CHAPTER ONE

INTRODUCTION: TYPES OF MOVEMENTS

ELEMENTARY MOVEMENTS

THE objective of economic dynamics is to describe and explain the fluctuations in economic magnitudes. In this first part of our study we shall limit ourselves to description. This description can be assisted greatly by the use of diagrams in which the movements, or changes, of the various economic magnitudes are represented by certain geometric figures. For a good understanding of the various types of movements it is necessary to have a minimum general knowledge of the geometric properties of these diagrams quite apart from the economic phenomena which they represent. By way of introduction we shall start out with these geometric characteristics.

Before we can study the relationship between the movements of one magnitude and the movements of one or more other magnitudes, it will be necessary to give a description of the movements of individual economic magnitudes. It is in the nature of the object of our study that these magnitudes vary; they are "variable magnitudes," or, briefly, "variables." The movements of these variables are often complicated. There is considerable advantage in separating the movements into components of a simpler character, which we may call "elementary movements." These we shall study first.

a) Systematic and random movements

A distinction has to be made between systematic and random movements. In the case of systematic movements the numbers representing successive magnitudes of one variable follow each other according to a design. Any such design is, however, absent from the magnitudes of successive observations of random

variables. Even in the very simple case in which the variable measured can have only two values (0 and 1) a distinction between systematic and random movements can be made. The following two series, for instance, represent systematic movements:

0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, etc.
0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, etc.

There is obviously an infinite number of systematic movements; more and more complicated systems of succession of the different values can be devised. As has been mentioned, in a random movement all design is absent in the way in which the successive values are arranged. Such a random movement will occur, for instance, when one tosses a coin and marks 1 for heads and 0 for tails. The results of an infinite repetition of this game will show random movements. A variable showing such movements is sometimes called a "random variable." A "normally distributed random variable" satisfies two conditions, namely, (1) there is no design in the succession of its values and (2) the various values satisfy the frequency distribution of the Gaussian law. The distribution under the Gaussian law, also called the law of normal distribution or of normal errors, may be expressed in a mathematical formula which is represented diagrammatically in the shape of a bell. In a normal distribution all possible values will occur—not only, as in the example just quoted, the values 0 and 1. In a normal distribution there will be a relatively large number of small deviations from the average and relatively few large deviations from the average. A normally distributed random variable will be obtained in more complicated games of chance in which the number of possible values is large, or, strictly speaking, in which the number of possible values is infinite. If, for instance, one were to toss not one coin but a hundred and were each time to count as the result of one game the number of heads that turned up, the result would approximate very closely the movement of a normally distributed random variable. In general, a variable will be a normally distributed random variable if its values are the sum of the values of a large number of random variables that are independent of each other. In nontechnical

language, the variable will appear as a normally distributed random variable if its fluctuations are due to a great number of independent small causes. The larger the number of causes, the more will the series of successive values approximate the movement of a normally distributed random variable.

b) Monotonic and periodic movements

There exist a great many different types of systematic movements. Within the scope of this book there is no need for a detailed mathematical treatment of them. A variety of functions and curves is discussed by textbooks in analysis and analytical geometry. One distinction, however, is of great importance, namely, that between monotonic and periodic movements. A monotonic movement is one which never reverses its direction. It may be either monotonically increasing or monotonically decreasing. In the first case, each successive value is greater than, or in the limiting case equal to, the previous value; in the second case, each successive value is smaller than, or in a limiting case equal to, the preceding value. A periodic movement, on the other hand, repeats itself exactly after a certain lapse of time. This lapse of time is called its "period." We also include in the category of periodic movements nonmonotonic movements which repeat themselves after a certain period in an enlarged or reduced form, in a certain proportion which we may indicate by a. Such movements will be called damped and antidamped movements, respectively. The following would be an example of a purely periodic movement:

4, 8, 12, 4, 8, 12, 4, 8, 12, etc.

An example of a damped periodic movement would be the following:

4, 8, 12, 2, 4, 6, 1, 2, 3, etc.

In both cases the period is three units of time; in the second case the movement repeats itself on a reduced scale with $a = \frac{1}{2}$.

We shall have to analyze some special cases of these two types of movements in somewhat greater detail.

The simplest example of a monotonic movement is a straight

line. This line may either rise or fall, with the horizontal line as the intermediate case.

For a straight line the difference in level of two successive time units (years, quarters, or months) is always the same. This difference is called the "slope," or the "rate of increase." In accordance with mathematical usage we shall use the term "increase" also for declining lines, that is, for negative increases. A straight line has a constant rate of increase.

A second important example of monotonic movement is the exponential curve. A mathematical property of an exponential

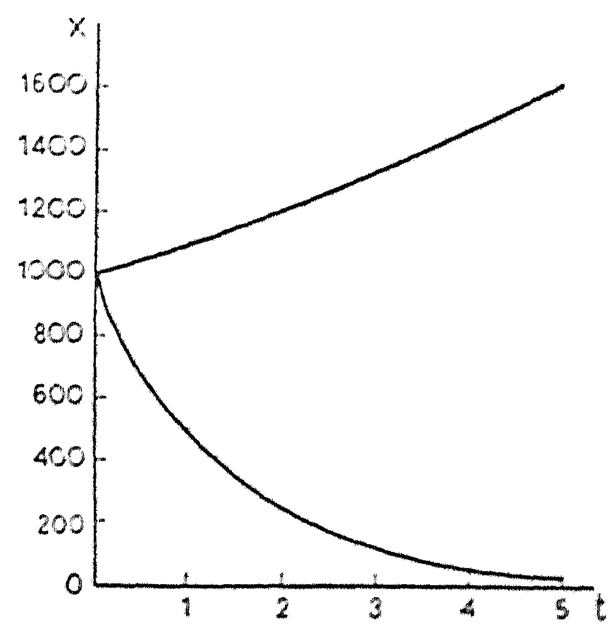

FIG. 1.—Exponential curves

curve is that its height at a series of successive equidistant points is indicated by a geometric series. That is to say that, in order to obtain a series of successive values, each at the same time distance from the preceding one, the height of the curve at each step has to be multiplied by a constant to obtain the next following step. Figure 1 shows two exponential curves. The first exponential curve is increasing. It is represented by the following figures:

Time	t =	0	1	2	3	4	5	...
Height	x =	1,000	1,100	1,210	1,331	1,464.1	1,610.51	...

The following series represents a declining exponential curve:

Time	t =	0	1	2	3	4	5	...
Height	x =	1,000	500	250	125	62.5	31.25	...

It will be noted that the slope, or rate of increase, of the exponential curve is not constant. It is proportional to the height attained by the curve. This may also be expressed by saying that the relative rate of increase is constant.

There are a number of other important monotonic movements. They cannot be treated without mathematical formulas. We mention here the parabolas of different degrees, represented by formulas of the form $x = at^n$, where n is the degree. These

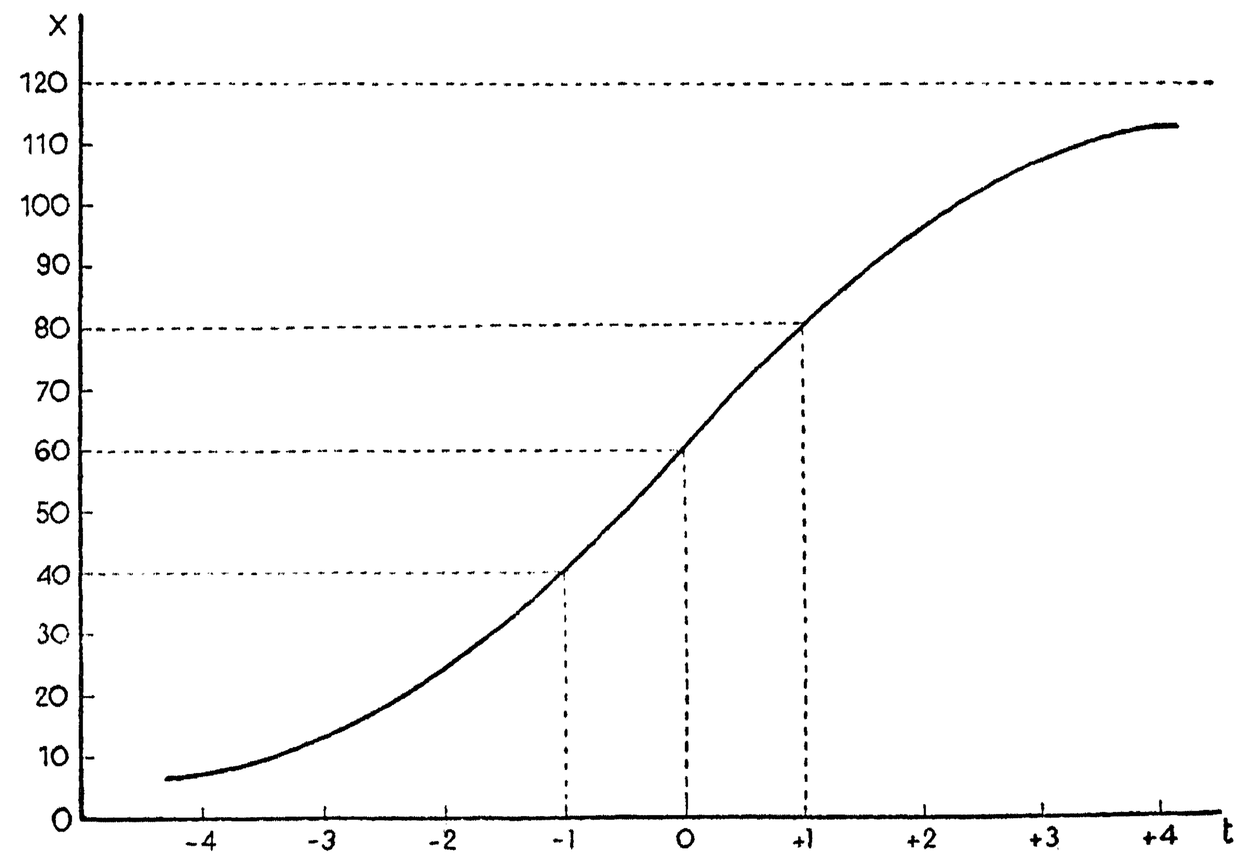

FIG. 2.—Logistic or growth curve ($a = 1$, $b = 2$, $k = 120$)

curves are monotonic for all positive values of t. Another monotonic curve of great importance is the logistic, or growth, curve: $x = k/(a + b^{-t})$. An example of the logistic curve is shown in Figure 2. A logistic curve will be described, for example, by the movement of a population, whose rate of growth per unit of time is proportional to (a) the size of the population already attained and (b) the size of the population for which, in addition to the population already present, means of subsistence are still available. The total means of subsistence are assumed to be such that a population of constant size can be maintained.

We count among the monotonic movements also certain movements that cannot be defined by one mathematical curve

for the entire period of time but that are defined, for instance, first, by a rising straight line and, after a certain period, by a horizontal straight line.

A very simple type of periodic movement is the sine curve shown in Figure 3. As this diagram shows, the sine curve may be described as the distance from the horizontal axis of a point which moves at constant speed around the circumference of a circle. This distance is measured vertically while time is measured horizontally. The distance *AB* is called the period; the distances *CD* and *EF* are also equal to the period. Half the distance *AG* is called the "amplitude."

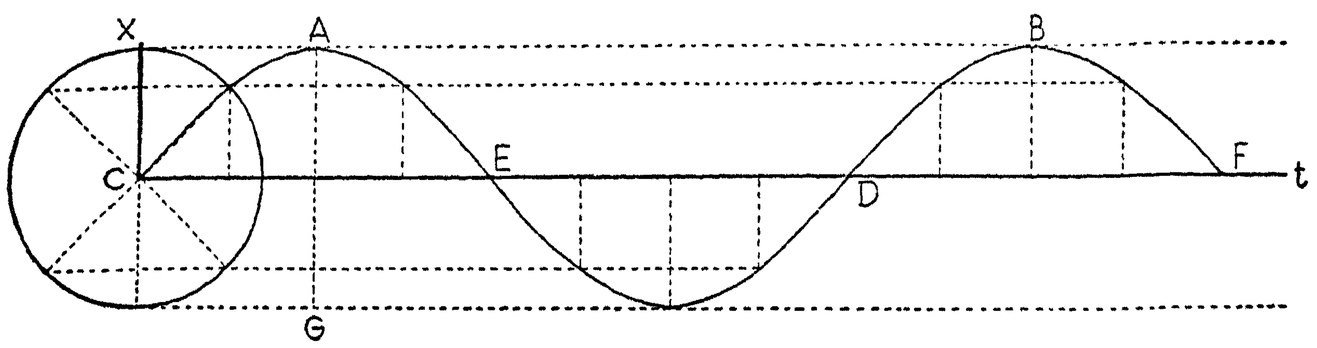

FIG. 3.—Sine curve (*C*, *A*, *E*, *D*, *B*, *F* . . .) indicating the distance from the horizontal axis of a point which moves around a circle with constant speed.

We shall often meet movements that are not exactly but approximately periodic in character. Such movements will show fluctuations repeating themselves approximately after roughly the same period. We shall sometimes refer to such movements as "fluctuating movements," or "fluctuations," or "waves." We may also call them "quasi-periodic movements." It should be noted here that random movements are also quasi-periodic. It may be proved that the quasi-period of these movements is three units of time, that is to say, that, on the average, one of every three values of such a series will be a peak and one a trough. The unit of time of such a series must be taken to be the distance between two successive independent observations.

c) Damped, undamped, and antidamped movements

As mentioned above, we include among the periodic movements those nonmonotonic movements that repeat themselves on an increasing or decreasing scale. We call periodic move-

ments "damped" if each successive fluctuation is on a smaller scale than the preceding one, that is to say, if the factor a mentioned earlier is smaller than 1. The rate of damping is indicated by the fraction $1/a$. Movements which show antidamping are sometimes called "explosive movements." The purely periodic movement is undamped.

The same distinction can be made for exponential movements. If in an exponential movement the ratio between two successive values is smaller than 1, the movement is called damped; if the ratio is greater than 1, the movement is called antidamped, or explosive.

THE COMPONENTS OF COMPOSITE MOVEMENTS

The movements shown by most economic phenomena are much more complicated than any one of the simple movements we have discussed. First, any actual series will at most be an approximation of any one of the elementary movements considered above. Purely periodic movements, for instance, are rarely met in reality; in successive periods the shape of the movements repeats itself only approximately; often the period which one can distinguish is by no means constant. Second, actual series often represent combinations of elementary movements. We are using the mathematically rather vague expression "combinations" to cover not only the sum of elementary movements but also their product and in some instances even more complicated combinations. Whatever the mathematical nature of these combinations, all have in common the attribute that the actual movement can be constructed out of elementary movements.

Statisticians have adopted very generally the practice of separating the movements of economic variables into various elementary components before studying these components in further detail. In many respects this is a useful procedure; but a number of objections, to which we will refer below, can be made against it.

Normally, a series is decomposed into four components, as follows: (*a*) the trend component, which indicates the general tendency of the movement, and which usually is represented by

a monotonic movement; (*b*) the cyclical component, consisting of fluctuations with a period of between three and eleven years; (*c*) the seasonal component, consisting of fluctuations with a period of one year, attributable to fluctuations of the natural and conventional seasons during the year; and (*d*) the random component, which covers both nonrecurrent changes, such as sudden changes in the level due to a "trend break," and fluctuations due to a large number of random causes, these latter fluctuations being usually of very short duration, e.g., those with a quasi-period of three months.

It is not advisable to apply this procedure mechanically, since there are many exceptions to the rule that these four components can be found in every economic series. We shall first mention a number of these exceptions and then treat the whole subject in a systematic manner.

Some fluctuations have a period in excess of eleven years. Among these should be counted the so-called "long cycles" and the fluctuations in certain individual markets. Depending on the subject and the period under consideration, such movements are classed under (*a*) or under (*b*); hence, a movement of type (*a*) cannot always be represented by a monotonic movement. There are, further, certain fluctuations which by their nature are equivalent to seasonal fluctuations but which have a shorter period, for instance, three months, a month, a week, or a day. On the other hand, random movements sometimes show quasi-periods that are considerably longer than three months. The fluctuations in crops, which must certainly to a large extent be considered as random, have a time interval of a year between two successive independent observations and hence may produce quasi-periods of three years. For these reasons it is often difficult to isolate components (*c*) and (*d*), or even (*b*) and (*d*), at all accurately.

On closer scrutiny, a variety of objectives may be discerned in the standard procedure. The first and clearest objective is to isolate movements with different periods. As a rule, all fluctuations with a period in excess of eleven years are comprised in the trend component. The monotonic movements of the trend component may be considered as parts of very long

cycles or even of movements with an infinitely long period. All movements with a period of between one and eleven years are considered to be part of the cyclical component; those with a period of one year represent the seasonal component; and those with a period of less than one year, the random component. As mentioned above, however, it is sometimes difficult to classify the actual movements in accordance with this scheme.

A second objective in the standard procedure is the separation of systematic and random movements, discussed earlier.

The third objective would appear to be to obtain a classification according to the causes of the movements. The separation of seasonal movements and trend movements would reflect this objective. The trend movement, for instance, may often be ascribed to the very slow movements of such data as the size of the population, technical knowledge, etc.

A fully satisfactory decomposition of an economic variable into its various elements can be achieved only on the basis of a complete theory of economic movements. At this stage in our analysis, therefore, we can consider this decomposition only as a provisional tool. After having dealt with economic theory in the second part of this study, we shall in certain simple cases be able to give very definite directions with respect to the separation of the various elements (see chap. xvi). It will be shown that the systematic and random components are often combined in a very intricate fashion which renders it logically impossible to separate them in any simple way.

Here, as in the case of the separation of movements according to causes, a clear distinction between direct and indirect causes will have to be made. An example may make this clear. If fluctuations in the quantity demanded of a particular commodity are due to (*a*) changes in income and (*b*) changes in price, while the changes in price are due to (*c*) changes in the price of raw materials, then we call (*c*) an indirect cause of changes in the quantity demanded. On the other hand, (*a*) and (*b*) are considered direct causes.

It is often useful to isolate the components on the basis of the various direct causes; in such cases, however, indirect causes should be kept clearly separated from direct causes. The sep-

aration according to causes may not at all agree with the separation according to periods: one direct cause may be the origin of movements of different periods; one and the same period can occur in movements of different direct causes. In some instances the separation by causes and by periods may coincide. This will often be the case for seasonal movements that have both a special cause and a special period which does not coincide with the periods of movements due to other causes.

THE RELATIONSHIP BETWEEN THE MOVEMENTS OF TWO SERIES

Having discussed the various types of movements and the combination of these types of movements into one series, we shall now deal with the relationship that may exist between the movements of two series. These movements may be simple or composite.

We refer purposely to the relationship that *may* exist between the movements of two series. There need not be any relationship between them. The movements may be entirely independent. In reality there would usually be a certain degree of relationship. The examples treated below are the ideal case of a perfect relationship which can only be approximated in reality. Two series of figures show an exact relationship (or, in mathematical language, a functional relationship) if for any given value of the one series there is always a precisely defined value of the other series. Often no such precise relationship is present, but instead any given value of the one series is always found to be accompanied by approximately the same value of the other series. In statistics such a relationship is called a "stochastic relationship." In the following pages we shall discuss some of the most common functional relationships.

To indicate these relationships, we may make use of two types of diagrams: one shows the movement of the two series in time, preferably using the same time scale for both series; the other shows both series on a "scatter diagram." Such a diagram consists of a number of points in a rectangular system of co-ordinates. Each point has two co-ordinates, of which one, the x-co-ordinate, represents a value of the one series; and the

other, the y-co-ordinate, the corresponding value of the other series. In such a diagram there are, therefore, as many points as there are values in each series. The time sequence of points cannot be traced in the diagram unless each point is labeled according to the time period to which it refers.

The simplest relationship between two series is that of equality. In that case the time diagrams of the two series cover each other in every point, if one adopts the same vertical scale for both series. In the scatter diagram, all points will lie on a straight line running through the origin at an angle of 45 degrees to both axes.

The next simplest relationship between two series is that of proportionality. In this case the time diagrams do not coincide, if the same vertical scale is used for both series. They can, however, be made to coincide by a special choice of the units for the vertical scale. If all figures of series Y are five times as large as those of series X, then the choice of the unit for Y at one-fifth of the unit for X will make the two series coincide. The scatter diagram still shows points on a straight line through the origin, but the slope of this line is now different. In this particular case the slope will be equal to a ratio of 5:1 (See Fig. 4).

One stage more complicated is the general linear relationship. In this relationship there is proportionality between the changes of the series but not between the absolute values of them; as a consequence, the ratio of the changes in the series is different from the ratio of their averages. The following two series give an example of this relationship:

X : 10, 12, 11, 14, 12, 13 (average 12), and
Y : 20, 22, 21, 24, 22, 23 (average 22).

Here the changes of the two series are equal, but the ratio of the averages is $\frac{1}{2}\frac{2}{2}$, not $= 1$.

A second example is shown in Figure 5. The figure used in this diagram are as follows:

X : 10, 12, 11, 14, 12, 13 (average 12), and
Y : 20, 26, 23, 32, 26, 29 (average 26),

in which each change in Y is three times as large as the corresponding change in X, whereas the ratio of the averages of the

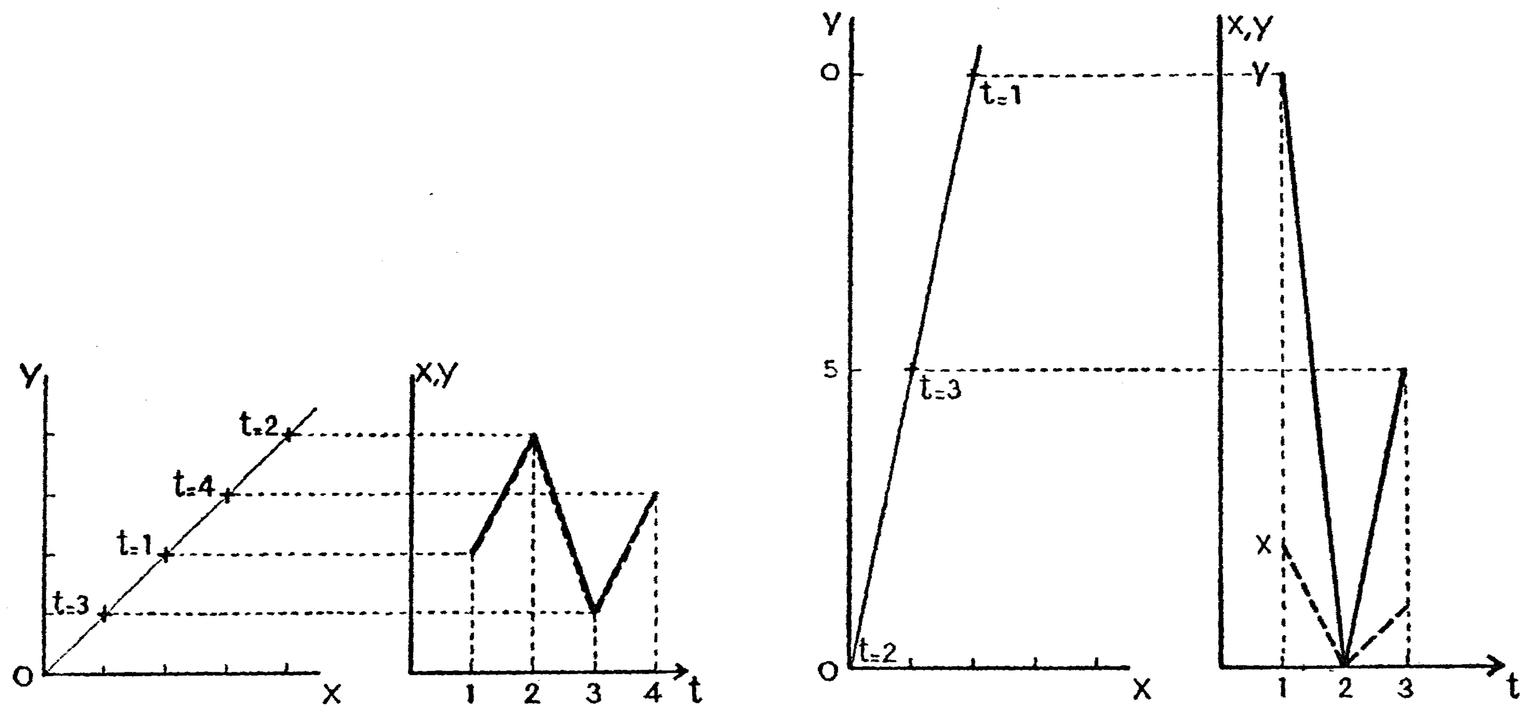

FIG. 4.—Equality (1st and 2d diagrams) and proportionality (3d and 4th diagrams) between X and Y. The first and third diagrams are scatter diagrams; the other two are time series.

two series is $26 \div 12$, obviously different from 3. If one wanted to make the graphs of these two time series coincide, it would not be sufficient to select different scales for them; it would also be necessary to draw the lines at different levels. In our first example it would be necessary to set the zero point for the Y series at the point -10 on the scale for X; in the second example it would be necessary to make the scale for Y one-third as great as the scale for X and, in addition, to set the zero point of the Y scale at the point $3\frac{1}{3}$ on the X scale.

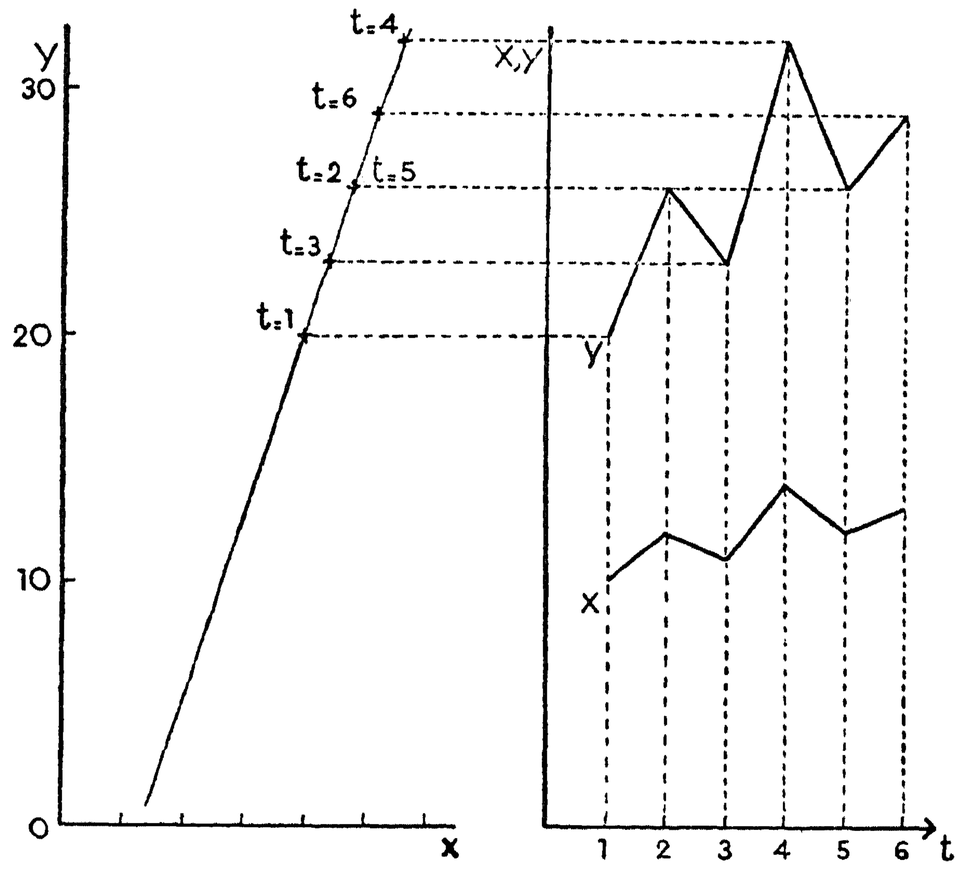

FIG. 5.—General linear relationship between X and Y

If two variables are related by a general linear relationship, it is sometimes said that their movement is parallel and also that there is a linear correlation between them. To distinguish this case from the one in which the relationship is only approximate in the stochastic sense, it is said that the correlation is perfect.

It will readily be realized that, depending on the absolute magnitudes of the fluctuations of the series and on their averages, one series may have larger absolute fluctuations while the other has larger relative fluctuations.

Equality, proportionality, and a general linear relationship

may occur with a negative sign. In such instances the movements of the two series are in opposite directions.

The three forms of functional relationship discussed can all be represented by a straight line in a scatter diagram. We have discussed them in some detail because it will be found that they occur very frequently. There is, further, an infinite number of curvilinear functions that can be represented by other than straight lines in a scatter diagram. Curvilinear relationships can have many different shapes, depending on the nature of the relationship between the two variables. One example is given in Figure 6, which shows the relationship between the price, p, of a

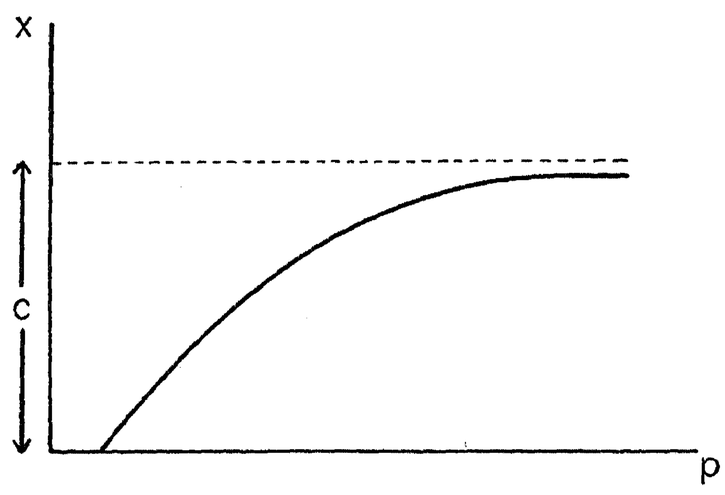

FIG. 6.—Example of a curvilinear relationship between two variables as represented by the supply curve (p = price, x = quantity supplied, and c = productive capacity).

commodity and the quantity supplied, x, on the assumption that there is a certain given productive capacity, c, and that no stocks are available. Starting from a relatively low price, every increase in price will initially produce a considerable increase in the quantity supplied. As the total capacity is approached, however, every further increase in the price by a constant amount will produce a smaller and smaller increase in the quantity supplied. The quantity supplied cannot exceed productive capacity and will approach this magnitude asymptotically; the horizontal line drawn at the distance c from the origin is called the "asymptote." In a time graph a curvilinear relationship of this nature will be recognized by the fact that the two series, though increasing and decreasing at the same time, show a pronounced difference in shape; the peaks in the p series will be much more pronounced and sharper than those

in the x series. It will not be possible to make these series coincide by an appropriate adjustment of the scales. This phenomenon will, in particular, be encountered with respect to the prices and production of mineral raw materials in some periods of boom, when production bottlenecks occur.

We have so far discussed only the case in which the corresponding values of the two series referred to the same moment of time for the two economic magnitudes. If, however, there is a causal relationship between the two variables and if the process of causation which links Y to X takes a certain time, T, there would be good reason to compare the value of X at a given moment with the value of Y at a moment T later than the corresponding value of X; hence one would compare the value of X at time t with the value of Y at time $t + T$. If the two variables are plotted against time in the normal manner, the fluctuations of Y will be shown to be lagging behind those of X; by shifting the diagram of Y to the left over a distance T, the two series can be made to coincide again, provided that appropriate scales have been chosen for X and Y. Similarly, in the scatter diagram the co-ordinates should be chosen in such a way that the value of X at time t is combined with the value of Y at time $t + T$, in order to obtain points which lie on a straight line. In such instances we say that Y shows a lag with respect to X, or that the relationship between Y and X is a lagged relationship (Fig. 7).

In the case of two series with irregular movements, the presence of a lag can readily be observed empirically. If, however, the two series have monotonic movements, the lag is almost impossible to establish. If the movement is purely periodic, it is not possible to establish from the data the direction of the lag, that is, whether Y lags behind X or X lags behind Y, unless one has separate information concerning the order of magnitude of the lag.

Besides lagged relationships, many other relationships are possible between two variables in which time plays a role. A very common relationship in economic analysis is that of cumulation. Series Y represents the cumulation of series X, if the nth value for Y is equal to the sum of all values from 1 to n

for X. Hence, the first value for Y (Y_1) is equal to the first value for X (X_1); the second value, Y_2, equals $X_1 + X_2$; Y_3 equals $X_1 + X_2 + X_3$; Y_4 equals $X_1 + X_2 + X_3 + X_4$, etc. This relationship will occur, for instance, if Y represents the stock of a certain commodity, while X represents the excess of production over consumption for each time period. If X is gold production, then Y represents the total gold stock (assuming

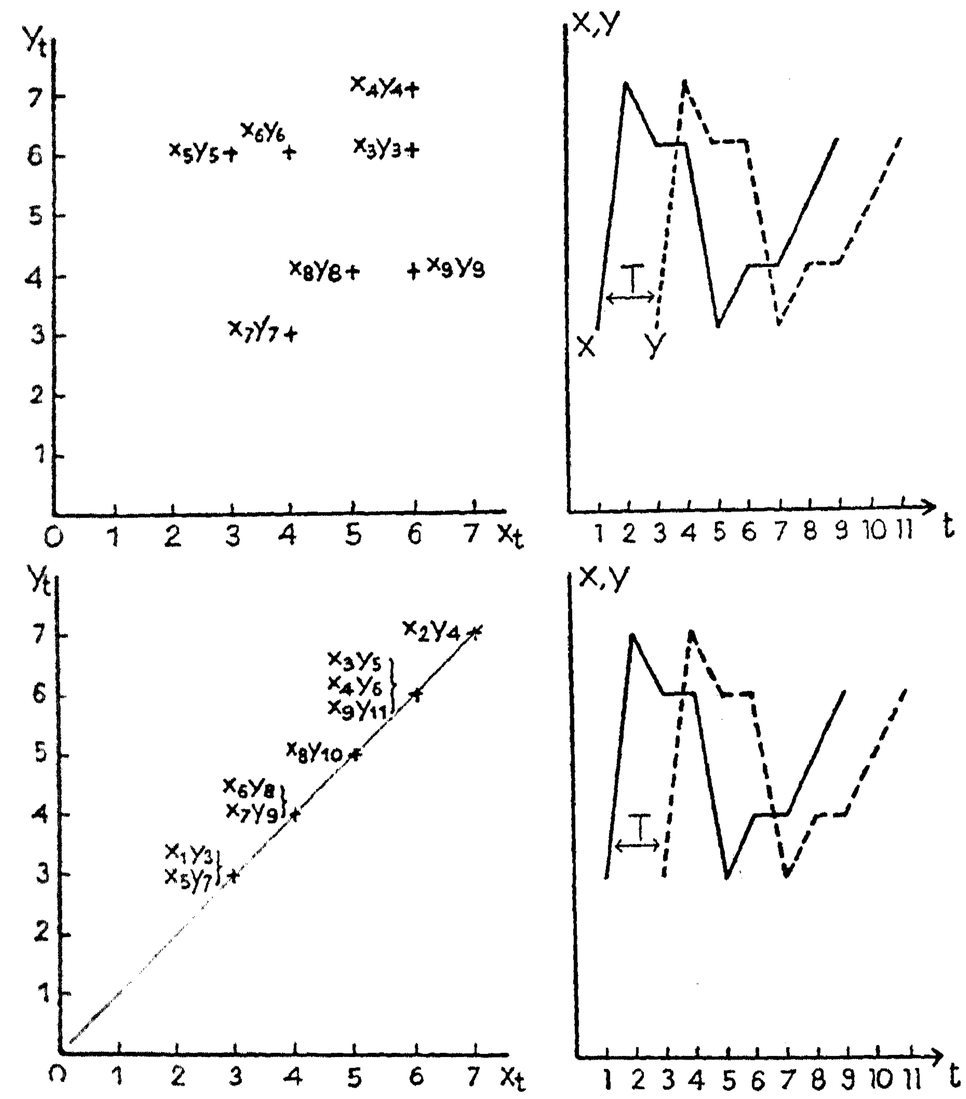

FIG. 7.—Example of a lag between two series X and Y. The top scatter diagram shows simultaneous value of X and Y; the bottom scatter diagram combines X_1 and Y_3, X_2 and Y_4, etc.

that the consumption of gold may be neglected); if X is the net increase of the number of houses during the year, then Y is the total stock of houses.

If X fluctuates, Y will fluctuate too. Normally, the fluctuations of Y will contain a trend component, unless the average value of X equals zero. If the series X is a sine curve with zero average, the Y curve will also be represented by a sine curve.

This may be proved mathematically and can also be seen from the following example computed on the basis of approximate values (compare Fig. 8):

t =	0	1	2	3	4	5	6	7	8	
X =	0,	0.7,	1,	0.7,	0,	−0.7,	−1,	−0.7,	0,	etc.
Y =	0,	0.7,	1.7,	2.4,	2.4,	1.7,	0.7,	0,	0,	etc.

Although Y also shows a sine curve, it is lagged in comparison to X; the peaks of Y are approximately one-fourth of the period of fluctuation behind those of X. Any numerical experiment will show that the cumulation of an irregular series will not reproduce a series of the same shape.

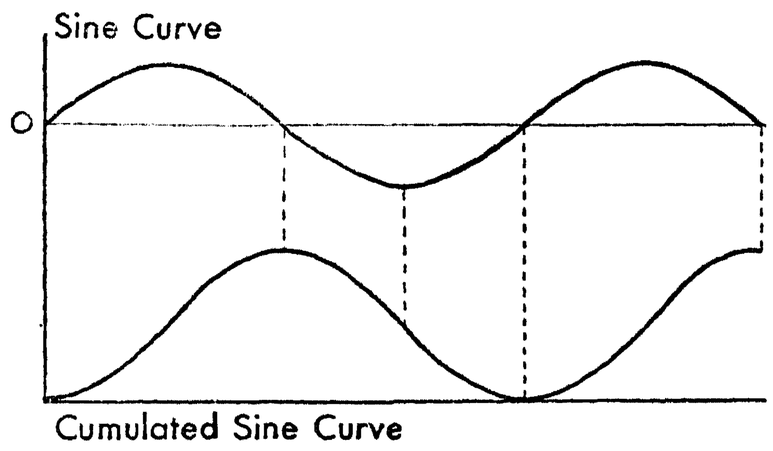

FIG. 8.—Sine curve and cumulation of sine curve

A purely empirical inspection of Figure 8 might lead one to believe that the relationship between X and Y was a lagged one, whereas in reality it is a cumulative relationship. Only theoretical knowledge of the real relationship between variables can prevent erroneous inductions of the type just mentioned.

THE RELATIONSHIP AMONG MOVEMENTS OF THREE OR MORE SERIES

In the preceding section we discussed the relationship between the movements of two series. The changes in two series will be exactly parallel when the changes in the one are caused by changes in the other and by nothing else. In the case of economic phenomena, however, fluctuations in one variable are usually due to fluctuations in more than one other variable. On this account it will normally not be possible to find a relationship between one of these causes and the variable in whose

causation one is interested. Instead of the relationship between two variables, we will now have to have recourse to more complicated relationships. The simplest of these occurs when the values of one variable are equal to the sum of the simultaneous values of two other variables. The numerical example shown in Example I may make this clear; in each year the value of

EXAMPLE I

Year	1	2	3	4	5	6	7	8	9
Series A.....	0	+2	0	0	−2	−1	−1	+2	0
Series B.....	0	+2	0	+1	0	0	0	−3	0
Series C.....	0	0	0	−1	−2	−1	−1	+5	0

series A is equal to the sum of the values of series B and C. A relationship of this nature can graphically best be represented by time graphs. Figure 9 shows this for Example I. It will be noted that certain of the characteristics of series B and certain of those of series C can be found in series A. Thus, the first peak in year 2 can be traced back to series B and the second peak in year 8 to series C. The second peak is less pronounced in A than it is in C because it is in part offset by a trough in series B. The irregular depression in series A in years 5–7 is also found in series C. In short, series A shows the joint effects of series B and C. Neither series B nor series C can by itself explain series A; only their sum can explain it.

The relationship between series A which we want to explain and the explanatory series B and C can be much more complicated in many respects. To take a relatively simple case, it may be that series B and C require to be multiplied each by a certain coefficient before they are added up. An example of this is worked out below and is also shown in Figure 9, in which series A' equals $2\,B + \frac{1}{2}\,C$.

EXAMPLE II

Year	1	2	3	4	5	6	7	8	9
Series A'.....	0	+4	0	+1.5	−1	−0.5	−0.5	−3.5	0
2×series B.....	0	+4	0	+2	0	0	0	−6	0
½×series C.....	0	0	0	−0.5	−1	−0.5	−0.5	+2.5	0

It will be seen that the values of B are multiplied by 2 and those of C by $\frac{1}{2}$ in order to yield A' by addition. Graphically, the depression of A' in the years 5–7 can again be found in series C, and the peak in year 2 is now more pronounced in series B. A', however, has a peak in year 4 which A did not have and which is due to B; in the first example this peak was offset by a trough in C. In year 8 the greater influence of series B now dominates the peak in C; hence, there is a trough in A', whereas in our first example there was a peak in A.

It would be possible to cite many more complicated cases. In the examples given so far the relationships were additive, that

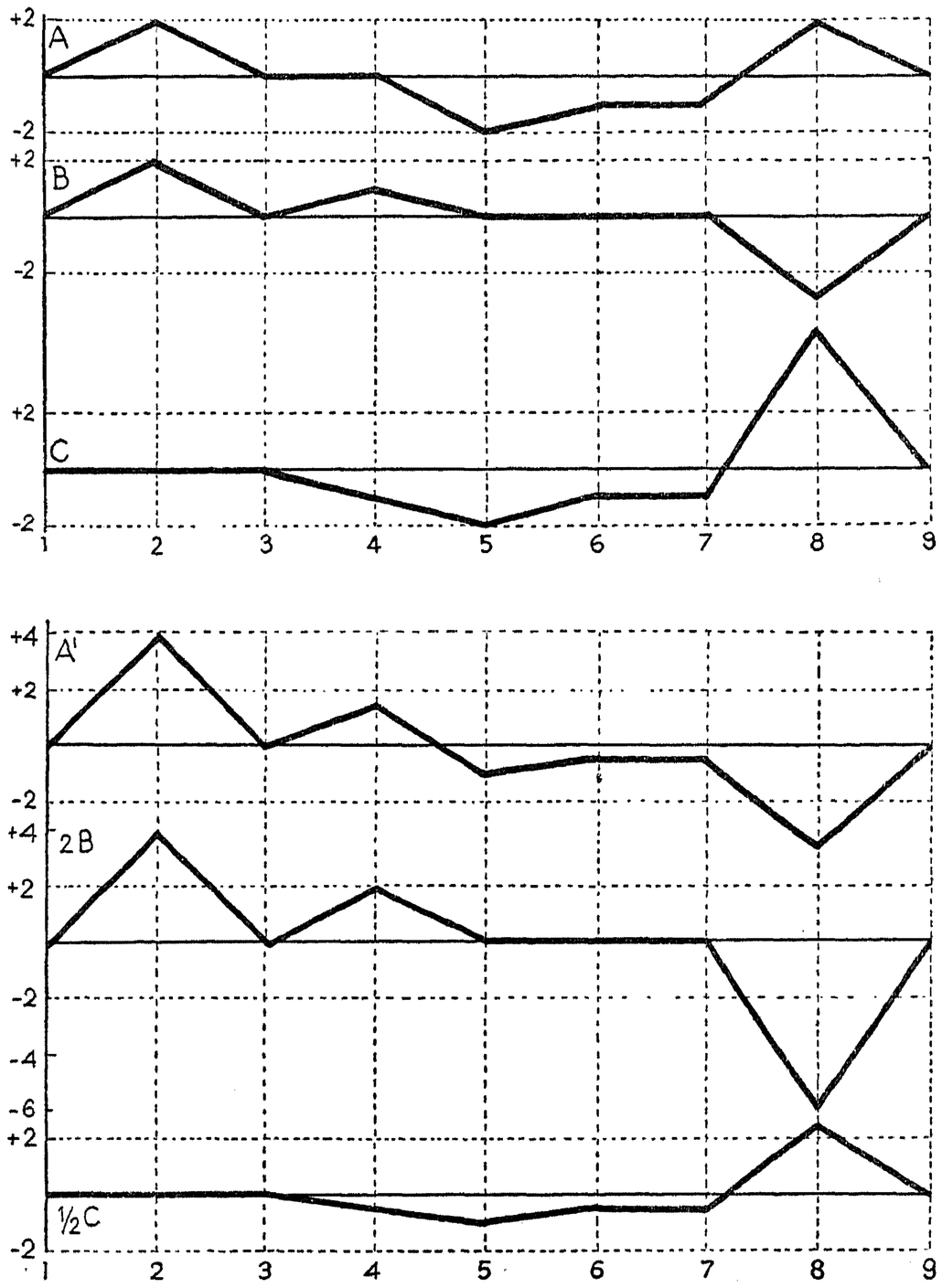

FIG. 9.—Examples of multiple correlation; $A = B + C$; $A' = 2B + \frac{1}{2}C$

is to say, A was obtained by the addition of B and C, A' by the addition of $2\,B$ and $\frac{1}{2}\,C$. Sometimes more complicated mathematical relationships (multiplicative, exponential, etc.) may prevail. It may, however, be proved that, for fluctuations that are relatively small in proportion to the average of the series, the more complicated operations will yield results only slightly different from the results of appropriate additions. Thus, for instance, the multiplication of two index numbers that deviate only slightly from 1, as shown in Table 1, may, with a high degree

TABLE 1

MULTIPLICATION OF TWO INDEX NUMBERS

Year	1	2	3	4	5	6	7	8	9
Series P	1.02	1.04	0.99	1.00	0.98	0.97	1.02	1.00	0.98
Series Q	1.01	0.99	.98	1.02	1.00	1.01	1.02	0.97	1.00
$P \times Q$	1.0302	1.0296	0.9702	1.0200	0.9800	0.9797	1.0404	0.9700	0.9800

TABLE 2

APPROXIMATION OF PRODUCT BY SUM

Year	1	2	3	4	5	6	7	8	9
Deviations:									
P	0.02	0.04	−0.01	0.00	−0.02	−0.03	0.02	0.00	−0.02
Q	.01	− .01	− .02	.02	.00	− .01	.02	− .03	.00
Sum	0.03	0.03	−0.03	0.02	−0.02	−0.02	0.04	−0.03	−0.02

of approximation, be replaced by adding to 1 the sum of the deviations of each of the two series P and Q, as shown by the figures in Table 2.

If the two series have other averages or if other mathematical operations have to be performed, the addition used as an approximation becomes somewhat different. But it will always be possible to give a suitable approximation by addition. For this reason, Example II has a very general significance. We shall therefore discuss it in slightly more detail.

The coefficients 2 and $\frac{1}{2}$ by which series B and C are multiplied may be called "influence coefficients." The series $2\,B$ and $\frac{1}{2}\,C$, given on the last two lines of Example II, are called the in-

fluence of B on A' and the influence of C on A'. If we desire to represent these influences by one single number, by which we may want to indicate whether the influence of B on A' is large or small, we can use for this number the average deviation of the standard deviation of $2B$ or $\frac{1}{2}C$, respectively, as is known from theoretical statistics. The theory of mathematical statistics provides an answer to the question of how to select the influence coefficients of the two series B and C in order to obtain as good an approximation as possible to any given series A''. This is of particular importance if we want to verify and give quantitative precision to a certain economic theory which would state that the movements of the variable A'' result from the movements of B and C but which does not indicate how large the relative influences are. This is the usual situation in economic theory, namely, that it can give qualitative but not quantitative indications about certain relationships. The coefficients found by the methods developed in theoretical statistics are usually called "regression coefficients." If A'' is exactly equal to the sum of a certain number times B and a certain number times C or, as we shall say, to the weighted sum of B and C (as is the case in Example I for series A and in Example II for series A'), then a perfect multiple correlation between A'', on the one hand, and B and C, on the other hand, is said to exist; if A'' is not exactly equal to the weighted sum of B and C for all periods of time, there is a certain degree of multiple correlation which may be expressed by the multiple-correlation coefficient. Reference is made to the textbooks on statistics for the conditions which have to be satisfied in order to estimate, with a stated degree of precision, regression coefficients on the basis of series of figures for A'', B, and C.

Example II, as shown in Figure 9, will be considered as the prototype of a multiple relationship among more than two variables. The number of explanatory variables need, however, not be limited to two; indeed, there may be any number. But, whatever their number, the explanatory series will always have to satisfy the condition that they must provide the explanation for the peaks and troughs in the series to be explained. Thus, not all the explanatory series can be straight lines. Applied to

economic fluctuations, this would mean, for example, that it would not be possible to explain cyclical fluctuations on the basis of series which themselves show only a trend movement. This statement would not seem to be redundant since, for example, in certain theories the recovery phase of the business cycle has been attributed to an increase in the population. Yet this series shows almost exclusively a trend movement. Similarly, changes in labor productivity cannot provide an explanation for cyclical fluctuations, since for the economy as a whole labor productivity normally shows a straight line or only a very slightly curved line.