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# Firm Formation with Complementarities: The Role of the Entrepreneur

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# Firm Formation with Complementarities: The Role of the Entrepreneur.

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**Abstract**

We model the emergence of organization forms in a game between prospective entrepreneurs. Complementary roles arise endogenously in a way that admits a stable assignment of workers to firms. This contrasts with existing work on job matching, where stability typically requires workers to be substitutes. Our approach demonstrates that the labor market selection of entrepreneurs and their profit-maximizing choices lead to specific technologies in which certain workers are substitutes and others are complements. We give a simple characterization of equilibrium firm memberships and organizations. We show that payoffs in our non-cooperative solution lie in the core of the corresponding cooperative game, and can be obtained in a decentralized process that reduces information and planning requirements for the entrepreneur.

How firms are formed is still imperfectly understood. Stable matchings of workers to employers have only been shown to exist when complementarities between workers are ruled out or severely restricted. We take the view that complement and substitute relationships in firms are not arbitrary or exogenous. They arise from the technology choices of entrepreneurs, who have an incentive (and face competitive pressure) to implement optimal organizations. In this paper, we derive under conditions of perfect information

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what these equilibrium organizations are, and we show that a unique stable assignment of workers to firms exists.

In standard job matching models, workers are exogenously substitutes (Kelso and Crawford [10], Martinez et al. [16] and [17], Hatfield and Milgrom [8], Hatfield and Kojima [7]).[1] I.e. the value of an additional worker to a firm declines whenever it makes a hire. Complementarity has been introduced through economies of scale that depend only on the number of workers the firm employs (Farrell and Scotchmer [6]) and through supermodularity (Sherstyuk [22]).[2] A new hire makes existing employees more valuable, and the size of the externality increases with every additional worker. Then no two workers are substitutes, and the symmetric nature of the complementarity raises the question whether firms would merge if they were not exogenous.

In this literature, firms are flat. Our firms are hierarchies: workers are not only hired into a firm, but also into a role with a level a seniority. Complement effects arise across levels (between an employee and his or her superior), while substitute effects arise at the same level. For a building company, for instance, different architects may be substitutes, whereas an architect and a construction worker are complements.

We associate every (ordered) pair of individuals $(i, j)$ with a value $v_{ij}$ that $j$ can create if assigned to $i$ in a firm's organization. In this case, we think of $i$ as $j$'s manager, and the pair is complementary. Hiring $h$ to replace $i$ as $j$'s manager reduces $i$'s value to the firm, so that $h$ and $i$ are substitutes. (I.e. substitutes arise from the constraint that a worker must have a unique manager.) Whether a given pair are complements or substitutes is an aspect

[1] In the salary adjustment process Kelso and Crawford [10] analyze, the best offer to a given worker must be repeated in the following round, while others may raise their bids. The central premise behind this approach is that firms will not want to withdraw a successful offer to one worker when competition for other workers intensifies. Hence the worker's value to the firm must not be diminished if co-workers are lost. Crawford and Knoer [2] assume that employee productivity is invariant to who else joins the firm. Kelso and Crawford [10] generalized to the "gross substitutes" property, which is imposed in a number of subsequent studies (e.g. Roth [20] and Ma [15]). In the Kelso and Crawford definition, workers are gross substitutes if higher salary offers to one do not adversely affect firms' willingness to hire the other.

[2] A related kind of complementarity appears in Kremer's [12] model of interdependent production tasks. Here, the likelihood of completing a job successfully increases in the skill of co-workers at their roles. A skilled individual bestows a symmetric externality on all colleagues. One implication that is not echoed in our model is that similarly skilled individuals tend to be hired into the same firms.

of the equilibrium organization technology, not a fundamental property.[3]

The organization with its levels and roles is created by an entrepreneur such that complementarities are maximized. The entrepreneur implicitly hires him- or herself into the role at the top. Because individuals are not uniformly suited to work together and manage, the firm's profitability depends on how each role is staffed, and in particular on the identity of the entrepreneur. Profitability under different staffing scenarios determines the wage offers a prospective entrepreneur can make, and thus whether he or she is ultimately successful in recruiting a workforce and starting a firm. Entrepreneurs are therefore also determined endogenously.

The managerial hierarchy is reminiscent of Rosen's [19] firms, in which the most skilled individuals are employed as managers and confer productive externalities on lower-level workers. A key difference with us is that the externality in Rosen depends only on the identity of the manager, not the worker. In fact, our agents cannot necessarily be ranked by "skill," since complementarities are specific to pairs. Two individuals may be highly effective managers in most cases, but not work well with each other. However, Rosen's explanation of high salaries for top managers, stemming from these hierarchical complementarities, partially carries over, since firms have an incentive to assign managers selectively to make them as productive as possible.[4]

Our approach is broadly in the spirit of Zame's [24] general-equilibrium framework with firm formation, built on earlier work with Ellickson et al. [5],

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[3] Pycia [18] derives a stable matching with complementarities if the equilibrium satisfies pairwise alignment: two members of a firm jointly benefit or jointly lose from adding any group of workers. This is a property of equilibrium payoffs that are, in Pycia's model (not ours), determined after workers are matched to firms. It does not hold in our setting, which differs in several respects (e.g. endogenous firms).

[4] It is worth pointing out how we differ from work on constrained cooperative games, where a network or hierarchy may determine which coalitions can form, or block allocations from the core. In Derks and Gilles [3], hierarchies imply veto rights for superiors against coalitions that subordinates might wish to enter into. Our agents join organizations voluntarily, and their powers to deviate are not in any way restricted.

Legros and Newman [14] also explained organization form and membership in firms. In their context, the organization form is a response to moral hazard: firms have a choice between investing in monitoring technology ($M$-firms) and writing incentive-compatible contracts ($I$-firms). We do not treat agency problems explicitly, and we refer to organization in another sense, as an assignment of employees to managers. Legros and Newman's setting is a risky world where firms have to borrow against the individual wealth of their members. We assume, in the tradition of Crawford and Knoer, that the value any hypothetical firm could create is known, so that funding issues do not arise.

which is not explicit about the process. He takes a feasible set of firm types as given. A firm is defined by the roles its workers need to fill, a stochastic production technology that depends on workers' skills and actions, and a contract that allocates net output among the workers. Workers may also make zero-sum transfers (wage payments) among themselves. A firm comes into existence when, in equilibrium, every role attracts an agent with appropriate skills. Hence, the agents coordinate on the equilibrium firm structure through their job choices. In Zame, there is no firm-building through personal initiative and no explicit mechanism through which coordination occurs. Our model casts every individual as a potential firm-builder, and the equilibrium firm structure as the outcome of active bidding for labor services.[5]

The next section describes the model, assumptions about valuations, and the nature of equilibrium. The valuations we admit include anything that could be derived from a spatial model, where agents are associated with points in $\mathbb{R}^n$ (e.g. professional characteristics), and the value one individual can create under another's management declines in the interpersonal distance. Then we discuss the membership and organization of equilibrium firms. They can be obtained from valuations by a simple algorithm. The third section gives a solution for the equilibrium payoffs and shows that it belongs to the core of the corresponding TU game. An example in terms of spatial valuations is given in the fourth section. We discuss the nature of complements and substitutes. Last, we consider the possibility of decentralization: a network game that implements the equilibrium organization structures without coordinated hiring by entrepreneurs. Proofs are collected in the appendix.

[5]If we view our firms from Zame's perspective, then "entrepreneur" is one of the roles each firm has to fill. The contract gives the entrepreneur a claim to all output, which can be valued at the equilibrium goods prices and treated as profit. In return, the entrepreneur transfers a sum to the other workers that is divided into wage payments for each role. Every role has a specific skill requirement and associated action. Our firm types can be described as sets of skills / actions that an entrepreneur may buy in the labor market. In contrast with Zame, these firms are subsets of a finite population, hence not necessarily small relative to the market, and their output is deterministic. We abstract from the interaction between firm formation and the goods markets, and we sidestep the moral hazard and adverse selection issues discussed by Zame. Hence we also do not concern ourselves with effects of hiring on the competitive structure of goods markets, as in Sasaki and Toda [21].

# 1 Economy

In a finite population $N$, every individual $i$ is assumed to have an exogenous valuation $v_{ij} \in \mathbb{R}_+$ for the labor services of any other individual $j$. The valuation refers to the profit (before wages) that $j$ can generate under $i$'s management.

**Assumption (A1): Exogeneity.** For a given pair of agents $i, j \in N$, $v_{ij} \in \mathbb{R}_+$ is constant.

Specifically, $j$'s productivity under $i$ is not affected by how many, and which, other individuals $i$ manages, or by who manages $i$. It also does not depend on the wage $j$ is paid. Nevertheless, A1 is not at odds with the principal-agent problem. The valuations may reflect, in addition to $j$'s skill at the job and $i$'s skill at designing tasks, how willingly $j$ exerts effort and how well $i$ monitors. If effort were unobservable, $v_{ij}$ could be interpreted as $j$'s expected performance under the optimal contract.[6]

We rule out equal valuations for the same person in the interest of efficient notation. (But one may have the same valuations for others.) The restriction is plausible if valuations are drawn from a continuous distribution. It does implies that, for everyone, there is someone with a strictly positive valuation for their services.

**Assumption (A2): Uniqueness.** For all $i, j, k \in N$, $v_{ik} = v_{jk}$ only if $i = j$.

If $j$ creates more value under $i$'s management than working independently, then perhaps $i$ is more knowledgeable about the task they perform. This reading suggests that $j$ is not an effective manager for $i$. We extend this logic to chains $v_{ij} \geq v_{jj}, v_{jk} \geq v_{kk}, \ldots, v_{lm} \geq v_{mm}$. We require that the first agent creates more value independently than under the management of the last, i.e. $v_{ii} \geq v_{mi}$.[7]

[6] This interpretation can be supported as long as the expected wage cost of inducing a given increase in $v_{ij}$ varies only with $j$, but not with the identity of the manager $i$.

[7] A3 could be replaced by a stronger "positive agency cost" axiom: for all $i, j \in N$, $v_{ii} \geq v_{ij}$, i.e. $i$ can manage self more effectively than others. This statement implies A3, e.g. $v_{ij} \geq v_{jj}$ and $v_{jk} \geq v_{kk}$ lead to $v_{ii} \geq v_{ij} \geq v_{jj} \geq v_{jk} \geq v_{kk} \geq v_{ki}$. Positive agency cost is plausible when management is top-down ($i$ sets tasks for $j$ without seeking $j$'s advice), and delegation may result in a loss from communication barriers and partial effort. The role of $j$ is then merely to carry out instructions as closely as possible.

In applications, it may be meaningful to infer valuations from distances between points

**Assumption (A3): Noncircularity.** For any indexing $t : N \to \{1, 2, \ldots, n\}$ of agents, if $v_{t(t+1)} \geq v_{(t+1)(t+1)}$ for all $t \leq T$, then $v_{11} \geq v_{(T+1)1}$.

In this paper, we treat valuations as public information.[8]

**Assumption (A4): Informedness.** Valuations $\{v_{ij}\}_{i,j \in N}$ are common knowledge.

The valuations are the the economy's data. Now we define strategy spaces and our equilibrium notion, which is a refinement of Nash's. A manager assignment is a function $r_i : N \times 2^N \to N$ such that $r_i(j, C) \in C$. It identifies whom (in $C$) $i$ would assign to manage $j \in C$.[9] Let $R_i$ be the set of such functions. Wage offers are a function $w_i : N \to \mathbb{R}_+$ that specifies a bid for everyone's labor services (including $i$'s own). Let $W_i$ be the set of such functions. Employer choice is a function $e_i : \mathbb{R}^n_+ \to N$ which names, for every profile of offers $w_1(i), w_2(i), \ldots, w_n(i)$ to $i$, the bidder $j \in N$ whose offer is accepted (possibly $j = i$). Let $E_i$ be the set of such functions.[10]

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associated with the individuals. These points could be attributes in a social or professional characteristics space, where distances represent communication barriers or skill mismatch. Positive agency cost is satisfied by valuations that are spatial in the following sense: there exists a mapping $f : N \to \mathbb{R}^l$ and a distance metric $d : N \times N \to \mathbb{R}$ such that, for all $i, j, k \in N$, $v_{ij} \geq v_{ik}$ if and only if $d(f(i), f(j)) \leq d(f(i), f(k))$. To verify that positive agency cost holds, note simply that $d(f(i), f(i)) = 0 \leq d(f(i), f(j))$ for all $j \in N$, so that $v_{ii} \geq v_{ij}$.

The converse, that valuations consistent with "positive agency cost" are spatial, is not true. For example, let (1) $v_{ii} > v_{ij} > v_{ik}$, (2) $v_{jj} > v_{jk} > v_{ji}$, (3) $v_{kk} > v_{ki} > v_{kj}$. While (1) and (3) would imply $d(f(i), f(j)) < d(f(i), f(k)) < d(f(j), f(k))$, (2) requires $d(f(j), f(k)) < d(f(i), f(j))$. By extension, A3 is also strictly more general than the "spatial property."

[8] It is possible to decentralize the game, either as a matching process or network formation game, and lessen information and coordination requirements.

[9] Notation is loose here. The domain of the function is implicity restricted to pairs $(i, C) \in N \times 2^N$ with $i \in C$.

[10] Several properties are implicit in the domains of these functions. The assignment of unique managers, in conjunction with non-circular valuations, implies that organization charts are trees. Holding multiple jobs is ruled out. Employer choice, as we have defined it, precludes a preference for working under specific managers. In practice, the best-paid job is not always chosen. It may be desirable to work with the supervisor that makes the agent most productive. (Dutta and Masso [4] study preferences over colleagues.) One may prefer to be one's own boss. A network of social and family relations may affect the benefits of a job. In our economy, social considerations are absent, i.e. job offers are evaluated only on wages.

A subtle restriction is hidden in the form of the wage offers. In general, $i$ would like to

Given a strategy profile $s \in \times_{i \in N} S_i$ (where $S_i = R_i \times W_i \times E_i$), a firm $F_i(s)$ consists of those inividuals who select $i$ as their employer:

$$F_i(s) = \{j \in N \text{ s.t. } e_j(w) = i\}.$$

Since everyone accepts exactly one wage offer, the collection of firms in the economy is a partition of $N$. Some firms may well be empty: if $F_i(s) = \varnothing$, we will call $i$ an employee; if $F_i(s) \neq \varnothing$, $i$ is an entrepreneur.

The profit that accrues to $i$ is the difference between value created (under the manager assignment) and wages paid by $F_i(s)$:

$$\pi_i(s) = \sum_{j \in F_i(s)} v_{r_i(j, F_i(s))j} - \sum_{j \in F_i(s)} w_i(j). \tag{1}$$

Note that the income of entrepreneurs, i.e. $i \in F_i(s)$, is invariant to the wages they pay themselves: $w_i(i) + \pi_i(s)$ is constant with respect to $w_i(i)$. Nevertheless, wage offers to self matter in a technical sense: they determine whether or not $i$ becomes self-employed. The invariance applies only after this choice is made.

**Definition: Economy.** The economy is a game $\Gamma = \left(N, \{v_{ij}\}_{i,j \in N}, \times_{i \in N} S_i, \{u_i\}_{i \in N}\right)$, with strategy space $S_i = R_i \times W_i \times E_i$ for each $i \in N$, valuations that satisfy A1-A4, and preferences represented by a utility function $u_i : \mathbb{R} \rightarrow \mathbb{R}^+$ that increases monotonically in income $w_{e_i(w)}(i) + \pi_i(s)$ for all $i \in N$.

We treat $\Gamma$ as a normal-form game: strategies are chosen simultaneously; in particular, every $i \in N$ plans the internal structure of any firm $i$ may run, makes wage offers to all $j \in N$, and decides how to select among wage offers $i$ will receive.

A solution of $\Gamma$ is a Nash equilibrium in undominated pure strategies that leads to well-structured firms in a sense we will explain. Strategy $s_i \in S_i$ is undominated if there exists no $s_i' \in S_i$ such that $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in \times_{j \in N \setminus \{i\}} S_j$, and $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for some $s_{-i} \in \times_{j \in N \setminus \{i\}} S_j$.

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offer a schedule of wages to each $j \in N$ that depends on the offers $j$ is making. Then $i$ can reward $j$ for competing less aggressively in the labor market. In particular, $i$ would prevent any employee $j$ from making the best alternative bid for another of $i$'s employees $k$, increasing $k$'s bargaining power with $i$. To this end, $i$ would offer $j$ a higher wage if $j$ bids zero for $k$. Because we do not allow such tie-ins (by forcing offers to be in $\mathbb{R}_+$), competing bids for $i$'s employees may come from within $i$'s firm. Internal competition, from potential spin-offs, is important in practice.

That is, if $s_i$ is not weakly dominated by, and in some situation strictly worse than, another strategy.

The rationale for ruling out equilibria in (weakly) dominated strategies is that agents can otherwise offer wages they are not prepared to pay, knowing they will be outbid. Entrepreneurs would have to pay unreasonably high wages - but might refuse to do so, in which case the overbidders would want to withdraw their offers. Such equilibria seem unstable.[11]

In principle, two employees of a firm could be assigned to manage themselves. This type of arrangement is problematic: no final authority exists to resolve coordination failures (admittedly, coordination is not required in the strict confines of our model). One might conjecture that $i$, as the designer of firm $F_i(s)$, would not adopt such a structure, unless it is strictly profitable to do so. Hence we focus on equilibria where, in each firm, only one individual reports to self. Moreover, in $F_i(s)$, it seems reasonable that this individual should be $i$.[12]

**Definition: Hierarchical Assignment.** Manager assignment $r_i$ is hierarchical if, for all $i, j \in N$, $r_i(j, F_i(s)) = j$ only if $i = j$.

Hierarchical assignments will not be an assumption, but a refinement property of equilibria. We eliminate no strategies and require solutions to be Nash equilibria on the full domain of the strategy space $\times_{i \in N} S_i$.[13] Not joining $F_i(s)$ or choosing a non-hierarchical assignment for $F_i(s)$, which are unilateral deviations for $i$, cannot be payoff-improving at a solution for any $i \in N$.

**Definition: Equilibrium.** Strategy profile $s^* \in \times_{i \in N} S_i$ is a (hierarchical) equilibrium of $\Gamma$ if, for every $i \in N$, $s_i^*$ is undominated, $r_i^*$ is hierarchical, and $u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i', s_{-i}^*\right)$ for all $s_i' \in S_i$.

Even though we have not "forbidden" strategies that lead to negative profits, it is easy to see that no agent can have a negative payoff in equilib-

[11] In English auctions, the private-values assumption prevents overbidding, since all bidders believe they have a positive probability of winning.

[12] If we only impose that there is a unique individual, not necessarily $i$, who reports to self in $F_i(s)$, we get permutations of firm names. The membership and structure of $F_i(s)$ migrate to $F_k(s)$ in alternate equilibria. Payoffs are not affected, but the division of entrepreneurial incomes into wages and profits is then restricted.

[13] The reason is partly technical: since hierarchical assignment requires $i \in F_i(s)$ or $F_i(s) = \varnothing$, $i$ could not make offers without committing to be an entrepreneur if the restriction were applied to the strategy space.

rium. Everyone has the option to be self-employed in a one-man firm and create non-negative value. Reservation wages are therefore non-negative; any negative profit could only arise because an entrepreneur overpays himself. A condition that forces profits to be non-negative could be introduced without changing any aspect of the outcome, except how entrepreneurs allocate their incomes between wages and profits.

# 2 Firms

Associated with an equilibrium $s^*$ is a partition of $N$ into firms $F_i(s^*)$. In this section we derive the unique membership and organization of (hierarchical) equilibrium firms. The requirement that equilibrium play is undominated imposes a few specific constraints. First, entrepreneurs always assign the best available manager to each employee. Second, workers join the firm that makes the highest wage offer to them.[14]

**Lemma (P1).** For all $i \in N$, $s_i \in S_i$ is an undominated strategy only if:
(i) for all $C \subseteq N$ and all $j \in C$, $r_i(j, C) = h$ only if $v_{hj} \geq v_{kj}$ for all $k \in C$;
(ii) $e_i(w) = h \neq i$ only if $w_h(i) \geq w_k(i)$ for all $k \in N \setminus i$.

*Proof.* p. 19.

Given the hierarchy requirement that only entrepreneurs can be assigned to themselves, they must join their own firms if they hire any employees in equilibrium.

**Lemma (P2).** For all $i \in N$, if $F_i(s^*) \neq \varnothing$, then $i \in F_i(s^*)$.

*Proof.* p. 20.

Intuitively, firms will be blocks of complementary individuals who can create value, i.e. effectively manage each other, independently of outsiders. Equilibrium firms can be characterized in terms of the set of individuals for whom $i$ has the highest valuation (is the ideal manager),

$$G_i = \{j \in N \text{ s.t. } v_{ij} \geq v_{kj} \text{ for all } k \in N\},$$

[14]That is, individuals accept the highest wage conditional on becoming workers. It must exceed a reservation level that reflects the option to be self-employed and contribute to value creation in one's own firm. Else, they become entrepreneurs and then may pay themselves less than their "market wage."

and its transitive closure,

$$\bar{G}_i = \left\{ \begin{array}{c} j \in N \text{ s.t. } j \in G_i \text{ or, for some } \{k_1, k_2, \ldots, k\} \subseteq N, \\ k_1 \in G_i,\ k_2 \in G_{k_1}, \ldots, j \in G_k \end{array} \right\}.$$

The latter is the set of individuals whose ideal manager is someone whose ideal manager is someone ... whose ideal manager is $i$. The ideal assignment of the entire population could be visualized as a group of trees, each branching out from an individual who is her own ideal manager (a likely entrepreneur) to members of "upper management" whose ideal manager is an entrepreneur, to members of "middle management" whose ideal manager is in upper management, etc. $\bar{G}_i$ contains everyone "under $i$," the subtree that begins with $i$.

**Lemma (P3).** For all $i, j, j' \in N$ such that $i \neq j \neq j' \neq i$,
(i) $G_i \cap G_j = \varnothing$;
(ii) $G_i \subseteq \bar{G}_i$;
(iii) if $j \in \bar{G}_i$, then (a) $i \notin G_j$, (b) $i \notin \bar{G}_j$, (c) $j \notin \bar{G}_j$, (d) $\bar{G}_j \subset \bar{G}_i$;
(iv) $\bar{G}_i \cap \bar{G}_j = \varnothing$ or $\bar{G}_i \subset \bar{G}_j$ or $\bar{G}_j \subset \bar{G}_i$, and if $j, j' \in G_i$, then $\bar{G}_j \cap \bar{G}_{j'} = \varnothing$;
(v) $G_i \cup \bigcup_{j \in G_i} \bar{G}_j = \bar{G}_i$.

*Proof.* p. 20.

If $j$ belongs to the firm $F_i(s^*)$ (where possibly $i = j$), then $j$'s complementary block $\bar{G}_j$ can create more value in $F_i(s^*)$ than anywhere else, since the ideal managers for members of $\bar{G}_j$ are themselves in $\bar{G}_j \cup j$. Hence, $j$'s employer is able to make the highest bid for $\bar{G}_j$.

**Lemma (P4).** For all $i, j \in N$, if $j \in F_i(s^*)$, then $\bar{G}_j \subseteq F_i(s^*)$.

*Proof.* p. 21.

We can now describe membership in equilibrium firms in terms of the complementary blocks.

**Proposition (P5).** For all $i \in N$, either $F_i(s^*) = \varnothing$ or $F_i(s^*) = \bar{G}_i$.

*Proof.* p. 22.

Nothing in P5 prevents firms from being empty. In particular, $F_i(s^*) = \varnothing$ if $i \notin \bar{G}_i$, i.e. (by P3ii) if $i \notin G_i$. The firms partition $N$ since $x \in \bar{G}_i$ and $i \in G_i$ imply $x \in \bar{G}_j$ only if $\bar{G}_j \subset \bar{G}_i$ (by inductive application of P3iiid).

The structure of the complementary blocks suggests a simple algorithm to solve for equilibrium firms. We define a function $f^0 : N \to N$ that maps to $i \in N$ the individual with the highest valuation for $i$.

$$f^0(i) = j \text{ s.t. } v_{ji} \geq v_{ki} \text{ for all } k \in N.$$

Iterations $f^{t+1}(i) = f(f^t(i))$ successively assign to $i$ the ideal manager, the ideal manager of $i$'s ideal manager, etc. The sequence $\{f^t\}_{t\in\mathbb{N}}$ converges because $N$ is finite and valuations are non-circular. Its limit, $f^\infty = f^t$ such that $f^t = f^{t+1}$, ranges over the set of individuals who are their own ideal managers. These are the entrepreneurs. One can express the firm run by $i$ as

$$F_i(s^*) = \{j \in N \text{ s.t. } f^\infty(j) = i\}.$$

On the basis of P5, we can say more about the equilibrium organization of firms. Since $j \in F_i(s^*)$ only if the largest complementary block that includes $j$ is in $F_i(s^*)$, $j$'s ideal manager, $k$ such that $j \in G_k$, is available. P1 says that $k$ must then be chosen to manage $j$ by all undominated strategies, hence in any equilibrium.

**Proposition (P6).** In any equilibrium, for all $i \in N$ and $j \in F_i(s^*)$, $r_i^*(j, F_i(s^*)) = k$ such that $j \in G_k$.

*Proof.* p. 23.

P6 is a prerequisite for efficiency: the value any given firm creates in equilibrium is the maximum it can achieve in any assignment of workers to managers. Given the equilibrium firm structure, which exploits all complementarities, the economy operates efficiently.

# 3 Payoffs

Wages are not uniquely determined in equilibrium because entrepreneurs have no preference between receiving their income in wages or profits, and between different wage offer schemes that leave the firm's profit unaffected. (Several are possible in equilibrium, as our example will show.) However, the entrepreneurs' equilibrium incomes (wages and profits combined) are unique.

Let $v_{(1)i}, v_{(2)i}, \ldots$ denote the highest, second-highest, etc. valuation for $i$ in the population.

**Proposition (P7).** There exists an equilibrium $s^*$ where the wage offers accepted by $i = 1, \ldots, N$ are

$$w^*_{e^*_i(w^*)}(s^*) = v_{(2)i} + \sum_{j \in G_i} \left(v_{(1)j} - v_{(2)j}\right).$$

*Proof.* p. 23.

The maximal value created by workers in $\bar{G}_j$ for the firm $F_i(s^*)$ depends solely on $j$; not on $j$'s manager, or even the entrepreneur $i$. Hence managers must be paid the entire profit made under their supervision, since that profit could be transferred to another firm. This is the principle underlying the equilibrium wages.

It is not obvious, or necessarily true, that the entrepreneur cannot pay some individuals a higher wage in equilibrium. The reason is that a wage increase for a group of employees reduces its incentive to defect and may therefore permit offsetting wage decreases for other employees (who could otherwise profitably attract the group through a unilateral change in wage offers). Hence there is no reason to believe that the equilibrium wages are unique.

Such redistributions must, however, leave the total wage bill of each equilibrium firm unchanged. Which wage scheme to implement is a matter of choice, not coincidence, given that the entrepreneur makes ultimatum offers. Because the equilibrium firm memberships are unique, any wage distribution that reduces the total bill is strictly preferred by the entrepreneur and applied in equilibrium. Entrepreneurial incomes thus follow from P7 to be, in any equilibrium,

$$w^*_{e^*_i(w^*)}(i) + \pi_i(s^*) = v_{(1)i} + \sum_{j \in G_i} \left(v_{(1)j} - v_{(2)j}\right)$$

for $i \in N$ with $i \in F_i(s^*)$.

Given that many related matching and firm-formation games are cooperative, it is interest to make an explicit connection. In the (transferable-utility) game $\left(N,\ V : 2^N \to \mathbb{R}\right)$ that corresponds to our non-cooperative game $\Gamma$,

$$V(C) = \sum_{i \in C} \max_{k \in C} v_{ki}$$

is the characteristic value (coalition $C$ can create by itself) for each $C \subseteq N$. The allocation $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ is in the core if there is no "blocking" coalition that can guarantee every member a higher payoff (for some $i \in C$ strictly higher). I.e. $u$ is in the core if $V(C) \leq \sum_{i \in C} u_i$ for all $C \subseteq N$.

**Proposition (P8).** If $s^*$ is an equilibrium of $\Gamma$, then $u$ such that $u_i = w^*_{e_i^*(w^*)}(i) + \pi_i(s^*)$ is in the core of the corresponding TU game.
*Proof.* p. 25.

# 4 Example

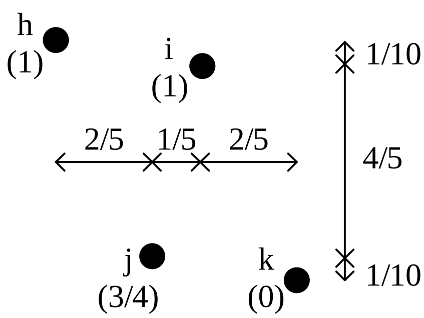
To derive specific valuations, consider a population of four individuals $h$, $i$, $j$ and $k$, who can be mapped to points in a two-dimensional space as in the top left corner of Figure 1. If we think of the dimensions as representing interest in sports and movies, then the minimum distance metric $d(x, y) = \min(|x_1 - y_1|, |x_2 - y_2|)$ quantifies a pair's ability to small talk. Suppose better socializers are also better collaborators, and the value the agent at $y$ creates under management by the agent at $x$ is:

$$v(x, y) = a(x) - d(x, y).$$

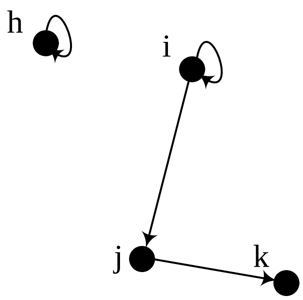
The function $a$ measures management ability; its values are given in Figure 1 in parentheses beneath the locations of $h$, $i$, $j$ and $k$.

On the right side of Figure 1, the distances and resulting valuations are tabulated. Recall that the value of any agent $y$ to potential employers varies only with $y$'s direct productivity $v_{xy}$ under the manager $x$ assigned by the employer. All indirect productivities that arise from workers the firm can profitably recruit and assign to $y$ are available to any employer, once she recruits $y$ (since she will be the highest bidder for these workers). The competition between potential employers demands that such gains are paid back to $y$ in wages. Thus, to determine who will hire $x$, we only need to consider $x$'s direct productivities under different managers, and identify the optimal manager and his employer.

The optimal managers for $h$, $i$, $j$ and $k$ can be found in the columns of Figure 1's valuation table. For $h$ and $i$, the column maxima occur at $h$ and $i$: both are their own best managers. However, $j$'s maximum is at $i$, and $k$'s at $j$. Hence equilibrium hiring and internal organization is given by the graphs in the lower left corner of Figure 1: $h$ employs self; $i$ employs $j$, $k$ and self, and assigns $j$ as manager to $k$ and self to $j$. The firm structure that emerges is $\{h\}$, $\{i, j, k\}$ with $h$ and $i$ entrepreneurs.




| d(.,.) | h | i | j | k |
|---|---|---|---|---|
| h | 0 | 1/10 | 2/5 | 1 |
| i | 1/10 | 0 | 1/5 | 2/5 |
| j | 2/5 | 1/5 | 0 | 1/10 |
| k | 1 | 2/5 | 1/10 | 0 |




| v(.,.) | h | i | j | k |
|---|---|---|---|---|
| h | 1 | 9/10 | 3/5 | 0 |
| i | 9/10 | 1 | 4/5 | 3/5 |
| j | 7/20 | 11/20 | 3/4 | 13/20 |
| k | -1 | -2/5 | -1/10 | 0 |

Figure 1: Spatial example

The wages $h$ and $i$ pay to themselves are immaterial, since they come out of profits and leave the entrepreneur's total income unchanged. How much does $i$ have to offer $j$ and $k$? Consider $k$ first. The only contribution $k$ makes to $i$'s firm is the direct one under $j$'s management, namely $v_{jk} = 13/20$. However, since $i$ makes an ultimatum offer, he only needs to bid up to the second-highest value $k$ could create. This happens to be $v_{ik} = 3/5$, $k$'s productivity under $i$'s management.

At first glance, it may seem odd that $i$ is constrained by his own valuation and pays $v_{ik}$, rather than 0. But note that, if $i$ lowers his bid for $k$, it is $j$ (not $i$) who benefits, since every potential employer of $j$'s would win the bid for $k$ at a lower price and must fully compensate $j$ for the higher value added. If $i$ still wants to hire $j$, he must increase $j$'s offer by the exact amount by which $k$'s offer is reduced.[15] Lowering the bid for $k$ is therefore not an improvement for $i$.

The argument extends generally to any scenario where an employee "withdraws" an offer to a future colleague in order to manipulate the distribution of value within the firm. If the entrepreneur took advantage of the situation by cutting the wage offer to $x$, every employee whose managerial productivity with $x$ exceeds the revised wage for $x$ experiences an increase in market value equal to the cut. Necessarily, the best and second-best manager for $x$ in the population have to be among those whose market value increases; else, if one of them does not belong to the firm, the entrepreneur cannot hire $x$ at the lower wage. But then the entrepreneur pays back at least twice the amount he saves on $x$ (if he is not the best or second-best manager for $x$ himself), so he will simply pass up the opportunity.

Thus $k$'s wage is 3/5, which implies that $j$ adds value $v_{jk} - w_i(k) = 1/20$ through his management. This amount is transferable to any other employer who would hire $j$ and $k$, in particular to $j$ who has the second-highest direct valuation for his own services, $v_{jj} = 3/4$. Clearly, $i$ must offer $j$ a wage of $3/4 + 1/20 = 4/5$; else $j$ would defect to become an entrepreneur and hire $k$. This leaves a profit of $v_{ii} + v_{ij} + v_{jk} - w_i(i) - w_i(j) - w_i(k) = 21/20 - w_i(i)$ to $i$; the total income is 21/20, slightly more than $i$ could obtain independently. Of course, $h$'s profit is $1 - w_h(h)$, and total payment is 1.[16]

[15] Put differently, the value created jointly by $j$ and $k$ is not affected by $i$'s bidding behavior. Thus, the total wage others are willing to pay for the package $\{j, k\}$ is fixed, which implies that payoffs can only be redistributed between the two.

[16] In principle, it is certainly possible that an employee (rather than the entrepreneur) receives the highest income in the firm, if she adds sufficient value by managing people

There are substitute and complement workers in firm $\{i, j, k\}$. For instance, $j$ and $k$ are clearly complements, since each increases the value of the other through their management relationship. But $i$ and $j$ are substitutes. The value $j$ adds to firm $\{i, j, k\}$ is the total value created by $\{i, j, k\}$ less the value of $\{i, k\}$, where $k$ is optimally assigned to be managed by $i$: $v_{ii} + v_{ij} + v_{jk} - v_{ii} - v_{ik} = 17/20$. In firm $\{j, k\}$, $j$ contributes $v_{jj} + v_{jk} - v_{kk} = 7/5$. Evidently, $j$'s value to the firm decreases if $i$ is present, so $i$ is a substitute for $j$. The value added by $i$ to firm $\{i, j, k\}$ is the value of $\{i, j, k\}$ less that of $\{j, k\}$, with $j$ optimally assigned to be self-managed: $v_{ii} + v_{ij} + v_{jk} - v_{jj} - v_{jk} = 21/20$. The value $i$ adds to firm $\{i, k\}$ is $v_{ii} + v_{ik} - v_{kk} = 8/5$. Hence $i$'s value to the firm decreases if $j$ is present, so that $j$ is conversely a substitute for $i$.

# 5 Complements and Substitutes

The coexistence of substitute and complement workers is made possible through the introduction of hierarchical organization forms. That it should be so is quite intuitive: the different roles in a firm are complementary, real substitutability only exists within a role. Two workers are complements in our model if they interact at different levels of the hierarchy: one is assigned to manage the other. On the other hand, they are substitutes if they compete on the same level of the hierarchy: one can replace the other as manager of a given group of employees.

Suppose firm $h$ increases its wage offer for employee $j$ of equilibrium firm $F_i(s^*)$. In case the wage offer is large enough to attract $j$ to $F_h(s^*)$, the effect on an employee $k \neq j$ of $F_i(s^*)$ can be of two kinds: $k$'s value added to $F_i(s^*)$ may weakly increase (making $j$ and $k$ substitutes) or weakly decrease (complements).

If $k$ leaves $F_i(s^*)$, then the value created by the group $G_k \subseteq F_i(s^*)$ is diminished, since $k$ is the best manager for its members. Also, the value of $j \in F_i(s^*)$ such that $k \in G_j$ is diminished, since $j$ is no longer required as the best manager for $k$. These are complement effects. On the other hand, $j$ could replace $k$ as managers for the individuals in $G_k$, if $j$ has the second-highest direct valuation for such an individual within the firm. This is a substitute effect.

directly.

More precisely, the value of worker $j$ to entrepreneur $i$ is

$$
\begin{aligned}
V_{i\{j\}}(s^*) &= \sum_{k\in F_i(s^*)\cup\{j\}} \max_{l\in F_i(s^*)\cup\{j\}} v_{lk} - \sum_{k\in F_i(s^*)\setminus\{j\}} \max_{l\in F_i(s^*)\setminus\{j\}} v_{lk} \\
&= \underbrace{\max_{l\in F_i(s^*)\cup\{j\}} v_{lj}}_{I} + \sum_{k\in F_i(s^*)\setminus\{j\}} \left( \underbrace{\max_{l\in F_i(s^*)\cup\{j\}} v_{lk} - \max_{l\in F_i(s^*)\setminus\{j\}} v_{lk}}_{II} \right) \qquad (2)
\end{aligned}
$$

Term I is the source of the complement effect: losing $k \in F_i(s^*)$ can lower the direct value of $j$ to $i$ if $k$ was $j$'s best manager. Both $k$ and $j$ will suffer wage reductions if they do not join the same firm. Term II is the source of the substitute effect: losing $k$ can increase the indirect value of $j$ to $i$, if $j$ replaces $k$ as the best manager for someone. As a result, $j$ can command a higher wage in a firm without $k$.

If complementarity is too strong, firms may "merge." Valuations are, in this context, supermodular if, for all $i, j \in N$, $v_{ii} + v_{jj} \leq v_{ij} + v_{ji}$ (pairs are more productive than individuals). Now if $v_{ii} > v_{ki}$ for all $k \neq i$, and also $v_{jj} > v_{kj}$ for all $k \neq j$, then $v_{ii} + v_{jj} > v_{ij} + v_{ji}$ unless $i = j$. Hence supermodularity implies a single equilibrium firm.

# 6 Decentralization

The model we have introduced makes unusally strong assumptions about the information available to entrepreneurs (they know not only their own valuations, but also everyone else's). Furthermore, entrepreneurs could only envision the equilibrium firms and make the correct wage offers if they were very sophisticated planners and confident that others play their equilibrium strategies. There are, however, alternative mechanisms that implement the memberships, organization and payoffs with knowledge of one's own valuations only, through sequential link formation. One can think of this as a development process where firms start out small and merge over time, or as a build-up of subcontracting relationships.)

The (non-directed) network game that corresponds to $\Gamma$ is $\left(N,\ V : 2^{N\times N} \to \mathbb{R}\right)$, where $R \subseteq N \times N$ is the endogenous network relation, and

$$V(R) = \sum_{i\in N} v_{R(i)i}$$

is the value population $N$ creates if it is connected by the links in $R$. (We adopt the usual notation $R(i) = j$ if $(i, j) \in R$.) The interpretation of $R(i) = j$ is that $i$ is managed by $j$.

An allocation rule $u : N \times 2^{N \times N} \to \mathbb{R}$ specifies how the network value is apportioned among individuals. We require that $u$ is feasible and consistent with $\Gamma$ setting in the following sense:

$$\sum_{i \in C} u(i, R) \leq \sum_{i \in C} v_{R(i)i}$$

for all $C \subseteq N$ and all $R \subseteq N \times N$. I.e. the total allocation to coalition $C \subseteq N$ cannot exceed the value created by $C$ in $\Gamma$. This condition presumes that individuals who do not belong to $C$ will not contribute to $C$.

The strongly stable networks with respect to an admissible allocation rule are those in which no coalition $C \subseteq N$ can make all members better off (and some $i \in C$ strictly) by severing links involving any of its members, and adding links between any two members. I.e. network $R$ is strongly stable if, for all $C \subseteq N$, for all $i \in C$ and all networks $R'$ obtainable from $R$ via $C$ such that $u(i, R') > u(i, R)$, there exists $j \in C$ such that $u(j, R') < u(j, R)$. Network $R'$ is obtainable from $R$ if $(i, j) \in R'$ and $(i, j) \notin R$ only if $\{i, j\} \subseteq C$, and $(i, j) \in R$ and $(i, j) \notin R'$ only if $\{i, j\} \cap C \neq \varnothing$.

**Proposition (P9).** If $s^*$ is an equilibrium of $\Gamma$, then $R^*$ such that $(x, y) \in R^*$ if and only if $x \in G_y$ is a strongly stable network with respect to an admissible allocation rule where $u(i, R^*) = u_i(s^*)$ for all $i \in N$.

*Proof.* p. 26.

The network game is distinctive in spirit from the non-cooperative formalization: firms organize in a decentralized fashion (through voluntary link formation between workers and managers), rather than by hiring and imposition of a structure from the top. Hence our equilibrium is stable in the further sense that its productive relationships could arise through delegated decision making in firms or subcontracting arrangements or mergers and acquisitions over time.

# 7 Conclusion

By introducing a natural type of complementarity, that between workers and their managers, and a process by which entrepreneurs create firms and implement organization designs, we have shown that firms with rich internal patterns of complementary and substitute relationships arise in non-cooperative or cooperative equilibrium. Firm formation is traditionally approached either from the "firm creation" (entrepreneurship) or the "firm design" (worker-to-job matching) side. The labor market is unlike other two-sided markets (men and women, students and colleges, doctors and hospitals) in that one has a choice to be a worker or an entrepreneur. In our view, this choice occurs simultaneously with - rather than before - a worker's choice of employer. We have developed an integrated model in which agents may organize firms based on the value they expect to create under a "business plan" that encompasses organization structure and labor-market bids for managerial talent.

Formally, this is a coalition-formation game, where the value of each coalition depends on a network it adopts. An entrepreneur is uniquely able to implement a network that covers the opportunity costs of its members (including his or her own). By resolving occupation choice (worker or entrepreneur) through labor market competition for employees, we account for opportunities as well as opportunity costs. In some cases, the entrepreneur is the sole member of a firm that is not very profitable, since no one else offers a better-paying job. Much of the economics of entrepreneurship has focused on personal qualities that affect only opportunity, such as skill (Laussel and Le Breton [13]), risk attitude (Kihlstrom and Laffont [11]) and access to wealth (Legros and Newman [14]). This leaves open why the labor market does not adequately compensate for such qualities. Our approach suggests that entrepreneurs can only realize their potential in organizations of a particular structure that do not yet exist.

# 8 Proofs

### P1

Replacing any $r_i$ with an optimal assignment of managers, i.e. $r_i(j, C) = h$ such that $v_{hj} \geq v_{kj}$ for all $k \in C$, can only be beneficial, and one may construct opposing strategy profiles $s_{-i}$ against which it is a strict improvement over any suboptimal assignment. (Specifically, let the person who is suboptimally assigned join $F_i(s)$.) If $i$ accepts someone else's wage offer, then $i$'s payoff increases directly with a higher wage.

■

**P2**

Let $F_i(s^*) \neq \varnothing$, and suppose $i \notin F_i(s^*)$. Take any $x_0 \in F_i(s^*)$, and label $k$ such that $r_i^*(x_0, F_i(s^*)) = k$ as $k = x_1$, $l$ such that $r_i^*(x_1, F_i(s^*)) = l$ as $l = x_2$, etc. Consider the sequence $\{x_t\}_{t\in\mathbb{N}}$. Because $F_i(s^*)$ is finite, it must be that $r_i^*(x_{t+\theta}, F_i(s^*)) = x_t$ for some $t$ and some non-negative integer $\theta$. Since assignments are hierarchical, and $i \notin F_i(s^*)$, there exists no $x_t \in F_i(s^*)$ such that $r_i^*(x_t, F_i(s^*)) = x_t$. Hence $\theta$ is not zero. P1i requires $r_i^*(x_{t+\theta}, F_i(s^*)) = x_t$ only if $v_{x_t x_{t+\theta}} \geq v_{y x_{t+\theta}}$ for all $y \in F_i(s^*)$. In particular $v_{x_t x_{t+\theta}} \geq v_{x_{t+\theta} x_{t+\theta}}$, which conflicts with A3 and A2.
■

**P3**

(i) A2 guarantees that $v_{ix} \geq v_{kx}$ for all $k \in N$ only if there exists no $j \in N$, $j \neq i$, such that $v_{jx} \geq v_{kx}$ for all $k \in N$. Hence if $x \in G_i$, then $x \notin G_j$.

(ii) If $j \in G_i$, then $j \in \bar{G}_i$ is immediate from the definition of $\bar{G}_i$.

(iii) If $j \in \bar{G}_i$, then there exists a sequence $\{k_1, k_2, \ldots, k\} \subseteq N$ such that $k_1 \in G_j$, $k_2 \in G_{k_1}$... $j \in G_k$. Thus $v_{ik_1} > v_{k_1k_1}$, $v_{k_1k_2} > v_{k_2k_2}$ ... $v_{kj} > v_{jj}$ (A2 makes the inequalities strict). Applying A3, we have $v_{ii} > v_{ji}$.

Hence it is not the case that $v_{ji} \geq v_{ki}$ for all $k \in N$, i.e. (a) $i \notin G_j$. If $i \in \bar{G}_j$, then $v_{ji} > v_{ii}$, which is also a contradiction, so (b) $i \notin \bar{G}_j$. If $j \in \bar{G}_j$, then $v_{jj} \geq v_{kj}$ for all $k \in N$; in particular $v_{jj} \geq v_{ij}$, which is at odds with $j \in G_i \subseteq \bar{G}_i$ and A2. Thus (c) $j \notin \bar{G}_j$. Let $x \in \bar{G}_j$. Then either $x \in G_j$ or there exists a sequence $\{k_1', k_2', \ldots, k'\} \subseteq N$ such that $k_1' \in G_j$, $k_2' \in G_{k_1'}$... $x \in G_{k'}$. In both cases, $j \in \bar{G}_i$ implies there is a sequence $\{l_1, l_2, \ldots, l\} \subseteq N$ such that $j \in G_i$, $l_1 \in G_j$, $l_2 \in G_{l_1}$ ... $x \in G_l$. Therefore $x \in \bar{G}_i$. So (d) $\bar{G}_j \subseteq \bar{G}_i$, and by (ii) and (iiic) $j$ is in $\bar{G}_i$ but not in $\bar{G}_j$, so the inclusion is strict.

(iv) Suppose there exists $x \in \bar{G}_i \cap \bar{G}_j$. Then there are sequences $K = \{k_1, k_2, \ldots, k\} \subseteq N$ such that $k_1 \in G_j$, $k_2 \in G_{k_1}$ ... $x \in G_k$ and $K' = \{k_1', k_2', \ldots, k'\} \subseteq N$ such that $k_1' \in G_{j'}$, $k_2' \in G_{k_1'}$ ... $x \in G_{k'}$. It follows from (i) that $x \in G_k \cap G_{k'} \neq \varnothing$ only if $k = k'$ etc. Therefore $K \subseteq K'$ or $K' \subseteq K$, and thus either $i \in K'$ or $j \in K$, i.e. either $i \in \bar{G}_j$ or $j \in \bar{G}_i$. By (iiid), $j \in \bar{G}_i$ implies $\bar{G}_j \subset \bar{G}_i$, and $i \in \bar{G}_j$ implies $\bar{G}_i \subset \bar{G}_j$.

If $j, j' \in G_i$, suppose $\bar{G}_j \cap \bar{G}_{j'} \neq \varnothing$, so that $\bar{G}_j \subset \bar{G}_{j'}$ or $\bar{G}_{j'} \subset \bar{G}_{j'}$. In the first case, $j \in G_i$ implies $i \in \bar{G}_{j'}$; in the second case, $j' \in G_i$ implies $i \in \bar{G}_j$ - either of which contradicts (iiib). We conclude $\bar{G}_j \cap \bar{G}_{j'} = \varnothing$.

(v) If $j \in G_i$, $j \neq i$, then $\bar{G}_j \subset \bar{G}_i$ by (iiid). Hence $\cup_{j \in G_i} \bar{G}_j \subseteq \bar{G}_i$. Moreover, $G_i \subseteq \bar{G}_i$ by (ii), which establishes the $\subseteq$ part of the equality. If $x \in \bar{G}_i$ and $x \notin G_i$, then there exists $\{k_1, k_2, \ldots, k\} \subseteq N$ such that $k_1 \in G_i$, $k_2 \in G_{k_1}$... $x \in G_k$. It follows that $x \in \bar{G}_{k_1}$ for some $k_1 \in G_i$, or $x \notin G_i$. Relabeling $k_1$ as $j$, we have $\bar{G}_i \subseteq G_i \cup \bigcup_{j \in G_i} \bar{G}_j$.

■

**P4**

We show: for all $i, j, k \in N$, if $j \in F_i(s^*)$ and $k \in G_j$, then $k \in F_i(s^*)$. This implies $j \in F_i(s^*)$ only if $G_j \subseteq F_i(s^*)$, and we apply P3 to argue $G_j \subseteq F_i(s^*)$ only if $\bar{G}_j \subseteq F_i(s^*)$.

Let $k \in G_j$, and suppose $s^*$ is such that $j \in F_i(s^*)$ while $k \in F_h(s^*)$, with $h \neq i$. Since $s^*$ is an equilibrium, the profit generated by $h$'s employees cannot be negative:

$$\sum_{x \in F_h(s^*) \backslash h} v_{r_h^*(x, F_h(s^*))x} - \sum_{x \in F_h(s^*) \backslash h} w_h^*(x) \geq 0; \tag{3}$$

else $h$ could strictly improve on $u_h(s^*)$ by offering $w_h(x) = 0$ to all $x \in F_h(s^*)$. If $h \in F_h(s^*)$, then

$$u_h(s^*) = v_{r_h^*(h, F_h(s^*))h} + \sum_{x \in F_h(s^*) \backslash h} v_{r_h^*(x, F_h(s^*))x} - \sum_{x \in F_h(s^*) \backslash h} w_h^*(x). \tag{4}$$

Suppose $i$ offered every one of $h$'s employees a slightly higher wage: $\tilde{w}_i(x) = w_h^*(x) + \varepsilon$ for all $x \in F_h(s^*) \backslash h$, with $\varepsilon > 0$. If $h \in F_h(s^*)$, suppose $i$ also offered $h$ a wage that exceeds the current payoff: $\tilde{w}_i(h) = u_h(s^*) + \varepsilon$. Any employer-choice function that would reject these offers is not undominated, hence cannot be part of an equilibrium strategy. (P1ii implies $x \in F_h(s^*)$ only if $h$ offered the highest wage to $x$ in $s^*$. After topping the offer, $i$ must be the high bidder and gain $x$.) We show that it is in fact an improvement for $i$ to offer these wages for some $\varepsilon > 0$.

The payoff for $i$ when running firm $F_i(\tilde{s}_i, s_{-i}^*) = F_i(s^*) \cup F_h(s^*)$ after increased offers $\tilde{w}_i$, with all else equal, is

$$\begin{aligned} u_i(\tilde{s}_i, s_{-i}^*) &= u_i(s^*) + \sum_{x \in F_h(s^*)} v_{r_i^*(x, F_i(\tilde{s}_i, s_{-i}^*))x} - \sum_{x \in F_h(s^*)} w_h^*(x) - \sum_{x \in F_h(s^*)} \varepsilon \\ &\geq u_i(s^*) + \sum_{x \in F_h(s^*)} \left( v_{r_i^*(x, F_i(\tilde{s}_i, s_{-i}^*))x} - v_{r_h^*(x, F_h(s^*))x} \right) - \sum_{x \in F_h(s^*)} \varepsilon \end{aligned} \tag{5}$$

if $h \notin F_h(s^*)$, and

$$\begin{aligned} u_i\left(\tilde{s}_i, s_{-i}^*\right) &= u_i(s^*) + \sum_{x \in F_h(s^*)} v_{r_i^*\left(x, F_i\left(\tilde{s}_i, s_{-i}^*\right)\right)x} - \sum_{x \in F_h(s^*) \setminus h} w_h^*(x) - u_h(s^*) - \sum_{x \in F_h(s^*)} \varepsilon \\ &\geq u_i(s^*) + \sum_{x \in F_h(s^*)} \left(v_{r_i^*\left(x, F_i\left(\tilde{s}_i, s_{-i}^*\right)\right)x} - v_{r_h^*(x, F_h(s^*))x}\right) - \sum_{x \in F_h(s^*)} \varepsilon \qquad (6) \end{aligned}$$

if $h \in F_h(s^*)$. Inequalities (5) and (6) derive, respectively, from (3) and (4).

For all $x \in F_h(s^*)$,

$$v_{r_i^*\left(x, F_i\left(\tilde{s}_i, s_{-i}^*\right)\right)x} \geq v_{r_h^*(x, F_h(s^*))x},$$

since $F_h(s^*) \subseteq F_i\left(\tilde{s}_i, s_{-i}^*\right)$. Because $s_i^*$ is undominated, P1i implies that the assignment $r_i^*$ is value-maximizing. Clearly, the maximal valuation for any $x \in F_h(s^*)$ must be at least as large in $F_i\left(\tilde{s}_i, s_{-i}^*\right)$ as in $F_h(s^*)$.

Since $k \in G_j$ and $j \notin F_h(s^*)$,

$$v_{jk} > v_{r_h^*\left(k, F_h^*\right)k}.$$

On the other hand $j \in F_i(s^*) \subseteq F_i\left(\tilde{s}_i, s_{-i}^*\right)$, so $r_i^*\left(k, F_i\left(\tilde{s}_i, s_{-i}^*\right)\right) = j$ and $v_{r_i^*\left(k, \tilde{F}_i\right)k} = v_{jk}$. Then $u_i\left(\tilde{s}_i, s_{-i}^*\right) \geq u_i(s^*)$ if

$$\varepsilon = \frac{v_{jk} - v_{r_h^*(k, F_h(s^*))k}}{n+1} > 0.$$

The deviation establishes that $k \in F_h(s^*)$ for any $h \neq i$ is not possible in equilibrium. Thus $k \in F_i(s^*)$, and we have demonstrated that $j \in F_i(s^*)$ leads to $G_j \subseteq F_i(s^*)$. Let $x \in \bar{G}_j$ and $x \notin G_j$. Then there exists $\{k_1, k_2, \ldots, k\} \subseteq N$ such that $k_1 \in G_j$, $k_2 \in G_{k_1}, \ldots, x \in G_k$. From $j \in F_i(s^*)$ and $k_1 \in G_j$ we have $k_1 \in F_i(s^*)$, applying our prior argument. Similarly, $k_1 \in F_i(s^*)$ and $k_2 \in G_{k_1}$ imply $k_2 \in F_i(s^*)$. Inductively, $k_1, k_2, \ldots, k \in F_i(s^*)$, and therefore $x \in F_i(s^*)$. It follows that $j \in F_i(s^*)$ entails $\bar{G}_j \subseteq F_i(s^*)$.

■

**P5**

Since $i \in F_i(s^*)$ by P2 if $F_i(s^*) \neq \varnothing$, P4 requires $\bar{G}_i \subseteq F_i(s^*)$. It remains to be shown that $F_i(s^*) \subseteq \bar{G}_i$, or equivalently $N \setminus \bar{G}_i \subseteq N \setminus F_i(s^*)$.

Suppose $x \in N \setminus \bar{G}_i$ and $x \in F_i(s^*)$. We relabel $x$ as $x_0$ and reconstruct the sequence $\{x_t\}_{t \in \mathbb{N}}$ as in the proof of P2. Observe that $i \neq x_t$ for any $t$; else we would have $x \in \bar{G}_i$. By our prior argument, $r_i^*(x_{t+\theta}, F_i(s^*)) = x_t$ for some $t$ and integer $\theta > 0$, which violates A3 unless $r_i^*(x_t, F_i(s^*)) = x_t$ for some $x_t \in F_i(s^*) \neq i$. But this does not satisfy the hierarchy requirement. Hence $x \in N \setminus F_i(s^*)$, and we have established $F_i(s^*) = \bar{G}_i$.

■

**P6**

Follows from P2 and the fact that $j \in F_i(s^*)$ only if $k \in F_i(s^*)$ such that $j \in G_k$, which is what we have to show. If $j \in F_i(s^*)$ and $j \in G_k$, but $k \in F_h^*$ with $h \neq i$, then $j \in F_h^*$: by P3ii $G_k \subseteq \bar{G}_k$, and by P4, $\bar{G}_k \subseteq F_h^*$. This contradicts the premise $j \in F_i(s^*)$.

■

**P7**

We construct the equilibrium $s^*$ as follows. Manager assignments $r^*$ are value-maximizing (satisfy P6), and employer choices $e^*$ select the highest wage offer (or, in case of a tie, the offer from the individual with the higher direct valuation). The high bid for each $i \in N$ is $w_{(1)}^*(i) = v_{(2)i} + \sum_{j \in G_i} \left(v_{(1)j} - v_{(2)j}\right)$, and is made by the person with the highest direct valuation of $i$, i.e. $h$ such that $v_{hi} = v_{(1)i}$. The high bid is matched by the person with the second-highest direct valuation of $j$, i.e. $h'$ such that $v_{h'i} = v_{(2)i}$.

The resulting firms are, for $i = 1, \ldots, N$, $F_i(s^*) = \bar{G}_i$ if $i \in G_i$ and $F_i(s^*) = \varnothing$ otherwise, which means $s^*$ is hierarchical. We argue that $s^*$ is also Nash. No one can have an incentive to deviate by reorganizing an efficient equilibrium firm (change $r_i^*$). Accepting the highest wage offer is always best for non-entrepreneurs and, given the form of the winning offers, implies that $i$ becomes an entrepreneur if and only if $i \in G_i$. In $F_i(s^*)$, $i$ adds at least $v_{ii} + \sum_{j \in G_i} \left(v_{(1)j} - v_{(2)j}\right)$ under the manager assignment $r_i^*$. If $i \in G_i$, then $v_{ii} = v_{(1)i}$, so $i$ can earn more income through contributing to profit in $F_i(s^*)$ than from the highest competing wage offer. Conversely, suppose $i \notin G_i$, but $i$ turns down the highest wage offer to become an entrepreneur. Because the entrepreneur's income is independent of the wage paid to self, this scenario is akin to an increase in wage offers. We may therefore confine ourselves to considering changes in wage offers.

Observe first that $i$ cannot profitably reduce wage offers. Suppose $i$ is an entrepreneur. Employing $j \in F_i(s^*)$ at wage $w_{(1)}^*(j)$ is strictly profitable for $i$, since $j \in \bar{G}_i$ and $j \in G_k$ implies $k \in \bar{G}_i$, so that $j$ is assigned to the best manager and directly adds $v_{(1)j} > v_{(2)j}$ to the firm $F_i(s^*)$. Moreover $G_j \subseteq \bar{G}_i$, hence $j$ indirectly adds at least $\sum_{x \in G_i} \left(v_{(1)x} - v_{(2)x}\right)$ to $F_i(s^*)$ as the best manager for the group $G_j$. Offering less than $w^*_{(1)}(j)$ loses $j$ to the previously second-highest bidder and therefore reduces $i$'s profit. If $i$ is not an entrepreneur, then none of $i$'s wage offers are accepted, and lowering them does not change anything for $i$.

No more can $i$ profitably increase wage offers. If $i$ is to benefit from raising offers, they must be accepted and add to membership in $F_i(s^*)$. Suppose $i$ attracts the group $C$ from outside $F_i(s^*)$. Then $i$ must offer strictly more than $w^*_{(1)}(j)$ to each $j \in C$:

$$\begin{aligned}\sum_{j \in C} w_i(j) \geq \sum_{j \in C} w^*_{(1)j} &= \sum_{j \in C} v_{(2)j} + \sum_{j \in C} \sum_{x \in G_j} \left(v_{(1)x} - v_{(2)x}\right) \\ &= \sum_{j \in C} v_{(2)j} + \sum_{x \in \cup_{j \in C} G_j} \left(v_{(1)x} - v_{(2)x}\right).\end{aligned}$$

Since $F_i(s^*)$ initially included all ideal managers for its employees, members of $C$ can only add value directly or through managing other members of $C$. I.e. their contribution to $F_i(s^*)$ is $\sum_{j \in C} \max_{k \in F_i(s^*)} v_{kj}$. Denote the subset of $C$ with best managers in $C$ by $C_0 \equiv \{x \in C \text{ s.t. } x \in G_j \text{ with } j \in C\}$. Because $F_i(s^*)$ already included anyone whose ideal manager is in $F_i(s^*)$, all other members of $C$, i.e. $j \in C \setminus C_0$, cannot make a direct contribution greater than $v_{(2)j}$ to $F_i(s^*)$. The contribution $C$ makes to $F_i(s^*)$ is therefore at most:

$$\sum_{j \in C_0} v_{(1)j} + \sum_{j \in C \setminus C_0} v_{(2)j} \geq \sum_{j \in C} \max_{k \in F_i(s^*)} v_{kj}.$$

Because $C_0 \subseteq \cup_{j \in C} G_j$,

$$\begin{aligned}\sum_{j \in C} w_i(j) &\geq \sum_{j \in C} v_{(2)j} + \sum_{x \in \cup_{j \in C} G_j} \left(v_{(1)x} - v_{(2)x}\right) \\ &= \sum_{j \in C_0} v_{(2)j} + \sum_{j \in C_0} \left(v_{(1)j} - v_{(2)j}\right) + \sum_{j \in C \setminus C_0} v_{(2)j} + \sum_{x \in \cup_{j \in C} G_j \setminus C_0} \left(v_{(1)x} - v_{(2)x}\right) \\ &= \sum_{j \in C_0} v_{(1)j} + \sum_{j \in C \setminus C_0} v_{(2)j} + \sum_{x \in \cup_{j \in C} G_j \setminus C_0} \left(v_{(1)x} - v_{(2)x}\right) \\ &\geq \sum_{j \in C} \max_{k \in F_i(s^*)} v_{kj} + \sum_{x \in \cup_{j \in C} G_j \setminus C_0} \left(v_{(1)x} - v_{(2)x}\right).\end{aligned}$$

This means $i$ would pay more for $C$ than its members can contribute to $F_i(s^*)$; raising bids is not profitable.

Hence individuals are optimizing in all three strategic components in $s^*$, and $s^*$ is a hierarchical equilibrium.

■

**P8**

Any $C \subseteq N$ can be partitioned into sets $\bar{G}_i \cap C$ for $i \in N$ such that $F_i(s^*) \neq \varnothing$ in a given equilibrium of $\Gamma$. (Since $F_i(s^*) = \bar{G}_i$ by P5 and the nonempty $F_i(s^*)$ partition $N$.) Hence, to establish $\sum_{i\in C} \max_{k\in C} v_{ki} \leq \sum_{i\in C} u_i$ for all $C \subseteq N$, it is sufficient to show that, for any $i \in N$ with $F_i(s^*) \neq \varnothing$,

$$\sum_{x\in\bar{G}_i\cap C} \max_{k\in C} v_{kx} \leq \sum_{x\in\bar{G}_i\cap C} u_i.$$

By assumption, for all $x \neq i$,

$$u_i = v_{(2)x} + \sum_{y\in G_x\cap C} \left(v_{(1)y} - v_{(2)y}\right).$$

For $x = i$,

$$u_i = v_{(1)x} + \sum_{y\in G_i\cap C} \left(v_{(1)y} - v_{(2)y}\right).$$

Since $\{G_x\}_{x\in\bar{G}_i}$ partitions $\bar{G}_i \setminus G_i$ by P3v, so that the union of $G_x \setminus \{i\}$ over $x \in \bar{G}_i$ is $\bar{G}_i \setminus G_i$,

$$\begin{aligned}
\sum_{x\in\left\{\bar{G}_i\setminus i\right\}\cap C} u_i \quad &\geq \quad \sum_{x\in\left\{\bar{G}_i\setminus i\right\}\cap C} v_{(2)x} + \sum_{x\in\left\{\bar{G}_i\setminus G_i\setminus i\right\}\cap C} \left(v_{(1)x} - v_{(2)x}\right) \\
&= \quad \sum_{x\in\left\{\bar{G}_i\setminus G_i\setminus i\right\}\cap C} v_{(1)x} + \sum_{x\in G_i\cap C} v_{(2)x}.
\end{aligned}$$

Now, if $i \notin \bar{G}_i \cap C$, then

$$\begin{aligned}
\sum_{x\in\bar{G}_i\cap C} u_x = \sum_{x\in\left\{\bar{G}_i\setminus i\right\}\cap C} u_x \quad &\geq \quad \sum_{x\in\left\{\bar{G}_i\setminus G_i\right\}\cap C} v_{(1)y} + \sum_{x\in G_i\cap C} v_{(2)y} \\
&\geq \quad \sum_{x\in\bar{G}_i\cap C} \max_{k\in C} v_{kx}
\end{aligned}$$

because $i \notin \bar{G}_i \cap C$ implies $i \notin C$ by design of the partition: $F_i(s^*) = \bar{G}_i$, and $i \in F_i(s^*)$ by P2, hence $i \in \bar{G}_i$. But then, for all $x \in G_i$, $\max_{k \in C} v_{kx} \leq v_{(2)x}$.

If $i \in \bar{G}_i \cap C$, then

$$\begin{aligned} \sum_{x \in \bar{G}_i \cap C} u_x &= u_i + \sum_{x \in \left\{\bar{G}_i \setminus i\right\} \cap C} u_x \\ &\geq \sum_{x \in \bar{G}_i \cap C} v_{(1)x} \geq \sum_{x \in \bar{G}_i \cap C} \max_{k \in C} v_{kx}. \end{aligned}$$

■

**P9**

By P8,

$$\sum_{i \in C} \max_{k \in C} v_{ki} \leq \sum_{i \in C} u_i(s^*)$$

for all $C \subseteq N$, and in particular for any coalition that deviates to induce a network $R'$ which is obtainable from $R^*$ through $C$. Since

$$\sum_{i \in C} u(i, R') \leq \sum_{i \in C} v_{R'(i)i} \leq \sum_{i \in C} \max_{k \in C} v_{ki}$$

(the first inequality from feasibility of the allocation $u$), we have

$$\sum_{i \in C} u(i, R') \leq \sum_{i \in C} u_i(s^*) = \sum_{i \in C} u(i, R^*),$$

which implies there is no deviation that makes every member of $C$ better off (and some $i \in C$ strictly).

■

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