Economic Lot-Sizing Problem with Bounded Inventory and Lost-Sales

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Econometric Institute Report EI 2009-01

In this paper we consider an economic lot-sizing problem with bounded inventory and lost-sales. Different structural properties are characterized based on the system parameters such as production and inventory costs, selling prices, and storage capacities. Using these properties and the results on the lot-sizing problems with bounded inventory, we present improved and new algorithms for the problem. Specifically, we provide algorithms for the general lot-sizing problem with bounded inventory and lost-sales, the lot-sizing problem with nonincreasing selling prices and the problem with only lost-sales.

Keywords: Economic lot-sizing; Lost-Sales; Inventory and Production; Algorithms

1. Introduction

A single item *economic lot-sizing problem with bounded inventory and lost-sales* (ELSP-BL) deals with how many items to produce (or procure) in each period and how to distribute the produced items to demands such that the sum of production costs, inventory costs and lost-sales costs is minimized under the limitation of warehouse storage capacity. The ELSP-BL is a generalization of the classical uncapacitated lot-sizing problem introduced by Wagner and Whitin (1958), which assumes unlimited storage capacity and requires all the demands to be met. The basic aim of the classical lot-sizing problem is to balance the fixed costs and holding costs such that total costs are minimized. However, in case of small demand sizes, it might be better to lose some of the demands or to have lost-sales, instead of satisfying all of them (Aksen et al. 2003). The storage constraint on stock replenishment is observed in various industries. In the process industry like refineries, petrochemical products are bounded by the oil tank size, which functions as a warehouse storage (Liu and Tu, 2007). In third-party logistics industry, which focuses on delivery and storage of items, the major concern in logistics planning is how to deal with warehouse space limitations (Jaruphongsa et al. 2004).

When storage capacity is unlimited in the ELSP-BL, it is called *lot-sizing problem with lost-sales* (ELSP-L). In this problem stockouts are allowed, which means that unsatisfied demand leads to a lost-sales or lost revenue. Lost-sales for a demand may even occur when enough items are available in a warehouse, because it can be more economical to lose the demand and to hold the items for more profitable demands in the following periods. The lost-sales or stockout cost for a demand is accounted by the opportunity cost of not satisfying it, i.e., the unit selling price times the demand size. If a production schedule needs to satisfy the warehouse storage capacity limitation but lost-sales are not allowed, then the ELSP-BL reduces to the *lot-sizing problem with bounded inventory* (ELSP-B).

The characteristics of the ELSP-BL problem can be described by three classes of parameters: costs, selling prices and storage capacities. The cost structure includes *concave*, *fixed-charge* and *nonspeculative costs*. The production function has nonincreasing marginal cost to represent economies of scale which can be described by concave functions. Each item kept in a storage incurs holding cost per unit per period. If the production cost function comprises of a fixed setup and unit production cost component, then it is called a fixed-charge cost structure. If each demand is fulfilled by production without speculative motive to hold inventory, the underlying cost structure is referred to as being nonspeculative. For selling prices, we consider two cases: the general case where selling prices can vary over time and the special case where the prices are nonincreasing over time. The storage capacities can have arbitrary values in each period for the ELSP-BL and are set to infinity for the ELSP-L. Most lot-sizing problems in the literature are solved based on two important policies: the *first come first service* (FCFS) policy where the first demand is satisfied by the first produced item, and the *zero-inventory-ordering* (ZIO) policy in which production or purchasing occurs whenever the inventory level goes down to zero. For the general ELSP-BL problem, neither the FCFS policy (because of difference in profitability between demands) nor the ZIO policy (because of storage capacity limitation) holds, which makes it a challenging problem (most research related with the ELSP-BL problems is concerned with the special cases where either FCFS policy or ZIO policy applies).

This study explores the ELSP-BL in general and the ELSP-L as a special case. We also solve the ELSP-BL and the ELSP-L for the case where selling prices are nonincreasing over time. In particular, we present an $O(T^4)$ general algorithm for the ELSP-BL problem, and special $O(T^2)$ and $O(T)$ algorithms for the problem with a concave and nonspeculative cost structure, respectively, under nonincreasing selling prices. The ELSP-L problem will be solved in $O(T^3)$ time in case of concave costs and in $O(T^2)$, $O(T\log T)$ and $O(T)$ time in case of a concave, fixed-charge and nonspeculative cost structure, respectively, under nonincreasing selling prices. The contributions of this paper are summarized in Table 1.

Table 1. Algorithms and Complexities

	Bounded Inventory and lost-sales (ELSP-BL)	Lost-sales (ELSP-L)
Concave	$O(T^4)$, $O(T^2)$ [a]	$O(T^3)$, $O(T^2)$ [a]
Fixed-charge		$O(T^2)$ [b], $O(T\log T)$ [a]
Nonspeculative	$O(T)$ [a]	$O(T)$ [a]

[a] Nonincreasing selling prices in periods
[b] Loparic et al. (2001), Aksen (2004)

In the next section, the current literature related to the ELSP-BL is surveyed. Section 3 formulates the ELSP-BL problem and presents key properties of an optimal solution. Section 4 presents the general algorithm for the ELSP-BL and Section 5 solves the ELSP-BL with nonincreasing selling prices. The ELSP-L then is solved in Section 6. We conclude the paper in Section 7.

2. Literature Review

Uncapacitated Lot-Sizing Problems. Since the uncapacitated lot-sizing problem of Wagner and Whitin (1958), there have been great efforts to solve it efficiently. Wagner and Whitin (1958) solved the problem for nonspeculative costs based on the ZIO policy in $O(T^2)$ and suggested a planning horizon theorem for possible improvements. Utilizing the monotonicity properties of the planning horizon theorem, Federgruen and Tzur (1991) developed improved $O(T\log T)$ and $O(T)$ algorithms for fixed-charge and nonspeculative cost structures, respectively. The same results were independently obtained by Wagelmans et al. (1992) and Aggarwal and Park (1993). The former found an optimum solution by applying geometric techniques and the latter by applying a matrix searching algorithm using the Monge property. Furthermore, Van Hoesel et al (1994) extended the idea of geometric techniques of Wagelmans et al. (1992) and provided generalized methods for solving dynamic programs.

Lot-Sizing Problems with Bounded Inventory. Love (1973) is the first to study the lot-sizing problem with bounded inventory (ELSP-B), which is closely related with the problem in this paper, and is known to have wide applicability, especially in remanufacturing, in lot-sizing with due dates and time windows, and lot-sizing with cumulative capacities (Van den Heuvel and Wagelmans 2008). By the property that each production quantity is equal to the sum of demands in a series of periods or equal to the storage capacity in that period, Love developed an $O(T^3)$ polynomial time algorithm for concave production and inventory holding costs. Another $O(T^3)$ algorithm is presented for the same problem by Gutiérrez et al. (2002), which is known to be very fast in practice. Under the fixed-charge cost structure, Toczylowski (1995) developed an $O(T^2)$ algorithm by reducing multi-graph edges in a shortest path graph. Furthermore, Liu (2007) presented $O(T^2)$ and $O(T)$ algorithms for fixed-charge and nonspeculative cost structures, respectively, based on the geometric techniques of Wagelmans et al. (1992). Recently, Atamtürk and Küçükyavuz (2008) showed that an $O(T^2)$ algorithm is possible even when set-up cost is included in the inventory holding costs. Gutiérrez et al. (2008) presented an improved $O(T\log T)$ algorithm for the ELSP-B with fixed-charge costs. Finally, the ELSP-B has been proven to be the same as the classical uncapacitated lot-sizing problem in terms of solution complexity (Hwang 2008). He showed that the ELSP-B with backlogging can be solved in $O(T^2)$, $O(T\log T)$ and $O(T)$ time for the concave, fixed-charge and nonspeculative cost structures, respectively.

Lot-Sizing Problems with Bounded Inventory and Lost-Sales. In the lot-sizing literature, Sandbothe and Thompson (1990) first studied the concept of lost-sales and provided an $O(T^3)$ algorithm for the constant production capacity case and an $O(2^T)$ algorithm for the time-varying capacity case. They further generalized the problem by incorporating storage capacity as well (Sandbothe and Thompson 1993). Under fixed-charge costs, the lot-sizing problem with no bound on storage capacity (ELSP-L) was solved in $O(T^2)$ time (Aksen et al. 2003). Loparic et al. (2001) solved a more general ELSP-L with fixed-charge costs and a limit on the number of lost-sales units. They provided an $O(T^2)$ algorithm and extended formulations for various situations. When selling prices are nonincreasing over time, it is shown that an $O(T^2)$ algorithm is possible for the ELSP-BL problem (Liu and Tu 2007). As we shall see later, under nonincreasing selling prices, the ELSP-BL problem is similar to the ELSP-B problem. In these specific problems, we apply known results for the ELSP-B to improve algorithms.

In the next section, notation and mathematical formulations for the lot-sizing problem with bounded inventory and lost-sales are presented.

3. Problem Formulation and Optimality

Let T denote the length of planning horizon. For each period $t \in \{1, 2, \ldots, T\}$ we define:

- d_t: demand in t;
- u_t: storage capacity in t; Without loss of generality, we assume that each storage capacity u_t satisfies $u_t \le d_t + u_{t+1}$. If not, we obtain this property by setting $u_t = \min\{u_t, d_t + u_{t+1}\}$ with $u_T = d_T$, *as this does not change the feasible set of solutions*;
- x_t: production level in t;
- y_t: stockout level (amount of unmet demand) in t;
- I_t: inventory on-hand level in t;
- $p_t(x_t)$: concave production cost function in t for the amount x_t with $p_t(0) = 0$. Under a fixed-charge cost structure, the production cost function $p_t(x_t)$ is represented as $K_t + p_t x_t$ for $x_t > 0$ where K_t and p_t denote setup and unit production costs in period t, respectively. In addition, a nonspeculative cost structure assumes that $p_t + h_t \ge p_{t+1}$;
- b_t: unit selling price or unit stockout (lost-sales) cost in t. (We use selling price as the cost being incurred from lost-sales)
- h_t: unit inventory holding cost in t;

For notational convenience, we let $v_{s,t} = v_s + v_{s+1} + \Lambda + v_t$ if $s \le t$ and $v_{s,t} = 0$ if $s > t$, for any series of values $v_1, v_2, \ldots, v_T$. Then, $x_{s,t}$ and $d_{s,t}$ represent the cumulative sums of productions and demands from s through t, respectively; and $h_{s,t}$ and $b_{s,t}$ represent the cumulative sums of unit holding costs and unit selling prices from period s through t, respectively. The economic lot-sizing problem with bounded inventory and lost-sales (ELSP-BL) is modeled as follows:

$$\text{Min } Z = \sum_{t=1}^{T} \left(p_t(x_t) + b_t y_t + h_t I_t \right) \tag{1}$$

Subject to

$$I_{t-1} + x_t + y_t = d_t + I_t, \quad t = 1, \ldots, T, \tag{2}$$

$$I_{t-1} + x_t \le u_t, \quad t = 1, \ldots, T, \tag{3}$$

$$y_t \le d_t, \quad t = 1, \ldots, T, \tag{4}$$

$$I_0 = I_T = 0, \tag{5}$$

$$x_t \ge 0,\ y_t \ge 0,\ I_t \ge 0, \quad t = 1, \ldots, T.$$

The objective function (1) is to minimize not only production costs but also lost-sales costs. Constraint (2) represents the inventory balance equation in association with lost-sales, and constraint (3) is used for feasibility of storage capacity limits. If stockouts are not permitted (or each $b_t = \infty$), the model (1)–(5) is a lot-sizing problem with bounded inventory (ELSP-B). On the other hand, when the storage capacity constraint is released (or each $u_t = \infty$), then the problem is a lot-sizing problem with lost-sales (ELSP-L). Note that the storage constraint (3) is imposed on the initial available quantity. Love (1973) deals with that constraint on the final inventory by imposing $I_t \leq u_t$. This constraint assumes the produced units in period t is directly delivered to customers without storage. In the ELSP-BL with $I_{t-1} + x_t \leq u_t$, however, we assume that each produced unit is stored in the warehouse before supplying to demands. We see that the ELSP-B imposed with the constraint of $I_{t-1} \leq u_t$ is equivalent to that with $I_{t-1} + x_t \leq u_t$ from the balance equation without lost-sales ($I_{t-1} + x_t = I_t + d_t$). Such equivalence, however, does not hold for the ELSP-BL. So, the solution procedures in this paper will not directly apply to the ELSP-BL with the constraint of $I_{t-1} \leq u_t$.

We now present some terminology to characterize different production, inventory and lost-sales levels. Any period t is called a *production period* if $x_t > 0$. By inventory level I_t, we introduce *renewal* and *warehouse periods*: If we have no inventory at the beginning of period t, i.e., $I_{t-1} = 0$, then we say t is a *renewal period*, and if we have item units up to storage capacity at the beginning of period t, i.e., $I_{t-1} + x_t = u_t$, then we say t is a *warehouse period*. Because it is possible that $I_{t-1} = 0$ and $I_{t-1} + x_t = u_t$, period t can be a warehouse period as well as a renewal period. We need to note that the *regeneration* period in most lot-sizing literature has been defined as the period with final inventory level being zero; that is, period t with $I_t = 0$. However, as the storage constraint (3) suggests, we focus on initial inventory levels and hence on renewal periods rather than on regeneration periods. If a period is either a renewal period or a warehouse period, it is also called an *inventory period*. Regarding lost-sales levels, we say that period t is a *lost-sales period* if $y_t > 0$. If $0 < y_t < d_t$, then we say that demand d_t is *partially satisfied* or *supplied*; if $y_t = 0$, demand d_t is called *fulfilled* and if $y_t = d_t$, then we just say demand d_t is *lost*.

We can view the problem (1)–(5) as a capacitated minimum concave cost flow network problem (Zangwill 1968; Sandbothe and Thompson 1990) as illustrated in Figure 1 for a 5-period problem.

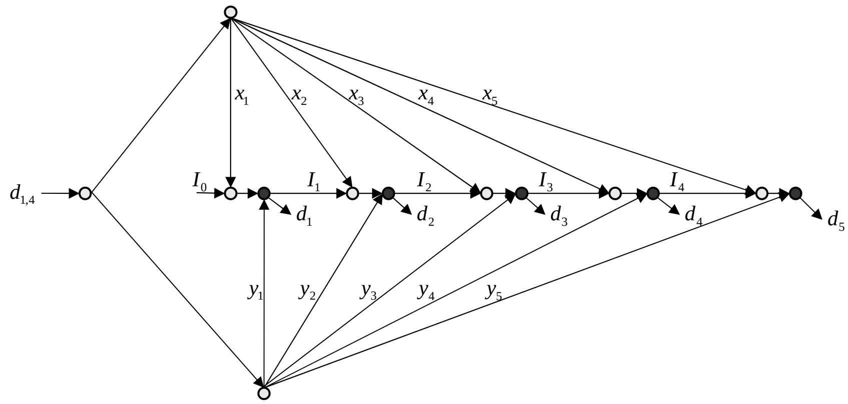

Figure 1. Network representation of the ELSP-BL problem

To better understand our problem in terms of minimum concave cost flow problem and to explore the structure of optimal extreme solutions, we present an example.

Example 1. Consider a five-period problem with zero unit holding and production costs. The setup cost in period 1 is zero but those in other periods are set to ∞, implying we can have production only in period 1. The storage capacities, demands and selling prices are given in Table 2.

Table 2. Demands and prices (Example 1).

Periods	1	2	3	4	5
Storage capacity u_t	100	60	60	40	30
Demand d_t	20	30	50	20	30
Unit price b_t	5	1	2	3	4

Since holding and production costs are all zero, selling prices play a key role in the construction of a production schedule. Considering the prices and the capacity, we can see that the following schedule is optimal.

Table 3. A production schedule (Example 1).

Periods	1	2	3	4	5
Production quantity x_t	80	0	0	0	0
Inventory level I_t	60	60	40	30	0
Lost quantity y_t	0	30	30	10	0

In this schedule, demands d_1 and d_5 are fulfilled, demands d_3 and d_4 are partially satisfied, and demand d_2 is lost. Observe that profitable demand d_5 is fulfilled while less profitable ones like d_2, d_3 and d_4 are not fully satisfied. This is because the storage capacity u_2 (= 60) prohibits the carrying of more than 60 units from period 1. We need to further note that the 40 units at the beginning of period 4 could not fulfill demand d_4 (= 20). Because demand d_5 is more profitable than d_4, demand d_5 has been first satisfied and then the remaining 10 units are supplied for demand d_4. Here, we finally observe that the partial satisfaction occurs at most once between inventory periods; that is, between periods 3 and 4 (periods in [3, 4)); and between 4 and 5 (periods in [4, 5)). Such observation can be generalized to any extreme solutions of minimum concave cost problems. An arc (in Figure 1) with *free flow*, a variable strictly between its lower and upper bound, is called *free arc*. Then any extreme point solution satisfies the 'no-cycle' property, that is, a subnetwork consisting of only free arcs has no cycle (Zangwill 1968; Ahuja et al. 1993). The subnetwork corresponding to the optimal extreme solution in Table 2 is shown in Figure 2.

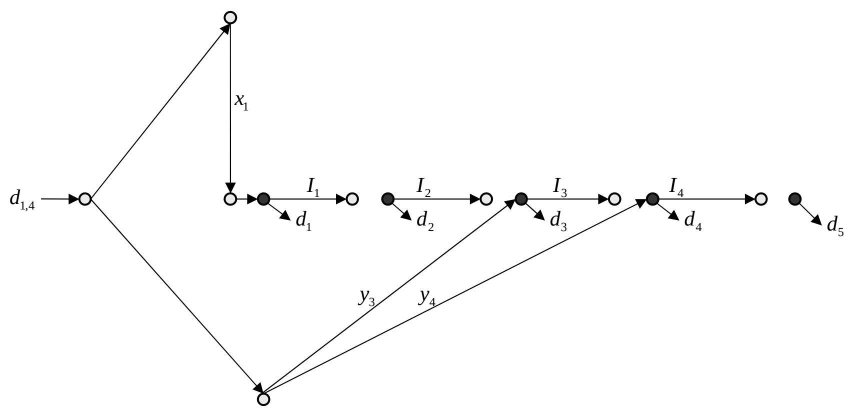

Figure 2. Subnetwork representation of free arcs for an extreme solution.

Consider any solution with consecutive inventory periods s and t+1 ($s < t$+1) where s is a renewal period but is not a warehouse period. If s has no production, then demand d_s is lost since we do not allow backlogging. Now we deal with the case that s is a production period. Note that if the next inventory period t+1 is a renewal period, then $I_i > 0$ and $I_{i-1} + x_i < u_i$ for periods $i = s, ..., t$; and if t+1 is a warehouse period, then $I_i > 0$ for periods $i = s, ..., t+1$ and $I_{i-1} + x_i < u_i$ for periods $i = s, ..., t$. The arcs corresponding to such free flows contribute to the associated subnetwork (See Figure 3. (a)). Note that any arc for a production or for a partial lost-sales will also contribute to the network. Then the no-cycle property implies that there is at most one production period without partial lost-sales or at most one partial lost-sales period without production. The production in period s further means that we have no partial lost-sales. We present the characteristics of an extreme solution in Property 1 being described schematically in Figure 3 (a).

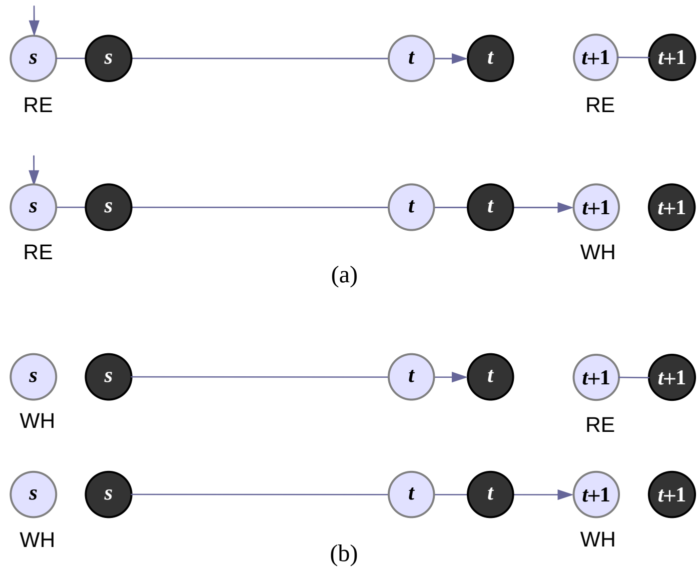

Figure 3. Subnetwork representation of free arcs in association with inventory periods.

Property 1. Suppose that we have consecutive inventory periods s and t+1 where s is a renewal and production period. Then, there exists an optimal schedule such that no partial supply occurs during {s, ..., t} and

(a) if t+1 is a renewal period, then no production occurs during {s+1, ..., t} and

(b) if t+1 is a warehouse period, then no production occurs during {s+1, ..., t+1}.

Note that a warehouse period s has items available up to the capacity u_s. Then the arc corresponding to $I_{s-1} + x_s$ equals to its upper bound and it is not a free arc. With this observation, using similar arguments of 'no-cycle' property as for Property 1, we have the following (See also Figure 3. (b)):

Property 2. Suppose that we have consecutive inventory periods s and t+1 where s is a warehouse period. Then, there exists an optimal schedule such that

(a) if t+1 is a renewal period, then at most one production occurs during {s+1, ..., t} or one partial supply occurs during {s, ..., t} (but not both of them).

(b) if t+1 is a warehouse period, then at most one production occurs during {s+1, ..., t+1} or one partial supply occurs during {s, ..., t} (but not both of them).

Even though Properties 1 and 2 provide a structure of an optimal schedule, we need further details for a complete schedule such as production and lost-sales quantities. Determination of production quantity requires information on which demand is satisfied and which one is lost. So, one of the important things in constructing a schedule is to identify lost-sales. We say that demand d_i is (*relatively*) *more profitable* than demand d_j with respect to period s if $b_i - h_{s,i-1} > b_j - h_{s,j-1}$ for $s \le i, j \le t$. In regard to relative profitability, we have the following property.

Property 3. Suppose that we have no inventory period during s+1, ..., t and demand d_i is more profitable than demand d_j with respect to period s for $s \le i, j \le t$. Then, there exists an optimal solution such that if at least one unit of demand d_j is satisfied, then demand d_i is fulfilled.
Proof. Appendix A.

Under the nonspeculative cost structure that $p_t + h_t \ge p_{t+1}$ for each $t = 1, 2, \ldots, T-1$, we have a useful structural property, allowing more efficient implementation.

Property 4 (Liu and Tub 2007). For the ELSP-BL problem with nonspeculative cost structure, there exists an optimal schedule such that $I_{t-1}x_t = 0$ for all $t = 1, 2, \ldots, T$.

Note that this property means that any production period is a renewal period.

4. Lot-Sizing with Bounded Inventory and Lost-sales: General Case

Based on Properties 1 and 2, we will decompose the problem by inventory periods. We introduce costs $F(s)$ and $G(s)$, by which we will obtain an optimal solution:

- $F(s)$ is the minimum cost in satisfying or losing demands $d_s, d_{s+1}, \ldots, d_T$ under the situation that period s is a renewal period. Note that the optimum cost is $F(1)$. We let $F(T+1) = 0$.
- $G(s)$ is the minimum cost in satisfying or losing demands $d_s, d_{s+1}, \ldots, d_T$ under the situation that u_s units are available at the beginning of period s; that is, s is a warehouse period. It should be noted that $G(s)$ does not include any (production) cost related with the u_s units by period s, These costs will be taken into account later. We let $G(T+1) = \infty$.

It should be emphasized that it may happen that period s in the terms $F(s)$ and $G(s)$ is both a renewal and warehouse period. We consider relative profitability of $b_i - h_{s,i-1}$ for demands d_i, $i = s, s+1, \ldots, t$ where we have no inventory period during s+1, s+2, ..., t. Suppose that periods $s, s+1, \ldots, t$ are arranged in nonincreasing order of $b_i - h_{s,i-1}$, ties being broken arbitrarily, and then stored in a list π. Thus, demand $d_{\pi[1]}$ is the most profitable one among the demands $d_s, d_{s+1}, \ldots, d_t$. The first k demands in the list are called k-

profitable with respective to period s. Property 3 suggests that if (k+1)th profitable demand is satisfied, then any k-profitable demand is also fulfilled. Hereafter in this section, we assume that the lists π for all periods s and t, $1 \le s \le t$, are preprocessed, which can be done at most in $O(T^3 \log T)$ time. For demands in relative profitability, we define the following: for $k = 0, 1, ..., t–s+1$, $1 \le s \le t \le T$,

- $d_{k|s,t}$: the cumulative sum of the k-profitable demands, defined as $d_{k|s,t} = \sum_{i=1}^{k} d_{\pi[i]}$. Note that $d_{0|s,t} = 0$.
- $h(k|s, t)$: the inventory holding cost incurred when satisfying k-profitable demands during $\{s, s+1, \ldots, t\}$ by $d_{k|s,t}$ units in period s, i.e., $h(k|s, t) = \sum_{i=1}^{k} h_{s,\pi[i]-1} \cdot d_{\pi[i]}$. Note that $h(0|s, t) = 0$. If it is not feasible to satisfy the k demands under storage capacity constraint, then we let $h(k|s, t) = \infty$.
- $h(Q, k|s, t)$: the inventory holding cost incurred when satisfying k-profitable demands during $\{s, s+1, \ldots, t\}$ and leaving $Q - d_{k|s,t} \ge 0$ units at the end of period t, i.e., $h(Q, k|s, t) = h(k|s, t) + h_{s,t}(Q - d_{k|s,t})$. If it is not feasible to satisfy the k demands and leave $Q - d_{k|s,t}$ units at the end of period $t - 1$, then we let $h(Q, k|s, t) = \infty$.
- $b(k|s, t)$: the lost-sales cost for giving up non-k-profitable demands during $\{s, s+1, \ldots, t\}$, i.e., $b(k|s, t) = \sum_{i=k+1}^{t-s+1} b_{\pi[i]} \cdot d_{\pi[i]}$. Thus, $b(0|s, t)$ denotes the total lost-sales cost when no demand is satisfied.

In Appendix B, it is shown that these values $d_{k|s,t}$, $h(k|s, t)$ and $b(k|s, t)$ for $1 \le k \le t - s + 1$ and $1 \le s \le t \le T$ are obtained in $O(T^3)$. In computing $h(k|s, t)$, we need to further check feasibility which takes additional $O(T)$ time. Hence, $h(k|s, t)$ for $1 \le k \le t - s + 1$ and $1 \le s \le t \le T$ are obtained in $O(T^4)$. From now on, we assume that all the values $d_{k|s,t}$, $h(k|s, t)$ and $b(k|s, t)$ are known by preprocessing. Then we note that the value $h(Q, k|s, t)$ is calculated immediately since we have the value of $h(k|s, t)$. So, the computation of $h(Q, k|s, t)$ only requires additional time of feasibility checking operation. That is, for given Q, we can obtain $h(Q, k|s, t)$ in $O(T)$.

4.1 Computation of $F(s)$

We provide a recursion procedure for determining $F(s)$. Recall that period s is a renewal period. If we have no production in period s, then as backlogging is prohibited, the demand d_s is being lost with $y_s = d_s$

incurring cost of $b_s d_s$. Since $I_s = 0$, the total cost during $\{s+1, s+2, \ldots, T\}$ is $F(s+1)$ by definition. Thus we have in this case

$$F(s) = b_s d_s + F(s+1).$$

Now we assume that we have production in period s. If period s is a warehouse period, i.e., $x_s = u_s$, then we have

$$F(s) = p_s(u_s) + G(s).$$

Finally we deal with the case that period s is a production and renewal period but not a warehouse period. Let t+1 be the next inventory period after the renewal period s. Then, the production in period s suggests, by Property 1, that each demand during $\{s, s+1, ..., t\}$ be either fulfilled or lost. We further suppose that among the demands d_s, d_{s+1}, …, d_t, only k-profitable demands are replenished in period s while the other ones are lost. With these assumptions in mind, we consider two cases of whether period t+1 is a renewal period or not.

Case 1. *t+1 is a renewal period*: Note that the cost for fulfilling the k-profitable demands is $p_s(d_{k|s,t}) + h(k|s, t)$ and that for giving up the sales is $b(k|s, t)$. Since the cost for the remaining demands d_{t+1}, d_{t+2}, …, d_T is $F(t+1)$ by definition, we have

$$F(s) = p_s(d_{k|s,t}) + h(k|s, t) + b(k|s, t) + F(t+1).$$

Case 2. *t+1 is not a renewal period*: In this case, t+1 is a warehouse period and it has no production by Property 1 (b). Thus, solely using the production in period s, we have to satisfy the k-profitable demands and then reserve u_{t+1} units at the end of period t (at the beginning of period t+1). Hence, the production quantity at period s should be $d_{k|s,t} + u_{t+1}$ with cost $p_s(d_{k|s,t} + u_{t+1})$. Note that the cost for satisfying the k-profitable demands and losing the non-k-profitable demands is $h(k|s, t) + b(k|s, t)$. Furthermore, we need to take into account of the cost for carrying u_{t+1} units from s through t, which is $h_{s,t}u_{t+1}$. Hence, with the fact that $h(d_{k|s,t}+u_{t+1}, k|s, t) = h(k|s, t) + h_{s,t}u_{t+1}$, the total cost during $\{s, s+1, ..., t\}$ is given as $p_s(d_{k|s,t} + u_{t+1}) + h(d_{k|s,t}+u_{t+1}, k|s, t) + b(k|s, t)$. We note that the total cost during $\{t+1, ..., T\}$ is $G(t+1)$ from the fact that u_{t+1} units are available at the beginning of period t+1. We, therefore, conclude $F(s)$ is obtained by

$$F(s) = p_s(d_{k|s,t} + u_{t+1}) + h(d_{k|s,t}+u_{t+1}, k|s, t) + b(k|s, t) + G(t+1).$$

Collecting the formulas for $F(s)$ above, with the initial condition of $F(T+1) = 0$, we have the following complete formula:

$$\begin{aligned} &F(T+1) = 0, \\ &F(s) = \\ &\min\begin{cases} b_s d_s + F(s+1), \\ p_s(u_s) + G(s), \\ p_s(d_{k|s,t}) + h(k\,|\,s,t) + b(k\,|\,s,t) + F(t+1) : 1 \le k \le t-s+1,\ s \le t \le T, \\ p_s(d_{k|s,t} + u_{t+1}) + h(d_{k|s,t} + u_{t+1}, k\,|\,s,t) + b(k\,|\,s,t) + G(t+1) : 0 \le k \le t-s+1,\ s \le t \le T. \end{cases} \end{aligned} \tag{6}$$

In this formula, it should be emphasized that the feasibility on the storage capacity is dealt with by the holding costs $h(k|s, t)$ and $h(d_{k|s,t} + u_{t+1}, k|s, t)$.

4.2 Computation of *G*(*s*)

Let $t+1 > s$ be the first inventory period after period s. As was done in the previous subsection, we deal with two cases of whether $t+1$ is a renewal period or not.

Case 1. *t+1 is a renewal period*: We further consider whether or not a production occurs during $\{s+1, ..., t\}$ as Property 2 (a) suggests. First suppose that we have no production during $\{s+1, ..., t\}$. In this case, by Property 2 (a), we might have a partial satisfaction during $\{s, ..., t\}$ since we have only u_s units for the demands d_s, d_{s+1}, ..., d_t, $u_s \le d_{s,t}$. To identify the demand with partial supply, we consider a list π of periods s, $s+1$, ..., t sorted by profitability with respect to period s. Let $k \ge 1$ be such that $d_{k-1|s,t} \le u_s \le d_{k|s,t}$, which can be found in $O(\log T)$ given the sorted list π. Then $(k-1)$-profitable demands during $\{s, s+1, .., t\}$ are fulfilled and the demand $d_{\pi[k]}$ may be partially fulfilled while remaining non-k-profitable demands are all lost. For the demand $d_{\pi[k]}$, the satisfied amount is $u_s - d_{k-1|s,t}$ and the lost one is $d_{k|s,t} - u_s$. For convenience, we denote the cost $G(s)$ in this case by $G_1(s)$. Then, taking into account of the cost of $F(t+1)$ during $\{t+1, ..., T\}$, we have

$$\begin{aligned} G_1(s) = \min\{&h(k-1|s, t) + b(k|s, t) + h_{s,\pi[k]-1}(u_s - d_{k-1|s,t}) + b_{\pi[k]}(d_{k|s,t} - u_s) + F(t+1): \\ &d_{k-1|s,t} \le u_s \le d_{k|s,t},\ 1 \le k \le t-s+1,\ s \le t \le T\}. \end{aligned}$$

Next consider the case that a production occurs during $\{s+1, ..., t\}$, which assures of no partial supply by Property 2 (a). Let i be the production period, $s + 1 \le i \le t$. We assume k-profitable demands are fulfilled and the other ones are lost during $\{s, ..., t\}$. Then it should hold that $u_s + x_i = d_{k|s,t}$, i.e., $x_i = d_{k|s,t} - u_s$. Among the k-profitable demands, suppose that we have k' demands during $\{s, ..., i-1\}$ and k'' demands during $\{i, ..., t\}$ where $k = k' + k''$. Then the holding and losing cost for the k' demands during $\{s, ..., i-1\}$ is $h(k'|s, i-1) + b(k'|s, i-1)$ and that cost for the k'' demands during $\{i, ..., t\}$ is $h(k''|i, t) + b(k''|i, t)$. Since $I_{i-1} = u_s - d_{k'|s,i-1} > 0$, we have to take into account the cost for carrying the quantity I_{i-1}, which is $h_{s,i-1}(u_s - d_{k'|s,i-1})$. Recall the definition of $h(u_s, k'|s, i-1) = h(k'|s, i-1) + h_{s,i-1}(u_s - d_{k'|s,i-1})$. Let the cost $G(s)$ in this case be $G_2(s)$. Then, we have a formula for $G(s)$ given as follows:

$$G_2(s) = \min\{h(u_s, k'|s, i-1) + b(k'|s, i-1) + p_i(d_{k|s,t} - u_s) + h(k''|i, t) + b(k''|i, t) + F(t+1): \\ 0 \le k \le t - s + 1, s < i \le t \le T\}.$$

Note that the numbers k' and k'' are derived from k for $1 \le k \le t - s + 1$. Hence for given period s, the value $G_2(s)$ is computed in time $O(T^3)$.

Case 2. $t+1$ *is not a renewal period*: Since period $t+1$ is an inventory period but is not a renewal period, it should be a warehouse period, which has u_{t+1} units at the beginning of period $t+1$. Similar to Case 1, we further consider whether or not a production occurs during $\{s+1, ..., t+1\}$ (See Property 2 (b)). First suppose that we have no production. Then, it must be the case that $u_s \ge u_{t+1}$ for feasibility and a partial supply might occur during $\{s, ..., t\}$. That is, we have $u_s - u_{t+1}$ units for satisfying demands $d_s, d_{s+1}, ..., d_t$. Let $k \ge 1$ be the index such that $d_{k-1|s,t} \le u_s - u_{t+1} \le d_{k|s,t}$. Using analogous arguments as for $G_1(s)$ in Case 1 and noticing the cost during $\{t+1, ..., T\}$ is $G(t+1)$, we have

$$G_3(s) = \min\{h(k-1|s, t) + b(k|s, t) + h_{s,\pi[k]-1}(u_s - u_{t+1} - d_{k-1|s,t}) + h_{s,t}u_{t+1} + b_{\pi[k]}(d_{k|s,t} - (u_s - u_{t+1})) + G(t+1): \\ d_{k-1|s,t} \le u_s - u_{t+1} \le d_{k|s,t}, 1 \le k \le t - s + 1, s \le t \le T\}.$$

We next consider the other case where we have production during $\{s+1, ..., t+1\}$. Let i be the production period, $s < i \le t+1$. We assume only k-profitable demands are fulfilled during $\{s, ..., t\}$. Furthermore, among the k-profitable demands, we suppose that k' demands belong to $\{s, ..., i-1\}$ and k'' demands to $\{i,$

..., $t\}$ where $k = k' + k''$. Let the cost $G(s)$ in this case be $G_4(s)$. Applying similar arguments as for $G_2(s)$ and $G_3(s)$, it is not hard to see that

$$G_4(s) = \min\{h(u_s, k'|s, i-1) + b(k'|s, i-1) + p_i(d_{k|s,t} - (u_s - u_{t+1})) + h(d_{k''|i,t}+u_{t+1}, k''|i, t) + b(k''|i, t) + G(t+1).$$
$$0 \le k \le t - s + 1, s < i \le t+1 \le T\}.$$

Combining the cost components $G_1(s)$, ..., $G_4(s)$, we have

$$G(T+1) = \infty,$$
$$G(s) = \min\{G_1(s), G_2(s), G_3(s), G_4(s)\}.$$

For given s, we note that the formulas $G_1(s)$, ..., $G_4(s)$ are all computed in $O(T^3)$. Thus all the values $G_1(s)$, ..., $G_4(s)$ for $s = 1, 2, ..., T$ are obtained in $O(T^4)$. Hence we can find an optimal solution $F(1)$ in $O(T^4)$.

5. Lot-Sizing with Bounded Inventory and Lost-sales: Nonincreasing Selling Prices

In this and the following Section 6, we consider special problems of the ELSP-BL, for which more efficient algorithms will be designed. We define more appropriate inventory holding cost terms for the algorithm efficiency than those of $h(k|s , t)$ and $h(Q, k|s , t)$. Note that the inventory carrying cost terms $h(k|s , t)$ and $h(Q, k|s , t)$ have two functions: the purely holding costs of associated supplies, and the feasibility checking. In the following sections, we will use not inventory holding costs for the second purpose of feasibility checking anymore, but use them only for the first purpose with notations of $h(s, t)$ and $h'(s, t)$:

- $h(s, t)$: the holding cost in fulfilling demands d_s, d_{s+1}, …, d_t with the units of $d_{s,t}$ available during period s, i.e., $h(s, t) = \sum_{i=s}^{t-1} h_i d_{i+1,t}$. Note that $h(s, t) = h(t - s +1|s, t)$.
- $h'(s, t)$: the holding cost in fulfilling demands d_s, d_{s+1}, …, d_t and reserving $u_s - d_{s,t} > 0$ units at the end of period t using u_s units available in period s, i.e., $h'(s, t) = \sum_{i=s}^{t} h_i (u_s - d_{s,i})$. We note that $h'(s, t) = h(u_s, t - s +1|s, t)$.

If the values $h(1, t)$, $h_{1,s-1}$ and $d_{1,t}$ are all preprocessed for $t = 1, ..., T$, the cost $h(s, t)$ can be obtained in constant time by the following recursion formula:

$$h(s, t) = h(1, t) - h(1, s) + h_{1,s-1}d_{1,s} - h_{1,s-1}d_{1,t},$$

We note that those values can be obtained in $O(T)$. In addition, since $h'(s, t) = h(s, t) + h_{s,t}(u_s - d_{s,t})$, we can also compute $h'(s, t)$ in constant time.

The algorithms in the previous section highly depend on the storage capacity constraint causing partial supplies to demands. In this section we consider a relaxed problem of ELSP-BL with nonincreasing selling prices, i.e., $b_1 \geq b_2 \geq \cdots \geq b_T$, which will simplify the designation of partial supplies. The assumption of nonincreasing selling prices also makes the ELSP-BL very similar to the ELSP-B, which will allow the application of the approach developed in the literature for the ELSP-B, in particular, that by Hwang (2008), which utilizes a matrix searching algorithm as in Aggarwal et al. (1987) and Aggarwal and Park (1993). The ELSP-BL with nonincreasing selling prices will be solved more efficiently using decomposition by production periods rather than by inventory periods. We define two sorts of production periods:

- Production period s is called a *complete* (*production*) *period* if a renewal period t+1 > s precedes any production period after s; no production period exists between the two periods s and t+1.
- Production period s is called a *successive* (*production*) *period* if a production period $t > s$ precedes any renewal period; no renewal period exists between the two periods s and t. As we shall see later, any successive period will be a warehouse period.

For a successive production period s with its next production period t, no renewal period exists between them but some warehouse period might exist. Since $I_i > 0$ for every $i = s, s+1, ..., t-1$, the supply from period s should cover demands $d_s, d_{s+1}, ..., d_{t-1}$, i.e., $I_{s-1} + x_s > d_{s,t-1}$. We recall that the assumption $u_t \leq d_t + u_{t+1}$ holds for the storage capacity. This assumption implies that, for the periods s and t, if s is not a warehouse period ($I_{s-1} + x_s < u_s$), then every period during $\{s+1, ..., t-1\}$ is also not a warehouse period ($I_{i-1} < u_i$ for all $i = s+1, ..., t-1$).

Subsection 5.1 presents appropriate optimality properties for the ELSP-BL with nonincreasing selling prices and Subsection 5.2 solves it by a usual $O(T^3)$ dynamic programming algorithm, which will be improved to $O(T^2)$ algorithm based on a matrix searching algorithm. Finally Subsection 5.3 solve the ELSP-BL with nonincreasing selling prices and nonspeculative costs in $O(T)$.

5.1 Properties from Nonincreasing Selling Prices

Under the assumption of nonincreasing selling prices, demand d_t is more profitable than any other following demand. Therefore, if a period t has on-hand inventory, then the demand d_t in that period is (partially) fulfilled. This is formalized in the following property.

Property 5. For the ELSP-BL with nonincreasing selling prices, there exists an optimal schedule such that $I_t y_t = 0$ for each $t = 1, 2, ..., T$.

Because of Property 5, we can simplify feasibility checking operations, which are performed during the computation of the inventory carrying costs $h(k|s\ , t)$ and $h(Q, k|s\ , t)$ in the general algorithm. In solving the ELSP-BL with nonincreasing selling prices, we will not use such implicit feasibility checking, but instead use explicit feasibility checking by the concept of the *minimum supply* and *maximum reachable periods*. For a given period i, its minimum supply and maximum reachable periods, denoted as $m(i)$ and $n(i)$, are defined as

$$m(i) = \min\{s: d_{s,i} \leq u_s,\ 1 \leq s \leq i \leq T\} \text{ and } n(i) = \max\{t: d_{i,t} \leq u_i,\ i \leq t \leq T\}.$$

If period s is successive with its next production period t, then demands d_s, d_{s+1}, ..., d_{t-1} are fulfilled because of Property 5 with $I_i > 0$ for $i = s$, s+1, ..., t–1. This means $d_{s,t-1} \leq u_s$, i.e., equivalently expressed in terms of period t with $m(t)$ or in terms of s with $n(s)$ as follows:

$$m(t-1) \leq s < t \text{ or } s < t \leq n(s) + 1. \tag{7}$$

Furthermore, if a complete production period s covers d_s, d_{s+1}, ..., d_t where t+1 is a renewal period, this situation is described as:

$$m(t) \leq s \leq t \text{ or } s \leq t \leq n(s).$$

With Property 5, we can also determine production quantities of successive production periods more precisely.

Property 6. For the ELSP-BL with nonincreasing selling prices, there exists an optimal schedule such that for each successive production period s, we have $I_{s-1} + x_s = u_s$.

Proof. Appendix A.

This property further suggests that any successive production period is a warehouse inventory period.

Now we consider the case that a production period is complete. Let s be a complete production period with its following renewal period t+1. If period t is no later than the maximum reachable period of s, i.e., $t \le n(s)$, then period t is feasible with respect to the complete production period s and hence all the demands during $\{s, s+1, \dots , t\}$ can be covered by the inventory and production in period s. In this case, we have $I_{s-1} + x_s = d_{s,t}$ by Property 5 since $y_i = 0$ (because of $I_i > 0$) for $i = s, s+1, ..., t$. Next consider the case of $t = n(s) + 1$. In this case period t is out of the feasible range $[s, n(s)]$, causing the demand d_t to be partially supplied. Note that $t = n(s) + 1$ means that $d_{s,t-1} \le u_s < d_{s,t}$. Thus of the demand d_t, the partial $u_s - d_{s,t-1}$ units are supplied but $d_{s,t} - u_s$ units are lost. Finally, we note that the case of $t > n(s) + 1$ is not under our discussion since it has a renewal period in period $n(s) + 1$, which is contradictory to the assumption that period t+1 is the first renewal period since period s. From these observations, we can determine more precisely the production quantity of a complete period.

Property 7. For the ELSP-BL with nonincreasing selling prices, there exists an optimal schedule such that for a complete production period s with its next renewal period t+1,

(a) if $t \le n(s)$, then $I_{s-1} + x_s = d_{s,t}$,

(b) if $t = n(s) + 1$, then $I_{s-1} + x_s = u_s$.

By applying Properties 6 and 7, we can easily see that the general algorithm in the previous section solves the ELSP-BL in $O(T^3)$ time. However, we can solve it more efficiently in $O(T^2)$ using the approach in Hwang (2008). This approach utilizes a matrix searching algorithm as in Aggarwal et al. (1987) and Aggarwal and Park (1993). Based on the Monge property inherent in concave cost functions, the algorithm efficiently determines the column or row minima of a matrix of numbers.

Before presenting solution algorithms, here we need to restrict the definition of $G(s)$ such that period s is not just a warehouse period but more exactly a successive production period in which u_s units are available (the production cost of period s (i.e., $p_s(x_s)$) is not included in $G(s)$). By Property 6, we see that s is also a warehouse period. We also define $G(s, i)$, which is the minimum cost of producing demands d_s, $d_{s+1}, \ldots, d_T$, $1 \le s < i \le T$, under the constraints that period s is successive with its next production period i, which is complete. Similar to $G(s)$, the cost $G(s, i)$ also does not include the production cost of period s. To apply the matrix searching algorithm we need to further define $g_i(s, t+1)$, $t \le n(i)$, which is the cost $G(s, i)$ when the renewal period after the complete production period i occurs at period $t+1$.

5.2 An $O(T^2)$ Algorithm for Nonincreasing Selling Prices

An optimal solution for this problem will be achieved by determining the values $F(s)$, $G(s)$, $G(s, i)$ and $g_i(s, t+1)$. We first show how to compute the cost $F(s)$.

5.2.1 Computation of $F(s)$

We note that period s is a renewal period in $F(s)$. As in the previous section for $F(s)$, if we have no production in period s, then $F(s) = b_s d_s + F(s+1)$. When period s has production, we need to consider whether s is successive or complete. Suppose that s is a successive production period. Then it is also a warehouse period by Property 6. Hence, we have $F(s) = p_s(u_s) + G(s)$. So, we consider the case that s is a complete production (and renewal) period. Let $t+1$ be the renewal period after s. First consider the case where $t \le n(s)$. Then we have no partial supply during $\{s, s+1, \ldots, t\}$ by Property 7 (a) and the production quantity is equal to $d_{s,t}$ (since $I_{s-1} + x_s = d_{s,t}$ and $I_{s-1} = 0$). Hence we have

$$F(s) = p_s(d_{s,t}) + h(s, t) + F(t+1).$$

Finally, we consider the case that $t = n(s) + 1$. In this case, by Property 7 (b) the demand d_t is partially satisfied with lost-sales cost $b_t(d_{s,t} - u_s)$. Note that $I_{s-1} + x_s = u_s$ by Property 7 (b). With the fact that $I_{s-1} = 0$, we have $x_s = u_s$ with associated cost $p_s(u_s)$. Using the u_s units, demands $d_s, d_{s+1}, \ldots, d_{t-1}$ are fulfilled and $u_s - d_{s,t-1}$ units are carried over from s through $t-1$, which costs $h'(s, t-1)$. Thus, we have

$$F(s) = p_s(u_s) + h'(s, t-1) + b_t(d_{s,t} - u_s) + F(t+1).$$

Incorporating the formulas developed so far, $F(s)$ is obtained by

$$F(T+1)=0,$$
$$F(s)=\min\begin{cases} b_s d_s + F(s+1), \\ p_s(u_s)+G(s), \\ p_s(d_{s,t})+h(s,t)+F(t+1): s\le t\le n(s), \\ p_s(u_s)+h'(s,t-1)+b_t(d_{s,t}-u_s)+F(t+1): t=n(s)+1. \end{cases} \tag{8}$$

5.2.2 Computation of $G(s)$

Let t be the next production period of s in $G(s)$. We first consider the case that the period t is also successive. Since s is a successive production period (and hence warehouse production period), a total of u_s units should be available during the period s, which will be used to supply demands d_s, d_{s+1}, …, d_{t-1} leaving $u_s - d_{s,t-1}$ units at the end of period t–1. The cost of carrying u_s units from s through t–1 and reserving $u_s - d_{s,t-1}$ units at the end of period t–1 is $h'(s, t-1)$. Note that the production cost in period s is not included in $G(s)$ by definition and hence it is not considered here. Then we consider period t. Since period t also has successive production, it must hold that $I_{t-1} + x_t = u_t$ (by Property 6), where the beginning inventory I_{t-1} of period t is $u_s - d_{s,t-1}$. Thus, its production quantity is $x_t = u_t - (u_s - d_{s,t-1})$ with associated cost $p_t(u_t - (u_s - d_{s,t-1}))$. Therefore, the cost $G(s)$ in this case is given by

$$G(s) = h'(s, t-1) + p_t(u_t - u_s + d_{s,t-1}) + G(t).$$

Now suppose that the next production is complete at period i for which it should hold $s < i \le n(s) + 1$ for feasibility. Then, by the definition of $G(s, i)$, we have

$$G(s) = G(s, i).$$

Combining the formulas for $G(s)$ leads to a comprehensive formula:

$$G(T+1)=\infty,$$
$$G(s)=\min\begin{cases} h'(s,t-1)+p_t(u_t-u_s+d_{s,t-1})+G(t): s<t\le n(s)+1, \\ G(s,i): s<i\le n(s)+1. \end{cases} \tag{9}$$

5.2.3 Computations of $G(s, i)$ and $g_i(s, t+1)$

Clearly, to calculate $G(s)$ in formula (9), we need to compute $G(s, i)$ for $s < i \le n(s) + 1$. For the purpose of convenience, especially for applying the matrix searching algorithm, we view $G(s, i)$ in terms of period i so that we compute them for $m(i–1) \le s < i$. Note that the inequality $s < i \le n(s) + 1$ is equivalent to $m(i–1) \le s < i$ as shown in (7). In the cost $G(s, i)$, suppose that t+1 is the renewal period of the complete period i. Note that the solution is feasible if the following relation between s, i and t holds:

$$m(i–1) \le s < i \le t \le n(i) + 1.$$

We further suppose that $t = n(i) + 1$. In this case all the demands d_s, d_{s+1}, ..., d_{t-1} are fulfilled but the last demand d_t is partially satisfied. Note that $I_{i-1} + x_i = u_i$ (Property 7 (b)) and $I_{i-1} = u_s - d_{s,i-1}$, resulting in $x_i = u_i + d_{s,i-1} - u_s$ with corresponding cost $p_i(u_i + d_{s,i-1} - u_s)$. Since the holding costs during $\{s, s+1, ..., t\}$ is $h'(s, i–1) + h'(i, t–1)$ and the lost-sales cost in period t is $b_t(d_{i,t} - u_i)$, in this case $G(s, i)$ is given as

$$G(s, i) = h'(s, i–1) + p_i(u_i + d_{s,i-1} - u_s) + h'(i, t–1) + b_t(d_{i,t} - u_i) + F(t+1).$$

We next consider the case where $t \le n(i)$. In this case the cost $G(s, i)$ is the same as $g_i(s, t+1)$. Hence, we have, for $1 \le m(i–1) \le s < i \le T$,

$$G(s,i)= \min\begin{cases} h'(s,i-1)+p_i(u_i+d_{s,i-1}-u_s)+h'(i,t-1)+b_t(d_{i,t}-u_i)+F(t+1): t=n(i)+1. \\ g_i(s,t+1): t\le n(i). \end{cases} \tag{10}$$

Now we focus on the computation of the cost $g_i(s, t+1)$. Recall that period s is successive, period t+1 is a renewal period and the complete production occurs at period i, for $m(i–1) \le s < i \le t \le n(i)$. Then, every demand d_s, d_{s+1}, …, d_{i-1} is fulfilled by the u_s units in period s (i.e., $I_{s-1} + x_s = u_s$) with associated cost $h'(s,$

i–1). Now we consider the cost during $\{i, i+1, \ldots, t\}$. Since $t \le n(i)$, each demand d_i, d_{i+1}, …, d_t is satisfied by the $d_{i,t}$ units in period i (i.e., $I_{i-1} + x_i = d_{i,t}$). Thus we have

$$g_i(s, t+1) = h'(s, i-1) + p_i(d_{s,t} - u_s) + h(i, t) + F(t+1). \tag{11}$$

With this formula, every $g_i(s, t)$, $1 \le s < t \le T$, is obtained in $O(T^3)$. Hence, we can find an optimum schedule in $O(T^3)$. However, by applying the approach by Hwang (2008), we can obtain an optimal solution in $O(T^2)$ time based on the matrix searching algorithm of Aggarwal et al. (1987) and Aggarwal and Park (1993).

5.2.4 Efficient $O(T^2)$ Algorithm

We will use the Monge property to develop an $O(T^2)$ algorithm. An $m \times n$ matrix $e = \{e(s, t)\}$ is called *Monge* if $e(s, t) + e(s+1, t+1) \le e(s+1, t) + e(s, t+1)$, and *inverse-Monge* if $e(s, t) + e(s+1, t+1) \ge e(s+1, t) + e(s, t+1)$ for all $1 \le s \le m$ and $1 \le t \le n$. It is well-known that we can find the row or column minima of any Monge (or inverse-Monge) matrix in $O(\max\{m, n\})$ time if each element $e(s, t)$ of the matrix can be computed in constant time. For given i, we can show that the matrix $g_i(s, t+1)$ is inverse-Monge.

Lemma 1. The matrix $\{g_i(s, t+1): m(i-1) \le s < i, i \le t \le n(i)\}$ is inverse-Monge for $i = 1, 2, \ldots, T$.
Proof. Appendix A.

We assume that $h(s, t)$ and $h'(s, t)$ and $d_{s,t}$ have been preprocessed, all of which can be done in $O(T)$. Thus, if we are given $F(i+1)$, $F(i+2)$, ..., $F(T+1)$, the computation of $g_i(s, t+1)$ in (11) is carried out in constant time. We use $G'(s, i)$ to denote a row minimum of the matrix $\{g_i(s, t+1): m(i-1) \le s < i, i \le t \le n(i)\}$; that is, $G'(s, i) = \min\{g_i(s, t): i \le t \le n(i)\}$ for each s with $m(i-1) \le s < i$. Then all the values $G'(s, i)$ for $m(i-1) \le s < i$ are obtained in $O(T)$ by application of the matrix searching algorithm in Aggarwal et al. (1987). We rewrite $G(s, i)$ in the formula (10) with $G'(s, i)$ as follows: for $1 \le m(i-1) \le s < i \le T$,

$$G(s,i) = \min\begin{cases} h'(s,i-1) + p_i(u_i + d_{s,i-1} - u_s) + h'(i,t-1) + b_t(d_{i,t} - u_i) + F(t+1) : t = n(i)+1. \\ G'(s,i). \end{cases} \tag{12}$$

Now we present a procedure for computing an optimal solution. The algorithm proceeds by complete periods $i = T, T-1, \ldots, 1$. At stage i, suppose that we are given the following values:

- $F(t)$ and $G(t)$ for $t = i+1, \ldots, T$.,
- $G(s, t)$ for $m(t-1) \le s < t$, $t = i+1, \ldots, T$.

Then stage i performs the following steps.

Step 1. Compute the row minima of the matrix $\{g_i(s, t+1): m(i-1) \le s < i, i \le t \le n(i)\}$ by the matrix searching algorithm, obtaining the values $G'(s, i)$ and $G(s, i)$ using formula (12) for each s, $m(i-1) \le s < i$

Step 2. Compute $G(i)$ by (9).

Step 3. Compute $F(i)$ by (8).

Note that the Steps 1–3 in each stage is executed in $O(T)$ and hence the overall complexity of the algorithm is $O(T^2)$.

5.3 ELSP-BL with Nonincreasing Selling Prices and Nonspeculative Cost Structure

In this subsection we consider the ELSP-BL where selling prices are nonincreasing and stored item units are not used for speculative motives. The nonspeculative cost structure allows the property of $I_{t-1}x_t = 0$ (Property 4), implying that no successive production period exists in an optimum solution. Thus, we can solve the problem without using $G(s)$ but only using $F(s)$. From formula (8), the recursion formula then becomes:

$$F(T+1) = 0,$$
$$F(s) = \min\begin{cases} b_s d_s + F(s+1), \\ K_s + p_s d_{s,t} + h(s,t) + F(t+1) : s \le t \le n(s), \\ K_s + p_s u_s + h'(s,t-1) + b_t(d_{s,t} - u_s) + F(t+1) : t = n(s)+1. \end{cases} \tag{13}$$

In this procedure, the key computation is in finding the minimum value of min{K_s + $p_s d_{s,t}$ + $h(s, t)$ + $F(t+1)$: $s \le t \le n(s)$}. In this formula, if every $n(s)$ is T, then it is the optimal procedure for the classical uncapacitated lot-sizing problem, which is shown to be solved in $O(T\log T)$ and $O(T)$ for fixed-charge and nonspeculative cost structures, respectively (Federgruen and Tzur 1991, Wagelmans et al. 1992, Aggarwal and Park 1993). In general, even when $n(s)$ is not equal to T, we can compute the minimum value by a geometric technique in $O(T\log T)$ and $O(T)$ for fixed-charge and nonspeculative cost structures, respectively. It is also worthwhile noting that using the geometric technique (with 2-3 tree data structure of storing linear functions) of Van Hoesel et al. (1994), the computation of min{$K_s + p_s(d_{s,t}) + h(s, t) + F(t+1)$: $s \le t \le n(s)$} is done in $O(\log T)$ and $O(1)$ under fixed-charge and nonspeculative cost structures, respectively (see also Hwang (2008)). Hence the ELSP-BL problem with nonincreasing selling prices and nonspeculative costs can be solved in $O(T)$ time.

Under fixed-charge costs, we cannot solve the problem only using $F(s)$ in formula (13) but have to solve it using the pair of $F(s)$ and $G(s)$. Hence, it takes $O(T^2)$ time using the algorithm in Subsection 5.2. Recently, it is shown that the ELSP-B is solved in $O(T\log T)$ by Gutiérrez et al. (2008) and Hwang (2008). It would be interesting to see whether an $O(T\log T)$ algorithm is possible for the ELSP-BL with fixed-charge costs (and nonincreasing selling prices) by applying the result, especially, of Hwang (2008).

6. Lot-Sizing with Lost-Sales

6.1 Algorithm for ELSP-L

In this section, we deal with the model with possible lost-sales but without storage capacity limits. Without the storage restriction, we have no warehouse period in every extreme solution. Hence we solve ELSP-L by renewal periods; that is, the ELSP-L can be solved by only using $F(s)$ in (6) and not using the pair $F(s)$ and $G(s)$. Thus removing the formulas for warehouse periods, the procedure $F(s)$ in (6) reduces to the following:

$$
\begin{aligned}
&F(T+1) = 0,\\
&F(s) = \min\begin{cases} b_s d_s + F(s+1), \\ p_s(d_{k|s,t}) + h(k \mid s,t) + b(k \mid s,t) + F(t+1) : 1 \le k \le t-s+1,\ s \le t \le T. \end{cases}
\end{aligned}
\tag{14}
$$

Note that the terms $d_{k|s,t}$, $h(k|s, t)$ and $b(k|s, t)$ are all obtained in $O(T^3)$ (Appendix B). With these values being preprocessed, the optimum solution $F(1)$ is computed in $O(T^3)$. In case of the fixed-charge cost structure $p_s(x) = K_s + p_s x$ for $x > 0$, Aksen et. al (2003) solve the problem in $O(T^2)$. We now show that the procedure (14) can also obtain an optimal solution in the same time $O(T^2)$. Note that most computation time in (14) is taken for computing the minimum value of $\min\{p_s(d_{k|s,t}) + h(k|s, t) + b(k|s, t) + F(t+1): 1 \le k \le t-s+1, s \le t \le T\}$, which is the cost when period s is not only a renewal period but also a production period. Note that the dispatch of demands d_s, d_{s+1}, ..., d_t by the production in period s have been done by relative profitability between demands, satisfying only k-profitable demands. Under fixed-charge cost structure, we can do this dispatch explicitly based on *absolute* profitability. The absolute profit of demand d_i with respect to production period s is the difference between its unit selling price and its unit production and holding cost: $b_i - p_s - h_{s,i-1}$ for $s \le i \le t$. For given two periods s and t, only demands with absolute profit $b_i - p_s - h_{s,i-1} \ge 0$ will be satisfied. Thus, the index k in computing the minimum value is the number of demands with nonnegative absolute profit. Hence, the minimum value is obtained in $O(T^2)$, implying the optimum solution $F(1)$ is computed in $O(T^2)$. One might think that the formula (14) could be implemented more efficiently using the geometric techniques developed in the literature (Federgruen and Tzur 1991, Wagelmans et al. 1992, Aggarwal and Park 1993, Van Hoesel et al. 1994). However, it seems not easy to obtain the values $d_{k|s,t}$, $h(k|s, t)$ and $b(k|s, t)$ in less than $O(T^2)$ time, and thus the overall complexity for finding $F(1)$ would be no less than $O(T^2)$.

6.2 ELSP-L with Nonincreasing Selling Prices

In this subsection, the problem with nonincreasing selling prices is considered, $b_1 \ge b_2 \ge \cdots \ge b_T$. This problem can be solved using formula (8) for the ELSP-BL with nonincreasing selling prices. Note that we have no warehouse period so that the formula (8) reduces to:

$$\begin{aligned} &F(T+1) = 0, \\ &F(s) = \min \begin{cases} d_s b_s + F(s+1), \\ p_s(d_{s,t}) + h(s,t) + F(t+1) : s \le t \le T. \end{cases} \end{aligned} \tag{15}$$

From this procedure, we can obtain an optimum solution $F(1)$ in $O(T^2)$. Under fixed-charge costs, the formula (15) is presented as follows:

$$F(T+1)=0,$$
$$F(s)=\min\begin{cases} d_s b_s + F(s+1), \\ K_s + p_s d_{s,t} + h(s,t) + F(t+1) : s \le t \le T. \end{cases} \quad (16)$$

We note that this formula is also obtained from that of (13) by removing the terms related with warehouse period. This procedure is solved in $O(T^2)$ by a usual implementation. Like the procedure (13), the formula (16) is solved in $O(T\log T)$ and $O(T)$ for fixed-charge and nonspeculative cost structures, respectively, by an application of the geometric technique in Van Hoesel et al. (1994). We, therefore, summarize that the ELSP-L problem with fixed-charge cost and nonspeculative cost structures is solved in $O(T\log T)$ and $O(T)$, respectively, under the assumption of nonincreasing selling prices.

7. Conclusion

We have presented improved and new optimal algorithms for a lot-sizing problem with bounded inventory and lost-sales. The algorithms are developed based on different properties of the system parameters such as cost structure, selling prices and storage capacities. The important point in developing the general algorithm of the ELSP-BL is the characterization of the relationship between demands' profitability and storage capacity. In the special case where selling prices are nonincreasing, the study chiefly focused on improving the efficiency of algorithms by applying known results in the literature, such as the matrix searching algorithm and geometric techniques for linear envelope functions.

The ELSP-BL in this paper concerns an upper bound on the number of items available at the beginning of each period. It seems that this problem is different from the problem with a bound on final inventory. However, it may not be hard to solve the ELSP-BL with bounded final inventory using the results developed in this paper. An interesting extension is to consider general inventory holding costs, for example, concave storage cost functions It seems that the concept of relative profitability between demands needs to be generalized in this case. Another extension of this paper is to consider production capacities and explore its relationship with other problem parameters.

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Appendix A. Proofs

Property 3. Suppose that we have no inventory period during s+1, ..., t and demand d_i is more profitable than demand d_j with respect to period s for $s \le i, j \le t$. Then, there exists an optimal soluiton such that if at least one unit of demand d_j is satisfied, then demand d_i is fulfilled.

Proof. To the contrary of the property, we suppose that at least one unit of demand d_j is satisfied but demand d_i is not fulfilled ($y_i > 0$). We will increase the supply quantity for demand d_i (decrease the level of y_i) by decreasing the one for demand d_j (increasing the level of y_j) while keeping feasibility. Since no inventory period exists during s+1, ..., t, it holds that $I_{k-1} > 0$ and $I_{k-1} + x_k < u_k$ for periods $k = s + 1, ..., t$. First suppose that period i precedes period j ($i < j$). Let $\delta = \min\{y_i, d_j - y_j, I_{j-1}\}$. Consider a new solution obtained by redistributing δ units from demand d_j to demand d_i. Because $b_i - h_{s,i-1} \ge b_j - h_{s,j-1}$, the perturbed solution is no worse than the original solution. Furthermore, by the definition of δ, we see that the new solution is feasible. If $\delta = y_i$, then demand d_i is fulfilled; if $\delta = d_j - y_j$, then demand d_j has no units satisfied; and if $\delta = I_{j-1}$, an inventory (renewal) period exists during $\{s+1, ..., t\}$. Hence, the new solution satisfies the property. Now suppose that period i follows period j ($i > j$). In this case, we set the perturbation quantity as

$$\delta = \min\begin{cases} y_i, d_j - y_j, \\ u_k - I_{k-1} - x_k : k = j, j+1, \ldots, i. \end{cases}$$

Note that the new solution is still feasible and no worse than the original one. If $\delta = y_i$ or $\delta = d_j - y_j$, we can apply the same arguments as above. On the other hand, if $\delta = u_k - I_{k-1} - x_k$ for some $k = j, j+1, ..., i$, then we have an inventory (warehouse) period at k. Thus, we see that the new solution in this case also satisfies the property. Repeatedly applying the perturbation process for any periods like i and j, we can finally obtain an optimal solution satisfying the property. □

Property 6. For the ELSP-BL with nonincreasing selling prices, there exists an optimal schedule such that for each successive production period s, we have $I_{s-1} + x_s = u_s$.

Proof. Assume the property is not true, i.e., $I_{s-1} + x_s < u_s$. Without loss of generality, we assume that the optimal schedule is an extreme solution whose corresponding subnetwork has no cycle. Let t be the next production period after period s. Then, the arcs corresponding to the quantities x_s, $I_{i-1} + x_i < u_i$ and $I_i > 0$ for $i \in \{s, s+1, \ldots, t-1\}$ and x_t consists in a cycle of free flows. This is a contradiction to the extreme point solution. Hence we have $I_{s-1} + x_s = u_s$. □

Lemma 1. The matrix $\{g_i(s, t+1)\colon m(i-1) \le s < i, i \le t \le n(i)\}$ is inverse-Monge for $i = 1, 2, \ldots, T$.

Proof. Aggarwal and Park (1993) shows that an $m \times n$ matirx $e = \{e(s, t)\}$ is inverse-Monge if each value $e(s, t)$ can be represented as $e(s, t) = a_s + a_t' + p(y_t' - y_s)$, where a_s, a_t', y_s and y_t' are constants with $y_1 \le y_2 \le \cdots \le y_m$ and $y_1' \le y_2' \le \cdots \le y_n'$, and p is a concave function. For a given period i, consider the value $g_i(s, t+1) = h'(s, i-1) + p_i(d_{s,t} - u_s) + h(i, t) + F(t+1)$. We rearrange it in terms of indices s and t as $g_i(s, t+1) = [h'(s, i-1) - p_i u_s] + [h(i, t) + F(t+1)] + p_i d_{s,t}$. For fixed i, the term $h'(s, i-1) - p_i u_s$ depends on period s while $h(i, t) + F(t+1)$ depends on period t. Only the production cost term $p_i(d_{s,t})$ relies on both periods s and t. We will divide the argument $d_{s,t}$ into two terms based on periods s and t. Note that $d_{s,t} = d_{1,t} - d_{1,s-1}$. Since $d_{1,1} \le d_{1,2} \le \cdots \le d_{1,t}$ and $d_{1,1} \le d_{1,2} \le \cdots \le d_{1,s-1}$, we can conclude the matrix $\{g_i(s, t+1)\colon m(i-1) \le s < i$ and $i \le t \le n(i)\}$ is inverse-Monge. □

Appendix B. Computations of Cost Data for Profitable Demands

In the computation of the data $d_{k|s,t}$, $h(k|s, t)$ and $b(k|s, t)$ for ELSP-BL, we suppose that periods $\{s, s+1, \ldots, t\}$ are arranged in a list π such that $b_{\pi(i)} - h_{s,\pi(i)-1} \ge b_{\pi(i+1)} - h_{s,\pi(i+1)-1}$ for $i = 1, 2, \ldots, t-s+1$. We further suppose that the data $d_{k|s,t}$, $h(k|s, t)$ and $b(k|s, t)$ have been calculated. With these data and the list π, we show how to compute the costs for periods $t+1$. We consider the next list π' for periods $\{s, s+1, \ldots, t+1\}$. Since π is sorted based on the values $b_{\pi(i)} - h_{s,\pi(i)-1}$ for $i = 1, 2, \ldots, t-s+1$, we can position the period $t+1$ in π and obtain a new list π' in $O(\log T)$ time. Then, by using the list π', we can compute the values $d_{k|s,t+1}$, $h(k|s,$

t+1), $b(k|s, t+1)$ immediately. Suppose that period t+1 is not k-profitable with respect to the periods s and t+1, which is easily checked by the list π'. Then the demand d_{t+1} is not supplied by period s but unsatisfied leading to lost-sales. Hence, the total cost of lost-sales during $\{s, s+1, \ldots, t+1\}$ is calculated by $b(k|s, t+1) = b(k|s, t) + b_{t+1}d_{t+1}$. Note that the values $d_{k|s,t+1}$, $h(k|s, t+1)$ are the same as $d_{k|s,t}$, $h(k|s, t)$, respectively. Next, suppose that period t+1 is k-profitable with respect to the periods s and t+1. Then, the period $\pi[k]$ is not k-profitable any more during $\{s, s+1, \ldots, t+1\}$, which has to be replaced by period t+1. Hence, in this case we have $d_{k|s,t+1} = d_{k|s,t} - d_{\pi[k]} + d_{t+1}$, $h(k|s, t+1) = h(k|s, t) - h_{s,\pi[k]-1}\, d_{\pi[k]} + h_{s,t}d_{t+1}$, and $b(k|s, t+1) = b(k|s, t) + b_{\pi[k]}d_{\pi[k]}$. We notice that the computation of $d_{k|s,t+1}$, $h(k|s, t+1)$, $b(k|s, t+1)$ is performed immediately by a simple recursion. Hence, given a period s, we can obtain all the values in $O(T^2)$ for $1 \le k \le t - s + 1$ and $t = s, s+1, \ldots, T$. Thus, all the values $d_{k|s,t+1}$, $h(k|s, t+1)$, $b(k|s, t+1)$ are obtained in $O(T^3)$.