Minimax results and finite dimensional separation

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Abstract

In this paper we review and unify some of the classes of generalized convex functions introduced by different authors to prove minimax results in infinite dimensional spaces and show the relations between those classes. We also list for the most general class already introduced by Jeyakumar (cf. [7]) an elementary proof of a minimax result. The proof of this result only uses a finite dimensional separation theorem and although this minimax result is already presented by Neumann (cf. [9]) and independently Jeyakumar (cf. [7]) we believe that the present proof is shorter and more transparent.

1 Classes of generalized convex functions and minimax theorems

Let A, B be nonempty sets and $f : A \times B \to I\!R$ some function, with $A \times B$ denoting the Cartesian product of the sets A and B. The so-called primal problem associated with the function f is given by

$$\inf_{b \in B} \sup_{a \in A} f(a, b) \tag{P}$$

while the so-called dual problem has the form

$$\sup_{a \in A} \inf_{b \in B} f(a, b) \tag{D}$$

If we denote by $v(P)$, resp. $v(D)$ the optimal objective value of the optimization problem (P), resp. (D) it is easy to verify and well-known that $v(D) \leq v(P)$. The difference $v(P) - v(D)$ is called the duality gap and in general this duality gap is strictly positive. One is now interested under which conditions on the function $f : A \times B \to I\!R$ the duality gap equals zero. Observe such results are called minimax theorems and a nice overview on the history of minimax theorems is given by Simons (cf. [13]). Moreover, applications of minimax results can be found in Aubin (cf. [2]). To introduce some classes of functions discussed within the minimax literature we need the following definition (cf. [1]).

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Definition 1.1 *A set $S \subseteq X$, with X a vector space, is called convex if $\alpha S + (1-\alpha)S \subseteq S$ for every $0 < \alpha < 1$. Moreover, the set $S \subseteq X$ is called nearly convex if $\alpha S + (1-\alpha)S \subseteq S$ for some $0 < \alpha < 1$.*

The next definition yields a modern set theoretic description (cf. [3] and the references mentioned in this paper) of some of the classes of functions $f : A \times B \to I\!R$ used within the minimax literature (cf. [13]). To introduce this definition we denote by $I\!R^S$ the vector space of real valued functions on the nonempty set S. Moreover, the set $I\!R^S_+ \subseteq I\!R^S$ represents the convex cone of all nonnegative real valued functions on S. Also, for any function $f : A \times B \to I\!R$ we introduce the mapping $F : B \to I\!R^A$ given by $F(b)(a) := f(a,b)$ for every $b \in B$ while $G : A \to I\!R^B$ is defined by $G(a)(b) := f(a,b)$ for every $a \in A$.

Definition 1.2 *The function $f : A \times B \to I\!R$ is called Ky-Fan convex (generalized König convex) in B if the set $F(B) + I\!R^A_+ \subseteq I\!R^A$ is convex (nearly convex). Moreover, the function $f : A \times B \to I\!R$ is called Ky-Fan concave (generalized König concave) in A if the set $G(A) - I\!R^B_+ \subseteq I\!R^B$ is convex (nearly convex).*

By the above definition it follows immediately that any Ky-Fan convex function $f : A \times B \to I\!R$ in B is also generalized König convex in B. Moreover, as will be shown by means of an example this inclusion is strict. It is now trivial to verify the following equivalent representation. This representation was actually used as a definition by Ky-Fan in 1953 (cf. [4]) while König (cf. [8]) and independently Simons (cf. [12]) used this representation for $\alpha = 1/2$. Finally Neumann (cf. [9]) generalized the definition of König to arbitrary $0 < \alpha < 1$ and so we call this class of functions generalized König convex.

Lemma 1.1 *A function $f : A \times B \to I\!R$ is Ky-Fan convex in B if and only if for every $b_1, b_2 \in B$ and $0 < \alpha < 1$ there exists some $b_0 \in B$ satisfying $f(a,b_0) \leq \alpha f(a,b_1) + (1-\alpha) f(a,b_2)$ for every $a \in A$. Moreover, the function $f : A \times B \to I\!R$ is generalized König convex in B if and only if there exists some $0 < \alpha < 1$ such that for every $b_1, b_2 \in A$ one can find some $b_0 \in B$ satisfying $f(a,b_0) \leq \alpha f(a,b_1) + (1-\alpha) f(a,b_2)$ for every $a \in A$.*

To define the most general class of functions $f : A \times B \to I\!R$ we introduce as follows a topology on the function spaces $I\!R^A$ and $I\!R^B$. A set $V \subseteq I\!R^A$ is called a neighbourhood of the element $\phi \in I\!R^A$ if there exists some $\varepsilon > 0$ such that the set $\{\psi \in I\!R^A : \psi(a) - \phi(a) < \varepsilon, \forall a \in A\}$ is a subset of V. It can easily be verified that these neighbourhoods induce a topology τ on the vector space $I\!R^A$. Observe the function space $(I\!R^A, \tau)$ is not a topological vector space (cf. [11]) since only the addition operator is continuous while the product operation with scalars is not. To introduce a topology η on the function space $I\!R^B$ a set $W \subseteq I\!R^B$ is called a neighbourhood of the element $\phi \in I\!R^B$ if there exists some $\varepsilon > 0$ such that the set $\{\xi \in I\!R^B : \xi(b) - \phi(b) > -\varepsilon\}$ is a subset of W and again these neighbourhoods induce a topology η on $I\!R^B$. By similar arguments the set $(I\!R^B, \eta)$ is not a topological vector space. Denote now the closure operators with respect to the topologies τ and η by cl_τ, resp. cl_η.

Definition 1.3 *The function $f : A \times B \to I\!R$ is called closely convex in B if the set $cl_\tau(F(B) + I\!R^A_+)$ is convex. Moreover, the function $f : A \times B \to I\!R$ is called closely concave in A if the set $cl_\eta(G(A) - I\!R^B_+)$ is convex.*

An equivalent representation of closely convex and closely concave is given by the following lemma. Observe the above class of functions is already discussed by Jeyakumar (cf. [7]) who uses the following equivalent representation as its definition.

Lemma 1.2 *The function $f : A \times B \to I\!R$ is closely convex in B if and only if for each $\varepsilon > 0$, $0 < \alpha < 1$ and $b_1, b_2 \in B$ there exists some $b_0 \in B$ such that*

$$f(a,b_0) \leq \alpha f(a,b_1) + (1-\alpha) f(a,b_2) + \varepsilon \tag{1}$$

for every $a \in A$. Moreover, the function $f : A \times B \to I\!R$ is closely concave in A if and only if for each $\varepsilon > 0$, $0 < \alpha < 1$ and $a_1, a_2 \in A$ there exists some $a_0 \in A$ such that

$$f(a_0,b) \geq \alpha f(a_1,b) + (1-\alpha) f(a_2,b) - \varepsilon \tag{2}$$

for every $b \in B$.

Proof: Let $f : A \times B \to I\!R$ be closely convex in B and consider arbitrary elements $b_1, b_2 \in B, 0 < \alpha < 1$ and $\varepsilon > 0$. By our assumption it follows that the function $\alpha F(b_1) + (1-\alpha)F(b_2)$ belongs to $cl_\tau(F(B) + I\!R^A_+)$. This implies by the definition of the topology τ that for $\varepsilon > 0$ given there exists some function $\phi \in F(B) + I\!R^A_+$ satisfying

$$\phi(a) - (\alpha F(b_1)(a) + (1-\alpha)F(b_2)(a)) = \phi(a) - \alpha f(a,b_1) - (1-\alpha) f(a,b_2) < \varepsilon \tag{3}$$

for every $a \in A$. Moreover, since ϕ belongs to $F(B) + I\!R^A_+$ one can find some $b_0 \in B$ and $r \in I\!R^A_+$ such that $\phi(a) = F(b_0)(a) + r(a) := f(a,b_0) + r(a)$ for every $a \in A$ and this implies by (3) that

$$f(a,b_0) - \alpha f(a,b_1) - (1-\alpha) f(a,b_2) < \varepsilon - r(a) \leq \varepsilon$$

for every $a \in A$. Hence the desired representation is verified and we now need to prove the reverse implication. To show this we consider arbitrary elements $\phi_i, i = 1,2$ belonging to $cl_\tau(F(B) + I\!R^A_+)$. By definition there exists for every $\varepsilon > 0$ some elements $\psi_i, i = 1,2$ belonging to $F(B) + I\!R^A_+$ such that $\psi_i(a) - \phi_i(a) < \frac{1}{2}\varepsilon$ for every $a \in A$ and $i = 1,2$. Also, due to $\psi_i \in F(B) + I\!R^A_+$ one can find elements $b_i \in B$ and $r_i \in I\!R^A_+$, $i = 1,2$ satisfying $\psi_i(a) = f(a,b_i) + r_i(a)$ and this implies by the previous inequality that

$$-\phi_i(a) < -f(a,b_i) - r_i(a) + \frac{1}{2}\varepsilon \tag{4}$$

for $i = 1,2$. Moreover, by relation (1) there exists some $b_0 \in B$ satisfying

$$f(a,b_0) \leq \alpha f(a,b_1) + (1-\alpha) f(a,b_2) + \frac{1}{2}\varepsilon \tag{5}$$

for every $a \in A$ and this implies by (4) and $r_0 := \alpha r_1 + (1-\alpha) r_2 \in I\!R^A_+$ that

$$f(a,b_0) + r_0(a) - \alpha\phi_1(a) - (1-\alpha)\phi_2(a) < f(a,b_0) - \alpha f(a,b_1) - (1-\alpha) f(a,b_2) + \frac{1}{2}\varepsilon$$

Finally by (5) the above inequality reduces to

$$f(a,b_0) + r_0(a) - \alpha\phi_1(a) - (1-\alpha)\phi_2(a) < \varepsilon$$

and this shows that $\alpha\phi_1 + (1-\alpha)\phi_2$ belongs to $cl_\tau(F(B) + I\!R^A_+)$ or equivalently the set $cl_\tau(F(B) + I\!R^A_+)$ is convex. The second part of the above lemma can be proved similarly with some obvious modifications and so we omit its proof. □

By the above representation for closely concave (convex) functions it is easy to show the following corollary.

Corollary 1.1 *If $f : A \times B \to I\!R$ is closely convex in B and $A' \subseteq A$ then the restriction of f with respect to the set $A' \times B$ is also closely convex in B (with respect to the topology τ induced in the space $I\!R^{A'}$). Moreover, if $f : A \times B \to I\!R$ is closely concave in A and $B' \subseteq B$ then the restriction of f to the set $A \times B'$ is closely concave in B (with respect to the topology η induced in the space $I\!R^{B'}$).*

Proof: If relation (1) is verified for all $a \in A$ it is also verified for every $a \in A' \subseteq A$ and so the assertion for closely convex follows by Lemma 1.2. A similar proof applies to closely concave. □

By applying Lemma 1.2 it is also possible to show that any generalized König convex function $f : A \times B \to I\!R$ in B (generalized König concave in A) is closely convex in B (closely concave in A). Before proving this result we introduce for notational convenience the convention that $0 \leq \beta \leq 1$ is called a generalized König convex constant if for every $b_1, b_2 \in B$ there exists some $b_0 \in B$ such that $f(a, b_0) \leq \beta f(a, b_1) + (1 - \beta) f(a, b_2)$ for every $a \in A$.

Lemma 1.3 *If the function $f : A \times B \to I\!R$ is generalized König convex in B (generalized König concave in A) then it is also closely convex in B (closely concave in A)*

Proof: We only give a proof of the first part since the proof of the second part is similar. To show the first part it is sufficient by Lemma 1.2 to verify that the set $D \subseteq [0, 1]$ given by

$$D := \{0 \leq \beta \leq 1 : \beta \text{ is a generalized König convex constant}\}$$

is dense in $[0, 1]$. Clearly the numbers 0 and 1 belong to D and since $f : A \times B \to I\!R$ is generalized König convex in B we obtain by Definition 1.1 and 1.2 that there exists some $0 < \alpha < 1$ belonging to D. Moreover, if the numbers $\beta_i, i = 1, 2$ belong to D it follows for every $b_1, b_2 \in B$ that there exists some elements $b(\beta_i) \in B, i = 1, 2$ satisfying

$$f(a, b(\beta_i)) \leq \beta_i f(a, b_1) + (1 - \beta_i) f(a, b_2) \tag{6}$$

for every $a \in A$ and $i = 1, 2$. Since the function $f : A \times B \to I\!R$ is generalized König convex in B one can find for the elements $b(\beta_i) \in B, i = 1, 2$ some $b_0 \in B$ such that $f(a, b_0) \leq \alpha f(a, b(\beta_1)) + (1 - \alpha) f(a, b(\beta_2))$ and this implies by relation (6) that

$$f(a, b_0) \leq (\alpha\beta_1 + (1 - \alpha)\beta_2) f(a, b_1) + (1 - \alpha\beta_1 - (1 - \alpha)\beta_2) f(a, b_2)$$

for every $a \in A$. This means for any $\beta_i \in D, i = 1, 2$ that $\alpha\beta_1 + (1 - \alpha)\beta_2$ belongs to D and in Green and Austin (cf. [5]) it is shown that such a set is dense in $[0, 1]$. □

As we will show by means of examples it can be shown that the class of Ky-Fan convex functions is strictly contained within the class of generalized König convex functions and the class of generalized König convex functions is strictly contained within the class of closely convex functions. However, if we introduce some additional topological properties these three classes coincide.

Lemma 1.4 *Let A, B be subsets of metric spaces and $f : A \times B \to I\!R$ a given function. If the set B is compact and for every $a \in A$ the function $b \to f(a,b)$ is lower semicontinuous then f is Ky-Fan convex in B if and only if f is closely convex in B.*

Proof: By the observation before Lemma 1.1 and Lemma 1.3 it follows that f is closely convex in B if f is Ky-Fan convex in B. To show the reverse implication let $f : A \times B \to I\!R$ be closely convex in B. By Lemma 1.2 there exists for every $n \in I\!N, 0 < \alpha < 1$ and $b_1, b_2 \in B$ some element $b_0^{(n)} \in B$ satisfying

$$f(a, b_0^{(n)}) \leq \alpha f(a, b_1) + (1-\alpha) f(a, b_2) + 1/n$$

for every $a \in A$. Since B is a compact subset of a metric space the set $\{b_0^{(n)} : n \in I\!N\}$ has a converging subsequence $\{b_0^{(n)} : n \in N \subseteq I\!N\}$ with limit $b_0 \in B$ and this implies by the lower semicontinuity of the mapping $b \to f(a,b)$ for every $a \in A$ that

$$f(a, b_0) \leq \liminf_{n \to \infty, n \in N} f(a, b_0^{(n)}) \leq \alpha f(a, b_1) + (1-\alpha) f(a, b_2)$$

Hence the function $f : A \times B \to I\!R$ is Ky-Fan convex in B and this shows the equivalence. $\Box$

A similar result can be stated for A a compact subset of a metric space and $a \to f(a,b)$ upper semicontinuos for every $b \in B$. In this case Ky-Fan concavity in A coincides with closely concavity in A. To conclude our discussion on Ky-Fan convexity and its generalizations we will present some examples showing that the class of Ky-Fan convex functions is strictly contained in the class of generalized König convex functions and the class of generalized König convex functions is strictly contained in the class of closely convex functions.

Counterexamples

1. (Generalized König convex function, but not Ky-Fan convex). Let $A := \{1, 2\} \subset I\!R$, $B := \{(s,t) \in I\!R^2 : s + t = 0,\ s \in \boldsymbol{Q}\}$ and $\boldsymbol{Q}$ denotes the set of all rational numbers. Introduce now the function $f : A \times B \to I\!R$ given by

$$f(1, (s,t)) := s, \ \ f(2, (s,t)) := t$$

for every $(s,t) \in B$. Regarding each function defined on $A = \{1,2\}$ as a two-components vector, i.e. identifying the space $I\!R^{\{1,2\}}$ with $I\!R^2$ it follows that $F(B) = B$ and so $F(B) + I\!R^2_+ = B + I\!R^2_+$. Clearly the elements $(1,-1)$ and $(0,0)$ belong to the set $F(B) + I\!R^2_+ = B + I\!R^2_+$, while for $\alpha = \sqrt{2}/2$ the element $(\sqrt{2}/2)(1,-1) + (1-\sqrt{2}/2)(0,0) = (\sqrt{2}/2,\ -\sqrt{2}/2)$ does not belong to this set. This shows that f is not Ky-Fan convex in B and on the other hand it is obvious that f is generalized König convex in B with $\alpha = 1/2$.

2. (Closely convex function, but not generalized König convex). Let B be the set $\{(s,t) \in I\!R^2 : s + t = 0,\ s \in I\!R \backslash \boldsymbol{Q}\}$ and take A and the function f as before. By the following observation one can show that f is not generalized König convex in B. Indeed, for every $0 < \alpha < 1$ take the irrational numbers $s_1 = \sqrt{2}$, if $\alpha \in \boldsymbol{Q}$ and $-1/\alpha$ otherwise, and $s_2 = \alpha/(\alpha - 1)s_1$. It is easy to check that we have $\alpha s_1 + (1-\alpha)s_2 = 0$. Denoting by t_i the irrational numbers $-s_i$ for $i = 1,2$, it is obvious that $(s_i, t_i) \in B + I\!R^2_+$, $i = 1,2$ while $\alpha(s_1, t_1) + (1-\alpha)(s_2, t_2) = (0,0)$ does not belong to $B + I\!R^2_+$. This shows that f is not generalized König convex in B. On the other

hand, for every $0<\alpha<1$, (s_1,t_1), $(s_2,t_2)\in B$ and $s_\alpha := \alpha s_1+(1-\alpha)s_2$, $t_\alpha := \alpha t_1+(1-\alpha)t_2$ one has

$$\alpha f(1,(s_1,t_1)) + (1-\alpha) f(1,(s_2,t_2)) = s_\alpha \tag{7}$$

and

$$\alpha f(2,(s_1,t_1)) + (1-\alpha) f(2,(s_2,t_2)) = t_\alpha. \tag{8}$$

Now for $\varepsilon > 0$ take the irrational numbers s_0 and t_0 such that $s_\alpha < s_0 < s_\alpha + \varepsilon$ and $t_0 = -s_0$. Clearly $(s_0,t_0)\in B$ and $t_0 < t_\alpha < t_\alpha + \varepsilon$. Since $f(1,(s_0,t_0)) = s_0$ and $f(2,(s_0,t_0)) = t_0$, this implies by relations (7), (8) and Lemma 1.2 that the function f is closely convex in B.

We will now present a proof of a minimax result for the most general class thus improving a similar result by Ky-Fan (cf. [4]), König (cf. [8]) and Neumann (cf. [9]). Although this result has already been verified by Jeyakumar (cf. [7]) by means of a theorem of the alternative in a finite dimensional space we believe that our proof is more transparent and direct. The next result is useful in establishing our minimax result. Observe the set Δ_n denotes the unit simplex in $I\!R^n$ given by $\Delta_n := \{\alpha \in I\!R^n : \sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0\}$.

Lemma 1.5 *If the function $f : A \times B \to I\!R$ is closely convex in B then it follows for every finite set $\{b_1,\ldots,b_n\} \subseteq B$ and $\alpha \in \Delta_n$ that $\sup_{a\in A}\sum_{i=1}^n \alpha_i f(a,b_i) \geq \upsilon(P)$.*

Proof: By induction and Lemma 1.2 it is easy to verify that $f : A\times B \to I\!R$ is closely convex in B if and only if for every $\varepsilon > 0$,$\alpha \in \Delta_n$ and every finite set $\{b_1,\ldots,b_n\}\subseteq B$ there exists some $b_0 \in B$ satisfying

$$\sum_{i=1}^n \alpha_i f(a,b_i) \geq f(a,b_0) - \varepsilon$$

for every $a \in A$. This implies that

$$\sup_{a\in A}\sum_{i=1}^n \alpha_i f(a,b_i) \geq \sup_{a\in A} f(a,b_0) - \epsilon \geq \upsilon(P) - \varepsilon$$

and since $\varepsilon > 0$ arbitrary the desired result follows. □

We will now give a short and elementary proof of the main result of this paper.

Theorem 1.1 *If the nonempty set A is a compact topological space and the function $f : A \times B \to I\!R$ is upper semicontinuous with respect to its first variable for every fixed $b \in B$ then the dual problem (D) has an optimal solution. Moreover, if additionally the function $f : A\times B \to I\!R$ is closely concave in A then it follows that $\upsilon(D) \geq t$ if and only if $\sup_{a\in A}\sum_{i=1}^n \alpha_i f(a,b_i) \geq t$ for every finite set $\{b_1,\ldots,b_n\}\subseteq B$ and $\alpha \in \Delta_n$.*

Proof: Since the infimum of upper semicontinuous functions is again upper semicontinuous it follows that the Lagrangian function $L : A \to I\!R$ given by $L(a) := \inf_{b\in B} f(a,b)$ attains its maximum on the compact set A and this shows the first part. To show the second part we observe for every $\alpha \in \Delta_n$ and $\{b_1, \ldots, b_n\} \subseteq B$ that $\sum_{i=1}^n \alpha_i f(a, b_i) \geq L(a)$ for every $a \in A$. This implies that

$$\sup_{a \in A} \sum_{i=1}^n \alpha_i f(a, b_i) \geq \sup_{a \in A} L(a) = v(D)$$

and hence we have verified that $\sup_{a \in A} \sum_{i=1}^n \alpha_i f(a, b_i) \geq t$ if $v(D) \geq t$. To show the reverse implication we assume by contradiction that $v(D) < t$. By the definition of the Lagrangian function we obtain that

$$\emptyset = \{a \in A : L(a) \geq c_1\} = \bigcap_{b \in B} \{a \in A : f(a, b) \geq c_1\}$$

for every $v(D) < c_1 < c_2 < t$. Since A is compact and $a \to f(a, b)$ is upper semicontinuous for every $b \in B$ it also follows that the set $\{a \in A : f(a, b) \geq c_1\}$ is compact for every $b \in B$ and hence by the finite intersection property for compact sets (cf. [10]) there exists a finite collection $\{b_1, \ldots, b_n\} \subseteq B$ satisfying

$$\bigcap_{i=1}^n \{a \in A : f(a, b_i) \geq c_1\} = \emptyset. \tag{9}$$

If we introduce the mapping $H : A \to I\!R^n$ given by

$$H(a) := (f(a, b_1) - c_2, \ldots, f(a, b_n) - c_2)$$

it is easy to verify due to $c_1 < c_2 < t$ that the set $cl_\eta(H(A) - I\!R^n_+)$ does not contain the zero vector 0. Indeed, if 0 belongs to $cl_\eta(H(A) - I\!R^n_+)$ then by the definition of the topology η there exists for $(1/2)(c_2 - c_1) > 0$ some $a_0 \in A$ and $r \in I\!R^n_+$ such that $f(a_0, b_i) - c_2 - r_i > (-1/2)(c_2 - c_1)$ for every $1 \leq i \leq n$ and this implies $f(a_0, b_i) > (1/2)(c_1 + c_2) > c_1$ contradicting relation (7). By Corollary 1.1 and $f : A \times B \to I\!R$ is closely concave in A we obtain that the set $cl_\eta(H(A) - I\!R^n_+)$ is convex and applying now a well-known separation result in finite dimensional spaces (cf. [6]) between a convex set and a point outside this set to the sets $cl_\eta(H(A) - I\!R^n_+)$ and $\{0\}$ one can find some $\alpha \in \Delta_n$ satisfying

$$0 \geq \sum_{i=1}^n \alpha_i (f(a, b_i) - c_2) = \sum_{i=1}^n \alpha_i f(a, b_i) - c_2$$

for every $a \in A$. This implies $\sup_{a \in A} \sum_{i=1}^n \alpha_i f(a, b_i) \leq c_2$ contradicting our initial assumption. Hence it must follow that $v(D) \geq t$ and this shows the result. □

Combining Lemma 1.5 and Theorem 1.1 we obtain the following minimax result. This minimax result was already discovered by Neumann (cf. [9]) using a version of the Hahn-Banach theorem in infinite dimensional spaces due to Mazur-Orlicz and independently by Jeyakumar (cf. [7]) using a more complicated "finite-dimensional" proof than ours. Observe by the first part of our paper that the minimax theorem of König and Ky-Fan is a special case of the next minimax theorem. Observe König (cf. [4]) also used the Mazur-Orlicz separation theorem to show this special case.

Theorem 1.2 *If A is a compact topological space and the function $f : A \times B \to I\!R$ is closely concave in A and closely convex in B and upper semicontinuous with respect to its first variable for every $b \in B$ then it follows that*

$$\inf_{b \in B} \sup_{a \in A} f(a, b) = \sup_{a \in A} \inf_{b \in B} f(a, b)$$

and the dual problem attains its optimal solution.

Proof: By Lemma 1.5 we obtain that $\sup_{a \in A} \sum_{i=1}^{n} \alpha_i f(a, b_i) \geq \upsilon(P)$ and this implies by Theorem 1.1 that $\upsilon(D) \geq \upsilon(P)$. Since always $\upsilon(D) \leq \upsilon(P)$ the equality follows. The second part is already shown in Theorem 1.1. $\square$

To conclude this paper we like to observe that for A a compact metric space it follows by Lemma 1.4 that the function f is actually Ky-Fan concave in A. Observe a similar observation does not hold for the closely convex part due the absence of topological properties.

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