APPENDIX A

DETAILS OF CALCULATIONS

1. *Trend Calculation.*

For pre-war periods, nine-year moving averages have been taken. In a few cases where seven-year moving averages were already available, the latter were used, as the differences were small. For post-war periods, which are shorter, rectilinear trends have been calculated. For short periods, the arbitrariness of this type of trend is less than for long periods, as the difference between a rectilinear trend and a moving average is in this case generally small. In addition, the rectilinear trend has the great advantage that no years need be left out of account, whereas, in the case of nine-year moving averages, four years are lost at each end of the series.

Let any series be represented by X_t, where t indicates the year (or any other unit period used) and assumes all values from 1 to N, N being the total number of years in the period considered. Let its rectilinear trend be $X_t^{(T)}$. The latter is then of the form

$$X_t^{(T)} = a + bt \qquad (1)$$

where a and b are constants, which should be chosen so as to obtain the " best " fit between X_t and $X_t^{(T)}$. This is usually done by applying the method of least squares, which prescribes that the expression

$$(X_1 - X_1^{(T)})^2 + (X_2 - X_2^{(T)})^2 + (X_3 - X_3^{(T)})^2 + \ldots$$
$$\ldots + (X_N - X_N^{(T)})^2 \quad \text{or} \quad \sum_1^N{}_t (X_t - X_t^{(T)})^2 \qquad (2)$$

be made as small as possible by a suitable choice of a and b. This problem can be solved most simply if t, X_t and $X_t^{(T)}$ are measured

as deviations from their mean values. If the mean values be indicated by $\overline{t}$, $\overline{X}$ and $\overline{X}^{(T)}$ respectively, and the deviations by t', x_t and $x_t^{(T)}$, we have:

$$\overline{t} = \frac{1}{N}\sum_1^N t\,t\,; \quad \overline{X} = \frac{1}{N}\sum_1^N t\,X_t\,; \quad \overline{X}^{(T)} = \frac{1}{N}\sum_1^N t\,X_t^{(T)} \qquad (3)$$

and

$$t' = t - \overline{t}\,; \quad x_t = X_t - \overline{X}\,; \quad x_t^{(T)} = X_t^{(T)} - \overline{X}^{(T)} \qquad (4).$$

It follows immediately that:

$$\Sigma t' = 0\,; \quad \Sigma x_t = 0\,; \quad \Sigma x_t^{(T)} = 0 \qquad (5).$$

Assuming now that the trend equation takes the form

$$x_t^{(T)} = a' + b't' \qquad (1')$$

the method of least squares requires by analogy with (2):

$$\sum_1^N t\,(x_t - x_t^{(T)})^2 \quad \text{minimum} \qquad (2').$$

Replacing $x_t^{(T)}$ by its value (1'), this becomes:

$$\sum_1^N t(x_t - a' - b't')^2 \quad \text{minimum.}$$

This expression depends on a' and b', since all other variates are given numbers drawn from observation. It is therefore a function $F(a', b')$ of a' and b', and its minimum value must, according to a well-known statement in differential calculus, obey the two relations:

$$\frac{\partial F(a', b')}{\partial a'} = 0 \qquad \frac{\partial F(a', b')}{\partial b'} = 0\,^{1} \qquad (6).$$

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[1] It can be proved that these equations yield, in this case, a minimum and not a maximum value of F.

These relations come to: [1]

$$\Sigma x_t - Na' - b'\Sigma t' = 0 \qquad \Sigma x_t t' - a'\Sigma t' - b'\Sigma t'^2 = 0.$$

Applying relation (5), they are much simplified and become:

$$a' = 0 \qquad \Sigma x_t t' = b'\Sigma t'^2 \quad \text{or} \quad b' = \frac{\Sigma x_t t'}{\Sigma t'^2} \qquad (7).$$

The trend, if measured in deviations from its mean value, is therefore of the simple shape:

$$x_t^{(T)} = t' \frac{\Sigma x_t t'}{\Sigma t'^2} \qquad (8).$$

To obtain its actual level, the easiest way is to introduce t' into equation (1), as follows:

$$X_t^{(T)} = a + b(t' + \overline{t}) = a + b\overline{t} + bt' \qquad (9).$$

As the average of $bt' = \frac{1}{N}\Sigma b t' = \frac{b}{N}\Sigma t' = 0$, it follows that $a + b\overline{t} = \overline{X}^{(T)}$.

On the other hand, it may be deduced directly from (2), by applying the principle of least squares, that $\overline{X}^{(T)} = \overline{X}$ (this, in fact, is self-evident).

Finally, from (4) and (8), it will be seen that $b = b'$: hence the trend equation in natural units will be:

$$X_t^{(T)} = \overline{X} + t' \frac{\Sigma x_t t'}{\Sigma t'^2} \qquad (10).$$

2. *Multiple Correlation; Calculation of Regression and Correlation Coefficients.*

The technique used in this case is much the same as that used in the trend calculations just described. The problem is to find coefficients b_1, b_{12}, b_{13}, etc., which give the best fit between any

[1] For simplicity, the suffixes to the sign Σ will in future be omitted.

given series X_{1t} and a linear function X'_{1t} of a number of "explanatory" series X_{2t}, X_{3t}, etc.[1]

$$X'_{1t} = b_1 + b_{12} X_{2t} + b_{13} X_{3t} + b_{14} X_{4t} \ldots \qquad (11).$$

All series may again be measured in deviations from their averages. (The suffix t will in future be omitted in the formulæ, as no ambiguities can arise.) The deviations will be denoted by x_1, x'_1, x_2, etc., and, as before, it follows from their definitions that

$$\Sigma x_1 = \Sigma x'_1 = \Sigma x_2 = \Sigma x_3 = \ldots \Sigma x_n = 0 \qquad (12).$$

By this choice, b_1 will become zero.

The problem is to find for b_{12}, b_{13} etc., values which make the expression

$$\Sigma x^2_{1.23\ldots n} = \Sigma (x_1 - x'_1)^2 = \Sigma (x_1 - b_{12} x_2 - b_{13} x_3 \ldots)^2 \quad (13)$$

a minimum. The quantities $x_{1.23\ldots n}$ will be called the residuals.

They have to fulfil the following conditions known as "normal equations" and perfectly analogous to equations (6):

$$\left.\begin{array}{l} \Sigma x_2 (x_1 - b_{12} x_2 - b_{13} x_3 \ldots) = 0 \quad \text{or} \quad \Sigma x_2 x_{1.23\ldots n} = 0 \\ \Sigma x_3 (x_1 - b_{12} x_2 - b_{13} x_3 \ldots) = 0 \quad \text{or} \quad \Sigma x_3 x_{1.23\ldots n} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \quad (14)$$

These may be written:

$$\left.\begin{array}{l} b_{12} \Sigma x_2^2 + b_{13} \Sigma x_2 x_3 + \ldots + b_{1n} \Sigma x_2 x_n = \Sigma x_1 x_2 \\ b_{12} \Sigma x_2 x_3 + b_{13} \Sigma x_3^2 + \ldots + b_{1n} \Sigma x_3 x_n = \Sigma x_1 x_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_{12} \Sigma x_2 x_n + b_{13} \Sigma x_3 x_n + \ldots + b_{1n} \Sigma x_n^2 = \Sigma x_1 x_n \end{array}\right\} \quad (15)$$

where $n - 1$ is the number of "explanatory series". Equations (15) are usually the most convenient to use in numerical calculations.

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[1] In Chapters III-V, where only the first regression has been considered, X' and x' have been replaced by X^* and x^*.

All the "sum-expressions" Σx_2^2, $\Sigma x_2 x_3$, etc.... are known from observation.

A general definition of the correlation coefficient between two series is given by the formula:

$$r_{12} = \frac{\Sigma x_1 x_2}{\sqrt{\Sigma x_1^2 \, \Sigma x_2^2}} \qquad (16).$$

This definition may also be applied to the series x and x_1', and then provides the "total correlation coefficient" $R_{1.23\ldots n}$. The result can be put in the form:

$$R_{1.23\ldots n} = \sqrt{\frac{b_{12}\Sigma x_1 x_2 + b_{13}\Sigma x_1 x_3 + \ldots + b_{1n}\Sigma x_1 x_n}{\Sigma x_1^2}}$$

which is convenient for numerical calculations.

If all series are measured in so-called normalised units—*i.e.*, in such units that their standard deviations become 1, then the normal equations may be given the form:

$$\left.\begin{array}{l} b_{12} + b_{13} r_{23} + \ldots + b_{1n} r_{2n} = r_{12} \\ b_{12} r_{23} + b_{13} + \ldots + b_{1n} r_{3n} = r_{13} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ b_{12} r_{2n} + b_{13} r_{3n} + \ldots + b_{1n} = r_{1n} \end{array}\right\} \qquad (17)$$

from which it will be seen that the regression coefficients in normalised units depend only on the system of correlation coefficients between all variates.

The coefficients b_{12}, b_{13}, etc., to be found from the normal equations (15) are the regression coefficients for the first elementary regression. If, for example, the second elementary regression is to be determined, the rôle of suffixes 1 and 2 should be interchanged and the coefficients in that regression —

$$x_2' = b_{21} x_1 + b_{23} x_3 + b_{24} x_4 + \ldots b_{2n} x_n \qquad (18)$$

— have to be calculated from:

$$\left.\begin{array}{l} b_{21}\Sigma x_1^2 + b_{23}\Sigma x_1 x_3 + \ldots b_{2n}\Sigma x_1 x_n = \Sigma x_2 x_1 \\ b_{21}\Sigma x_1 x_3 + b_{23}\Sigma x_3^2 + \ldots b_{2n}\Sigma x_3 x_n = \Sigma x_2 x_3 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_{21}\Sigma x_1 x_n + b_{23}\Sigma x_3 x_n + \ldots b_{2n}\Sigma x_n^2 = \Sigma x_2 x_n \end{array}\right\} \quad (19).$$

Equation (18) should be transformed into:

$$x_1'' = \frac{1}{b_{21}}(x_2 - b_{23}x_3 - b_{24}x_4 - \ldots - b_{2n}x_n) \qquad (20)$$

by writing x_2 instead of x_2', and x_1'' instead of x_1 if it is to be used as a second estimate for the "explanatory equation" for x_1 (Frisch's method).

The regression used in this publication is the first elementary regression; for the most important cases, however, bunch maps (*cf.* 3 below) have been added.

3. *Construction of Bunch Maps.*

Here the technique outlined by Professor Frisch, who proposed the method, in his "Statistical Confluence Analysis by Means of Complete Regression Systems" [1] has been almost exactly followed. As has been pointed out in § 2, the regression coefficients in normalised units can be determined from the system of all correlation coefficients between the variates considered. Starting with these correlation coefficients, it is conceivable that every possible regression formula is calculated in the way indicated *sub* 2. Frisch's method is, however, far more efficient, as any repetition of operations is avoided. The general idea is that all symmetrical minors

[1] Publication No. 5 of the Economic Institute of the University of Oslo (Universitetets Økonomiske Institutt), Oslo, 1934.

of the determinant of the correlation coefficients:

$$\Delta_{12\ldots n} = \begin{vmatrix} r_{11} r_{12} \ldots r_{1n} \\ r_{21} r_{22} \ldots r_{2n} \\ \vdots \\ r_{n1} r_{n2} \ldots r_{nn} \end{vmatrix} \tag{21}$$

should be calculated, and this is done in a systematic way.

As any minor of a two-rowed determinant is itself again an element of that determinant, no calculations are needed for such cases. We begin, therefore, with the minors of three-rowed determinants. Take, as an example, the determinant

$$\Delta_{123} = \begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix} \tag{22}.$$

The minor $\hat{r}_{11}$ (this notation being provisional only) corresponding with the element r_{11} is found from $\hat{r}_{11} = r_{22} r_{33} - r_{23} r_{32} = 1 - r_{23}^2$. Similarly, $\hat{r}_{22} = 1 - r_{13}^2$ and $\hat{r}_{33} = 1 - r_{12}^2$. The minor $\hat{r}_{12}$ corresponding with r_{12} equals

$$- r_{21} r_{33} + r_{23} r_{31} = - r_{12} + r_{13} r_{23} \, .$$

Similarly,

$$\hat{r}_{13} = - r_{13} + r_{12} r_{23}$$
$$\hat{r}_{23} = - r_{23} + r_{12} r_{13} \, .$$

The figures for r_{ik} and for $\hat{r}_{ik}$ are written in a table showing them in the following way:

r_{ik}	1	2	3
1	r_{11}	r_{12}	r_{13}
2		r_{22}	r_{23}
3			r_{33}

$\hat{r}_{ik}$	1	2	3
1	$\hat{r}_{11}$	$\hat{r}_{12}$	$\hat{r}_{13}$
2	$\hat{r}_{12}$	$\hat{r}_{22}$	$\hat{r}_{23}$
3	$\hat{r}_{13}$	$\hat{r}_{23}$	$\hat{r}_{33}$

Because of the symmetry, the figures below the diagonal may be left out, as has been done in the left-hand table. The figures $\hat{r}_{ik}$ can now be easily checked, as there are three ways of calculating the value of Δ_{123}:

$$\begin{aligned}\Delta_{123} &= r_{11}\hat{r}_{11} + r_{12}\hat{r}_{12} + r_{13}\hat{r}_{13} \\ &= r_{12}\hat{r}_{12} + r_{22}\hat{r}_{22} + r_{23}\hat{r}_{23} \\ &= r_{13}\hat{r}_{13} + r_{23}\hat{r}_{23} + r_{33}\hat{r}_{33}.\end{aligned}$$

a well-known identity. This means simply that corresponding figures in the two above tables have to be multiplied and the results belonging to one and the same column to be added up. The same result must then be obtained, whatever column be taken.

The desired information concerning all possible regression coefficients is now obtainable from table $\hat{r}_{ik}$. The regression coefficient of variate 2 in the "explanation" of variate 1 is equal to $-\frac{\hat{r}_{12}}{\hat{r}_{11}}$ if the direction of minimising is that of variate 1; $-\frac{\hat{r}_{22}}{\hat{r}_{12}}$ if the direction of minimising is that of variate 2; and $-\frac{\hat{r}_{23}}{\hat{r}_{13}}$ if the direction of minimising is that of variate 3. Similarly, the regression coefficients of variate 3 in the explanation of variate 1 are respectively $-\frac{\hat{r}_{13}}{\hat{r}_{11}}$, $-\frac{\hat{r}_{23}}{\hat{r}_{12}}$ or $-\frac{\hat{r}_{33}}{\hat{r}_{13}}$ in the three cases mentioned. Each group of three coefficients supplies the material for the construction of one bunch map. Taking the first group, the number $-\frac{\hat{r}_{12}}{\hat{r}_{11}}$ indicates the slope of the beam to be indicated with $\odot$ 1, $-\frac{\hat{r}_{22}}{\hat{r}_{12}}$ that of beam $\odot$ 2 and $-\frac{\hat{r}_{23}}{\hat{r}_{13}}$ that of beam $\cdot$ 3; the symbol $\odot$ being used for the "leading beams"—*i.e.*, for cases where minimising has taken place either in the direction of the variate to be "explained" or in that of the "explanatory" variate.

By the same technique, all "three-sets"—*i.e.*, groups of three variates—can be analysed.

Here it must be emphasised that the symbols used above, $\hat{r}_{13}$, etc., are, stricty speaking, incomplete and should be written $\hat{r}_{13(123)}$, indicating that it is the minor within Δ_{123} corresponding with the element r_{13}. Otherwise there would be no possibility of distinguishing it from, *e.g.*, $\hat{r}_{13(134)}$.

In order to make bunch maps for the next higher stage—*i.e.*, "four-sets"—the minors $\hat{r}_{12(1234)}$, etc., have to be calculated. Those relating to elements on the diagonal—viz., $\hat{r}_{11(1234)}$, $\hat{r}_{22(1234)}$, etc.—are, owing to their definition, simply equal to Δ_{234}, Δ_{134}, etc. Those not relating to an element on the diagonal have to be calculated with the formula:

$$\hat{r}_{ij(\alpha,\beta\ldots\gamma)} = -\Sigma_{k=\alpha,\beta\ldots)j(\ldots\gamma}\hat{r}_{ik(\alpha,\beta\ldots)j(\ldots\gamma)}r_{kj}$$
$$(i \neq j)\,.$$

The meaning of the suffixes is the following:

α, $\beta \ldots \gamma$ indicate the set studied.
i indicates the row, and
j the column of the element under consideration.
k is a "current suffix"—*i.e.*, it assumes all values indicated under the sum-sign.
$)j($ means that k must not assume the value j, whereas the $\hat{r}_{ik}$'s have to be taken as corresponding to the set without j.

As an example, $\hat{r}_{12(1234)}$ may be taken. Here $i = 1$, $j = 2$, α, $\beta \ldots \gamma = 1, 2, 3, 4$, and k evidently has to assume the values 1, 3, 4. We find:

$$\hat{r}_{12(1234)} = -\hat{r}_{11(134)}r_{12} - \hat{r}_{13(134)}r_{32} - \hat{r}_{14(134)}r_{42}\,.$$

No further new elements have to be introduced in the calculations.

4. *Calculation of Standard Error of Regression Coefficients* (Classical method).

Under the hypotheses formulated in Chapter II, the standard error $\sigma_{b_{1k}}$ for any regression coefficient b_{1k} must be calculated

by means of the formula:

$$\sigma^2_{b_{1k}} = \frac{N}{N-n}\sigma^2_{1.23\ldots n}\frac{M_{11.kk}}{M_{11}} \qquad (23)$$

in which the symbols have the following meaning:

(*a*) $\sqrt{\frac{N}{N-n}}\,\sigma_{1.23\ldots n}$ is the standard deviation of the residuals $x_{1.23\ldots n}$, corrected for the number of coefficients in the regression formula (11) and therefore equal to:

$$\sqrt{\frac{1}{N-n}\Sigma x^2_{1.23\ldots n}} = \sqrt{\frac{1}{N-n}\Sigma(x'_1 - x_1)^2} \qquad (24).$$

The easiest way of calculating this standard deviation is given by:

$$\sigma^2_{1.23\ldots n} = \sigma^2_1(1 - R^2_{1.23\ldots n})$$

where σ_1 is the standard deviation of x_1.

(*b*) M_{11} is the determinant:

$$\begin{vmatrix} \Sigma x_2^2 & \Sigma x_2x_3 & \Sigma x_2x_4 & \ldots & \Sigma x_2x_n \\ \Sigma x_2x_3 & \Sigma x_3^2 & & & \vdots \\ \vdots & & & & \vdots \\ \Sigma x_2x_n & \ldots & \ldots & \ldots & \Sigma x_n^2 \end{vmatrix}$$

i.e., the determinant formed by all moments of the " explanatory " series. In those cases in which calculations for bunch maps are already available, it can very easily be obtained from:

$$M_{11} = N^{n-1}\sigma_2^2\sigma_3^2 \ldots\ldots \sigma_n^2 \begin{vmatrix} 1 & r_{23} & r_{24} & \ldots\ldots & r_{2n} \\ r_{23} & 1 & \ldots\ldots & & \vdots \\ \vdots & \vdots & & & \vdots \\ r_{2n} & \ldots & \ldots & \ldots & 1 \end{vmatrix}$$

$$= N^{n-1}\sigma_2^2\sigma_3^2 \ldots \sigma_n^2\Delta_{23\ldots n}$$

where $\sigma_2 \ldots \sigma_n$ are the standard deviations of $x_2 \ldots x_n$, N is the number of observations, and Δ is the symbol introduced in Appendix A, 3.

(*c*) M_{11kk} is the determinant remaining if the kth column and the kth row in M_{11} are dropped; *e.g.*:

$$M_{1133} = \begin{vmatrix} \Sigma x_2^2 & \Sigma x_2 x_4 \ldots & \Sigma x_2 x_n \\ \Sigma x_2 x_4 & \Sigma x_4^2 & \Sigma x_4 x_n \\ \vdots & & \\ \Sigma x_2 x_n & \ldots\ldots\ldots & \Sigma x_n^2 \end{vmatrix}.$$

This, again, can be calculated by:

$$M_{1133} = N^{n-2} \sigma_2^2 \sigma_4^2 \ldots \sigma_n^2 \begin{vmatrix} 1 & r_{24} & \ldots\ldots & r_{2n} \\ r_{24} & 1 & \ldots\ldots & \vdots \\ \vdots & & & \vdots \\ r_{2n} & \ldots\ldots & & 1 \end{vmatrix}$$

$$= N^{n-2} \sigma_2^2 \sigma_4^2 \ldots \sigma_n^2 \Delta_{24\ldots n}$$

5. *Calculation of Limits to the Error of Weighting in Regression Coefficients.*

The formulæ devised by Koopmans for computing limits to the error of weighting apply only when the signs of corresponding coefficients in all of the elementary regression equations are the same (after solving for the explained variate x_1). In terms of the bunch map, this means that in each of the maps referring to the complete set of variates, all beams should lie in one of the four right-angles formed by the axes.

If a possible regression equation is denoted by

$$x_1^* = b_{12} x_2 + b_{13} x_3 + b_{14} x_4$$

for the case of three explaining variates, where the variates are expressed in normalised units, a first set of limits to each of the b's is formed by the two ultimate beams in the corresponding bunch map involving the complete set of variates. Thus, b_{12} must lie between the largest and smallest of the four coefficients

$$b_{12}^{(1)} = -\frac{\hat{r}_{12}}{\hat{r}_{11}}, \quad b_{12}^{(2)} = -\frac{\hat{r}_{22}}{\hat{r}_{21}}, \quad b_{12}^{(3)} = -\frac{\hat{r}_{32}}{\hat{r}_{31}}, \quad b_{12}^{(4)} = -\frac{\hat{r}_{42}}{\hat{r}_{41}} \tag{25}$$

Additional limits to the error of weighting in b_{12}, etc., can often be imposed from the following considerations. To the first elementary regression corresponds the assumption that disturbances occur only in the first variate. In this case, the standard deviation of these disturbances is estimated, according to (24), as

$$\sqrt{\frac{N}{N-n}}\,\sigma_{1.234} = \sqrt{\frac{N}{N-n}\,(1 - R_{1.234}^2)} \tag{26}$$

the standard deviation σ of the variate x_1 itself being equal to unity. Similarly, if only the second variate is subject to disturbances, their standard deviation is estimated by

$$\sqrt{\frac{N}{N-n}}\,\sigma_{2.134} = \sqrt{\frac{N}{N-n}\,(1 - R_{2.134}^2)} \tag{27}$$

and so on. The expressions containing the four multiple correlation coefficients R are easily computed from

$$1 - R_{1.234}^2 = \frac{\Delta_{1234}}{\Delta_{234}}, \quad 1 - R_{2.134}^2 = \frac{\Delta_{1234}}{\Delta_{134}}, \quad \text{etc.}$$

If any one of the standard deviations (26), (27), etc., of disturbances seems too large to be accepted, *a priori* considerations may lead one to adopt limits, say,

$$\rho_1', \quad \rho_2', \quad \rho_3', \quad \rho_4', \tag{28}$$

that should not be exceeded by the estimated standard deviations of the disturbances in the corresponding variate.

Somewhat complicated calculations or diagrams are required to find exactly which values of b_{12}, etc., are still admitted by these limitations. For practical purposes, the uncertainty concerning the error of weighting may, however, already be considerably reduced, in a number of cases, by the following thumb rule,[1] which excludes a considerable part of those values of b_{12}, etc., that are incompatible with the limitations (28), but not always all of them. This rule imposes two additional limits to each of the regression coefficients b_{12} ..., the interval between these limits partly overlapping, or falling entirely within, the interval defined by the figures (25). Only values of b_{12} ... common to both intervals, then, have to be admitted.

The additional limits for b_{12} according to this rule are the largest and smallest of the quantities

$$j_{12}^{(1)} = -\frac{\hat{n}_{12}}{\hat{n}_{11}}, \quad j_{12}^{(2)} = -\frac{\hat{n}_{22}}{\hat{n}_{21}}, \quad j_{12}^{(3)} = -\frac{\hat{n}_{32}}{\hat{n}_{31}}, \quad j_{12}^{(4)} = -\frac{\hat{n}_{42}}{\hat{n}_{41}} \qquad (29).$$

Here, $\hat{n}_{12}$, etc., are the minors of the elements n_{12}, etc., in a determinant, obtained from the determinant Δ of the correlation coefficients by replacing the diagonal elements r_{11}, r_{22}, etc. (which equal unity), by

$$n_{11} = r_{11} - \rho_1, \quad n_{22} = r_{22} - \rho_2, \quad \text{etc.},$$

where

$$\rho_1 = \frac{N - n}{N} \rho_1', \quad \rho_2 = \frac{N - n}{N} \rho_2', \quad \text{etc.}$$

Mutatis mutandis, similar limits for b_{13} and b_{14} are found, which may then be converted from normalised units to the units in which the variates were originally expressed.

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[1] This thumb rule, which has not yet been published, has been communicated to the author by Dr. KOOPMANS.

If greater precision is required, it may be useful to determine the full consequences of the limitations (28) on the coefficients b_{12}.... These can best be formulated in geometrical terms. A set of coefficients, b_{12}, b_{13}, b_{14}, may be represented by a point C in three-dimensional space having these coefficients as its rectangular co-ordinates. Then, instead of the simple limits (25) for b_{12}, and similar limits for b_{13} and b_{14}, the more restrictive proposition holds that the point C is confined to the tetrahedron formed by the four points $(b_{12}^{(1)}, b_{13}^{(1)}, b_{14}^{(1)})$, $(b_{12}^{(2)}, b_{13}^{(2)}, b_{14}^{(2)})$, etc. Further, as a consequence of the limitations (28), the point C must in addition be confined to the tetrahedron formed by the four points $(j_{12}^{(1)}, j_{13}^{(1)}, j_{14}^{(1)})$, $(j_{12}^{(2)}, j_{13}^{(2)}, j_{14}^{(2)})$, etc. Thus, the point C is confined to the common part of the two tetrahedra. In the case of four (or less) variates, the extension of this common part is most easily read from a geometrical figure, if necessary, using orthogonal projections.

In devising the maximum amounts ρ_1', ρ_2', etc., admitted for the standard deviations of the disturbances in each variate, it should be borne in mind that the disturbances in the dependent variate x_1 are also due to the omission of explaining variates of minor importance. The safest procedure, therefore, is to choose ρ_1' equal to, or at a round figure somewhat exceeding, the amount represented by (26). In this case, the limit ρ_1' does not contribute to a restriction of the possibilities left to the regression coefficients b_{12}, etc., by the remaining limitations, but only serves as an aid in computing the effect of the choice of ρ_2', ρ_3', ρ_4'.

For the computation of the errors of sampling corresponding to any of the regressions admitted by the limits to the error of weighting, reference may be made to the original publication.