{1, \mu(J) - \tau(J)\}$. The proof of Theorem 7 is given in Section 6.4. From this proof one can see that an optimal solution to the Max-Degree problem corresponds to an optimal schedule for the associated instance of 2MFSP-MR. As was noted earlier, 2MFSP-MR can be solved by Johnson's rule. Thus the Max-Degree problem can be solved by Johnson's rule as well. The following example serves as a illustration of Theorem 7.

EXAMPLE 2. Let the companion based function W be defined by

$$W(\lambda) = \begin{bmatrix} 1 & \dfrac{1}{(\lambda+1)^4(\lambda-1)^6} \\ 0 & \dfrac{\lambda^3}{(\lambda+1)(\lambda-1)^2} \end{bmatrix}.$$

Then $p_W(\lambda) = (\lambda + 1)^4(\lambda - 1)^6$ and $p_W^{\times}(\lambda) = (\lambda + 1)^3\lambda^3(\lambda - 1)^4$. Thus in the notation introduced before, $k = 3$ and $\beta_1, \beta_2, \beta_3 = -1, 0, 1$. The associated instance J of 2MFSP-MR consists of the jobs $(s_1, t_1) = (3, 4)$, $(s_2, t_2) = (3, 0)$, and $(s_3, t_3) = (4, 6)$ and has $\tau(J) = 10$. By applying Johnson's rule, we obtain the optimal permutation schedule $(1, 3, 2)$ which is shown in Figure 2. The sets V_1 and V_2 appearing in Johnson's rule (see the end of Section 3) are $V_1 = \{1, 3\}$ and $V_2 = \{2\}$. Note that $I_1 = 2$, $I_2 = 0$, and $I_3 = 2$, which gives $\gamma_2(J) = 2$.

The optimal permutation schedule $(1, 3, 2)$ for J corresponds to orderings

$$-1, -1, -1, -1, 1, 1, 1, 1, 1, 1 \quad \text{and} \quad -1, -1, -1, 1, 1, 1, 1, 0, 0, 0$$

of the zeros of p_W and $p_W^{\times}$ respectively. This implies that $\gamma_1(W) = 3$ and

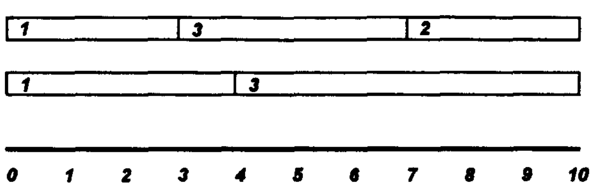

FIG. 2. The optimal permutation schedule $(1, 3, 2)$ for the instance J.

that W admits a minimal factorization

$$W(\lambda) = \begin{bmatrix} 1 & \dfrac{r_1(\lambda)}{(\lambda+1)^3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_2}{\lambda+1} \\ 0 & \dfrac{\lambda-1}{\lambda+1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{r_3(\lambda)}{(\lambda-1)^3} \\ 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \dfrac{c_4}{\lambda-1} \\ 0 & \dfrac{\lambda}{\lambda-1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_5}{\lambda-1} \\ 0 & \dfrac{\lambda}{\lambda-1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_6}{\lambda-1} \\ 0 & \dfrac{\lambda}{\lambda-1} \end{bmatrix},$$

where $r_1(\lambda) = -\frac{1}{32}(29\lambda^2 + 68\lambda + 41)$, $c_2 = 4$, $r_3(\lambda) = \frac{1}{32}(29\lambda^2 - 68\lambda + 41)$, $c_4 = -4$, $c_5 = 1$, and $c_6 = -1$. In this factorization the poles and zeros of each factor are implied by the orderings of the zeros of p_W and $p_W^{\times}$. This factorization is optimal for the Max-Degree problem.

Next, Theorem 8 describes the connection between the Number problem for companion based matrix functions and 2MFSP-TR. Again, the proof is given in Section 6.4.

THEOREM 8. *Let W be a companion based matrix function, let J be an instance of 2MFSP-TR, and assume W and J are associated. Then $\delta(W) = \nu_1(W) + \nu_2(J)$.*

Since $\delta(W) = \tau(J)$, the conclusion of Theorem 8 can also be written as $\nu_1(W) + \nu_2(J) = \tau(J)$. Theorem 8 is illustrated by Example 3.

EXAMPLE 3. Let the companion based matrix function W be the same as in Example 2. Then the associated instance J of 2MFSP-TR consists of the jobs $(s_1, t_1) = (3, 4)$, $(s_2, t_2) = (3, 0)$, and $(s_3, t_3) = (4, 6)$ and has $\tau(J) = 10$. The optimal permutation schedule for J is $(3, 1, 2)$, which is shown in Figure 3. Here $I_1 = 0$, $I_2 = 0$ and $I_3 = 3$, which gives $\nu_2(J) = 3$. Note that the total reduced infeasibility of the schedule shown in Figure 2 equals 4.

The optimal permutation schedule $(3, 1, 2)$ for J corresponds to orderings

$$1, 1, 1, 1, 1, 1, -1, -1, -1, -1 \quad \text{and} \quad 1, 1, 1, 1, -1, -1, -1, 0, 0, 0$$

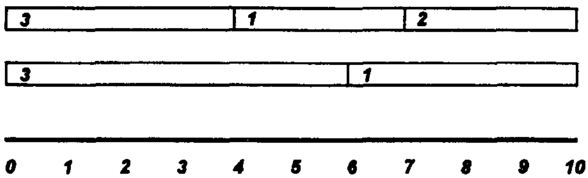

FIG. 3. The optimal permutation schedule (3, 1, 2) for the instance J.

of the zeros of p_W and $p_W^\times$. This implies that $\nu_1(W) = \delta(W) - \nu_2(J) = 10 - 3 = 7$ and that W admits a minimal factorization

$$W(\lambda) = \begin{bmatrix} 1 & \dfrac{r(\lambda)}{(\lambda-1)^4} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_1}{\lambda-1} \\ 0 & \dfrac{\lambda+1}{\lambda-1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_2}{\lambda-1} \\ 0 & \dfrac{\lambda+1}{\lambda-1} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \dfrac{c_3}{\lambda+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_4}{\lambda+1} \\ 0 & \dfrac{\lambda}{\lambda+1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_5}{\lambda+1} \\ 0 & \dfrac{\lambda}{\lambda+1} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{c_6}{\lambda+1} \\ 0 & \dfrac{\lambda}{\lambda+1} \end{bmatrix},$$

where $r(\lambda) = \frac{1}{32}(99\lambda^3 - 345\lambda^2 + 411\lambda - 169)$, $c_1 = \frac{699}{64}$, $c_2 = -\frac{381}{16}$, $c_3 = -\frac{1}{64}$, $c_4 = 12$, $c_5 = 3$, and $c_6 = 1$. Again the poles and zeros of each factor are implied by the orderings of the zeros of p_W and $p_W^\times$. This factorization is optimal for the number problem.

Results analogous to Theorems 7 and 8 also exist for the problems 2MFSP-M and 2MFSP-T, where one deals with the nonreduced infeasibilities of the jobs. For details, see Sections 8 and 9 of Bart and Kroon [8].

6.4. *Proofs of the Results*

In this subsection we provide the proofs of the Theorems 7 and 8. We start out with some definitions and two auxiliary results.

Thus, let W be a companion based matrix function, let J be an instance of 2MFSP-MR or 2MFSP-TR with k jobs (s_j, t_j) and deadline $\tau(J)$, and assume W and J are associated. Assume further that σ is a (possibly) preemptive schedule for J where all preemptions occur at integer time instants. Then with σ we associate the following orderings $\alpha_1, \dots, \alpha_m$ of the zeros of p_W and $\alpha_1^\times, \dots, \alpha_m^\times$ of the zeros $p_W^\times$:

$\alpha_t = \beta_j$, if the second machine is processing O_j^2 in the time interval $(t-1, t)$;

$\alpha_t^\times = \beta_j$, if the first machine is processing O_j^1 in the time interval $(t-1, t)$.

Furthermore, we say that the integer time instant $\tau \in \{0, \dots, m\}$ is *skipped by job j with respect to the schedule* σ if

$$S(O_j^2) < \tau < F(O_j^1). \tag{14}$$

Note that the number of integer time instants that are skipped by job j equals the (reduced) infeasibility I_j of job j. Further, an integer time instant is said to be *skipped with respect to the schedule* σ if it is skipped by at least one job j. In the following the qualification "with respect to the schedule σ" will be omitted. Note that the time instants 0 and m are never skipped.

The motivation for these definitions is the following. Suppose $\alpha_1, \dots, \alpha_m$ and $\alpha_1^\times, \dots, \alpha_m^\times$ are the orderings of the zeros of p_W and $p_W^\times$ associated with σ, and let the polynomials p_τ and $p_\tau^\times$ be defined by

$$p_\tau(\lambda) = (\lambda - \alpha_1)\cdots(\lambda - \alpha_\tau), \quad p_\tau^\times(\lambda) = (\lambda - \alpha_{\tau+1}^\times)\cdots(\lambda - \alpha_m^\times). \tag{15}$$

Then the integer time instant τ is skipped if and only if the polynomials p_τ and $p_\tau^\times$ have at least one common zero. Thus Theorem 2 implies that, if the integer time instant τ is skipped, then W does *not* admit a minimal factorization $W = UV$ such that $p_U = p_\tau$ and $p_V^\times = p_\tau^\times$. Conversely, Theorem 2 also implies that, if the integer time instant τ is *not* skipped, then W admits a unique minimal factorization $W = UV$ such that $p_U = p_\tau$ and $p_V^\times = p_\tau^\times$. In other words, a skipped time instant corresponds to a "missed opportunity for factorization" of W.

Given the schedule σ as above, let $\{\tau_i \mid i = 0, \dots, r\}$ be the set of all integer time instants that are not skipped, and suppose $0 = \tau_0 < \tau_1 < \cdots < \tau_{r-1} < \tau_r = m$. Then, by repeating the above argument several times, it follows that W admits a unique minimal factorization

$$W = W_1 \cdots W_r, \tag{16}$$

where the factor W_i has pole polynomial $p_i(\lambda) = (\lambda - \alpha_{\tau_{i-1}+1})\cdots(\lambda - \alpha_{\tau_i})$

and zero polynomial $p_i^\times(\lambda) = (\lambda - \alpha^\times_{\tau_{i-1}+1}) \cdots (\lambda - \alpha^\times_{\tau_i})$ $(i = 1, \dots, r)$. This minimal factorization is called the *minimal factorization associated with* σ. If t is an integer time instant with $(t-1, t) \subset (0, m)$, then the unique factor W_i in (16) such that $(t-1, t) \subset (\tau_{i-1}, \tau_i)$ is called the *factor corresponding* to the interval $(t-1, t)$. Now we are ready to prove the following auxiliary result.

LEMMA 9. *Let W be a companion based matrix function, let J be an instance of 2MFSP-MR or 2MFSP-TR, and assume W and J are associated. Suppose σ is a permutation schedule for J with the property $S(O_j^1) \leqslant S(O_j^2)$ and $F(O_j^1) \leqslant F(O_j^2)$ for $j = 1, \dots, k$, and suppose t is an integer time instant with $(t-1, t) \subset (0, \delta(W))$. If (16) is the minimal factorization associated with σ, then the following statements hold:*

(i) *If in the time interval $(t-1, t)$ the machines are processing operations of different jobs, then the factor W_i in (16) corresponding to the interval $(t-1, t)$ has type 1.*

(ii) *If in the time interval $(t-1, t)$ the machines are processing operations of the same job j, then the factor W_i in (16) corresponding to the interval $(t-1, t)$ has type 2 with degree $I_j + 1$.*

Proof. Let $\alpha_1, \dots, \alpha_m$ and $\alpha_1^\times, \dots, \alpha_m^\times$ be the orderings of the zeros of p_W and $p_W^\times$ associated with σ, as described before.

(i): If in the time interval $(t-1, t)$ the machines are processing operations of different jobs, then this implies $\alpha_t \neq \alpha_t^\times$. Now the nonpreemptive character of σ, together with the property $S(O_j^1) \leqslant S(O_j^2)$ and $F(O_j^1) \leqslant F(O_j^2)$ for $j = 1, \dots, k$, implies that the time instants $t-1$ and t are not skipped. Hence, if W_i is the factor in (16) corresponding to the interval $(t-1, t)$, then W_i has pole polynomial $p_i(\lambda) = (\lambda - \alpha_t)$ and zero polynomial $p_i^\times(\lambda) = (\lambda - \alpha_t^\times)$ where $\alpha_t \neq \alpha_t^\times$. Thus W_i has type 1.

(ii): If in the time interval $(t-1, t)$ the machines are processing operations of the same job j, then all integer time instants s with $S(O_j^2) < s < F(O_j^1)$ are skipped. Furthermore, if $\alpha_t = \alpha_t^\times = \alpha$, then the nonpreemptive character of σ, together with the property $S(O_j^1) \leqslant S(O_j^2)$ and $F(O_j^1) \leqslant F(O_j^2)$ for $j = 1, \dots, k$ implies $\alpha_s = \alpha_s^\times = \alpha$ for $s = S(O_j^2) + 1, \dots, F(O_j^1)$. These facts also imply that the time instants $S(O_j^2)$ and $F(O_j^1)$ are not skipped. Hence, if W_i is the factor in (16) corresponding to the interval $(t-1, t)$, then W_i has pole polynomial p_i and zero polynomial $p_i^\times$ such that $p_i(\lambda) = p_i^\times(\lambda) = (\lambda - \alpha)^{c_i}$ where $c_i = F(O_j^1) - S(O_j^2) = I_j + 1$. Thus W_i has type 2 with degree $I_j + 1$. ∎

Next, suppose W admits a minimal factorization (16) where all factors are irreducible, and hence have type 1 or type 2. Then this minimal factorization uniquely determines a (possibly) preemptive schedule σ for J where all preemptions occur at integer time instants. Indeed, if the integers τ_i are given by $\tau_i = \sum_{j=1}^{i}\delta(W_j)$ for $i = 0, \dots, r$, then σ is defined as follows:

If the factor W_i in (16) has type 1 with pole polynomial $p_i(\lambda) = \lambda - \alpha$ and zero polynomial $p_i^{\times}(\lambda) = \lambda - \alpha^{\times}$ where $\alpha \neq \alpha^{\times}$, then in the time interval (τ_{i-1}, τ_i) the first machine carries out job j which is such that $\beta_j = \alpha^{\times}$, and the second machine carries out job k which is such that $\beta_k = \alpha$.

If the factor W_i in (16) has type 2 with pole polynomial and zero polynomial $p_i(\lambda) = p_i^{\times}(\lambda) = (\lambda - \alpha)^{c_i}$, then in the time interval (τ_{i-1}, τ_i) both machines are carrying out job j which is such that $\beta_j = \alpha$.

The obtained schedule is called the schedule *associated with the minimal factorization* (16). Note that Theorem 2 implies that in this schedule the time instants τ_i are not skipped. Now we can prove our second auxiliary result.

LEMMA 10. *Let W be a companion based matrix function, let J be an instance of 2MFSP-MR or 2MFSP-TR, and assume W and J are associated. Suppose* (16) *is a minimal factorization of W where all factors are irreducible, and hence have type* 1 *or type* 2. *Let $\tau_0, \dots, \tau_r$ be defined as before, and let σ be the schedule associated with* (16). *Then the following statements hold*:

(i) *If the factor W_i in* (16) *has type* 1, *then σ does not contain any job j with $S(O_j^2) \leqslant \tau_{i-1}$ and $F(O_j^1) \geqslant \tau_i$. In this case $\tau_{i-1} = \tau_i - 1$.*

(ii) *If the factor W_i in* (16) *has type* 2, *then σ contains a unique job j with $S(O_j^2) \leqslant \tau_{i-1}$ and $F(O_j^1) \geqslant \tau_i$. In fact, for this job j we have $S(O_j^2) = \tau_{i-1}$ and $F(O_j^1) = \tau_i$.*

Proof. (i): If the factor W_i in (16) has type 1, then the machines are carrying out different jobs in the time interval (τ_{i-1}, τ_i) where $\tau_{i-1} + \delta(W_i) = \tau_{i-1} + 1 = \tau_i$. Now suppose σ contains a job j with $S(O_j^2) \leqslant \tau_{i-1}$ and $F(O_j^1) \geqslant \tau_i$. Since the time instants τ_{i-1} and τ_i are not skipped, we have $S(O_j^2) = \tau_{i-1}$ and $F(O_j^1) = \tau_i$. However, this implies that both machines are carrying out the same job j in the time interval (τ_{i-1}, τ_i). From this contradiction we conclude that σ does not contain any job j with $S(O_j^2) \leqslant \tau_{i-1}$ and $F(O_j^1) \geqslant \tau_i$.

(ii): If the factor W_i in (16) has type 2, then there exists a unique job j such that both machines are carrying out job j in the time interval (τ_{i-1}, τ_i). Thus $S(O_j^2) \leqslant \tau_{i-1}$ and $F(O_j^1) \geqslant \tau_i$. Since the time instants τ_{i-1} and τ_i are not skipped, we have $S(O_j^2) = \tau_{i-1}$ and $F(O_j^1) = \tau_i$. Furthermore, if σ

contained any other job k with $S(O_k^2) < \tau_{i-1}$ and $F(O_k^1) > \tau_i$, then the time instants τ_{i-1} and τ_i would be skipped. However, the latter is not the case. This completes the proof of the lemma. ■

Now we are ready to prove the Theorems 7 and 8, which describe the connections between the Max-Degree problem and the Number problem for companion based matrix functions on one hand, and 2MFSP-MR and 2MFSP-TR on the other.

Proof of Theorem 7. There exists an optimal schedule σ for J with $F(O_j^1) - S(O_j^2) \leqslant \gamma_2(J) + 1$ for $j = 1, \dots, k$. Without loss of generality, σ is a permutation schedule with the property $S(O_j^1) \leqslant S(O_j^2)$ and $F(O_j^1) \leqslant F(O_j^2)$ for $j = 1, \dots, k$ (cf. Lemma 6). Let (16) be the minimal factorization of W associated with σ, and let t be an integer with $(t-1, t) \subset (0, \delta(W))$. Then we can apply Lemma 9. If in the time interval $(t-1, t)$ the machines are processing operations of different jobs, then the factor W_i in (16) corresponding to the time interval $(t-1, t)$ has type 1. Furthermore, if in the time interval $(t-1, t)$ the machines are processing operations of the same job j, then the factor W_i in (16) corresponding to the time interval $(t-1, t)$ has type 2 with degree $I_j + 1$. These statements hold for all time intervals $(t-1, t)$. Thus $\gamma_1(W) \leqslant \gamma_2(J) + 1$.

Conversely, we assume, without loss of generality, that W admits a minimal factorization (16) where all factors W_i are irreducible, and hence have type 1 or type 2 with degree $c_i \leqslant \gamma_1(W)$ for $i = 1, \dots, r$. Let σ be the schedule associated with this minimal factorization. Recall that the time instants τ_i are defined by $\tau_i = \sum_{j=1}^{i} \delta(W_j)$ $(i = 0, \dots, r)$. According to Lemma 10, if the factor W_i in (16) has type 1, then σ does not contain any job j with $S(O_j^2) \leqslant \tau_{i-1}$ and $F(O_j^1) \geqslant \tau_i$. If the factor W_i in (16) has type 2 with degree $c_i \geqslant 1$, then σ contains a unique job j with $S(O_j^2) = \tau_{i-1}$ and $F(O_j^1) = \tau_i$. For this job we have $I_j + 1 = c_i \leqslant \gamma_1(W)$. This implies $\gamma_2(J) + 1 \leqslant \gamma_1(W)$. ■

Proof of of Theorem 8. Suppose σ is an optimal schedule for J with total reduced infeasibility $\nu_2(J)$, and let (16) be the minimal factorization of W associated with σ. Then the total number of skipped time instants equals $[\delta(W) + 1] - (r + 1) = \delta(W) - r$. Since the number of time instants skipped by job j equals I_j, the total number of skipped time instants does not exceed $\nu_2(J)$. Thus $\delta(W) - r \leqslant \nu_2(J)$. Furthermore, $\nu_1(W) \geqslant r$. Combining these inequalities, we find $\nu_1(W) + \nu_2(J) \geqslant \delta(W)$.

Conversely, suppose we have a minimal factorization (16) with $\nu_1(W)$ factors. Obviously, each factor W_i is irreducible, and hence has type 1 or type

2 with degree $c_i \geqslant 1$. This gives the following equalities for the number of factors $\nu_1(W)$ and for the McMillan degree $\delta(W)$:

$$\sum_{i:W_i \text{ of type 1}} 1 + \sum_{i:W_i \text{ of type2}} 1 = \nu_1(W),$$

$$\sum_{i:W_i \text{ of type 1}} 1 + \sum_{i:W_i \text{ of type 2}} c_i = \delta(W).$$

Now let σ be the associated schedule for J. Recall that the time instants τ_i are defined by $\tau_i = \sum_{j=1}^{i} \delta(W_j)$ $[i = 0, \dots, \nu_1(W)]$. According to Lemma 10, if the factor W_i in (16) has type 1, then σ does not contain any job j with $S(O_j^2) \leqslant \tau_{i-1}$ and $F(O_j^1) \geqslant \tau_i$. Thus W_i does not contribute to the total reduced infeasibility of the schedule. If the factor W_i in (16) has type 2 with degree $c_i \geqslant 1$, then σ contains a unique job j with $S(O_j^2) = \tau_{i-1}$ and $F(O_j^1) = \tau_i$. Thus W_i contributes $I_j = c_i - 1$ units to the total reduced infeasibility of the schedule. As a consequence, we have the following (in)equalities:

$$\begin{aligned}\nu_2(J) &\leqslant \sum_{i:W_i \text{ of type 2}} (c_i - 1) = \sum_{i:W_i \text{ of type 2}} c_i - \sum_{i:W_i \text{ of type 2}} 1 \\ &= \sum_{i:W_i \text{ of type 2}} c_i - \left(\nu_1(W) - \sum_{i:W_i \text{ of type 1}} 1\right) = \delta(W) - \nu_1(W).\end{aligned}$$

By combining the obtained inequalities, it follows that $\nu_1(W) + \nu_2(J) = \delta(W)$. ■

7. FURTHER RESULTS

We begin this final section by discussing a generalization of Corollary 4. In fact, for a companion based matrix function we describe the smallest possible McMillan degree of a nontrivial factor that can appear in a minimal factorization, either as an arbitrary middle factor, as a left factor, or as a right factor. In order to make things precise, let W be a rational matrix function.

Then $\theta(W)$, $\theta_1(W)$, and $\theta_2(W)$ are defined by

$$\theta(W) = \min\{\delta(U) \mid W = U_1 U U_2 \text{ is a minimal factorization and } \delta(U) \geqslant 1\}$$

$$\theta_1(W) = \min\{\delta(U) \mid W = U U_2 \text{ is a minimal factorization and } \delta(U) \geqslant 1\},$$

$$\theta_2(W) = \min\{\delta(U) \mid W = U_1 U \text{ is a minimal factorization and } \delta(U) \geqslant 1\}.$$

Here we allow U_1 and U_2 to be trivial factors (i.e. identically equal to the appropriate identity matrix). In general, it is difficult to determine $\theta(W)$, $\theta_1(W)$, or $\theta_2(W)$ for an arbitrary rational matrix function W. However, for companion based matrix functions we have Proposition 11, which is a generalization of Corollary 4.

PROPOSITION 11. *Let W be a companion based matrix function with pole polynomial p and zero polynomial $p^\times$ given by* (9) *and* (10). *Then*

$$\theta(W) = \begin{cases} \min\{s_j \mid j = 1,\dots,k\} & \textit{if } s_j = t_j \textit{ for } j = 1,\dots,k, \\ 1 & \textit{otherwise}, \end{cases}$$

$$\theta_1(W) = \begin{cases} \min\{s_j \mid s_j \leqslant t_j\} & \textit{if } s_j > 0 \textit{ for } j = 1,\dots,k, \\ 1 & \textit{otherwise}, \end{cases}$$

$$\theta_2(W) = \begin{cases} \min\{t_j \mid s_j \geqslant t_j\} & \textit{if } t_j > 0 \textit{ for } j = 1,\dots,k, \\ 1 & \textit{otherwise}. \end{cases}$$

Proposition 11 can be proved by applying methods similar to the ones used in Section 6.4. For details we refer to Bart and Kroon [8]. Note that $\min\{s_j \mid j = 1,\dots,k\} = \min\{t_j \mid j = 1,\dots,k\}$ if $s_j = t_j$ for $j = 1,\dots,k$. Furthermore, $\sum_{j=1}^k s_j = \sum_{j=1}^k t_j = \delta(W)$. Thus $\{j \mid s_j \leqslant t_j\} \neq \varnothing \neq \{j \mid s_j \geqslant t_j\}$.

EXAMPLE 4. Let the companion based matrix function W be the same as in Examples 2 and 3. Both examples show that W admits a minimal factorization $W = U_1 U_2$ such that $\delta(U_2) = 1$. This implies $\theta(W) = \theta_2(W) = 1$. Furthermore, Example 2 also shows that $\theta_1(W) = 3$.

We conclude this paper with some final remarks. First, recall from Section 6.1 that in Mitten's variant of 2MFSP (cf. Mitten [18]) one is interested in a permutation schedule with a minimum makespan, where the infeasibility of each job does not exceed a certain job-specific upper bound. It turns out that, just as 2MFSP-MR and 2MFSP-TR can be related to the Max-Degree problem and the Number problem for companion based matrix functions, there exists a connection between Mitten's variant of 2MFSP and another factorization problem for companion based matrix functions. In this factorization problem one is looking for a minimal factorization of a companion based matrix function where the size of each factor of type 2 does not exceed a certain pole-specific upper bound. For details on this subject, see Bart and Kroon [8].

Further, Zuidwijk [20] has shown that every nontrivial rational matrix function W admits a (possibly nonminimal) factorization into elementary factors. Such a factorization is called a *quasicomplete* factorization if the number of factors involved is minimum. This minimum number of elementary factors is denoted by $\rho(W)$. Obviously, $\rho(W) \geqslant \delta(W)$. Furthermore, Zuidwijk has proved that $\rho(W) < 2\delta(W)$, where $\delta(W)$ is the McMillan degree of W. Now let W be a companion based matrix function, let J be an instance of 2MFSP, and assume W and J are associated. We have indications that quasicomplete factorizations of W and the number $\rho(W)$ are closely related to the combinatorial properties of the instance J of 2MFSP. Zuidwijk and the authors will return to this topic in a forthcoming paper.

Another topic that will be a subject for further research is to find out whether the flow shop problem for more than two machines also has a counterpart in terms of factorization problems for rational matrix functions of a special type.

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