2
Mathematical Formulation

§ 4. The Function of Mathematical Economics

We have already seen that in econometric research, mathematical economics and statistics go hand in hand. For a good understanding of the working methods of econometrics, it is, nevertheless, necessary to distinguish clearly between the functions exercised by each of them. The order of succession in which they are applied is not always the same. "Living" research work constantly adapts itself to the circumstances. The investigation sometimes is like a game of tennis between the economist and the statistician—even if they are one and the same person—or a discussion in which each makes a contribution in turn. In general, the investigation will begin with an economic analysis of the problem. After that, in many cases, a statistical testing of the theory set up concerning the phenomenon under consideration will follow. After the testing, there is often occasion to draw further conclusions with the help of the theory. But it is also possible that the theory has to be revised as a consequence of the testing. The revised theory, then, has to be tested again, before further conclusions can be drawn; the tennis ball goes back and forth twice.

We shall begin with what happens in the first stage and what eventually may repeat itself in the later intermediary stages.

The role of mathematical economics also is a many-sided one in the first stage, and does not easily take on a definite form. We shall only try to give a more or less systematic anthology from the experience acquired in this work.

Speaking very generally, it is the task of mathematical economics to derive a relationship from the premises by which it is determined and to specify that relationship in greater detail.

The problem usually is to explain the changes in the magnitude of a phenomenon, X, from the changes in the size of other phenomena, Y, Z, and so on, which we may call the causes of X. We often speak in this connection of the *phenomenon to be explained* as X and the *explaining phenomena* as Y, Z, and so on. In case X relates to a great complex of homogeneous phenomena—for instance the total demand for consumption goods exerted by all laborers—we may speak of a *macroeconomic* problem as opposed to a *microeconomic* one as in the case where we are concerned with the demand of one laborer for bread. This boundary, again, cannot be drawn sharply; in every case we can distinguish various degrees of macroeconomic research, depending on whether we are investigating more or less comprehensive totals of elementary phenomena.

A distinction of more importance, about which much discussion has been going on, is the one between a *static* and a *dynamic* method of treatment. We may not use these words in an economic argument without first defining them; they have already been used in too many different meanings. In our analysis, we shall follow Frisch's definition and speak of dynamic research only when a relation is sought between economic phenomena referring to various moments of time. So, if today's price determines the quantity to be supplied a month or a year later, we speak of a dynamic relation; if we make use of that relation in our investigation, the investigation is dynamic.

The usual method of analysis of equilibrium economics is, therefore, nondynamic. It is assumed that price and quantity supplied are stationary and the relation between them which will keep them stationary is determined. To that end, the price must be equal to the marginal cost pertaining to that quantity supplied under competitive conditions. With stationary magnitudes, it is not necessary to take a possible time difference into account, as it makes no difference whether we consider the quantity supplied now or next year.

According to our definition, however, the relation between two nonstationary magnitudes is also nondynamic if the movements of the one immediately adapt themselves to those of the

other; if, therefore, the quantity supplied depends on the price at the same moment. In that case, both magnitudes may very well change in time. Many writers call this a case of "dynamics." With our definition, however, this case, like the previous one, has a static character, because the situation is not at any moment determined by what has happened before. Should we assume that the quantity supplied also depends on the size of the production plant, which itself is a cumulation of investments in earlier periods, then we have a dynamic relation. Finally, we speak of a dynamic relation when a phenomenon, X, depends on the speed with which a phenomenon, Y, changes, because we can only ascertain a speed by comparing two figures at different moments.

For a long time it has been held that statics is simpler than dynamics and therefore economics has been built up in a static form, from which attempts were made later to derive dynamics by reducing the degree of abstraction. In many respects, this can be done. In other respects, however, statics is less simple, less lucid, than dynamics, because the direction of the causal relation is hidden.

Applying a dynamic method of treatment, we can distinguish between the following two cases, which are intended here as examples only:

1. With a given price, y, the supplier determines the quantity he supplies, x, in such a way, that the marginal cost is equal to the price.

2. With a given amount of orders determining the supply, x, the supplier fixes his price, y, in such a way that his marginal cost is covered.

In static terms, there will be no difference between the cases 1 and 2: in both cases, the price is equal to the marginal cost. In dynamic terms, in case 1, y is the cause and x, the consequence, in case 2, x is the cause and y, the consequence. As every reaction, in principle, requires a certain amount of time —even though it may be very small—in case 1, x will lag behind y, whereas in case 2, y will lag behind x.

In both cases, the application of statics is correct by way of

an approximation; the picture of reality, however, is somewhat poorer. It does not matter when the stationary situation has actually been reached: cases 1 and 2, then, cannot be distinguished any more. But under changing circumstances, there is a difference between the two cases and the market mechanism is different. By considering the problem dynamically, it is set out more clearly, which becomes evident particularly if one has to go back to the premises in order to take account of a factor neglected thus far.

This will become more obvious after our discussion of complete economic systems or models.

The kind of relationships which can be brought into discussion in an economic investigation can be of widely different types. They can be of a relatively trivial nature if they only reflect the *definition* of the magnitude considered or if the relations are of a *natural, technical,* or *institutional* nature. An example of a definition is: Value of turnover is equal to price multiplied by quantity, or profits are equal to the proceeds minus costs. An example of a natural, or technical, relation is the connection between the quantity produced and the quantities of productive services used. An institutional relation is the one between income and taxes to be paid. Although the production relation in particular can be very important and though even a definition can sometimes be tested (we remind the reader of the various methods of determining profits), these relations, nevertheless, are less important to the economist than the second sort of relations which we want to distinguish: the *reaction relations.* These relations depict the reaction of a person, a business enterprise, or another economic subject or group of subjects to the given circumstances. They reflect the psychological or institutional efforts of those subjects; i.e., the attempt to reach a maximum satisfaction or a maximum profit, for instance, depending on the organization of the economy. In ordinary economic language, these reaction relations are the *demand* or *supply* relations; however, they can also assume different forms. They usually constitute the core of the economic

model. The majority of our examples, in fact, will be concerned with ordinary demand and supply equations.*

§ 5. Which Variables Should Be Included in the Relation?

The size of the phenomena X, Y, Z, etc., which we shall denote by x, y, z, etc., are called the *magnitudes,* or—as in mathematics—the *variables* in the relation considered. The first question to be answered in setting up the relation required is: Which variables appear in it? Or, expressing the question more simply: Which phenomena influence the phenomenon to be explained? As will be discussed later in another connection, we have in mind here the direct influences; we therefore ask for the proximate causes of changes in x.

Generalizations cannot easily be made; moreover, they have little value. Concrete examples are necessary to clarify the analysis.

A very important question is whether one wants to determine a *demand* or a *supply relation;* i.e., whether one wants to consider the quantities sold from the viewpoint of the buyers or from that of the sellers. In the first case, one is only allowed to include variables which are relevant for the buyers; in the second case, only variables which have importance to the sellers may be taken into account. The quantity bought, x, for instance, may be explained from the demand side by the price, y, the incomes, z, and the price of a competing article, u. There is no sense, however, in also including the prices of the raw materials, v. Those are of interest to the sellers, not to the buyers. Insofar as they affect the buyers, they find expression in the price, y, of the product itself.

In considering the additional variables which might play a role in the *demand relation,* one will have to take into account the characteristics of the commodity dealt with, the motives of the buyers, and the organization of the market.

The characteristics of certain commodities are subject to

* In compliance with mathematical practice, the word "equation" here is synonymous with "relation," or "connection"; one could also speak of a "law," according to economic practice.

changes. For instance, those of the commodity called "share of stock." To the buyer, one of the most important characteristics is the yield of the share and, in many cases, also the price rise which can be expected. The buyer's appraisal and, as a consequence, also the price, will, among other things, depend on the yield and on the expected price movement.

It may happen quite frequently that the characteristics of the good sought by the buyer are also present in other goods. In that case, therefore, there are *substitute* goods, and it will depend on the price of these goods in relation to that of the goods concerned how high the demand will be. The same is true when the commodity is *complementary* to another one; this means that it is used in combination with that other commodity. In the first case, the two prices will exert a contrary influence, whereas in the second case, their influences will work into the same direction.

Finally, a demand relation may be influenced by quite a number of factors which are associated with special peculiarities of the good or the market concerned. In every case to be analyzed, the particulars of the market should be studied thoroughly before a general theory can be developed. As an example, see § 29.

In case we are concerned with a *supply equation,* the variables which play a role in the first place will be—except the price of the product supplied—the *production costs,* which may be split up into wage and raw material costs, depending on the prices of those factors and on the quantities. These quantities will be proportional to the productivity of the enterprises concerned, while, on the other hand, the marginal productivities determine *marginal costs,* with which we are concerned in a supply function. In general, the marginal productivities are lower as production is higher, and as a rule they depend on the latter.

In certain circumstances—at full employment for instance—the factor determining supply will be the *productive capacity.* This magnitude itself can practically be taken as a datum; how-

ever, if one wants to make sharper distinctions, it can in its turn be considered dependent on other factors.

Another factor of importance, both to the demand and the supply equation, is the *organization of the market.* Up to this point, the assumption of free competition has been made, where demanding and supplying economic subjects consider the price as a datum which they cannot influence and there is only one price. When competition is limited, the situation may be different. In some cases, there are more prices than one; the buyer then has to choose between various prices of different suppliers. The situation in this case is similar to the one of competing goods; the demand for the goods of each of the sellers depends both on his own and on his competitor's price. A monopolist's supply curve, moreover, will depend on the price in a different way from the supply curve of a seller competing with others (see § 9).

§ 6. The Mathematical Form of the Relation

A question deserving separate consideration is, what should be the mathematical form of the relation? In the case of the demand function, should the quantity demanded be determined by the first power of the price or by its second power, or by some other function of the price? Of course, this is of some importance for the process of calculation, sometimes even of great importance. Economic theory tells us very little about this; it usually does not know the answer itself. In a few special cases, it is possible by reasoning to deduce something more definite about the form of the function. For instance, if the demand for a certain good is connected with the quantity of another good, by a purely technical relation, and is, say, proportional to that quantity, then at least the mathematical relation with respect to the quantity is determined. When it is certain that somebody will spend a certain amount for a certain commodity, then the quantity he will buy is inversely proportional to the price. These cases, however, are rare. Usually, this sort of information is lacking, and precisely in such cases theoretical eco-

nomics will only be able to make headway through econometric research. Sometimes it is possible on the basis of further reasoning to detect certain characteristics of the mathematical relation sought after. The way in which demand for a certain good depends on income is a good instance of this. We can let income vary from zero to infinity and ascertain that with zero income, nothing can be spent on the article in question. Further, we can never spend more than the amount of the whole income. In the case of an article which is one of the necessaries of life and has no substitution possibilities of any importance, it can be assumed that there is a certain saturation quantity, above which demand does not rise.

Another consideration which can sometimes be applied is the following: For the real satisfaction of a buyer, it makes no difference if all prices and his income are twice or n times as high as they actually are. Therefore, in such a situation, his reaction will not differ from the one actually in existence. This means, among other things, that also the quantity demanded of each good remains the same, if all prices and money amounts with which the subject is concerned, increase in the same proportion. When this is the case, we say that the quantity demanded is a *homogeneous function* (of zero degree) of the variables mentioned. It should be kept in mind that this homogeneity is subject to the conditions that the changes referred to really take place in all prices and money amounts and that the buyer is not subject to the so-called "money illusion," through which the absolute value of the prices have an independent significance for him. This condition means that the money amounts not mentioned explicitly and which are of some importance for the buyer must also be taken into account.

From such considerations, some general characteristics of the function sought after can be deduced. This limits the choice of functions to some extent (see § 26).

A very general mathematical theorem is of advantage in all these cases; namely, that within a small range of variation, nearly every function can be approximated by a *linear one.*

For that range, the curve representing the relation is replaced by part of the tangent. If the changes in the variables concerned are small, a linear function of those variables will always suffice; in other words, an attempt can be made to approximate x by an expression $x' = a + by + cz + du \ldots$, where a, b, c, etc., are constants. This does not mean, by the way, as is sometimes thought, that a proportionality between x′ and y is assumed. The only proportionality assumed is between a small increment of y, for instance, and a corresponding increment of x′, z, u, etc., remaining constant. If we represent these increments, as is usual in mathematics, by Δy and $\Delta x'$ respectively, then by virtue of our assumption, the equation

$$\Delta x' = b\Delta y \tag{6.1}$$

holds good. This assumption, however, does not go as far as one of proportionality between x′ and y itself.

There are still other aspects of the mathematical relation which, even with small changes in the variables, cannot be determined in advance and which, therefore, deserve attention. Economic theory does not always make it clear whether a magnitude x depends on y, or on a *cumulation* of y (in mathematical language: on the "integral over time" of y). What is meant by the cumulation of y will be explained briefly with the help of a numerical example. If a series, y, is given in the course of time, for instance:

$$y = 5, 3, 6, 8, 4 \ldots.$$

then by the cumulation of y is meant the series, the first term of which is 5, the second, $5 + 3 = 8$, the third, $5 + 3 + 6 = 14$, the fourth, $5 + 3 + 6 + 8 = 22$, etc. As an economic example of the uncertainty referred to above, the question may be asked, How does the rate of interest influence the stocks of certain commodities? Two solutions are conceivable; either the rate

of interest determines the stock or it determines
or negative additions to the stock. Economic litera
respect, is far from clear, and often one hesitates l
contrary opinions. If the first one is true, the stock
mined by the rate of interest, y. In case the second
however, the stock, x, being itself a cumulation
to the stock in previous periods, is determined by a
of the values of the rate of interest, y.

It makes a very considerable difference whether
the other assumption is made. Before testing a th
type, therefore, one has to decide which assumptic

§ 7. The Way in Which the Time Factor Appear in the Relation

We have now come to a further problem; namel
tion in what way the time factor plays its role in t
concerned. The appearance of a cumulated varial
means a form of influence of the time factor; in tl
we have seen, we have a dynamic relation. The latt
make its appearance in a simpler way, if a variable, z,
x, not at the same moment, but with a certain *time-*
can say, then, that

$$x_t = a + cz_{t-T}$$

where the subscripts of the magnitudes x and z in
period of time for which those variables should be t
sometimes called the "lag" which x shows with respe
the "lead" which z shows with respect to x. This is p
simplest form of dynamic relation conceivable. Ano
tion is whether economic theory can also give a decisi
about the way the time factor should appear in the
when the exact correlation of interrelated phenome
sidered. Such a simple time-lag undoubtedly may pres
namely, in case there is a causal factor of a technic
which requires a constant lapse of time for a certain
to have its effect. The length of a production process
rise to such a lag.

It may also happen, however, that dynamic relations possess quite different forms if they have a psychological origin. Let us suppose that according to the circumstances prevailing at the moment, t, the demand for a certain good must be equal to x_1 in order to give maximum satisfaction to the buyer. If, now, the circumstances suddenly change, the most desirable demand will perhaps be x_2. But some time passes before the buyer is able to realize this. With a small change in circumstances, he will probably not notice that changing x_1 will be advantageous to him. When the change is greater, this will gradually become clear to him. He will now aim at proceeding from x_1 to x_2. Various obstacles, however, may counteract this and therefore the adjustment does not take place immediately. A process of adjustment will, however, develop. This may happen in such a way that the speed with which he moves from x_1 to x_2 is proportionate to the distance from x_2—the "power of attraction," one could say—and, therefore, decreases slowly. At first it may also increase, and so on. Practically no investigations have yet been made in this field.

In the course of that process of adjustment, the circumstances again change, the process is broken off and replaced by a new one. The result of such a series of adjustment processes is a curve quite different from that of one single adjustment process; as an approximation, however, it may very well result in a line showing a lagged relationship.

It can also happen that the magnitude, z, influences the magnitude, x, in different ways and that each of these has a different lag. If we assume that these lags are 1, 2, 3, etc., time units respectively then

$$x_t = a + c_1 z_{t-1} + c_2 z_{t-2} + c_3 z_{t-3} + \ldots$$

If, now, z undergoes a change, Δz, which is maintained for some time, the change Δx in x which appears consequently, can be calculated as follows

$$\Delta x = (c_1 + c_2 + c_3 + \ldots)\Delta z$$

It can be said now that $c_1 + c_2 + c_3 + \ldots$ represents the *long-term influence of z on x*. The change in x which has taken place after one time unit, however, is

$$\Delta_1 x = c_1 \Delta z$$

In certain cases it is useful to name the coefficient c_1 the *short-term influence of z on x;* in particular, for instance, when c_2, c_3, etc., are each small but when their sum total is important compared to c_1. Of course, we can also put the boundary between short and long term at some other point; and as a matter of fact, the exact description of z's influence on x can only be given by enumerating all the coefficients c_1, c_2, c_3, etc.

There is still quite a different way in which the time factor plays a role. When striving after maximum satisfaction, which underlies all economic action, it makes a difference over how long an interval of time one looks ahead. Expressed in everyday language, the immediate interests sometimes do not coincide with the interests of a later period, and often it is unwise to be shortsighted. In discussions of a more scientific character, use has sometimes been made of the concept of *horizon;* i.e., the period over which somebody looks ahead. The length of the horizon sometimes influences supply curves and demand curves considerably. The simplest example is the case of an excessively large crop. If the farmer or the merchant would not look further ahead than one year, he would take the whole crop to the market and satisfy himself with an extremely low price. If, however, he looks two years ahead, he will take account of the probability that there will be an average crop in the second year; and if, now, he distributes the total crop of those two years equally over the two years, he will have a greater total yield. He will do well, therefore, not to take the whole surplus crop of the current year to the market, but to keep half the surplus back. The length of the horizon here has a clear and, for simple cases, a calculable influence on the amount supplied.

The same occurs in the valuation of shares. If a company does not pay out dividends in a certain year, a shareholder

who does not look further ahead than that year would argue that the share was worth nothing to him; it does not give him any revenue. Usually, however, his calculations will cover a longer period of time in the future and in his valuation he will take account of the possibility that after some time, dividends will be paid again. On the other hand, if, in one year, the dividend is three times the normal amount, the price of that share will not necessarily be tripled. Fig. 4B (p. 70) shows this very clearly. There the dividend percentage is measured along the horizontal axis, whereas the "valuation" is measured vertically. The latter concept is defined as "share price divided by nominal value of the share and multiplied by the long-term interest rate" (the yield of government bonds). This magnitude indicates, as it were, interest-bearing bond which is considered to be equivalent to the share. If the share price is twice the nominal amount and the yield of government bonds is 4 per cent, then apparently the share is considered to be equivalent in value to an imaginary bond yielding 8 per cent. If one were to be governed entirely by the momentary dividend percentage, a dividend of 8 per cent would also entail a valuation of 8 per cent. In reality, we do not assume this relation, but rather a relation which depends much less strongly on the momentary dividend percentage. We can say roughly that a rise of 1 per cent in the dividend rate causes an increase in valuation of only ½ per cent, so that with a dividend of 4 per cent a valuation based on 4 per cent is made, whereas with a dividend of 8 per cent the valuation is based on 6 per cent. Again, this is an example of the influence of the horizon on the behavior of economic subjects.

Finally, there is a way in which "the time factor" makes itself felt, if the relation sought after (demand or supply relation, etc.) gradually changes in the course of time. Strictly speaking, this is not an influence of the time factor, because the direct influence originates from other phenomena. Examples of these are a gradual increase of technical knowledge, a gradual change of tastes, and many other phenomena. But if it is difficult to

find other criteria to measure these phenomena—often there are many present at the same time—and if we are primarily interested in the short-term fluctuations, we often include time as a variable in the relation investigated. This can also be expressed by saying that we add a trend component to represent the influence of the gradually changing factors mentioned above. Of course, this is only correct as long as these factors actually grow gradually and not jerkily. Examples are the influence of the temperance movement—in its divergent direct and indirect forms—on the consumption of alcoholic beverages, and the influence of increasing highway traffic on the demand for railway transport.

§ 8. Connection with Other Relations

The mathematical formulation of economic relations is of particular importance if one wants to illustrate the connection with other, more or less allied relations. It is useful to consider several examples. Let us choose, as a first example, the various relations which are usually distinguished in the description of a single "Walrasian" market, or rather, which are sometimes not distinguished sharply, to the detriment of the analysis! We have already mentioned the demand and the supply relation. If p represents the price and u the quantity sold, both these relations are equations in which u is, among other things, explained by p. But the supply relation explains u from the point of view of the sellers and therefore, beside p, there appear magnitudes in it which we could call *supply factors* and which we shall indicate by s_1, s_2. . . . Similarly, in the demand equation, the demand factors d_1, d_2 . . . appear.

Written in full, they are as follows:

$$\text{Supply equation:} \quad u = f\ (p, s_1, s_2 \ldots) \tag{8.1}$$
$$\text{Demand equation:} \quad u = g\ (p, d_1, d_2 \ldots) \tag{8.2}$$

By equating demand and supply we obtain a new equation

$$f\ (p, s_1, s_2 \ldots) = g\ (p, d_1, d_2 \ldots)$$

in which u does not appear any more. From this equation, we can solve p in the majority of cases important to us:

$$p = h\,(s_1, s_2, \ldots d_1, d_2 \ldots) \tag{8.3}$$

We shall call this the *price equation.* It expresses the price in terms of the supply and demand factors without the help of the quantity sold. However, we can also solve p both from (8.1) and (8.2) and equate the two expressions obtained in this way; in other words we can eliminate p from (8.1) and (8.2); we then get an expression in u, s_1, s_2, d_1, d_2 . . . from which we can solve u:

$$u = k\,(s_1, s_2 \ldots d_1, d_2 \ldots) \tag{8.4}$$

We can speak here of a *turnover equation.* The equations (8.3) and (8.4) together are equivalent to (8.1) and (8.2); they express the market quantities—p and u—in this simple problem, the "unknowns"—in terms of the market data s_1, s_2, . . . d_1, d_2 . . . For that matter, we can take any pair to describe the market. But it is incorrect, for instance, to try to explain p by *all* the factors u, s_1, s_2, . . . d_1, d_2 . . . . Such a relation has no sense, because u alone can be derived from s_1, s_2, d_1 d_2 . . . as is shown in (8.4) and therefore it would be duplication to include u in an equation for p in addition to s_1, s_2, d_1, d_2 . . .

As (8.3) and (8.4) can take the place of (8.1) and (8.2), there must also be a relation between the coefficients of those equations. This can be seen most easily, if one makes use of a linear approximation or of an exponential form of the equations. Let, for instance,

$$u = v_{11}p + v_{12}s \text{ be the supply relation and} \tag{8.1'}$$
$$u = v_{21}p + v_{22}d \text{ the demand relation,} \tag{8.2'}$$

where, for the sake of simplicity, we have assumed only one demand factor and one supply factor. Introducing more would not be difficult.

The price equation then becomes

$$(v_{11} - v_{21})p = v_{22}d - v_{12}s$$

or

$$p = \frac{v_{22}d - v_{12}s}{v_{11} - v_{21}} \tag{8.3'}$$

and the turnover equation

$$(v_{21} - v_{11})u = v_{21}v_{12}s - v_{11}v_{22}d$$

or

$$u = \frac{v_{21}v_{12}s - v_{11}v_{22}d}{v_{21} - v_{11}} = \frac{v_{11}v_{22}d - v_{21}v_{12}s}{v_{11} - v_{21}} \tag{8.4'}$$

from which it becomes very clear how the coefficients of the last two equations are connected with those of the first two. If it is better for some reason to test statistically the equations (8.3′) and (8.4′) rather than (8.1′) and (8.2′) then with the help of the expressions given above, it is possible to determine the coefficients of the demand and supply relation from the former two equations.

We remind the reader that the coefficients v_{11} and v_{21} are closely connected with the elasticity coefficients of demand and supply, respectively. By elasticity is meant the ratio of a change in demand or supply, expressed as a percentage, to a given price change, also expressed as a percentage, from which the change in demand or supply results; strictly speaking, the changes should be infinitely small, but such small changes are difficult to deal with in practice. If we designate a small price change by Δp, and the resulting change in quantity sold by Δu, the elasticity is:

$$\epsilon = \frac{\Delta u}{u} : \frac{\Delta p}{p} = \frac{\Delta u}{\Delta p}\frac{p}{u}$$

In order to find the supply elasticity, Δu must be derived from the supply equation; we then have:

$$\Delta u = v_{11}\Delta p \text{ or } \frac{\Delta u}{\Delta p} = v_{11} \text{ and } \epsilon_s = v_{11}\frac{p}{u}$$

To determine the demand elasticity, one has to derive Δu from the demand equation, after which one finds:

$$\epsilon_d = v_{21}\frac{p}{u}$$

In this formula, p and u must have values belonging together; i.e., u must follow from p according to the demand or supply equation.

If the *exponential* approximation is used, the supply equation is represented by

$$u = p^{v_{11}}s^{v_{12}} \text{ or } \log u = v_{11} \log p + v_{12} \log s \qquad (8.1'')$$

and the demand equation by

$$u = p^{v_{21}}d^{v_{22}} \text{ or } \log u = v_{21} \log p + v_{22} \log d \qquad (8.2'')$$

The price and turnover equation, now, can be obtained by reading in (8.3′) and (8.4′) log p, log s, log d, and log u instead of p, s, d, and u.

The elasticity in this case is simpler; it is equal to v_{11} for supply and v_{21} for demand and therefore is independent of the values of u and p; i.e., of the appropriate place on the curve. This is the reason why demand and supply curves of the exponential type are usually assumed if no more details are known about their shape. For other purposes, namely for the use of such relations in so-called "complete systems," the application of linear equations, again, is simpler and therefore those are also often used. As has been pointed out, both types are intended as approximations over small intervals and as a rule they are definitely wrong if one moves too far from the starting point.

§ 9. Supply Equation and Price-Fixation Equation

A second example of connections with other relations and of the use of a mathematical method of treatment in investigating them is obtained when it is thought necessary to dig

deeper and to investigate the origin of the supply and demand curves themselves.* According to Walras, both are derived from aiming at maximum satisfaction, and therefore the coefficients of the functions (8.1) and (8.2) must be connected with the coefficients of the ophelimity function or of the profit function. An example of the first one will be discussed later (§ 26), whereas a very simple example of the second will be dealt with now, at the same time illustrating another point, namely, how the market organization can influence the coefficients of the supply function. Let us imagine a branch of industry producing a commodity in quantity u, and the demand function of which is as follows:

$$u = V_0 - V_1 p \tag{9.1}$$

Let production costs exist only as a constant part k_0 and a proportional part $k_1 u$, where k_0 represents total fixed costs for the whole production and k_1 the variable costs per unit of the product.

We first assume that the whole production is in the hands of a *monopolist*. Then the profit, Z, of this monopolist at a price, p, will amount to:

$$\begin{aligned} Z &= pu - k_0 - k_1 u \\ &= (p - k_1)\, u - k_0 \\ &= (p - k_1)\,(V_0 - V_1 p) - k_0 \end{aligned}$$

The profit will be a maximum if the price is fixed at the amount p_M, satisfying the requirement that Z be as great as possible. It can be ascertained, as can easily be seen, that:

$$p_M = \frac{V_0 + V_1 k_1}{2V_1} = \frac{V_0}{2V_1} + \frac{k_1}{2} \tag{9.2}$$

If, now, in the course of time, the demand curve shows parallel shifts, i.e., if V_0 is changeable, while V_1 remains unchanged,

* This is generally not necessary; the econometricians here actually follow Cassel, who wishes to consider the demand and supply curves as the basic data.

the course of the price is indicated by (9.2). This equation, however, is not a supply equation, because it is not exclusively in terms of p plus supply factors. The magnitude k_1 is a supply factor, and V_1 is a constant; but V_0 is a demand factor. On the other hand, u does not appear in equation (9.2), which would be the case in a supply equation. In order to find the supply function in this case, we must replace V_0 by u, if possible. This is indeed feasible, with the help of the demand equation. We then obtain:

$$p_M = \frac{u + V_1 p_M}{2V_1} + \frac{k_1}{2}$$

which can be written as follows:

$$u = V_1 (p_M - k_1) \tag{9.3}$$

This supply equation must not be understood in the strict Walrasian sense of an equation, which determines the behavior of the suppliers independently of the demand side. But judging by its form, it is such an equation. The remarkable thing is that it contains a constant from the demand side, namely V_1. It shows that a monopolist's supply elasticity depends on V_1 and is equal to $V_1 \frac{p}{u}$. This makes it possible to test statistically whether the supply on a given market is controlled monopolistically.

If free competition prevails, the supply function would look quite different, namely

$$p = k_1 \tag{9.4}$$

If, finally, some form of limited monopolistic competition prevails, still different supply functions would be valid. At this moment, however, we shall not go further into these details.

This simple example forms a good starting-point for a further consideration of the supply function and its significance. If one looks into what exactly is going on in a market, and in par-

ticular, if one looks at the sequence in time of the separate "economic actions"—i.e., if one proceeds dynamically—it will happen in many cases that the supplier does not consider the price as a datum and that he does not use it as a basis for his decisions on how much to produce. If he produces to satisfy market needs through the depletion of inventories, this can approximately be the case. If, however, he produces to order, it is certainly not so. In that case, his production is based rather on the amount ordered; i.e., on the quantity demanded, whereas he fixes the price more or less independently. Of course, he is forced in the long run to follow his competitors when fixing his price; too great a deviation would make his customers disappear after some time. But for a short period he is autonomous in his price fixing, at which time he will take his costs into account. These depend on the supply factors mentioned earlier; namely, the wage rate, the price of raw materials, and the labor productivity in his enterprise, and in addition, on the level of his production. The latter determines which marginal machines and laborers are just needed and what organization of labor can be used. Should we wish to reproduce the reaction relations describing this situation, we could write:

$$u = v_{21}p + v_{22}d$$

being the demand equation described earlier, representing the buyer's reaction on a given price p, and

$$p = g\,(s, u) \tag{9.5}$$

which represents the seller's reaction, namely the *fixing of the price,* which takes place on the basis of the supply factors mentioned, and on the level of production. The only factor still neglected in these equations is the possible difference in time between the variables. This does not concern us at the moment; we only want to demonstrate now that equation (9.5) is a supply equation according to the variables included; it is a relation between the magnitudes u, p, and s. If we neglect time differences, therefore, there is no distinction, and a supply equation

can also be interpreted as a *price-fixation equation*. In many cases, this will lead to a more realistic method of treatment.

§ 10. Macroeconomic Investigations

As we have already mentioned, these refer to *groups* of persons or markets, for instance, to the total demand for a certain article exerted by all consumers, or to the total demand for all consumption goods exerted by all consumers. If we start by confining ourselves to the first case, and assume that free competition exists in the market, with an equal price to each of the buyers, it is not difficult to determine the dependence of the total demand for the good considered on the price. If the individual demand functions are

$$u_1 = f_1(p),\ u_2 = f_2(p) \ldots \tag{10.1}$$

where the subscript refers to a certain person, the total demand is $u = u_1 + u_2 + \ldots = f_1(p) + f_2(p) + \ldots = f(p)$, i.e., again a function of the price.

The situation is different if we want to determine the total demand for consumption goods as a whole. In that case, there are various prices; namely, one for each of the goods, say p^1, p^2, $p^3 \ldots$ etc., where the superscripts refer to the various goods. The demand for each separate good depends on each of these prices, or perhaps, as a first approximation, only on the price of the good itself. In both cases, the total demand for all goods depends on all the prices mentioned. In macroeconomic investigations, however, we usually do not consider all the prices separately, but only one average price of consumption goods, since the very idea of macroeconomic investigations is to simplify the picture. It is by no means certain beforehand, however, that a function of the prices p^1, p^2, $p^3 \ldots$ may also be taken as a function of some average of those prices:

$$p = \frac{a^1p^1 + a^2p^2 + \ldots}{a^1 + a^2 + \ldots} \tag{10.2}$$

Of course, there are functions for which this is the case. For instance, the function $F(p^1, p^2 \ldots)$, which is defined as follows:

$$F(p^1, p^2 \ldots) = A\ (a^1p^1 + a^2p^2 + \ldots) \qquad (10.3)$$

is equal to $A\ (a^1 + a^2 + \ldots)p$. But another arbitrarily chosen function as a rule cannot be written as a function of p alone. Does this prove that such macroeconomic methods are inadmissible? It does not; a few conditions can be enumerated, under which macroeconomic procedures are very useful as approximations.

The situation is simplest when a single relation is involved. When the changes in the variables are small, it is always permissible to use linear functions. The total demand for all goods, therefore, may also be written as follows:

$$u = b^1p^1 + b^2p^2 + \ldots + c \qquad (10.4)$$

It is not certain, however, that now the coefficients b^1, b^2, etc., of the terms p^1, p^2 . . . are related in the same way as the coefficients a^1, a^2 . . . in the definition of p in (10.2). Otherwise there would be no objection against defining p differently, namely with the coefficients b^1, b^2 . . . instead of a^1, a^2. . . . On the contrary, this is an example of the fact that index numbers should always be constructed in such a way that they are suitable to the problem we want to solve with their help.

We shall illustrate this with a concrete example. We assume that the demand for every separate good depends only on its own price and also that these relations have a linear form; for instance,

$$u^1 = b^1p^1 + c^1,\ u^2 = b^2p^2 + c^2,\ u^3 = b^3p^3 + c^3 \qquad (10.5)$$

The magnitudes b^1, b^2 . . . depend on two circumstances. A certain b^1 is greater as the quantity consumed of that good is greater, and it is also greater as the demand for that good is more elastic. If the elasticity is indicated by ϵ^1, then, as we have seen before (§ 8),

$$\epsilon^1 = b^1\frac{\bar{p}^1}{\bar{u}^1} \qquad (10.9)$$

where $\bar{p}^1$ and $\bar{u}^1$ are the values of p^1 and u^1 at some reference point. It follows that

$$b^1 = \frac{\epsilon^1 \bar{u}^1}{\bar{p}^1} \tag{10.7}$$

Instead of (10.5), therefore, we can write

$$u^1 = \frac{\epsilon^1 \bar{u}^1}{\bar{p}^1} p^1 + c^1, \; u^2 = \frac{\epsilon^2 \bar{u}^2}{\bar{p}^2} p^2 + c^2 \ldots$$

From this it can be seen that the total demand

$$u = \frac{\epsilon^1 \bar{u}^1}{\bar{p}^1} p^1 + \frac{\epsilon^2 \bar{u}^2}{\bar{p}^2} p^2 + \ldots + c^1 + c^2 + \ldots \tag{10.8}$$

and also that the definition of the general price level in this problem must be such that the weights used are equal to

$$\frac{\epsilon^1 \bar{u}^1}{\bar{p}^1}, \; \frac{\epsilon^2 \bar{u}^2}{\bar{p}^2},$$

etc. These weights are different from the ones which are generally used in a price index number. There we usually take $\bar{u}^1$, $\bar{u}^2$, etc., as weights. In our case, goods with a strongly elastic demand are weighted extra heavily.*

As we have said, there is no objection against this procedure, as long as we are dealing with only one relation. If, however, in another relation, the price level p also has to play a role, it can happen that the weighting system which would be most suitable does not coincide with the weighting system required for the first relation. Strictly speaking, another index number p′ would in such case be necessary for the second relation. This index number will possibly not coincide with p, and then the intended

* For a practical application, the reader is referred to my "Business Cycles in the United States of America, 1919-1932" (League of Nations edition), § 3.4, which has been worked out by Dr. J. J. Polak.

simplification is partly illusory. If, however, the movements of the separate prices do not diverge strongly, the difference between p and p′ will remain within very narrow limits, so that as an approximation we may put $p = p'$. In many cases, this parallelism of the separate prices is actually present. This is particularly the case if the systematic factors which are the cause of the price movement are present in all prices, for instance, wages as an element of costs or general overproduction as a cause of declining prices. The component parts of price indices have, for the most part, moved parallel to one another. This does not alter the fact that, in principle, it is always desirable to investigate whether the conditions are satisfied.

A good example of the complications which a macroeconomic concept may entail can be given if we investigate in greater detail the dependence of demand on *income*.*

Let income for an individual family be i and the demand for consumption goods, c; further, let the total income of all families be I and the total demand for consumption goods, C. If c is a linear function of i, C will depend on I in a simple way; if, namely,

$$c = a + bi \text{ (a and b constant)}$$

and if these relations are added for all families,

$$C = aN + bI$$

where N is the number of families. But as soon as c depends on i in a more complicated way, the sum total of all c's is no longer dependent on I only. This makes the question, how an increase of the income I influences the demand for consumption goods, a difficult one to solve. An increase of I can take place in so many ways that it is no longer a matter of indif-

* The following is quoted from an article by P. de Wolff: Income elasticity of demand, a micro-economic and a macro-economic interpretation, *The Economic Journal*, 1940.

ference how it actually happens. Further computations in the article cited above indicate that an unambiguous relation between I and C exists only in those cases where the income distribution contains only two "parameters," and secondly that even then there is no sense in supposing that C depends also on the distribution of incomes. By "parameters" we mean those magnitudes which "describe" a frequency distribution; a well-known example is the one of Pareto's law of income distribution, which contains as one of the parameters the magnitude a. Although the details of the argument cannot readily be clarified without higher mathematics, we call the reader's attention to it as a good example of what mathematical analysis can achieve here.

Also in the case mentioned before of linear dependence of c on i, we have an example of an unambiguous dependence between I and C and of the absence of any influence of income distribution on the total demand. It is a happy circumstance for macroeconomic research that actually within wide limits c depends linearly on i (see § 26).

§ 11. The Description of Complete Systems

We shall now go somewhat deeper into the contribution of mathematical economics to what we have called the last stage of the investigation; i.e., after the testing of the relations discussed in the previous section has taken place. In order to see the sense of the following, we have to recall what is the ultimate purpose of all economic analysis. We formulate this purpose twofold: (1) to explain the historical course of economic phenomena from their noneconomic causes, or "data," and (2) to ascertain how a change of these data—for instance, a certain form of economic policy—will have its effect on that course. So far, we have only been occupied with one single relation between economic phenomena. Changes in one phenomenon have been reduced to changes in other economic phenomena and to certain data of a natural, a technical, or an institutional character. If, however, we want to explain the whole of economic activity, we have to consider not one relation but a

number of these together, enough, in fact, so that the number of phenomena to be explained is equal to the number of relations. Only in this case can we hope really to explain the course of economic phenomena from noneconomic phenomena. Expressed more concretely: it is not sufficient to derive the demand for consumption goods from incomes and prices; the latter, in their turn, must also be explained.

In order to do this, the network of causal relations which together form the economic structure of a certain area should be investigated systematically. Some preliminary defining of the concepts used cannot be avoided here. We think this can best be carried out by using a "system of arrows" (see Fig. 1).

In this figure, every dot represents a certain economic phenomenon during a certain time-unit, say during a certain quarter. Horizontally, the same phenomenon has been symbolized in subsequent quarters. Vertically we find the various phe-

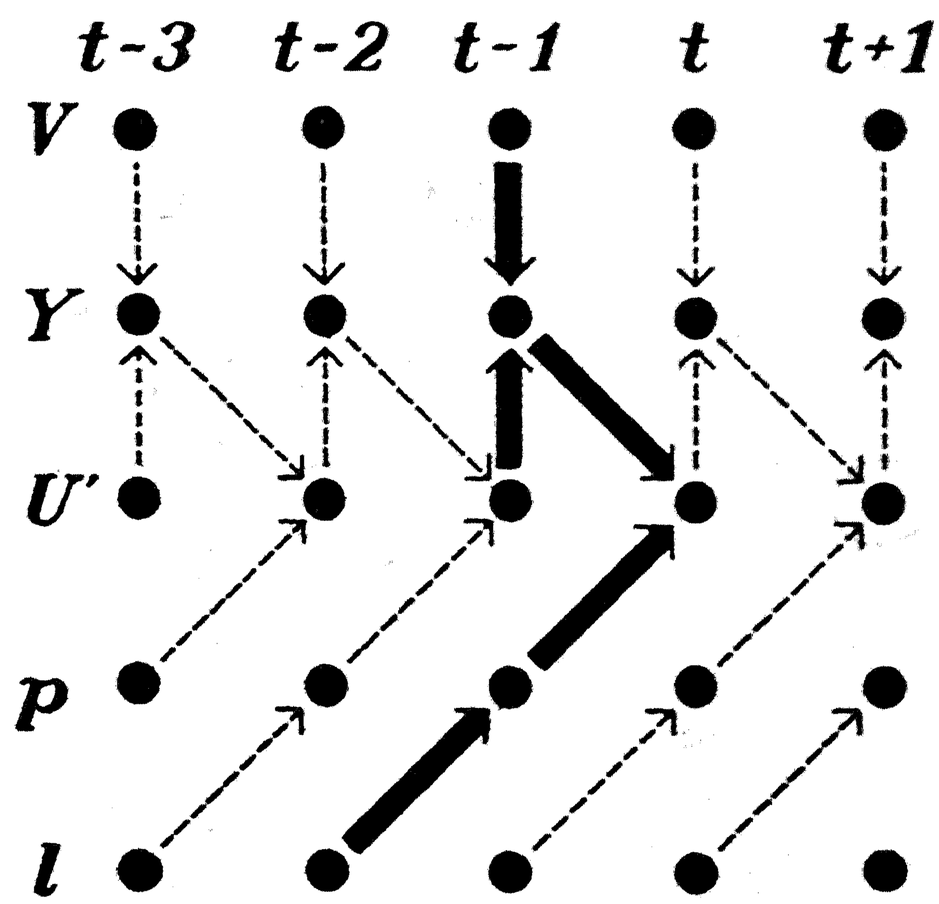

Fig. 1. System of arrows.

nomena taken into consideration for the same quarter. Now the value of a certain phenomenon in a certain quarter, say of the demand for consumption goods U'_t during the quarter t,

will be determined by the values of some other phenomena during the same or another quarter. To clarify this, let us assume that it is determined by the level of income in the previous quarter, Y_{t-1}' and by the prices in that same quarter, p_{t-1}. We indicate this by the arrows running from Y_{t-1} and p_{t-1} to U'_t. These arrows indicate a direct causal relation. They repeat themselves quarterly, if the structure of the economy is assumed constant. The values of U', Y, and p do not necessarily repeat themselves—that would only be the case in a stationary situation —but the causal relation does repeat itself.

In their turn, the incomes Y again depend on other magnitudes, say on the value of consumption U' and on the value of investments V during the same quarter. That will be indicated by arrows from U' and V to Y, without lags. The prices p and the investments V' will also have their determining factors. In this way we obtain a whole scheme of arrows which repeat themselves. By inquiring again and again after the direct causes of phenomena which have not yet been explained, a growing number of relations is obtained. In the beginning we shall also have to consider a growing number of phenomena. But at some point this becomes unnecessary: now, phenomena which have already been considered, again act as causes; at that point the phenomena not yet explained have all been reduced to explained ones. If the number of phenomena then is equal to the number of relations, we have a complete system, or *model,* which word stresses the fact that such a system is always more or less *simplified* in comparison to reality. In such a model, unexplained noneconomic phenomena can still appear as data, but it is not the task of economics to explain these.

The entire set of arrows now forms the *logical structure* of our system. These arrows can be arranged in two ways. We can consider all arrows ending in one point; we then see at a glance the various *causes* of one phenomenon, characterized by that point. We can also take together all arrows which start from one point; we then consider the *influence* of one phenomenon on the others. This last arrangement is, so to speak, directed forward; it is used in most verbal arguments. The first

arrangement is the one which corresponds with our equations: the various terms of the latter indicate the various arrows which end in the variable to be explained. With both methods of arrangement, of course, in each column every point should be considered successively, i.e., all phenomena should be taken into consideration. Moreover, as we have already pointed out, the whole picture obtained in this way repeats itself in time.

In both arrangements mentioned, we work exclusively with direct causal relations. This is sufficient because each arrow may be considered in turn. It must not be inferred, however, that the *indirect*—i.e., the "deeper"—causes are neglected. But every direct cause at the same time is an indirect cause. If p has a direct influence on U′ and if the wage-level l has a direct influence on p, then l has an indirect influence on U′. Going backward along the arrow ending in U′ and coming from p, we first arrive at p, and subsequently we can go further back to l, etc. Thus we can detect the secondary and tertiary causes, etc. The system, therefore, is completely defined by the equations which we have introduced indicating the "elementary laws" which connect the variables. We shall speak, consequently, of the *elementary equations* or of the *structural equations.*

We have to go a little further into the form of these equations: What kinds of variables appear in them? As an example, we again take the demand equation for consumption goods:

$$U'_t = v_1 Y_{t-1} + v_2 p_{t-1} + v_3(t) + v_4 + R_t^{U'}$$

We have added a few terms, which will be discussed presently. The following variables appear now:

1. the *variable to be explained,* U′ (demand for consumption goods);

2. the *explanatory variables* Y′ and p (income and the price level);

3. a number of *constant magnitudes:* coefficients v_1, v_2, v_4 and lags (of one time-unit in this case), which together determine the *structure* of the system (we shall discuss this term later);

4. a term $v_3(t)$ depending on time, which reflects the changes

of *systematically* changing *data* or of explanatory phenomena which have not been included separately, such as changes in technique, seasonal influences, influences coming from foreign countries or from the rate of interest;

5. a term $R_t^{U'}$ which has not been discussed earlier, representing the *nonsystematic* changes of *data* or explanatory variables which might occur and which are not included elsewhere. Here we have to think first of a number of influences, each of which is small in itself, and which has been omitted in the simplification process; and second, of the influence of less important data which are usually considered as having a random character. The demand for consumption goods may be lower because more people happen to be ill or because more rain has fallen than usual, etc. Such magnitudes have an irregular influence, but also a small one, and as a rule they do not interest us. By way of exception, influences of some more importance may be included, such as, for instance, crop yields and possible technical improvements of great importance, or the influence of a strike, a political event of major importance, and so on.

We have used the word *structure* in particular when referring to statement 3 preceding. As a matter of fact, by the structure of an economy we mean in particular the permanent characteristics of it. Much may be said, however, in favor of including also the systematically changing variables mentioned under statement 4, particularly the natural, technical, and institutional factors which are included in that group. With an "open" economy, however, we also have under statement 4 those variables which may perhaps not be considered as data and consequently not as structural magnitudes in economic science taken as a whole.*

In our example, v_1 and v_2 have been considered as structural constants, i.e., the coefficients by which income and price level influence consumption. As we have remarked earlier, we can also dig deeper here and consider those magnitudes which lie behind the first ones as structural, as for instance the coefficients

* For instance, the world price level.

in the ophelimity functions and the technical coefficients. It is only for purposes of exposition that the coefficients of the demand curve themselves have been considered as data; a device to be compared to Cassel's working method.

As we have seen above, in the system of structural equations describing a certain economic model, the following kinds of magnitudes appear:

1. variables at the moment t
2. variables at earlier moments
3. data

The equations for the period t may be, for instance:

$$u_t = \upsilon_1 Y_{t-1} + \upsilon_2 p_{t-1} + \upsilon_3 T_{t-1} + R_t^u$$
$$Y_t = \eta_1 u_t + \eta_2 p_t + R_t^Y$$
$$p_t = \pi_1 Y_{t-1} + \pi_2 q_{t-1} + R_t^p$$

where:

u_t = the quantity demanded of all goods and services
p_t = the average price of these
Y_t = national income
T_t = the temperature and
q_t = the crop.

The first equation is the demand equation for all goods and services taken together; the second is the definition equation of national income put into a linear form, and the third is the price-fixation equation. It has been assumed that the total demand has also been influenced by the temperature, and the price by the crop size.

The categories under (2) and (3) together are sometimes called the *predetermined variables*. They form the whole of those magnitudes which are known at the beginning of the period t and which determine the magnitudes with t-subscripts except the R's. The three equations actually enable us to determine these magnitudes because there are precisely three such magnitudes. It may be noticed that each of these appears in several of the equations. If the system of structural equations

is *solved* for the variables of period t, they assume the so-called *reduced form.*

In order to clarify the importance of the reduced form of the equations, we consider another example of a very simple system of equations, namely the demand and supply equations which together describe a certain market. The variables in such a system are the price p and the sales u; all the other magnitudes may be considered as data. The reduced form now is obtained by solving the demand and supply equations for p and u; as we have seen in § 8, we then have the so-called price equation and the turnover equation. These together form the reduced form of the structural equations of one market.

§ 12. The Movements of Complete Systems

The elementary equations taken together, now, enable us to learn the movements of all economic phenomena, if we only know:

1. the structure of the system, embodied in the given coefficients, lags, and trend terms;
2. the values of the other systematically changing data for each period of time;
3. the values of the nonsystematically changing data, R, and
4. certain initial values of the economic variables. What these values are can only be determined in a concrete case.

If we possess these data, we can determine the course of the final figures by a process of computation. We shall demonstrate this with the help of a very simple example referring to a single market which is in isolation and forms a system of its own. Let the demand u_t be dependent only on the price p_t, and without lag:

$$u_t = -p_t + 200 + t \tag{12.1}$$

Let, further, the supply likewise be dependent on the price only, but with a lag of three time-units:

$$u_t = p_{t-3} \tag{12.2}$$

In the demand equation we have also included a gradually increasing term, brought about, say, by population growth. The positive constant, here taken equal to 200, is necessary, because otherwise for every positive value of the price p at the moment O, a negative demand would appear, which is unacceptable.

As has been said, we also have to know a few initial values of one of the magnitudes in order to be able to compute the further course of the figures. In this case it can easily be seen that we must know the course of one of the magnitudes during the first three periods, say for $t = 1$, 2, and 3. We therefore assume that we know u_1, u_2, and u_3, which respectively amount to:

$$u_1 = 100; \; u_2 = 120; \; u_3 = 130$$

In fact, with the help of (12.1), we can now calculate p for $t = 1$, 2, and 3 and with the help of these, we can determine the values of u for $t = 4$, 5, and 6; from (12.2); then again, with (12.1) we can find p for $t = 4$, 5, and 6, etc. The result of this *numerical extrapolation* is as follows:

$t =$	1	2	3	4	5	6	7	8	9	10	11	12
$u_t =$	100	120	130	101	82	73	103	123	133	104	85	76
$p_t =$	101	82	73	103	123	133	104	85	76	106	126	136

The peculiarities of this movement will be dealt with presently. The calculations do not offer particular difficulties, but the method, in general, has two disadvantages. First, in order to become acquainted with the movement of one of the magnitudes, we always have to compute those of the other one as well. With systems containing a great number of variables, this involves a considerable amount of work. Second, in the method used above we can only learn something about the type of movement, and its dependence on the assumptions made, after much laborious work. The results are obtained only after a large amount of experimentation. Mathematics, however, offers us the means to overcome both objections.

In order to be able to leave undesired variables out of consideration, we must and can *eliminate* them, an operation well-

known in elementary algebra. In the simple case of our example, we only have to substitute (12.2) into (12.1):

$$p_{t-3} = -p_t + 200 + t \tag{12.3}$$

This means that from a consideration of "first causes" we change over to the study of "second causes"; in other words, that we change over from elementary economic laws to *"derived economic laws."* There is quite a hierarchy of such derived laws, namely, derived laws obtained by combining 2, 3, etc. elementary equations. It is not easy to speak about these laws in words instead of in formulas, because most of them have not been given separate names. For this reason the method has sometimes been rejected as "streamlined analysis." The only thing to do about this is to choose the system as simple as possible, but of course, its closeness to reality will suffer by this procedure.

The end of this elimination process is reached when we are left with one equation with only one variable. It is not always possible to get as far as this; but it can be done with linear equations and similarly with a number of other types. In this *final equation* the remaining variable in general appears with values pertaining to various periods; the equation is, therefore, *dynamic*. Its coefficients are combinations of the coefficients appearing in the elementary equations. Moreover, in this final equation, various random disturbances from the elementary equations will be included. We have neglected these in our example, but they can easily be included in the model.

Both for the final equation and for the system as a whole, we can compute the further course by numerical extrapolation, if we know a few initial values of the variable. We shall now consider more systematically which results we can expect when doing so. To this purpose we first write equation (12.3) somewhat differently:

$$p_t = -p_{t-3} + 200 + t \tag{12.4}$$

Disregarding the random disturbance terms, which are neg-

lected here, we now have three kinds of terms in the right-hand side of the equation.

The first kind of term for which we ask attention is the constant; in our example, the term 200. If this were the only term, the value of the variable p_t would also be constant, i.e., a stationary situation would exist and we would be dealing with *statics* in the narrower sense. This constant value of p would depend on the constant additive terms in the elementary equations and on the constant coefficients (for instance, elasticity coefficients and so on). It would be what we usually call the *equilibrium value* of p. If in the final equation (12.4) also random deviations R appear, p will show random deviations, which, however, group themselves around this equilibrium value.

The second kind of term is that *in which time appears explicitly* and which is a consequence of systematically changing data, such as increasing population and technical knowledge, or seasonal fluctuations or, also, changes originating abroad. If these appear—whether accompanied by those of the first kind, or not—the variable p cannot be constant any longer—i.e., a static equilibrium is impossible—but p will change in the course of time and, consequently, show systematic movements. These movements take place *simultaneously with the movements in the data* which give rise to them. On account of the absence of lag terms like p_{t-3}, the system immediately adapts itself to changes in data. If these changes of data were not present, no movement would be possible, i.e., *endogenous movements could not exist.* This system is not static in the narrower sense; it is static, however, according to our definition, and the corresponding theory is sometimes called *comparative statics.* The same formulas can be used for mutual comparison of two stationary situations, in which the data possess different values.

Finally, let us consider the influence of the third kind of term, namely, the term in which the variable appears for other moments of time. Let us first imagine that these terms—p_{t-3} in our example—are the only ones, with or without the constants. We can now conclude that in this case movements are possible

without changes of data, i.e., *endogenous movements may occur.* We can only speak of this sort of movement, however, if this third kind of term is present; in other words, the system must be dynamic.

It can be shown mathematically that the movements which follow from the presence of terms with variables at various moments of time form a special type. They consist of one or more of the following movements, which may manifest themselves with various degrees of relative intensity:*

1. damped, nondamped, or antidamped periodic movements,
2. one-sided movements, gradually approaching a certain limit (convergent one-sided movements),
3. one-sided movements leading further and further away from the initial situation.

These one-sided movements are of certain mathematical types, but we shall not go into this further. The periodic movements may have various *periods.* These periods, however, are determined by the structural constants; they show some analogy to the so-called self-vibrations of various mechanical systems, the periods of which are also determined by their mass, their dimensions, elasticities, deflection resistances, etc.

In our example it can easily be seen that the movement is a purely periodic one (i.e., nondamped) with a period of six time-units. Omitting the last two terms, the equation becomes:

$$p_t = - p_{t-3}$$

From this it follows that

$$p_{t+3} = - p_t = p_{t-3}$$

which means that the value of p_{t-3} is equal to that of $p_t{}^{+}{}_3$. This is true for every arbitrary t, and therefore p is equal for any two periods of time with a mutual distance of six units. If, instead of $p_t = - p_{t-3}$ we have

* For exact proofs we refer to "Einige Grundfragen der mathematischen Konjunkturtheorie," Archiv für mathematische Wirtschafts- und Sozialforschung III (1937), pp. 1, 83.

$$p_t = -a\,p_{t-3}$$

and therefore $$p_{t+3} = a^2 p_{t-3}$$

a damped movement would originate with $a < 1$ and an anti-damped movement with $a > 1$, as can easily be shown. In this simple case the period only depends on one of the structural data, namely the "lag" of three time-units, which, for instance, could represent the duration of a production process. In most cases the period will depend on other structural data as well, which will make the relation less simple.

The *amplitude* of the fluctuations, however—i.e., the size of the maximum deviations which appear—depends on the deviation from equilibrium existing in the beginning; this could be called an inheritance of history, although it need not always be old history. In general, influence will be exerted by the intensities with which the random deviations—for instance, the consequences of crop changes, possible wars, etc.—make themselves felt.

What happens if all terms are present at the same time? In case we are dealing with a linear equation, which, as a first approximation, can often be assumed, it can be shown that the movement is the *sum total* of:

1. an equilibrium level,
2. an endogenous movement as described above and
3. "exogenous" movements brought about by the change of data, with the same period as the movements of the data, insofar as these are periodic, and which show a linear growth in time if the data show a linear growth.

To these, we have to add the possible random deviations which are cumulated in a certain way. We shall explain this in more detail. For the sake of simplicity, let us assume that only the first term of the right side of (12.4) and a random deviation are present; we then have:

$$p_t = -p_{t-3} + R_t^p$$

If p_{t-3} is given, p follows from this equation. For p_{t+3} we have:

$$p_{t+3} = -p_t + R^p_{t+3} = p_{t-3} - R^p_t + R^p_{t+3}$$

Here the combined influence of two random deviations can be seen. For p_{t+6} we would have:

$$p_{t+6} = -p_{t-3} + R^p_t - R^p_{t+3} + R^p_{t+6}$$

and so on.

If we survey the result, it becomes clear that the movements shown by economic phenomena, under the influence of changes in data and of the dynamic character of the relations, show the following components:

1. one-sided movements brought about by one-sided changes of the data, for instance, growth of the population;
2. one-sided movements brought about by internal dynamics, for instance, capital formation;
3. periodic movements brought about by periodic movements of the data, for instance, foreign business-cycle influences and seasonal fluctuations;
4. periodic movements brought about by internal dynamics, for instance, the movement of the business-cycle;
5. random movements brought about by random (irregular) changes of the data, for instance, crop changes and many small structural changes.

§ 13. Stable and Unstable Equilibria. The Purpose of Economic Policy

We have already seen that the economic systems with which we have occupied ourselves thus far possess no static equilibrium. Nevertheless, the purpose of economic policy, especially of business-cycle policy, is often formulated as a more balanced development of economic life. In the light of the preceding we have to understand by this, a development where the movements under 3, 4, and 5 are restricted as far as possible. The aim of econometric research in this field, therefore, is to determine whether such an aim can be attained by the measures which are being proposed from various sides. The investigations in this respect are only in their first stage; a few pre-

liminary results will be given in § 41. In this methodologic chapter it is appropriate to indicate briefly the concepts of *equilibrium* and of *stability* and *instability* of equilibria. Whereas by equilibrium in the limited static sense, a situation is meant which can be maintained, such a definition has only little value for our systems as it requires a situation which cannot exist. There is more sense in the definition of an *equilibrium development,* indicating a movement as gradual as possible, or, more precisely: that movement of all possible movements of the system which conforms itself most closely to a straight line. From the preceding it can be inferred that, for a given system, these movements are certain well-defined ones, namely the components 1 and 2 of the series mentioned at the end of § 12. The movements taking place in reality and containing also the components 3, 4, and 5, are fluctuations around the former ones.

If these fluctuations are damped, we can call the equilibrium development *stable*. In case they are nondamped or antidamped, we call the equilibrium development *unstable*. It is the task of business-cycle policy to change the structure of the national economy—or the world economy—in such a way that its stability becomes as great as possible; in other words, that the *degree of damping becomes as great as possible*. Expressing this in everyday language, but perhaps easier to understand: we must learn to nip the fluctuations in the bud and, if possible, not incidentally but systematically, i.e., by changing the structure. Such a structural change will perhaps have to be found in the financial or monetary government policy, but possibly also elsewhere. Here the econometrician has no vote: he only has to investigate the problems put before him.

This problem, however, cannot be separated from the other aims of general economic policy, namely the requirement to reach the highest possible average level of prosperity. If it would turn out that a greater stability could only be reached at the cost of a lower average level of prosperity, a choice would have to be made on welfare grounds.