Chapter 3

PUSH and PULL control: Mathematical modeling

To analyze the effects of remanufacturing on system performance we would like to select a combination of system characteristics and control strategies such that *(i)* mathematical analysis is possible, and *(ii)* all important effects that occur in practice could be represented. Unfortunately, conditions *(i)* and *(ii)* are impossible to satisfy simultaneously: constraint *(i)* limits the selection to relatively simple systems, operating under relatively simple strategies, whereas *(ii)* forces to investigate complex real-life systems. As a compromise we have chosen to analyze a single-product hybrid system by means of which the influence of a number of typical process variables and process characteristics discussed in Chapter 1 could be evaluated. The specific characteristics regarding the system are outlined in Section 3.1.

To control the system, we investigate two different continuous review operating strategies, one being a PUSH strategy, the other one being a PULL strategy. Our motivation for choosing continuous review rather than periodic review is that the time-independent steady-state system behaviour of continuous review models is easier to describe and understand than the time-dependent system behaviour of periodic review models. Also, these strategies are closely related to the control policies that are nowadays widely accepted and used in practice to control inventory systems in situations without remanufacturing.

This chapter is further outlined as follows. The PUSH and PULL strategy are analyzed in Section 3.2. The enumerative procedure that has been implemented to search for optimal strategy parameters is discussed in Section 3.3. Finally, to further investigate the effects of specific assumptions re-

garding process variables and system characteristics, we discuss in Section 3.4 how we can relax these assumptions.

3.1 System characteristics

The system that will be considered in this chapter is shown in Figure 3.1. It is basically a simplification of the system that has been implemented at the copier manufacturer mentioned in Chapter 1. The first main simplification consists herein that our system applies to a single module remanufacturable product, rather than to a multi-component product like a photocopier. Consequently, we do not consider disassembly operations to disassemble returned products into modules, and we have only two stocking points: one for remanufacturable products and one for serviceable products, i.e., products used to fulfill customer demand. The second main simplification consists herein that we assume that all returned products satisfy the quality requirements for remanufacturing.

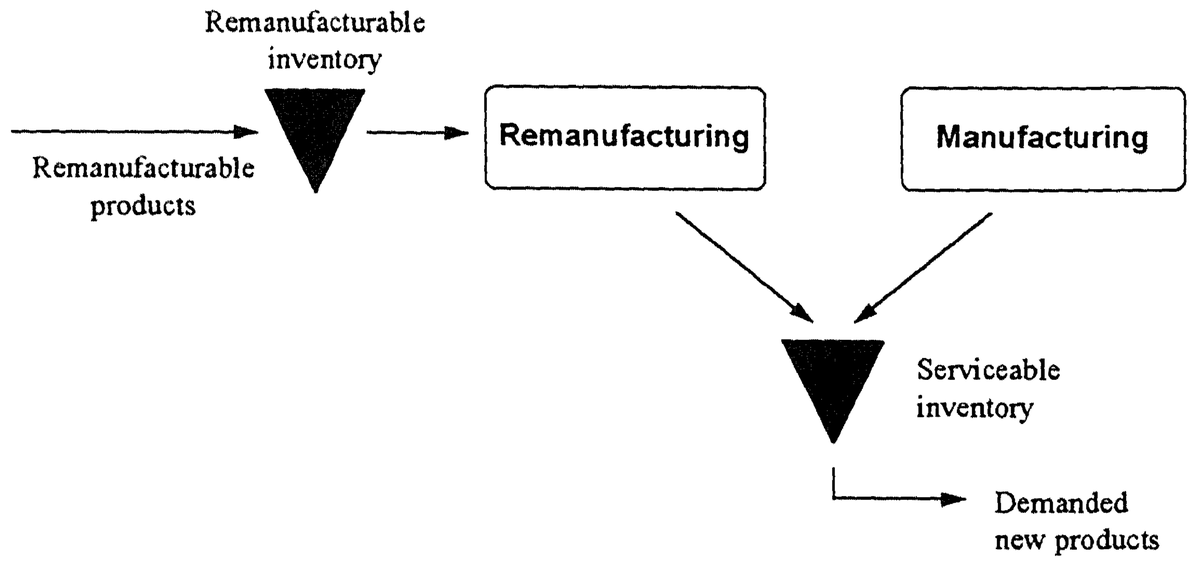


Figure 3.1. *A hybrid system with manufacturing and remanufacturing operations, and stocking of remanufacturable and serviceable products.*

In detail the system characteristics are:

- *Demand and return process.* To evaluate the influence of the uncertain demand and return process on system performance we assume that unit sized demands and unit sized remanufacturable returns have *exponentially* distributed inter-occurrence times. The average time between two subsequent remanufacturable returns is $\frac{1}{\lambda_R}$, and the average time between two subsequent product demands equals $\frac{1}{\lambda_D}$.

Furthermore, the intensity of remanufacturable returns λ_R is smaller than the demand intensity λ_D, and demands and remanufacturable returns are uncorrelated.

- *Testing process.* It is assumed that every returned item is already tested and satisfies the quality requirements for remanufacturing.

- *Remanufacturing process.* The remanufacturing process has unlimited capacity; the remanufacturing lead-time L_r, which is the time that passes between the time at which a remanufacturing order is placed and the time of actual delivery, is a random variable with mean μ_{L_r} and variance $\sigma^2_{L_r}$; the fixed remanufacturing costs are c_r^f, and the variable remanufacturing costs are c_r^v per product. A remanufacturing operation moves a certain number of remanufacturables from the remanufacturable inventory to 'Work In Process' (WIP). After remanufacturing all products enter the serviceable inventory.

- *Outside procurement process.* New products are manufactured in an outside procurement facility. The manufacturing costs consist of a fixed component c_m^f, and a variable component of c_m^v per product. The manufacturing capacity is unlimited, and the manufacturing lead-time L_m, which is the time that passes between the time at which a manufacturing order is placed and the time of actual delivery, is a random variable with mean μ_{L_m} and variance $\sigma^2_{L_m}$. Manufactured products enter the serviceable inventory.

- *Inventory process.* There exist two stocking points in the system, one to keep remanufacturable inventory and one to keep serviceable inventory. The holding costs in remanufacturable inventory are c_r^h per product per time-unit, and the holding costs in serviceable inventory are c_s^h per product per time-unit. Both stocking points have unlimited capacity.

- *Disposal process.* All returned products enter the remanufacturing process, i.e, disposal of remanufacturables does not occur. Strategies for *planned* disposal will be considered in Chapter 6.

- *Customer service.* Demands that can not be fulfilled immediately are backordered against backorder costs c_b per product per unit of time.

The notation with respect to the system characteristics is summarized in Table 3.1.

λ_D	=	The intensity of product demands, i.e. the average number of demands per unit of time.
λ_R	=	The intensity of product returns, i.e. the average number of returns per unit of time.
L_m	=	The manufacturing lead-time, which is a random variable with mean μ_{L_m} and variance $\sigma^2_{L_m}$.
L_r	=	The remanufacturing lead-time, which is a random variable with mean μ_{L_r} and variance $\sigma^2_{L_r}$.
c_m^f	=	The fixed manufacturing costs per manufacturing order.
c_r^f	=	The fixed remanufacturing costs per remanufacturing order.
c_m^v	=	The variable manufacturing costs per manufactured product.
c_r^v	=	The variable remanufacturing costs per remanufactured product.
c_s^h	=	The holding costs in serviceable inventory per product per unit of time.
c_r^h	=	The holding costs in remanufacturable inventory per product per unit of time.
c_b	=	The backorder costs per backordered product per unit of time.

Table 3.1. *Notation with respect to system characteristics.*

The long-run average system costs per unit of time under PUSH control and under PULL control are denoted by the functions $\overline{C}_{PUSH}(.)$ and $\overline{C}_{PULL}(.)$ respectively. These functions both are the summation of the following components:

c_s^h	×	average serviceable inventory per time unit
c_r^h	×	average remanufacturable inventory per time unit
c_r^v	×	average number of remanufactured products per time unit
c_r^f	×	average number of remanufacturing batches per time unit
c_m^v	×	average number of manufactured products per time unit
c_m^f	×	average number of manufacturing batches per time unit
c_b	×	average backordering position per time unit

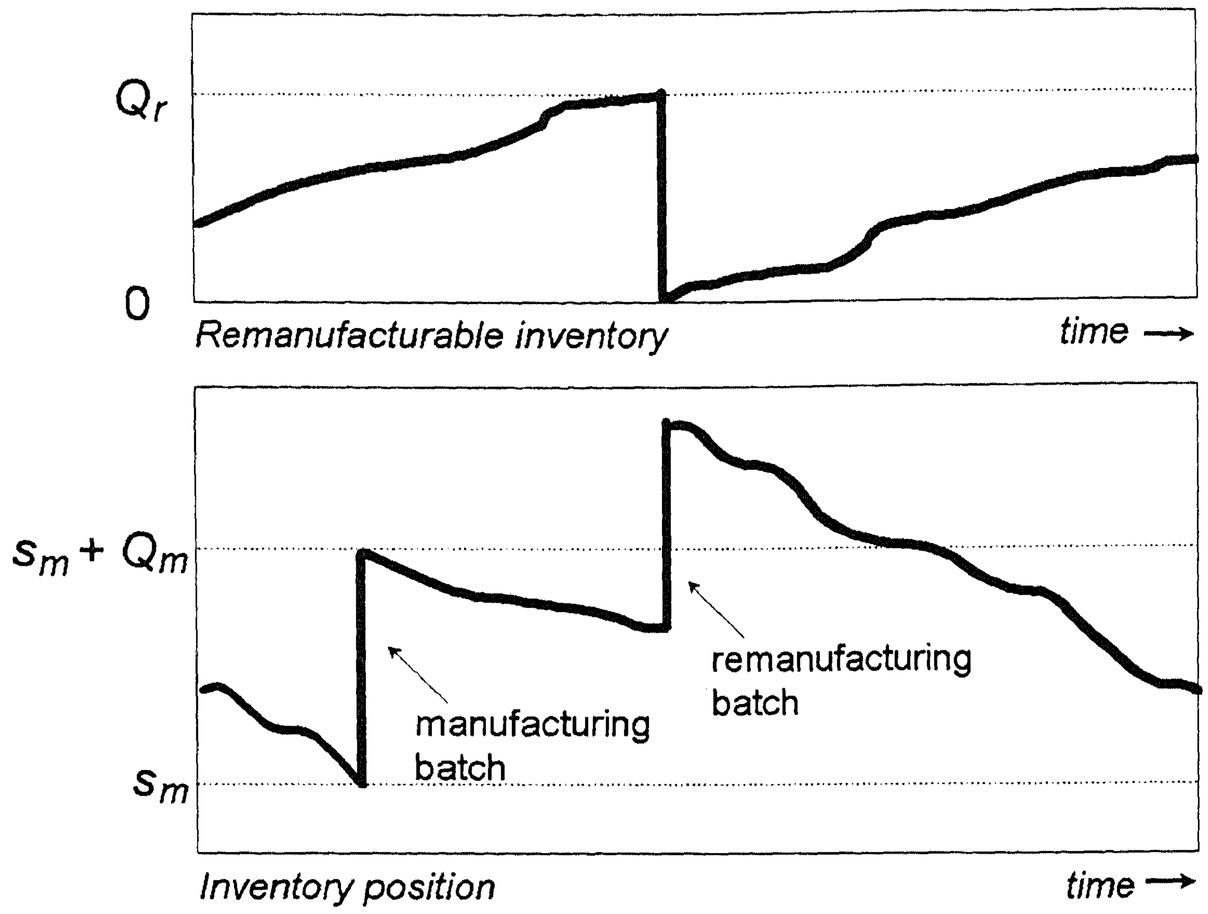
The definition and analysis of the PUSH and PULL control strategies will be considered in the next section.

3.2 Strategy definitions and analysis

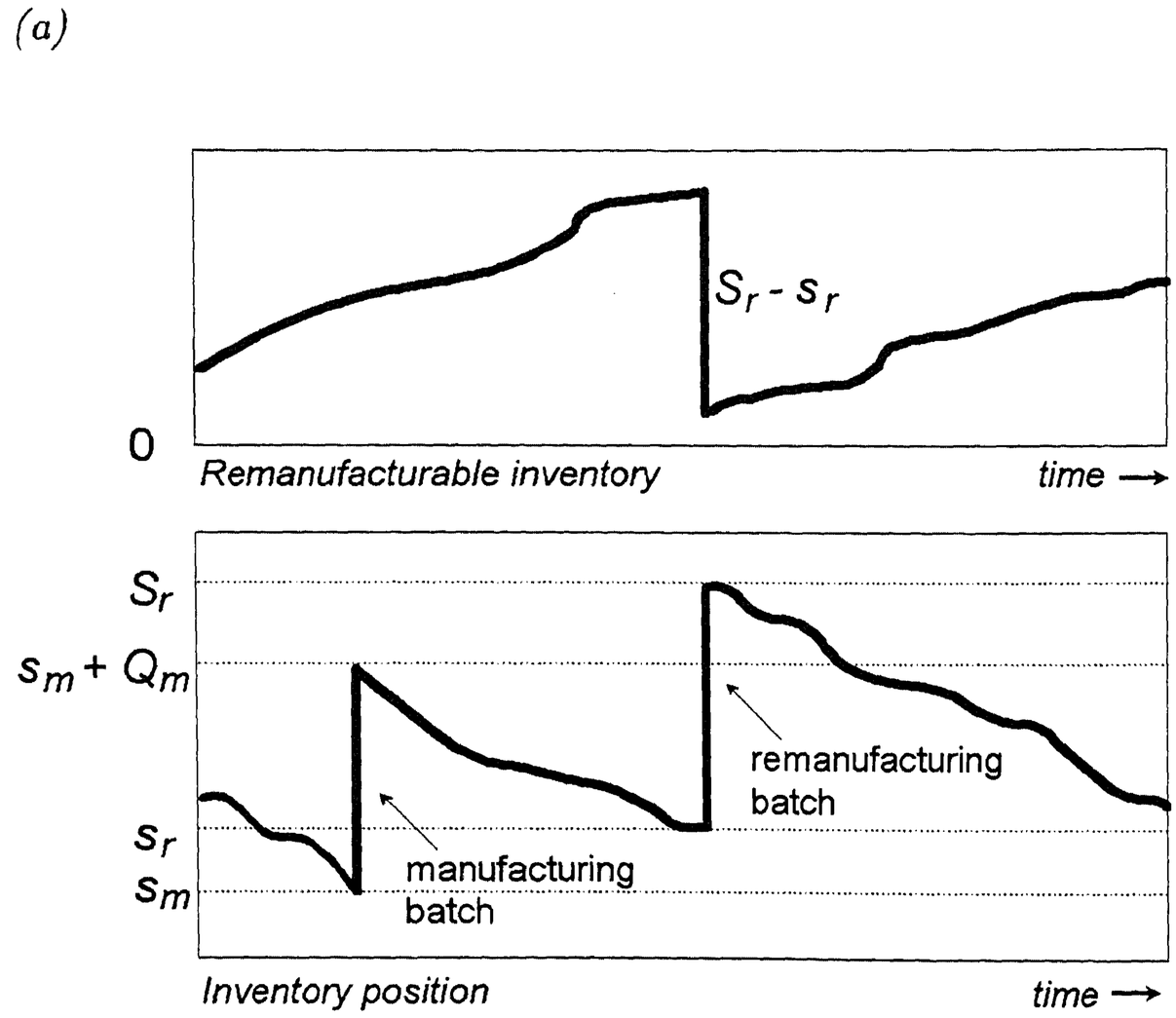
The PUSH and PULL strategies that we consider here rely on the strategies that were observed at the copier manufacturer of Chapter 1. More formally the strategies are defined as follows.

- the (s_m, Q_m, Q_r) PUSH strategy (Figure 3.2a). In this strategy remanufacturing starts as soon as the on-hand inventory of remanufacturables contains exactly Q_r used products. At that time, a remanufacturing order is placed for Q_r products. This amount is then transferred from the remanufacturable inventory to the remanufacturing Work In Process, the remanufacturable inventory drops to zero and the serviceable inventory position (serviceable inventory minus backlog plus all products in (re)manufacturing Work In Process) increases with Q_r. The remanufacturing batch will arrive at the on-hand serviceable inventory after a (stochastic) lead-time. Manufacturing takes place in batches of size Q_m, and starts whenever the serviceable inventory position drops to the level s_m. At that time, a manufacturing order is transferred to the manufacturing Work In Process and the inventory position is increased with Q_m. The manufacturing order will arrive at the on-hand serviceable inventory after a (stochastic) lead-time. The strategy is named *PUSH* strategy since used products are pushed into the remanufacturing process as soon as possible, independently from the actual demands and from the on-hand serviceable inventory.

- the (s_m, Q_m, s_r, S_r) PULL strategy (Figure 3.2b). In this strategy remanufacturing starts whenever the serviceable inventory position is at or below s_r, and sufficient remanufacturable inventory exists to increase the serviceable inventory position to S_r. The remanufacturing batch is exactly the amount that is necessary to increase the inventory position to S_r. If we denote this amount by Q then a remanufacturing order decreases the remanufacturable inventory with Q products, and increases the remanufacturing Work In Process and the inventory position with Q products. The remanufacturing order will arrive at the on-hand serviceable inventory after a (stochastic) lead-time. Manufacturing starts whenever the serviceable inventory position drops to the level $s_m (\leq s_r)$. At that time, a manufacturing order of size Q_m is transferred to the manufacturing Work In Process and the inventory position is increased with Q_m. The manufacturing order will arrive at



(a)



(b)

Figure 3.2 *A schematic representation of the PUSH strategy (a), and the PULL strategy (b).*

the on-hand serviceable inventory after a (stochastic) lead-time. The strategy is named *PULL* strategy since remanufacturable inventory is pulled into the remanufacturing process only when needed to fulfill customer demands for serviceables.

As can be concluded from the above, the most important difference between PUSH and PULL control is the timing of the remanufacturing operations. With PUSH control the start of a remanufacturing operation is solely based on the number of products in remanufacturable inventory, whereas under PULL control the start depends both on the inventory position and on the number of products in remanufacturable inventory.

Although it has been shown in [59] that neither the PUSH strategy, nor the PULL strategy is optimal within the class of all control strategies, our motivation to investigate these strategies is that they are not too complex to be analyzed numerically, and more importantly, they are actually used in practice.

In [26] Inderfurth proves for a much simpler system than ours, that a simple PULL strategy is optimal. The characteristics of this system include periodic review, zero fixed ordering costs, and equal non-stochastic lead-times for manufacturing and remanufacturing. Additionally, he argues that under stochastic lead-times and/or fixed ordering costs the structure of the optimal control strategy will be very complex. Even if the optimal structure could be identified, it is highly questionable whether the optimal strategy could ever be implemented in practice, because of its complexity.

In Section 3.2.1 and Section 3.2.2 we outline a procedure to calculate the long run average costs $\overline{C}_{PUSH}(s_m, Q_m, Q_r)$ and $\overline{C}_{PULL}(s_m, Q_m, s_r, S_r)$ respectively. The notation that we use in this outline is specified in Table 3.2. In accordance with the system assumptions made in Section 3.1 we have restricted the scope of the outline to the situation with uncorrelated and exponentially distributed demand and return inter-occurrence times, and non-stochastic lead-times. However, we will relax these assumptions at a later stage.

3.2.1 Analysis of the PUSH strategy

It should be noted that the (s_m, Q_m, Q_r) PUSH strategy generalizes the (s_p, Q_p) strategy of Muckstadt and Isaac [37] in that fixed remanufacturing

$I_s^{net}(t)$	=	The net serviceable inventory at time t, defined as the number of products in on-hand serviceable inventory minus the number of products in backorder at time t
$I_s(t)$	=	The serviceable inventory position at time t, defined as the net serviceable inventory plus the number of products in manufacturing work-in-process plus the number of products in remanufacturing work-in-process
$I_r^{OH}(t)$	=	The number of products in remanufacturable on-hand inventory at time t
$W_m(t)$	=	The number of products in manufacturing Work In Process at time t
$W_r(t)$	=	The number of products in remanufacturing Work In Process at time t
$D(t_0, t_1)$	=	The demands in the time-interval $(t_0, t_1]$
$Z(t-t_0, t-t_1)$	=	The number of products ordered to be (re)manufactured in the time-interval $(t-t_0, t-t_1]$ that enter serviceable inventory at or before time t *minus* the demands in the interval $(t-t_0, t-t_1]$
$\overline{B}$	=	The average backordering position
$\overline{I}_s^{OH}$	=	The average on-hand serviceable inventory
$\overline{I}_r^{OH}$	=	The average on-hand remanufacturable inventory
$\overline{O}_r$	=	The average number of remanufacturing orders
L^{min}	=	The minimum of all possible realisations of the manufacturing and remanufacturing lead time
L^{max}	=	The maximum of all possible realisations of the manufacturing and remanufacturing lead time

Table 3.2. *Notation used for the analysis.*

costs can be taken into account by allowing to batch remanufacturing orders. On the other hand we assume that remanufacturing capacity is unlimited. Furthermore, our analysis to calculate the costs $\overline{C}_{PUSH}(s_m, Q_m, Q_r)$ is exact rather than approximative.

The state transitions of the manufacturing/remanufacturing system defined in Section 3.1 under the PUSH strategy can be formulated as a continuous time Markov chain. This Markov chain, $\mathcal{M}_1$ say[1], has a two-dimensional

[1]The subscript '1' denotes that we are considering the PUSH strategy. Later we will

state variable

$$X_1(t) = \left(I_s(t), I_r^{OH}(t)\right).$$

By definition, $X_1(t) = (i_s, i_r^{OH})$ whenever $I_s(t) = i_s$ *and* $I_r^{OH}(t) = i_r^{OH}$. As a result, all realizations (i_s, i_r^{OH}) of $X_1(t)$ are elements of the two-dimensional state space

$$\mathcal{S}_1 = \{s_m + 1, s_m + 2, \ldots, \infty\} \times \{0, 1, \ldots, Q_r - 1\}. \tag{3.1}$$

To explain (3.1) we make the following observations:

- The inventory position $I_s(t)$ can never be smaller than $s_m + 1$ since in this state a demand would trigger a manufacturing operation and the inventory position is immediately raised to $s_m + Q_m$. Also, remanufacturing operations may eventually raise the inventory position to an arbitrary level. So, all possible realizations of $I_s(t)$ are elements of the set $\{s_m + 1, s_m + 2, \ldots, \infty\}$.

- The remanufacturable inventory $I_r^{OH}(t)$ can never exceed $Q_r - 1$ since in this state, a product return would trigger a remanufacturing operation and the inventory is decreased to zero. So, all possible realizations of $I_r^{OH}(t)$ are elements of the set $\{0, 1, \ldots, Q_r - 1\}$.

Formally, the transition rates $\nu_{s^{(1)},s^{(2)}}$ which are related to transitions from state $s^{(1)} \in \mathcal{S}_1$ to $s^{(2)} \in \mathcal{S}_1$ are defined as

$$\begin{aligned}
&(i) \quad \nu_{(i_s,i_r^{OH}),(i_s,i_r^{OH}+1)} = \lambda_R, && i_r^{OH} < Q_r - 1,\\
&(ii) \quad \nu_{(i_s,i_r^{OH}),(i_s+Q_r,0)} = \lambda_R, && i_r^{OH} = Q_r - 1,\\
&(iii) \quad \nu_{(i_s,i_r^{OH}),(i_s-1,i_r^{OH})} = \lambda_D, && i_s > s_m + 1,\\
&(iv) \quad \nu_{(i_s,i_r^{OH}),(s_m+Q_m,i_r^{OH})} = \lambda_D, && i_s = s_m + 1.
\end{aligned}$$

These transition rates can be explained as follows:

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use the subscript '2' to denote the PULL strategy.

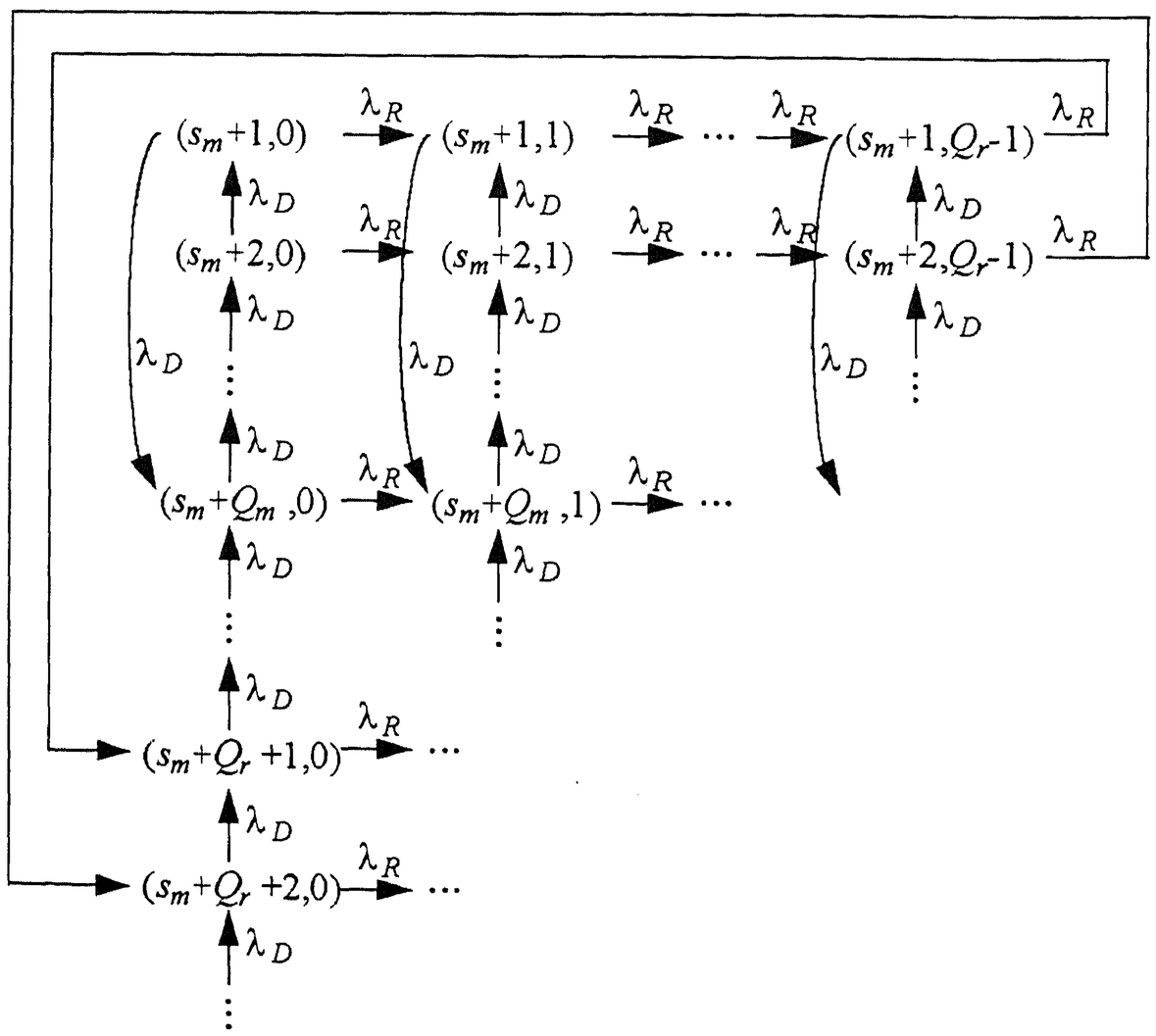
(i): This rate is associated with a product return that increases the remanufacturable inventory with one, while the inventory position remains unchanged.

(ii): Here a product return triggers a remanufacturing order. The remanufacturable inventory drops to zero, while the inventory position is increased with Q_r products.

(iii): With this rate a product demand decreases the inventory position with one product, while the remanufacturable inventory remains unchanged.

(iv): This rate is associated with a product demand that triggers a manufacturing order. The inventory position is increased with Q_m products, while the remanufacturable inventory remains unchanged.

The following flow-diagram further clarifies the set of transition rates and the mechanism of the PUSH strategy.



The limiting joint probability distribution $\pi_1(i_s, i_r^{OH})$, defined as

$$\pi_1(i_s, i_r^{OH}) = \lim_{t\to\infty} \Pr\{I_s(t) = i_s, I_r^{OH}(t) = i_r^{OH}\} \tag{3.2}$$

is obtained by solving the balance equations (inflow equals outflow) that are associated with each node in the flow-diagram. Although these balance equations are easily written down, it is quite difficult to find a closed form expression for (3.2). For a special case, i.e $Q_r = 1$, the two-dimensional state space $\mathcal{S}_1$ reduces to the one-dimensional state space of inventory position. In this case a closed form expression is available (see e.g. Muckstadt and Isaac, [37]), but in general we have to rely on numerical procedures.

Since the remanufacturing process is independent of the inventory position, it is easily seen that the marginal distribution of remanufacturable inventory, $\pi_1^{(r)}$, is a uniform distribution on $\{0, 1, \ldots, Q_r - 1\}$. In formula this reads

$$\pi_1^{(r)}(i_r^{OH}) = \begin{cases} \frac{1}{Q_r}, & 0 \leq i_r^{OH} < Q_r, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Using (3.3), the cost components of $\overline{C}_{PUSH}(s_m, Q_m, Q_r)$ that are shown in Table 3.3 are now easily obtained.

average remanufacturable inventory per time unit	=	$\frac{Q_r - 1}{2}$
average number of remanufactured products per time unit	=	λ_R
average number of remanufacturing batches per time unit	=	$\frac{\lambda_R}{Q_r}$
average number of manufactured products per time unit	=	$\lambda_D - \lambda_R$
average number of manufacturing batches per time unit	=	$\frac{\lambda_D - \lambda_R}{Q_m}$

Table 3.3. *Calculation of cost components in $\overline{C}_{PUSH}(s_m, Q_m, Q_r)$, apart from the average serviceable inventory and the average backorder position.*

More complicated is the calculation of the average on-hand serviceable inventory and the average backorder position, i.e.,

$$\overline{I}_s^{OH} = \sum_{i_s^{net} > 0} i_s^{net} \lim_{t \to \infty} \Pr\{I_s^{net}(t) = i_s^{net}\}, \tag{3.4}$$

and

$$\overline{B} = - \sum_{i_s^{net} < 0} i_s^{net} \lim_{t \to \infty} \Pr\{I_s^{net}(t) = i_s^{net}\}. \tag{3.5}$$

These can not be calculated directly using a Markov chain formulation, since the transitions of I_s^{net} are not Markovian. However, we can derive an expression for $I_s^{net}(t)$ from which we can calculate its long run distribution.

By definition, we have

$$I_s^{net}(t) = I_s(t) - W_m(t) - W_r(t). \tag{3.6}$$

Also, we have the following relation for net inventory:

$$\begin{aligned} I_s^{net}(t) = \; & I_s^{net}(t - L^{max}) \\ & + \left\{ \begin{array}{l} \text{every ordered product that} \\ \text{is delivered in } (t - L^{max}, t] \end{array} \right. \\ & - \text{ every demand that arrives in } (t - L^{max}, t] \end{aligned} \tag{3.7}$$

Here we mean by 'every ordered product' all products that were ordered for manufacturing *and* remanufacturing.

The number of ordered products that arrive in $(t - L^{max}, t]$ can be split up into two groups: The products that are in (re)manufacturing WIP at time $t - L^{max}$ (these will all arrive before or at time t, since lead-times are never larger than L^{max}), and the number of products that were ordered during the interval $(t - L^{max}, t]$ *and also* arrived before time t. So, (3.7) becomes

$$\begin{aligned} I_s^{net}(t) = \; & I_s^{net}(t - L^{max}) \\ & + \left\{ \begin{array}{l} \text{every product in (re)manufacturing} \\ \text{WIP at time } t - L^{max} \end{array} \right. \\ & + \left\{ \begin{array}{l} \text{every product that is both ordered} \\ \textit{and} \text{ delivered in } (t - L^{max}, t] \end{array} \right. \\ & - \text{ every demand that arrives in } (t - L^{max}, t] \end{aligned}$$

$$\begin{aligned} = \; & I_s^{net}(t - L^{max}) + W_m(t - L^{max}) + W_r(t - L^{max}) \\ & + \left\{ \begin{array}{l} \text{every product that is both ordered} \\ \textit{and} \text{ delivered in } (t - L^{max}, t] \end{array} \right. \\ & - \text{ every demand that arrives in } (t - L^{max}, t] \end{aligned} \tag{3.8}$$

sing (3.6) we simplify (3.8) as

$$\begin{aligned} I_s^{net}(t) = \quad & I_s(t - L^{max}) \\ & + \left\{ \begin{array}{l} \text{every product that is both ordered} \\ \textit{and} \text{ delivered in } (t - L^{max}, t] \end{array} \right. \\ & - \text{ every demand that arrives in } (t - L^{max}, t] \end{aligned}$$

lote that all products that are ordered after time $t - L^{min}$ arrive *after* time . So finally we have

$$\begin{aligned} I_s^{net}(t) = \quad & I_s(t - L^{max}) \\ & + \left\{ \begin{array}{l} \text{every product ordered in } (t - L^{max}, t - L^{min}] \\ \textit{and} \text{ delivered before time } t \end{array} \right. \\ & - \text{ every demand that arrives in } (t - L^{max}, t - L^{min}] \\ & - \text{ every demand that arrives in } (t - L^{min}, t] \\ = \quad & I_s(t - L^{max}) \\ & + Z(t - L^{max}, t - L^{min}) \\ & - D(t - L^{min}, t) \end{aligned} \tag{3.9}$$

Ve can use relation (3.9) to derive the long run distribution of $I_s^{net}(t)$ f we take into account the following stochastic (in)dependencies between $_s(t - L_{max})$, $I_r^{OH}(t - L_{max})$, $Z(t - L_{max}, t - L_{min})$, and $D(t - L^{min}, t)$:

- $Z(t-L_{max}, t-L_{min})$ is correlated with $I_s(t-L_{max})$ and $I_r^{OH}(t-L_{max})$,
- $I_s(t - L_{max})$ and $I_r^{OH}(t - L_{max})$ are correlated,
- $D(t - L^{min}, t)$ is uncorrelated with $I_s(t - L_{max})$, $I_r^{OH}(t - L_{max})$, and $Z(t - L_{max}, t - L_{min})$.

Substituting $u = t - L^{max}$ and $\Delta L = L^{max} - L^{min}$, the net inventory is derived from (3.9) as

$$\begin{aligned}
&\lim_{t\to\infty} \Pr\{I_s^{net}(t) = i_s^{net}\} \\
&\quad = \sum_{\Omega} \lim_{t\to\infty} \Pr\{D(t - L^{min}, t) = d\} \\
&\qquad \times \lim_{u\to\infty} \Pr\{I_s(u) = i_s, I_r^{OH}(u) = i_r^{OH}, Z(u, u + \Delta L) = z\} \\
&\quad = \sum_{\Omega} \exp^{-\lambda_D L^{min}} \frac{(\lambda_D L^{min})^d}{d!} \\
&\qquad \times \pi_1(i_s, i_r^{OH}) \times h_{z|(i_s, i_r^{OH})}(\Delta L),
\end{aligned} \tag{3.10}$$

where

$$\Omega = \{(i_s, i_r^{OH}, z, d) | i_s + z - d = i_s^{net}\}, \tag{3.11}$$

and

$$\begin{aligned}
&h_{z|(i_s, i_r^{OH})}(\Delta L) = \\
&\qquad \lim_{u\to\infty} \Pr\{Z(u, u + \Delta L) = z | I_s(u) = i_s, I_r^{OH}(u) = i_r^{OH}\}.
\end{aligned} \tag{3.12}$$

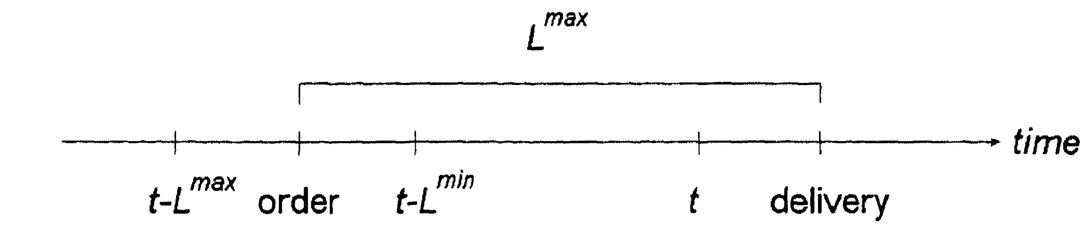
Note that up to this point we have not used the assumption that the lead-times are fixed constants. The further analysis for stochastic lead-times though will be proceeded in Chapter 5. Here we will proceed with the analysis for the case that lead-times are fixed constants.

If the manufacturing and remanufacturing lead-times are fixed constants, then the special case $\mu_{L_m} = \mu_{L_r}$ results in $Z(u, u + \Delta L) \equiv 0$, since then $\Delta L = 0$. Moreover, $h_{z|(i_s, i_r^{OH})}(\Delta L) = 1$, if $z = 0$ and zero otherwise. As a first step in the procedure to calculate the conditional probability $h_{z|(i_s, i_r^{OH})}(\Delta L)$ if $\mu_{L_m} \neq \mu_{L_r}$, we define a Markov chain $\mathcal{M}_1'$ with state space

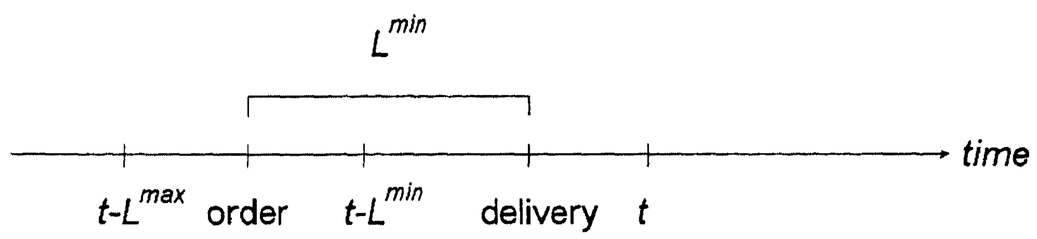
$$\mathcal{S}_1' = \{s_m + 1, s_m + 2, \ldots, \infty\} \times \{0, 1, \ldots, Q_r - 1\} \times \{-\infty, \ldots, \infty\},$$

and state variable

$$X_1'(\tau) = \left(I_s(\tau), I_r^{OH}(\tau), Z(u, \tau)\right).$$



(a)



(b)

'igure 3.3 *Products that are ordered in the interval* $(t-L^{max}, t-L^{min}]$ *will never rrive before or at time* t *if they have lead-time* L^{max} *(a), and will always arrive efore or at time* t *if they have lead-time* L^{min} *(b).*

3y definition, $X_1'(\tau) = (i_s, i_r^{OH}, z)$ whenever $I_s(\tau) = i_s$, *and* $I_r^{OH}(\tau) = i_r^{OH}$, *nd* $Z(u,\tau) = z$, with $\tau \in (u, u+\Delta L]$.

f the manufacturing lead time is *larger* than the remanufacturing lead time, .e. $\mu_{L_m} = L_{max}$ and $\mu_{L_r} = L_{min}$, *all* of the remanufacturing batches and *ione* of the manufacturing batches that were ordered during time $(u, u + L_{max} - L_{min}]$, will arrive before or at time $t = u + L^{max}$ (see Figure 3.3).

As a consequence, $Z(u,\tau)$ will only change due to a manufacturing order f $\mu_{L_m} < \mu_{L_r}$, and only due to a remanufacturing order if $\mu_{L_r} < \mu_{L_m}$. We nodel this by defining the binary variable δ, which is assigned the value . if the manufacturing lead time is larger than the remanufacturing lead ime, and 0 if it is shorter. The transition rates of $\mathcal{M}_1'$ are then as follows.

$$
\begin{aligned}
&(i) && \nu_{(i_s,i_r^{OH},z),(i_s,i_r^{OH}+1,z)} = \lambda_R, && i_r^{OH} < Q_r - 1,\\
&(ii) && \nu_{(i_s,i_r^{OH},z),(i_s+Q_r,0,z+\delta Q_r)} = \lambda_R, && i_r^{OH} = Q_r - 1,\\
&(iii) && \nu_{(i_s,i_r^{OH},z),(i_s-1,i_r^{OH},z-1)} = \lambda_D, && i_s > s_m + 1,\\
&(iv) && \nu_{(i_s,i_r^{OH},z),(s_m+Q_m,i_r^{OH},z+(1-\delta)Q_m-1)} = \lambda_D, && i_s = s_m + 1.
\end{aligned}
$$

These transition rates can be explained as follows:

(i): This rate is associated with a product return that increases the remanufacturable inventory with one, while the inventory position and $Z(u,\tau)$ remain unchanged.

(ii): Here a product return triggers a remanufacturing order. The remanufacturable inventory drops to zero, while the inventory position is increased by Q_r products, while $Z(u,\tau)$ is only increased by Q_r products if the remanufacturing lead-time is smaller than the manufacturing lead-time, i.e. if $\delta = 1$.

(iii): A product demand decreases the inventory position with one. The remanufacturable inventory remains unchanged, while $Z(u,\tau)$ is decreased by one.

(iv): A product demand triggers a manufacturing order. The inventory position is increased by Q_m products. The remanufacturable inventory remains unchanged, while $Z(u,\tau)$ is increased with $(1-\delta)Q_m$ products minus one demand.

Note that we are not interested here in the steady-state probability that the system is in some state $s \in \mathcal{S}_1'$, but in the probability that the system is in state $s^{(1)}$ at time $u - \Delta L$ given that the system was in state $s^{(0)}$ at time u. To calculate this probability we evaluate the *transient* behaviour of the Markov chain $\mathcal{M}_1'$, using a uniformization technique (see Tijms [56]). The uniformization technique transforms $\mathcal{M}_1'$ into an equivalent discrete time Markov chain $\overline{\mathcal{M}}_1'$. The one-step transition probabilities $\overline{q}^{(1)}_{s^{(1)}|s^{(0)}}$ in $\overline{\mathcal{M}}_1'$ are calculated as

$$\overline{q}^{(1)}_{s^{(1)}|s^{(0)}} = \begin{cases} \frac{\nu_{s^{(0)},s^{(1)}}}{\nu}, & s^{(1)} \neq s^{(0)}, \\ 1 - \frac{\nu_{s^{(0)}}}{\nu}, & s^{(0)} = s^{(1)}, \end{cases}$$

where $\nu_{s^{(0)}} = \sum_{s^{(1)} \in \mathcal{S}_1'} \nu_{s^{(0)},s^{(1)}}$, and the constant ν is chosen such that $\nu = \max_{s^{(0)} \in \mathcal{S}_1'} \{\nu_{s^{(0)}}\}$. The m-step transition probabilities $\overline{q}^{(m)}_{s^{(1)}|s^{(0)}}$ are derived from the recursive relation

$$\overline{q}^{(m)}_{s^{(1)}|s^{(0)}} = \sum_{s^{(2)} \in \mathcal{S}_1'} \overline{q}^{(m-1)}_{s^{(2)}|s^{(0)}} \overline{q}^{(1)}_{s^{(1)}|s^{(2)}}, \quad m > 1.$$

If we define $q_{s^{(1)}|s^{(0)}}(\Delta L)$ as the conditional probability that during time

ΔL the initial system state changes from $s^{(0)} \in \mathcal{S}_1'$ into $s^{(1)} \in \mathcal{S}_1'$, we have

$$q_{s^{(1)}|s^{(0)}}(\Delta L) = \sum_{m=0}^{\infty} \exp^{-\nu(\Delta L)} \frac{(\nu(\Delta L))^m}{m!} \overline{q}^{(m)}_{s^{(0)},s^{(1)}}.$$

Finally, the conditional probability $h_{z|(i_s,i_r^{OH})}(\Delta L)$ which appears in (3.10) is obtained from the relation

$$h_{z|(i_s,i_r^{OH})}(\Delta L) = \sum_{k=s_m+1}^{\infty} \sum_{\ell=0}^{Q_r-1} q_{(k,\ell,z)|(i_s,i_r^{OH},0)}(\Delta L).$$

Substituting (3.10) in (3.4) and (3.5) enables to calculate the average serviceable inventory per time unit and the average backorder position per time unit.

3.2.2 Analysis of the PULL strategy

Our motivation to analyze the (s_m, Q_m, s_r, S_r) PULL strategy in addition to the (s_m, Q_m, Q_r) PUSH strategy is, that the PULL strategy enables to control the timing of remanufacturing operations more accurately than the PUSH strategy, by pulling batches from remanufacturable inventory into the remanufacturing process only if the serviceable inventory position becomes too low to satisfy future expected customer demands adequately.

The analysis for the PULL strategy proceeds analogously to the analysis for the PUSH strategy if we replace the Markov chains $\mathcal{M}_1$ and $\mathcal{M}_1'$ with appropriate Markov chains $\mathcal{M}_2$ and $\mathcal{M}_2'$, which will be discussed next.

The Markov chain $\mathcal{M}_2$ is defined by the state variable

$$X_2(t) = \left(I_s(t), I_r^{OH}(t)\right).$$

By definition, $X_2(t) = (i_s, i_r^{OH})$ whenever $I_s(t) = i_s$ *and* $I_r^{OH}(t) = i_r^{OH}$. Consequently, all realizations (i_s, i_r^{OH}) are elements of the two-dimensional state space

$$\mathcal{S}_2 = \{s_m + 1, s_m + 2, \ldots, U\} \times \{0, 1, \ldots, \infty\}, \tag{3.13}$$

where

$$U = \max\{s_m + Q_m, S_r\}.$$

To explain (3.13) we make the following observations:

- The inventory position $I_s(t)$ can never be smaller than $s_m + 1$ since in this state a demand would trigger a manufacturing operation and the inventory position is immediately raised to $s_m + Q_m$. Remanufacturing operations may eventually raise the inventory position to S_r. So, all possible realizations of $I_s(t)$ are elements of the set $\{s_m + 1, s_m + 2, \ldots, U\}$.

- There is no limit to the level of remanufacturable inventory, so all possible realizations of $I_r^{OH}(t)$ are elements of the set $\{0, 1, \ldots, \infty\}$.

The transition rates $\nu_{s^{(1)},s^{(2)}}$ related to a transition from state $s^{(1)} \in \mathcal{S}_2$ to $s^{(2)} \in \mathcal{S}_2$, are defined as

$$
\begin{array}{lll}
(i) & \nu_{(i_s,i_r^{OH}),(i_s,i_r^{OH}+1)} = \lambda_R, & i_s > s_r \text{ or } i_r^{OH} < S_r - i_s - 1, \\
(ii) & \nu_{(i_s,i_r^{OH}),(S_r,0)} = \lambda_R, & i_s \leq s_r \text{ and } i_r^{OH} = S_r - i_s - 1, \\
(iii) & \nu_{(i_s,i_r^{OH}),(i_s-1,i_r^{OH})} = \lambda_D, & i_s > s_r + 1 \text{ or} \\
 & & \left\{ i_s > s_m + 1 \text{ and } i_r^{OH} \leq S_r - i_s \right\}, \\
(iv) & \nu_{(i_s,i_r^{OH}),(S_r,i_r^{OH}-(S_r-s_r))} = \lambda_D, & i_s = s_r + 1 \text{ and } i_r^{OH} \geq S_r - s_r, \\
(v) & \nu_{(i_s,i_r^{OH}),(S_r,i_r^{OH}-(S_r-s_m-Q_m))} = \lambda_D, & i_s = s_m + 1 \text{ and } s_r \geq s_m + Q_m \\
 & & \text{and } i_r^{OH} \geq S_r - s_m - Q_m, \\
(vi) & \nu_{(i_s,i_r^{OH}),(s_m+Q_m,i_r^{OH})} = \lambda_D, & i_s = s_m + 1 \text{ and } i_r^{OH} < S_r - s_m \\
 & & \text{and } \left\{ i_r^{OH} < S_r - s_m - Q_m \right. \\
 & & \quad \left. \text{or } s_r < s_m + Q_m \right\}.
\end{array}
$$

These transition rates can be explained as follows:

(i): A product return increases the remanufacturable inventory with one. The inventory position remains unchanged.

(ii): A product return triggers a remanufacturing order. The remanufacturable inventory drops to zero, while the inventory position is raised to S_r.

(iii): A product demand decreases the inventory position with one. The remanufacturable inventory remains unchanged.

(iv): A product demand triggers a remanufacturing order. The inventory position is raised to S_r, while the remanufacturable inventory is decreased by $S_r - s_r$ products.

(v): A product demand triggers a manufacturing order *and* a remanufacturing order. This is possible if the manufacturing order increases the inventory position to a level smaller than or equal to s_r, i.e., $s_m + Q_m \leq s_r$, and the remanufacturable inventory contains enough products to increase the inventory position further to S_r. The inventory position is increased to S_r, while the remanufacturable inventory is decreased with $S_r - (s_m + Q_m)$ products.

(vi): A product demand triggers a manufacturing order. The inventory position is raised to $s_m + Q_m$, while the remanufacturable inventory remains unchanged.

The limiting joint probability distribution $\pi_2(i_s, i_r^{OH})$ is obtained by solving the associated balance equations. Note that here, in contrast with expression (3.3), we do not have an analytical expression for the marginal distribution of remanufacturable inventory. Instead, we obtain this distribution numerically as

$$\pi_2^{(r)}(i_r^{OH}) = \sum_{i_s=s_m+1}^{U} \pi_2(i_s, i_r^{OH}). \tag{3.14}$$

Using (3.14), all components of $\overline{C}_{PULL}(s_m, Q_m, s_r, S_r)$ apart from the average number of remanufacturing batches per time unit, the average serviceable inventory per time unit, and the average backorder position per time unit, are obtained from Table 3.4.

average remanufacturable inventory per time unit	$=$	$\sum_{i_r^{OH}=1}^{\infty} i_r^{OH} \pi_2^{(r)}(i_r^{OH})$
average number of remanufactured products per time unit	$=$	λ_R
average number of manufactured products per time unit	$=$	$\lambda_D - \lambda_R$
average number of manufacturing batches per time unit	$=$	$\frac{\lambda_D - \lambda_R}{Q_m}$

Table 3.4. *Cost components of $\overline{C}_{PULL}(s_m, Q_m, s_r, S_r)$ apart from the average number of remanufacturing batches per time unit, the average serviceable inventory, and the average backorder position.*

Using $\pi_2(i_s, i_r^{OH})$ and the PASTA (Poisson Arrival See Time Averages) property (see e.g. Wolff [67]) we derive the average number of remanufacturing batches as

$$\begin{aligned}
\overline{O}_r &= \lambda_R \times \Pr\{\text{a return triggers a remanufacturing batch}\} \\
&\quad + \lambda_D \times \Pr\{\text{a demand triggers a remanufacturing batch}\}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i_s=s_m+1}^{s_r} \pi_2(i_s, S_r - i_s - 1)\lambda_R \\
&\quad + \sum_{i_r^{OH}=S_r-s_r}^{\infty} \pi_2(s_r+1, i_r^{OH})\lambda_D \\
&\quad + \begin{cases} \displaystyle\sum_{i_r^{OH}=S_r-s_m-Q_m}^{S_r-s_m} \pi_2(s_m+1, i_r^{OH})\lambda_D, & s_r \geq s_m + Q_m \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.15}$$

The calculation of the average serviceable inventory (3.4) and the average backorder position (3.5) again is based on the distribution of the net inventory. Similar to (3.10) we have

$$\lim_{t\to\infty} \Pr\{I_s^{net}(t) = i_s^{net}\} = \sum_{\Omega} \exp^{-\lambda_D L^{min}} \frac{(\lambda_D L^{min})^d}{d!} \times \pi_2(i_s, i_r^{OH}) \times h_{z|(i_s,i_r^{OH})}(\Delta L),$$

where Ω and $h_{z|(i_s,i_r^{OH})}(\Delta L)$ are defined in (3.11) and (3.12). The procedure to derive the conditional distribution $h_{z|(i_s,i_r^{OH})}(\Delta L)$ proceeds analogously to the analysis in the previous section, although instead of the Markov chain $\mathcal{M}_2$ we use the Markov chain $\mathcal{M}_2'$, with state space

$$\mathcal{S}_2' = \{s_m + 1, \ldots, U_2\} \times \{0, \ldots, \infty\} \times \{-\infty, \ldots, \infty\}$$

and state variable

$$X_2(\tau)' = \left(I_s(\tau), I_r^{OH}(\tau), Z(u,\tau)\right).$$

The transition rates of $\mathcal{M}_2'$ are

$$\nu_{(i_s,i_r^{OH},z),(i_s,i_r^{OH}+1,z)} = \lambda_R, \qquad \begin{array}{l} i_s > s_r \text{ or} \\ i_r^{OH} < S_r - i_s - 1, \end{array}$$

$$\nu_{(i_s,i_r^{OH},z),(S_r,0,z+\delta(S_r-i_s))} = \lambda_R, \qquad \begin{array}{l} i_s \leq s_r \text{ and} \\ i_r^{OH} = S_r - i_s - 1, \end{array}$$

$$\nu_{(i_s,i_r^{OH},z),(i_s-1,i_r^{OH},z-1)} = \lambda_D, \qquad \begin{array}{l} i_s > s_r + 1 \text{ or} \\ \{i_s > s_m + 1 \text{ and} \\ i_r^{OH} \leq S_r - i_s\}, \end{array}$$

$$\nu_{(i_s,i_r^{OH},z),(S_r,i_r^{OH}-(S_r-s_r),z+\delta(S_r-s_r)-1)} = \lambda_D, \qquad \begin{array}{l} i_s = s_r + 1 \text{ and} \\ i_r^{OH} \geq S_r - s_r, \end{array}$$

$$\nu_{(i_s,i_r^{OH},z),(S_r,i_r^{OH}-S_r+s_m+Q_m,z+c(\delta))} = \lambda_D, \qquad \begin{array}{l} i_s = s_m + 1 \text{ and} \\ s_r \geq s_m + Q_m \text{ and} \\ i_r^{OH} \geq S_r - s_m - Q_m, \end{array}$$

$$\nu_{(i_s,i_r^{OH},z),(s_m+Q_m,i_r^{OH},z+(1-\delta)Q_m-1)} = \lambda_D, \qquad \begin{array}{l} i_s = s_m + 1 \text{ and} \\ i_r^{OH} < S_r - s_m \text{ and} \\ \{i_r^{OH} < S_r - s_m - Q_m \\ \text{or } s_r < s_m + Q_m\}. \end{array}$$

where $c(\delta) = (1-\delta)Q_m + \delta(S_r - s_m - Q_m)$. The final step in the analysis of the PULL strategy is the transient analysis of Section 3.2.1.

3.3 Optimization with respect to control parameters

In the previous section we outlined procedures to calculate the cost functions $\overline{C}_{PUSH}(s_m, Q_m, Q_r)$ and $\overline{C}_{PULL}(s_m, Q_m, s_r, S_r)$ for *arbitrary* sets of control parameters. However, to analyze system behaviour, we are interested in the particular set of strategy parameters under which the cost functions are *minimized.*

In order to find

$$\overline{C}^*_{PUSH} = \min_{(s_m,Q_m,Q_r)} \overline{C}_{PUSH}(s_m, Q_m, Q_r),$$

and

$$\overline{C}^*_{PULL} = \min_{(s_m,Q_m,s_r,S_r)} \overline{C}_{PULL}(s_m, Q_m, s_r, S_r)$$

we have implemented an enumerative search procedure. Although an extensive numerical study has indicated that for every problem instance only one single local minimum exists, we did not succeed in constructing a formal proof. Despite of the latter, it must be noted that the objective of this chapter is to analyze real system behaviour and *not* to develop fast, approximative optimization algorithms. Our focus is therefore on accuracy rather than on computation time.

3.4 Numerical aspects and validation

Although the analysis of our HMR system under the PUSH and PULL strategy is *exact*, we have to evaluate the cost functions *numerically*. The main problem here involves the truncation of infinite sums. While in specific situations there exist general rules for truncation, they did not always apply for our procedures. In these cases we had to commit ourselves to heuristical bounds and stopping criteria. The numerical procedures that were used in the numerical studies in Chapter 4, 5, and 6, were extensively tested and validated by repeated experiments. Moreover, all outcomes presented were validated by simulation.

3.5 Relaxation of system assumptions

The analysis in the foregoing sections is limited in the sense that it only applies to systems with uncorrelated Poisson arrivals of demands and returns, and to systems with constant lead-times for (re)manufacturing. To further study the influences of process interactions and process uncertainties on system performance, we will investigate the system defined in Section 3.1 under relaxation of the following assumptions:

- *Uncertainties in returns and demands.* To evaluate the influences of uncertainties in the timing of demands and the timing and quality of returns in more detail, the assumption of exponentially distributed demand and return inter-occurrence times has been generalized to Coxian-2 distributed demand and return inter-occurrence times. Besides that the Coxian-2 distribution enables to fit a first and second

moment of processes, rather than a first moment only, it is also possible to model these processes as a Markov chain. In this way, we can adapt our procedures to more realistic demand and return processes. The required modifications to calculate $\overline{C}_{PUSH}(s_m, Q_m, Q_r)$, and $\overline{C}_{PULL}(s_m, Q_m, s_r, S_r)$ under Coxian-2 distributed demand and return inter-occurrence times are outlined in Section 4.1.

- *Correlation between returns and demands.* The correlation between returns and demands is modeled by the coefficient ρ_{RD}, which indicates the fraction of product returns that instantaneously induce a demand for a new product to replace the returned product. The introduction of correlations between returns and demands changes the calculation of $\overline{C}_{PUSH}(s_m, Q_m, Q_r)$ and $\overline{C}_{PULL}(s_m, Q_m, s_r, S_r)$. For further details we refer to Section 4.2.

- *Lead-time uncertainty.* The system defined in Section 3.1 can be evaluated in presence of stochastic lead-times for (re)manufacturing. Since we have devoted a separate chapter on lead-time effects, we will address the mathematical analysis to evaluate the cost functions $\overline{C}_{PUSH}(s_m, Q_m, Q_r)$ and $\overline{C}_{PULL}(s_m, Q_m, s_r, S_r)$ under lead-time uncertainty in Section 5.1.