A Multi-Item Inventory Model With Joint Setup And Concave Production Costs

Z.P. Bayındır, S.I. Birbil and J.B.G. Frenk

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Address	Erasmus Research Institute of Management (ERIM) Rotterdam School of Management / Rotterdam School of Economics Erasmus Universiteit Rotterdam P.O. Box 1738 3000 DR Rotterdam, The Netherlands Phone: +31 10 408 1182 Fax: +31 10 408 9640 Email: info@erim.eur.nl Internet: www.erim.eur.nl

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A Multi-item Inventory Model with Joint Setup and Concave Production Costs

Z. P. Bayındır[†], Ş. İ. Birbil[‡], and J. B. G. Frenk[†*]

[†]*Econometric Institute, Erasmus University Rotterdam*

Postbus 1738, 3000 DR Rotterdam, The Netherlands.

{bayindir, frenk}@few.eur.nl

[‡]*Erasmus Research Institute of Management, Erasmus University Rotterdam*

Postbus 1738, 3000 DR Rotterdam, The Netherlands.

sibirbil@few.eur.nl

ABSTRACT. The present paper discusses an approach to solve the joint replenishment problem in a production environment with concave production cost functions. Under this environment, the model leads to a global optimization problem, which is investigated by using some standard results from convex analysis. Consequently, an effective solution procedure is proposed. The proposed procedure is guaranteed to return a solution with a predetermined quality in terms of the objective function value. A computational study is provided to illustrate the performance of the proposed solution procedure with respect to the running time.

Keywords. inventory, multi-item, concave production cost, joint setup, Lipschitz optimization, concave-convex programming

1 Introduction

Inventory planning for multi-item systems with a joint setup cost, is a well-analyzed area of research in Management Science. The problem is referred to as the joint replenishment problem. The main issue, as the name suggests, is investigating the economies of scale due to the coordination of the replenished items that have a common fixed (ordering/setup) cost. Apart from this major fixed cost, there is an individual minor fixed cost associated with each item. The overall objective is to minimize total average cost by determining a common basic cycle time along with the ordering frequency for each item. Setting the ordering frequencies to integer values ensures the coordination of replenishment of items. As a result of this coordination, savings on the major fixed cost are obtained.

There exists a vast amount of literature on the deterministic joint replenishment problem [see 1–17]. In the literature, one of the main environmental conditions employed is the instantaneous replenishment assumption, which reflects the fact that either items are purchased from an outside supplier or there is no capacity restriction on the production facility in terms of the production rate. Indeed in a production

[*]Corresponding author.

setting, where the items are produced with a finite rate, a joint replenishment problem exists for families of items sharing the same major production setup cost. Another issue overlooked in the literature is the effect of replenishment order size on unit costs, and consequently, on the inventory value of items. In real life, unit costs under certain environments decrease as the replenishment order size increases. In case of purchasing, this occurs when quantity discounts are offered by the suppliers. Similarly, when there are economies of scale in a production process, the marginal cost of each unit produced decreases as the lot size increases. Economies of scale can be observed when the production process involves learning or when the inputs like energy, raw material are subject to cost discounts on the volume used.

In this paper we investigate the joint replenishment problem in a production environment, where the items are manufactured by a finite rate. Specifically, we focus on the case where the total variable cost of production is concave in the lot size. Our main contributions are the following;

- ◇ Investigation of a single item production/inventory system under general concave production costs.
- ◇ Analyzing the structure of the joint replenishment problem under the assumed environment, along with characterizing the main difficulties for the solution.
- ◇ Utilizing convex analysis to reformulate both single and multi-item models to develop an efficient global optimization algorithm.
- ◇ Discussing the implementation details for the proposed algorithm.

The outline of the paper is as follows. In Section 2, we start explaining the environment that is considered in the paper. This is followed, in Section 3, by the analysis of a single item production/inventory system under concave production cost. The joint production setup cost comes into the picture with the multi-item model, which is discussed thoroughly in Section 4. Finally we explain, in Section 5, our computational experience to solve the multi-item model.

2 Environment

We consider a family of n items, where the demand process for each is deterministic and stationary over time. The demand rate for item i is denoted by $\lambda_i > 0$, $1 \leq i \leq n$. Unfilled demand is assumed to be completely backordered, and unit backorder cost per unit time is b_i for item i. The considered family of items have a joint production setup cost; regardless of the types and quantities of items to be produced, a fixed cost, A, is incurred each time production starts. In addition to this major setup cost, there exists a minor setup cost a_i for each item i. We assume that the production rate of item i is fixed and given by $\mu_i > \lambda_i$. All possible capacity restrictions arising from the system load are ignored. Therefore, the production rate of each item is independent of both its lot size and the number of outstanding orders of other items. Moreover, the variable production cost of each item depends on the corresponding lot size and it is given by $c_i(Q)$, whenever a lot size of Q is produced for item i. There exists an economies of scale situation; that is, the marginal cost of each additional item included in the lot decreases as the lot size increases. Hence, the production cost function $c_i(\cdot)$, satisfying $c_i(0) = 0$ and $c_i(\infty) = \infty$, is in general concave, continuous and strictly increasing. We assume that the unit holding cost for each item consists of the following two components:

1. **Unit out-of-pocket holding cost:** This component includes real costs; like insurance cost or warehouse rent. For each item i held in the inventory, a cost of h_i is incurred per unit time.

2. **Unit opportunity cost of holding:** This component reflects the opportunity cost of tiding up money into inventories. We consider unit production cost as the cost added to each item. Since unit production cost depends on the lot size, inventory value of each item included in a certain lot is not identical. Since it is not possible to differentiate the items physically, the average costing principle is utilized. Therefore, under the traditional way of setting holding cost rates, when the inventory carrying charge is r, an opportunity cost of $\frac{c_i(Q)}{Q}r$ is incurred per unit time for each item i produced in a lot of size Q.

An (S_i, T_i) type of inventory control rule is considered. According to this rule, the net inventory level of item i is raised up-to level S_i at every T_i time units. Due to the complete backordering assumption, a production order of $\lambda_i T_i$ is given for item i at every T_i time periods. The net inventory level under this policy is depicted in Figure 1.

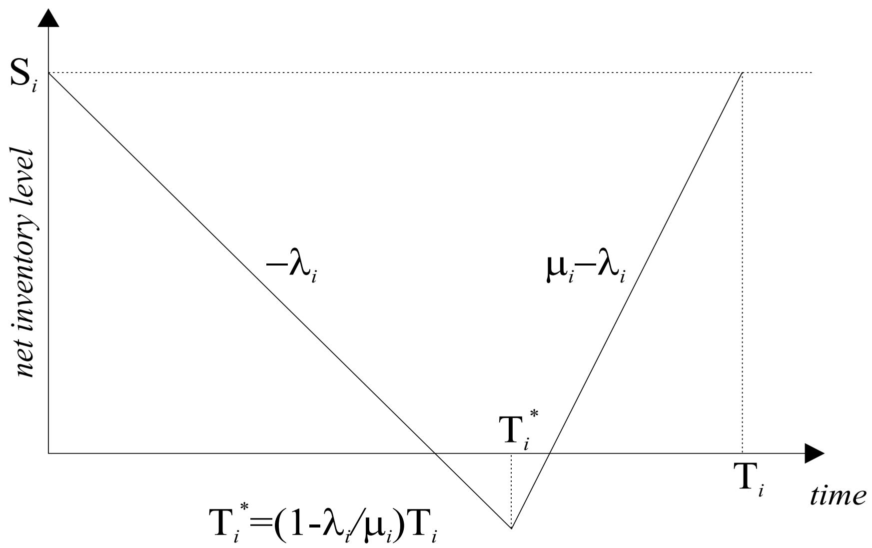

Figure 1: Inventory level for item i over time.

Using Figure 1, it follows in a cycle of length T_i and an order-up-to level S_i that the total inventory and penalty costs in one cycle is given by

$$\int_0^{T_i^*} f_i(T_i, S_i - \lambda_i t)dt + \int_0^{T_i - T_i^*} f_i(T_i, S_i - (\mu_i - \lambda_i)t)dt, \tag{2.1}$$

where $T_i^* = (1 - \frac{\lambda_i}{\mu_i})T_i$, and $f_i(\cdot, \cdot)$ denotes the so-called cost rate function introduced in [2]. After some easy calculations it follows from relation (2.1) that the total inventory and penalty costs in one cycle is equal to

$$\int_0^{T_i} f_i(T_i, S_i - \sigma_i t)dt,$$

where $\sigma_i = \lambda_i(1 - \frac{\lambda_i}{\mu_i})$. Consequently, if we take into account the minor ordering and production costs, then the average cost under an (S_i, T_i) inventory rule is given by

$$\frac{a_i + c_i(\lambda_i T_i) + \int_0^{T_i} f_i(T_i, S_i - \sigma_i t)dt}{T}.$$

Hence, in case of a general cost rate function, we have to solve for each item i, $1 \le i \le n$ the optimization problem

$$\min\{\Phi_i(T_i) : T > 0\}, \tag{2.2}$$

where

$$\Phi_i(T_i) := \frac{a_i + c_i(\lambda_i T_i) + \phi_i(T_i)}{T_i} \tag{2.3}$$

and

$$\phi_i(T_i) := \min_{0 \le S_i \le \sigma_i T_i} \left\{ \int_0^{T_i} f_i(T_i, S_i - \sigma_i t) dt \right\}.$$

In this setting, the cost rate function has the form

$$f_i(T_i, x) := \begin{cases} (h_i + \frac{c_i(\lambda_i T_i)}{\lambda_i T_i} r)x & \text{for } x \ge 0 \\ -b_i x & \text{for } x < 0. \end{cases} \tag{2.4}$$

If we introduce the multi-item joint replenishment problem for the above family, we need to consider the coordination of these items. In this case, all items have the same basic cycle time T and for each item i production starts within every $k_i T$ time units. It is shown by Dagpunar [1], that this boils down to solving the following optimization problem

$$\min \left\{ \frac{A\Delta(k)}{T} + \sum_{i=1}^{n} \Phi_i(k_i T) : T > 0, k_i \in \mathbb{N} \right\},$$

where $\Delta(k)$ denotes the correction factor that keeps track of the empty replenishments. This factor is given by

$$\Delta(k) := \sum_{i=1}^{n} (-1)^{i+1} \sum_{\gamma \subset \{1, \cdots, n\}; |\gamma| = i} (\text{lcm}(k_{\gamma_1}, \cdots, k_{\gamma_i}))^{-1}$$

with $\text{lcm}(\cdot)$ denoting the least common multiple of the integer arguments.

Solving now the above problem is complicated due to the correction factor $\Delta(k)$. Moreover, Goyal criticizes the formulation of Dagpunar and proposes to set the correction factor equal to one [3]. This means that the possibility of having empty replenishment occasions are ignored, and so, implicitly there exists at least one item produced within each basic cycle. This simplifies the problem by setting $\Delta(k) = 1$. This approach is followed in many papers (see overviews of Goyal and Satir in [4] and Kaspi and Rosenblatt in [5]). We therefore rconsider the optimization problem

$$\min \left\{ \frac{A}{T} + \sum_{i=1}^{n} \Phi_i(k_i T) : T > 0, k_i \in \mathbb{N} \right\}. \tag{2.5}$$

The model considered in the current study is an extension of a similar model with linear production costs discussed in [2, 6]. Contrary to the model in [2, 6], our analysis is not related to a special convex programming problem. It is in general a global optimization problem, and this naturally affects our solution strategy. However, it is still possible to use tools from convex analysis. As a byproduct, we will show that the optimization problem associated with the inventory control of a single item is actually a C-programming problem [18]. To start with our analysis, we first present in the next section a detailed investigation of a single item production/inventory model with economies of scale in production.

3 Single Item Production/Inventory Model

In this section we concentrate on the single item production/inventory model introduced in the previous section. Therefore, the subscript i is suppressed in the subsequent analysis. Recall from relation (2.2) that for the determination of the optimal policy parameters (S, T), we need to solve the optimization problem

$$z_1 := \min_{T > 0} \left\{ \frac{a + c(\lambda T) + \phi(T)}{T} \right\}$$

with

$$\phi(T) := \min_{0 \le S \le \sigma T} \left\{ \int_0^T f(T, S - \sigma t) dt \right\}. \tag{3.1}$$

Using the specific form of the function $f(\cdot,\cdot)$ given in relation (2.4), we obtain by standard arguments that the optimal solution $S(T)$ of the optimization problem in relation (3.1) is given by

$$S(T) = \frac{b\sigma T}{h + \frac{rc(\lambda T)}{\lambda T} + b}.$$

Moreover, substituting $S(T)$ into (3.1) gives

$$\phi(T) = \frac{b}{h + \frac{rc(\lambda T)}{\lambda T} + b} \frac{(h + \frac{rc(\lambda T)}{\lambda T})\sigma T^2}{2}, \tag{3.2}$$

and after some simple calculations, this implies

$$\Phi(T) = \frac{a + c(\lambda T)}{T} + \frac{b\sigma T}{2} - \frac{\lambda\sigma(bT)^2}{2\lambda(h+b)T + 2rc(\lambda T)}. \tag{3.3}$$

Consequently, the proposed single item production/inventory problem boils down to the following optimization problem

$$z_1 = \min\{\Phi(T) : T > 0\}. \tag{P_1}$$

Example 3.1 *In case no shortages are allowed ($b \uparrow \infty$), it follows by relation (3.2) that*

$$\phi(T) = \frac{(h + \frac{rc(\lambda T)}{\lambda T})\sigma T^2}{2}.$$

Denoting the objective function for the no shortages case by $\Psi(\cdot)$, we obtain

$$\Psi(T) = \frac{a + c(\lambda T)}{T} + \frac{\sigma(h\lambda T + rc(\lambda T))}{2\lambda}, \tag{3.4}$$

and so, the corresponding optimization problem becomes

$$z_0 := \min\{\Psi(T) : T > 0\}. \tag{3.5}$$

Since the production cost function $c(\cdot)$ is concave, the optimization problem (P_1) belongs to the field of global optimization [19]. Therefore, solving this problem might be quite difficult. We next investigate whether it is possible to rewrite the problem, and then verify under which additional conditions on the concave cost function $c(\cdot)$, problem (P_1) can be solved efficiently. To do this, we use the well-known dual representation of a continuous concave function by means of its biconjugate function. Moreover, in Section 4 this dual approach will immensely simplify solving the multi-item joint replenishment problem.

In Appendix A we show for every $x \ge 0$ (see Lemma A.1) that

$$c(x) = \inf_{\omega \in \Omega} \{x\omega - c^*(\omega)\}, \tag{3.6}$$

where $\Omega := [c'_-(\infty), c'_+(0)]$ is a compact interval with $c'_-(\cdot)$ and $c'_+(\cdot)$ denoting the left and the right derivatives of the function $c(\cdot)$, respectively. Moreover the function $c^*(\cdot)$ denotes the conjugate of $c(\cdot)$, given by

$$c^*(\omega) = \inf_{x \ge 0} \{\omega x - c(x)\}.$$

Defining the function $F : [0, \infty) \times (0, \infty) \to \mathbb{R}$ by

$$F(x, T) := \frac{a + x}{T} + \frac{b\sigma T}{2} - \frac{\lambda\sigma(bT)^2}{2\lambda(h+b)T + 2rx}, \tag{3.7}$$

and using relation (3.3) lead to

$$\Phi(T) = F(c(\lambda T), T). \tag{3.8}$$

Since the function $x \to F(x,T)$ is increasing on $(0,\infty)$ for every T, we obtain by relations (3.6) and (3.8) that

$$\begin{aligned}\Phi(T) &= F(\min_{\omega\in\Omega}\{\lambda T\omega - c^*(\omega)\}, T) \\ &= \min_{\omega\in\Omega} F(\lambda T\omega - c^*(\omega), T).\end{aligned} \tag{3.9}$$

This implies that problem (P_1) can be rewritten as

$$\begin{aligned} z_1 = \min_{T>0}\Phi(T) &= \min_{T>0}\min_{\omega\in\Omega} F(\lambda T\omega - c^*(\omega), T) \\ &= \min_{\omega\in\Omega}\min_{T>0} F(\lambda\omega T - c^*(\omega), T) \\ &= \min_{\omega\in\Omega}\min_{T>0} F(\lambda\omega T^{-1} - c^*(\omega), T^{-1}).\end{aligned} \tag{3.10}$$

Note that in the last step we replaced the variable T by T^{-1}. It is easy to see that such a transformation does not change the optimal objective function value. However, after this simple transformation the inner minimization problem

$$\min_{T>0} F(\lambda\omega T^{-1} - c^*(\omega), T^{-1}), \tag{$P_1(\omega)$}$$

becomes a convex optimization problem for each $\omega\in\Omega$. This is an important observation since computing the optimal objective function value z_1 requires the solution of problem ($P_1(\omega)$) for $\omega\in\Omega$. We next give a formal proof of this important observation.

Lemma 3.1 *For each $\omega\in\Omega$ the optimization problem ($P_1(\omega)$) is a convex programming problem.*

Proof. It is easy to compute that

$$F(\lambda\omega T - c^*(\omega), T) = \lambda\omega + \frac{a - c^*(\omega) + \psi_\omega(T)}{T}$$

with

$$\psi_\omega(T) = \frac{b\sigma T^2(\lambda(h + r\omega)T - rc^*(\omega))}{2\lambda(h+b+r\omega)T - 2rc^*(\omega)}.$$

Since $\psi_\omega(\cdot)$ is the ratio of a squared convex function (note that $c^*(\omega) \le 0$) and a linear function, it is convex on $(0,\infty)$ [20]. This implies that the function $T \to \psi_\omega(T^{-1})T$ is also convex on the same set [21]. Consequently, the function $T \to F(\lambda\omega T^{-1} - c^*(\omega), T^{-1})$ is convex on $(0,\infty)$, and hence the desired result follows. □

Remark 3.1 *In Section 4 we will focus on the joint replenishment problem, where the overall optimization problem has to be solved for multiple items. The optimal solution of the problem ($P_1(\omega)$) will play an important role in solving the multi-item (joint replenishment) problem.*

As the next example shows, one can give for the no shortages case an analytical expression for the optimal objective value of the inner optimization problem ($P_1(\omega)$).

Example 3.2 *If shortages are not allowed ($b \uparrow \infty$), we know by relation (3.4) that the objective function is given by*

$$\Psi(T) = \frac{a + c(\lambda T)}{T} + \frac{\sigma(h\lambda T + rc(\lambda T))}{2\lambda}.$$

Using again the dual representation of the cost function $c(\cdot)$, we can rewrite the optimization problem in relation (3.5) as

$$z_0 = \min_{T>0} \Psi(T) = \min_{\omega\in\Omega} \min_{T>0} G(\lambda T\omega - c^*(\omega), T),$$

where $G : [0, \infty) \times (0, \infty) \to \mathbb{R}$ is given by

$$G(x, T) := \frac{a + x}{T} + \frac{\sigma(h\lambda T + rx)}{2\lambda}.$$

Now, the corresponding (inner) optimization problem

$$\min_{T>0} G(\lambda T\omega - c^*(\omega), T)$$

can be solved analytically with optimal solution value $T(\omega)$ given by

$$T(\omega) = \sqrt{\frac{2(a - c^*(\omega))}{\sigma(h + r\omega)}}.$$

Substituting $T(\omega)$ into the objective function yields

$$G(\lambda T(\omega)\omega - c^*(\omega), T(\omega)) = \lambda\omega - \frac{r\sigma c^*(\omega)}{2\lambda} + \frac{a - c^*(\omega)}{T(\omega)} + \frac{\sigma(h + r\omega)T(\omega)}{2}.$$

In this case, the overall optimization problem (3.5) becomes

$$z_0 = \min_{\omega\in\Omega} G(\lambda T(\omega)\omega - c^*(\omega), T(\omega)).$$

Before we discuss the general structure of problem (3.10), we introduce the following auxiliary function

$$v(\omega, -c^*(\omega)) := \min_{T>0} F(\lambda\omega T - c^*(\omega), T) - \lambda\omega. \quad (3.11)$$

It is important to note that although the inner problem $v(\omega, -c^*(\omega))$ can be computed efficiently, the overall problem as given by relation (3.10) is, in general, a difficult problem to solve. Nevertheless, the problem has a special structure that is worth mentioning. In Appendix B, we show in Lemma B.1 that $v(\cdot, \cdot)$ is a concave function. Moreover, the arguments of this function ω and $-c^*(\omega)$ are both convex functions. Therefore, the objective function of problem (3.10) is a composition of concave and convex functions. As a particular subfield of global optimization, C-programming methods are specialized to find stationary points of such optimization problems [18]. We also note, since the objective function of the problem is one dimensional, that powerful one dimensional Lipschitz optimization methods can be applied to solve the problem. One of these approaches will be further elaborated in Section 5.

As mentioned above solving the overall problem can be quite difficult, unless the cost function possesses some special properties. In the next example, we illustrate a case where the function $c(\cdot)$ is polyhedral concave; *i.e.*, piecewise linear and concave. In this case solving the overall problem is equivalent to solving a finite number of convex optimization problems.

Example 3.3 *Consider an environment where the production cost function $c(\cdot)$ is a polyhedral concave function. This means*

$$c(\lambda T) := \min_{1\le j\le m} \{\alpha_j \lambda T + \beta_j\}, \quad (3.12)$$

where $\alpha_m < \alpha_{m-1} < \cdots < \alpha_1$ and $0 = \beta_1 < \beta_2 < \cdots < \beta_m$ (see Figure 2). For the function listed in (3.12), it is easy to check that $\Omega = [c'_-(\infty), c'_+(0)] = [\alpha_m, \alpha_1]$. Moreover, it is well-known that $c^*(\cdot)$ is

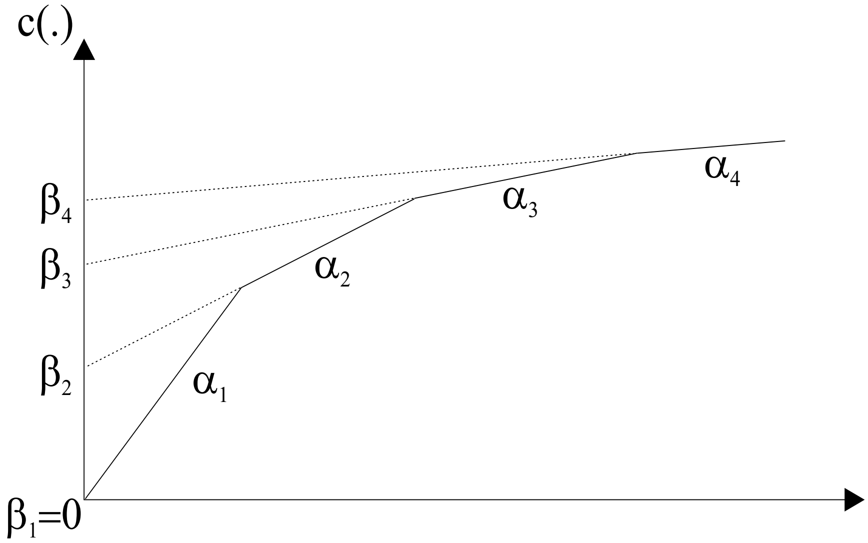

Figure 2: An example polyhedral concave function $c(\cdot)$ where $m = 4$.

a polyhedral concave function with breaking points $\alpha_1, \cdots, \alpha_m$ *and so, the function* $\omega \to \lambda\omega T - c^*(\omega)$ *is a polyhedral convex function with the same set of breaking points for every* $T > 0$. *Since the function*

$$x \to \frac{\lambda\sigma(bT)^2}{2\lambda(h+b)T + 2rx}$$

is convex on $(0, \infty)$ *for every* $T > 0$, *we obtain by relation (3.7) that the function* $x \to F(x, T)$ *is concave on* $(0, \infty)$ *for every* $T > 0$. *This implies using the polyhedral convexity of* $\omega \to \lambda\omega T - c^*(\omega)$ *with breaking points* $\alpha_1, \cdots, \alpha_m$ *that*

$$\min_{\omega\in\Omega} F(\lambda T\omega - c^*(\omega), T) = \min_{1\leq j\leq m} F(\lambda T\alpha_j - c^*(\alpha_j), T).$$

Since $c^*(\alpha_j) = -\beta_j$, *we obtain by relation (3.10) that the overall problem becomes*

$$z_1 = \min_{1\leq j\leq m}\left\{\lambda\alpha_j + \min_{T>0}\left\{\frac{a+\beta_j}{T} + \frac{b\sigma T}{2} - \frac{\lambda\sigma(bT)^2}{2(h+b+r\alpha_j)\lambda T + 2r\beta_j}\right\}\right\}.$$

Notice that solving m *convex optimization problems, and then taking the minimum of their optimum objective function values gives* z_1. *Moreover, if we also consider the no shortages case (see Example 3.2), then the problem can be solved analytically*

$$z_0 = \min_{1\leq j\leq m}\left\{\lambda\alpha_j + \frac{a+\beta_j}{T_j} + \frac{\sigma(h+r\alpha_j)T_j}{2} + \frac{r\sigma\beta_j}{2\lambda}\right\}$$

where

$$T_j := \sqrt{\frac{2(a+\beta_j)}{\sigma(h+r\alpha_j)}}.$$

The polyhedral concave production cost function is frequently used in the literature, especially when quantity discounting is applied to the cost structure. In fact, it exactly represents the incremental discounting case [22]. It is also important to note that the polyhedral concave functions can be used to approximate the general concave cost functions.

4 Multi-item Joint Replenishment Problem

We are now ready to analyze the multi-item joint replenishment problem (2.5). In this section the subscript i, ranging from 1 to n, is used to denote the items. Before analyzing the problem, we observe that the multi-item problem (2.5) is separable with respect to $k_i \in \mathbb{N}$ for $1 \leq i \leq n$. Therefore, we can rewrite the problem as

$$z_n := \min_{T>0} \left\{ \frac{A}{T} + \sum_{i=1}^{n} \{ \min_{k_i \in \mathbb{N}} \Phi_i(k_i T) \} \right\}.$$

If we further define the function $\Theta : [0, \infty) \to \mathbb{R}$ by

$$\Theta(T) := \frac{A}{T} + \sum_{i=1}^{n} \{ \min_{k_i \in \mathbb{N}} \Phi_i(k_i T) \}, \tag{4.1}$$

then the multi-item joint replenishment problem becomes

$$z_n = \min\{\Theta(T) : T > 0\}. \tag{P_n}$$

We are now interested in solving the above one-dimensional optimization problem. To achieve this, we should be able to compute $\Theta(T_0)$ for any given $T_0 > 0$, and so, by relation (4.1) we need to solve the inner optimization problem

$$\min_{k_i \in \mathbb{N}} \Phi_i(k_i T_0). \tag{4.2}$$

Unfortunately, the function $c(\cdot)$ in relation (2.3) is concave. Therefore, the objective function $\Phi_i(\cdot)$, in general, does not have a unimodal structure and hence, it might be difficult to solve the problem in (4.2). A remedy for this problem is to reformulate the function $\Phi_i(\cdot)$ by the dual approach as done in the previous section. If we use the subscript i for an item, relation (3.7) is revised as

$$F_i(x, T) := \frac{a_i + x}{T} + \frac{b_i \sigma_i T}{2} - \frac{\lambda_i \sigma_i (b_i T)^2}{2\lambda(h_i + b_i)T + 2rx}, \tag{4.3}$$

and by relations (3.9) and (4.3), we obtain

$$\Phi_i(T) = \min_{\omega \in \Omega_i} F_i(\lambda_i T \omega - c_i^*(\omega), T). \tag{4.4}$$

To simplify the exposition of the function $\Theta(\cdot)$, we define the function $H_i : \Omega_i \times (0, \infty) \to \mathbb{R}$ by

$$H_i(\omega, T) := F_i(\lambda_i T \omega - c_i^*(\omega), T). \tag{4.5}$$

Using now relations (4.1), (4.4) and (4.5) leads to

$$\begin{aligned} \Theta(T) &= \tfrac{A}{T} + \textstyle\sum_{i=1}^{n} \{ \min_{k_i \in \mathbb{N}} \min_{\omega \in \Omega_i} H_i(w, k_i T) \} \\ &= \tfrac{A}{T} + \textstyle\sum_{i=1}^{n} \{ \min_{\omega \in \Omega_i} \min_{k_i \in \mathbb{N}} H_i(w, k_i T) \}. \end{aligned} \tag{4.6}$$

If we further denote the optimal solution of the inner optimization problem by $k_i(\omega, T) \in \mathbb{N}$; *i.e.,*

$$H_i(\omega, k_i(\omega, T)T) = \min_{k_i \in \mathbb{N}} H_i(\omega, k_i T), \tag{4.7}$$

then problem (4.2) can be rewritten as

$$z_n = \min_{T>0} \Theta(T) = \min_{T>0} \left\{ \frac{A}{T} + \sum_{i=1}^{n} \{ \min_{\omega \in \Omega_i} H_i(w, k_i(\omega, T)T) \} \right\}. \tag{4.8}$$

Since we want to evaluate $\Theta(T_0)$, we should be able to compute for given $\omega \in \Omega_i$ the value $k_i(\omega, T_0)$ for all $1 \leq i \leq n$. Recall now from Lemma 3.1 that the function $T \to H_i(\omega, T^{-1})$ is a convex function on $(0, \infty)$ for every $\omega \in \Omega_i$, and so, for given $\omega \in \Omega_i$ the problem

$$\min_{T>0} H_i(\omega, T^{-1}) \tag{4.9}$$

is a one-dimensional convex optimization problem. If we denote the optimal solution of problem (4.9) by $\bar{T}_i(\omega)$ and the optimal solution of problem

$$\min_{T>0} H_i(\omega, T) \tag{4.10}$$

by $T_i(\omega)$, we obtain that $T_i(\omega) = 1/\bar{T}_i(\omega)$. It is clear by the convexity of the function $T \to H_i(\omega, T^{-1})$ that the function $T \to H_i(\omega, T)$ is increasing for $T > T_i(\omega)$ and decreasing for $T < T_i(\omega)$. Thus, for $T_0 > T_i(\omega)$ it is clear that $k_i(\omega, T_0) = 1$. On the other hand, for $T_0 < T_i(\omega)$ we need to find two integers k_i^- and k_i^+ (see Figure 3) such that

$$k_i^- T_0 \leq T_i(\omega) \leq k_i^+ T_0.$$

This leads to

$$k_i^- = \left\lfloor \frac{T_i(\omega)}{T_0} \right\rfloor \text{ and } k_i^+ = \left\lceil \frac{T_i(\omega)}{T_0} \right\rceil, \tag{4.11}$$

and consequently, for given $T_0 > 0$, we have

$$k_i(\omega, T_0) = \arg\min\{H_i(\omega, k_i^- T_0), H_i(\omega, k_i^+ T_0)\}. \tag{4.12}$$

Using relations (4.12) and (4.6), it is easy to evaluate $\Theta(T_0)$ whenever the sets Ω_i, $1 \leq i \leq n$ are finite. In Example 4.1 we demonstrate this observation for the case of polyhedral concave production cost functions.

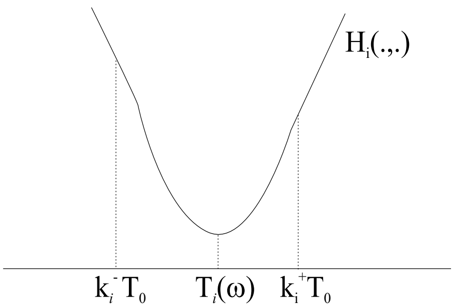

Figure 3: Calculating $k_i(\omega, T_0)$

Example 4.1 *As in Example 3.3, suppose that for each item i, $1 \leq i \leq n$, the cost functions $c_i(\cdot)$ are polyhedral concave. This implies that*

$$c_i(\lambda_i T) = \min_{1 \leq j \leq m_i} \{\alpha_{ij} \lambda_i T + \beta_{ij}\},$$

with $\alpha_{im_i} < \alpha_{im_i - 1} \cdots < \alpha_{i1}$ and $0 = \beta_{i1} < \beta_{i2} < \cdots < \beta_{im_i}$. In this special case, the multi-item joint replenishment problem (4.6) can be written as

$$z_n = \min_{T>0} \left\{ \frac{A}{T} + \sum_{i=1}^{n} \{ \min_{1 \leq j \leq m_i} H_i(\alpha_{ij}, k_i(\alpha_{ij}, T)T)\} \right\}.$$

Notice that the evaluation of $\Theta(T_0)$ for a given T_0 then becomes straightforward, since it only requires $\sum_{i=1}^{n} m_i$ calculations to find the values of $H_i(\alpha_{ij}, k_i(\alpha_{ij}, T_0)T_0)$ for all $1 \leq j \leq m_i$ and $1 \leq i \leq n$.

Although the multi-item replenishment problem (P_n) is, in general, a global optimization problem, it is to our advantage that the problem is only one-dimensional. To solve such a problem, there exist various efficient global optimization methods. Here we focus on an important class, namely, the Lipschitz optimization methods [23]. The main reason behind our choice is that for a given parameter ϵ, the Lipschitz optimization methods are guaranteed to return an ϵ-optimal objective function value; *i.e.*, the difference between the value found by the Lipschitz optimization method and the actual optimal objective function value, is less than or equal to ϵ.

There are three requirements for applying the one-dimensional Lipschitz method to an optimization problem. Firstly, there should be an oracle returning the value of the objective function for a given point. In the multi-item joint replenishment problem (P_n), this corresponds to calculating the value of $\Theta(\cdot)$ for any given $T > 0$. This issue has been already discussed. Secondly, the optimal solution of the problem should belong to a known bounded interval. Thirdly, on this interval the objective function should be Lipschitz with a known Lipschitz constant. We show in Appendix C that the optimal solution of the multi-item joint replenishment problem (P_n) indeed belongs to an interval denoted by $[T_l, T_u]$, and also we prove in the same appendix that the objective function (4.1) of this problem is Lipschitz on $[T_l, T_u]$ with a computable Lipschitz constant L.

We now outline an algorithm to solve problem (P_n) when the sets Ω_i, $1 \leq i \leq n$ are finite (see also Example 4.1). The parameter ϵ is predetermined by the decision maker and for this parameter, the proposed algorithm returns an ϵ-optimal objective function value.

Algorithm 4.1 Solving problem (P_n) for finite Ω_i, $1 \leq i \leq n$

1. Evaluate $T_i(\omega)$, for all $\omega \in \Omega_i$, $1 \leq i \leq n$.
2. Find the interval $[T_l, T_u]$.
3. Calculate the Lipschitz constant L for the objective function $\Theta(\cdot)$.
4. Apply a Lipschitz optimization algorithm [23] to problem (P_n).

5 Computational Results

We devote this section to present the computational results of the proposed solution procedure and to show that it is rather easy to implement. By solving various examples, we also present the sensitivity of the algorithm to different problem parameters. To implement the solution procedure, we have used MATLAB 6.1 on a Pentium III - 1 GHz personal computer. This is a straightforward implementation of Algorithm 4.1, which serves well for our purposes. Therefore, other than simple observations for performance improvement, we have not concentrated on improving the efficiency of the computer code. Our results have shown that even with this straightforward implementation, the procedure generates solutions in little time.

It is clear that the computational burden of Algorithm 4.1 comes from Step 4, where a Lipschitz optimization procedure is needed. In our implementation we have used the simplest Lipschitz optimization procedure, also called the *passive algorithm* [23], that evaluates the function at successive points $T_l + \epsilon/L,\ T_l + 2\epsilon/L,\ T_l + 3\epsilon/L, \cdots$, and returns the point at which the minimum value is found. We have also improved the passive algorithm by implementing the idea suggested by Evtushenko [see 23]; that is, at any iteration the next step is taken larger than ϵ/L, whenever the best function value realized up to the current iteration is at least ϵ above the function value found at the current iteration. Since the values $T_i(\omega)$ and $\Theta(T_i(\omega))$ are already calculated in Step 1 and Step 2 of Algorithm 4.1, at the first iteration we can set the best function value to $\min\{\Theta(T_i(\omega)) : T_i(\omega) \in [T_l, T_u],\ \ 1 \leq i \leq n\}$. Moreover, we observe by relations (C.7) and (C.9) that the Lipschitz constant depends on the values T_l and T_u. Therefore, if we can partition the interval $[T_l, T_u]$ into subintervals, for each interval we can calculate the corresponding Lipschitz constants and apply the Lipschitz optimization procedure within these subintervals. Notice that in this case the Lipschitz constants within each interval can be computed by the formulas given in Appendix C after replacing T_l and T_u by the lower and upper bounds of the subinterval, respectively. To partition the interval $[T_l, T_u]$, we can again use the $T_i(\omega) \in [T_l, T_u]$ values found in Step 1. In addition to Evtushenko's idea, we have also implemented this simple observation.

As mentioned in the previous section, in real life problems the polyhedral concave functions are used frequently. Since our main purpose is to illustrate the applicability of the proposed method, we have generated examples with polyhedral concave cost functions (see Example 4.1).

In the experimental setting we have considered 4 levels for both the number of items, n, and the major setup cost, A. For every combination of n and A, 25 problems have been randomly generated. The range of values for the parameters as well as the levels of the two factors; *i.e.*, n and A, are given in Table 1. Note that except the parameter m_i, all randomly generated parameters are continuous. The inventory carrying charge, r has been fixed to 0.15 in all problems. Each generated problem instance has been solved under three values of the precision parameter ϵ; 0.1, 0.01, 0.001.

Number of items	$n \in \{5, 10, 25, 50\}$
Major setup cost	$A \in \{1, 5, 8, 12\}$
Demand rate	$\lambda_i \sim U(0.1, 5)$ for $1 \leq i \leq n$
Production rate	$\mu_i \sim U(\lambda_i, \lambda_i + 5)$ for $1 \leq i \leq n$
Minor setup cost	$a_i \sim U(0.01, 5)$ for $1 \leq i \leq n$
Out-of-pocket holding cost	$h_i \sim U(0.01, 2)$ for $1 \leq i \leq n$
Number of breakpoints	$m_i \sim DU(2, 5)$ for $1 \leq i \leq n$
Slopes†	$\alpha_{ij} \sim U(0.01, 2)$ for $1 \leq i \leq n$, $1 \leq j \leq m_i$
Intercepts‡	$\beta_{i1} = 0$ and $\beta_{ij} \sim U(1, 10)$ for $1 \leq i \leq n$, $2 \leq j \leq m_i$
Backorder cost	$b_i = h_i + 20r\alpha_{i1}$ for $1 \leq i \leq n$

$U(lb, ub)$: Uniform distribution on $[lb, ub]$

$DU(lb, ub)$: Discrete uniform distribution on $\{lb, ub\}$

† For each i, $1 \leq i \leq n$, the values α_{ij} are sorted in ascending order with respect to j, $1 \leq j \leq m_i$

‡ For each i, $1 \leq i \leq n$, the values β_{ij} are sorted in descending order with respect to j, $2 \leq j \leq m_i$

Table 1: Experimental setting for the factors and the randomly generated parameters.

The raw results of our experimental setting are available upon request. To show the validity of our results, we have taken 5 problem instances with $n = 25$. Table 2 shows the optimal objective function values together with the optimal basic cycle lengths under different major setup cost values. The figures in Table 2 confirm the basic tradeoff between the major setup cost and the basic cycle time. That is, as the major setup cost becomes higher, the optimal basic cycle lengths increase. Naturally, the corresponding cost values also increase.

	Optimal Basic Cycle Length				**Optimal Average Cost**			
Instance	$\mathbf{A = 1}$	$\mathbf{A = 5}$	$\mathbf{A = 8}$	$\mathbf{A = 12}$	$\mathbf{A = 1}$	$\mathbf{A = 5}$	$\mathbf{A = 8}$	$\mathbf{A = 12}$
1	2.22	5.69	5.79	6.12	63.66	64.74	65.26	65.93
2	2.46	4.20	4.61	4.82	65.97	67.46	68.11	68.96
3	2.96	5.39	5.49	5.65	62.49	63.48	64.03	64.75
4	1.95	2.85	3.50	4.91	51.99	53.71	54.68	55.69
5	2.51	4.62	5.07	5.17	79.78	80.86	81.46	82.25

Table 2: The results for testing the validity of the proposed procedure ($n = 25$, $\epsilon = 0.01$).

Table 3 gives average running times in seconds (over 25 runs) for each combination of n, A and ϵ. As the figures in Table 3 show, no clear effect of the major setup cost on average running times can be observed. This is due to the fact that although increasing A increases both the lower bound, T_l and the upper bound, T_u, it may also increase the Lipschitz constants (see Appendix C). However, in many cases, especially the cases with large number of items, high values of A reduces the running time slightly.

On the other hand, increasing the precision and the number of items, yields high running times as expected. Moreover, with respect to n and ϵ, the growth in the running time is almost linear, and in all problems the solution is found within less than a minute. At this point, we note that the running times can be further reduced by efficient implementations. For instance, rewriting the procedure with a programming language such as `C++` could easily improve the performance.

		$\mathbf{A = 1}$	$\mathbf{A = 5}$	$\mathbf{A = 8}$	$\mathbf{A = 12}$
$\epsilon = \mathbf{0.1}$	$\mathbf{n = 5}$	0.64	0.62	1.04	1.64
	$\mathbf{n = 10}$	1.44	1.27	1.24	1.22
	$\mathbf{n = 25}$	8.85	7.13	6.85	6.91
	$\mathbf{n = 50}$	29.46	28.36	27.46	27.33
$\epsilon = \mathbf{0.01}$	$\mathbf{n = 5}$	1.78	1.05	1.43	2.08
	$\mathbf{n = 10}$	2.85	2.28	2.00	1.99
	$\mathbf{n = 25}$	18.86	10.00	9.29	8.90
	$\mathbf{n = 50}$	42.88	37.39	32.42	32.07
$\epsilon = \mathbf{0.001}$	$\mathbf{n = 5}$	8.34	2.10	2.39	3.35
	$\mathbf{n = 10}$	6.01	4.23	4.61	4.41
	$\mathbf{n = 25}$	56.42	17.78	15.89	15.34
	$\mathbf{n = 50}$	59.56	55.48	51.27	51.73

Table 3: The average solution times in seconds.

Remember that polyhedral concave functions are frequently used in applications for approximating general concave functions. As the number of affine functions, or equivalently the number of breakpoints, in the polyhedral concave function increases, the quality of the approximation improves. Therefore, from a practical point of view it is important to see in which way the number of breakpoints (m_i) affect the performance of the proposed procedure. To test the efficiency of the procedure with respect to varying number of breakpoints, we have selected three distinct ranges for the discrete uniform distribution. The values of A and ϵ are 8 and 0.01, respectively. For each combination of range and n, 25 problem instances have been generated using the setting given in Table 1 for the remaining parameters.

	$\mathbf{n = 5}$	$\mathbf{n = 10}$	$\mathbf{n = 25}$	$\mathbf{n = 50}$
$\mathbf{m_i \sim DU(2, 5)}$	1.35	1.88	9.58	31.38
$\mathbf{m_i \sim DU(6, 10)}$	2.84	6.41	35.40	136.86
$\mathbf{m_i \sim DU(11, 20)}$	6.19	18.09	114.56	368.52

Table 4: The average solution times in seconds ($A = 8$, $\epsilon = 0.01$).

The average running times are reported in Table 4, where each row corresponds to a different range and each column represents the number of items. It can be seen from Table 4 that there is a monotone relationship between the running time and the number of breakpoints. In addition to increasing the number of breakpoints, if we also increase the number of items, then the running times grow faster. This is due to the computational effort invested in the double loops (n by m_i) that have to be considered in the procedure. Nevertheless, using many breakpoints to get fairly accurate approximations, does not cost much with respect to running time even for the problems with a large number of items.

APPENDIX

A Conjugate Function

Since the function $c : [0, \infty) \to \mathbb{R}$ is concave, it follows that the function $c(\cdot)$ is continuous on $(0, \infty)$ and there are only a countable number of points where the function is nondifferentiable. Moreover, for every $y > 0$ both the right derivative

$$c'_+(y) := \lim_{x \downarrow y} \frac{c(x) - c(y)}{x - y}$$

and the left derivative

$$c'_-(y) := \lim_{x \uparrow y} \frac{c(x) - c(y)}{x - y}$$

exist. Also, for every $y_1 > y_2 > 0$ it holds that

$$c'_+(y_1) \leq c'_-(y_1) \leq c'_+(y_2) \tag{A.1}$$

and the functions $c_+(\cdot)$ and $c_-(\cdot)$ are right and left continuous on $(0, \infty)$, respectively. By the monotonicity property, expressed in relation (A.1), this implies that $c'_+(0) := \lim_{y \downarrow 0} c'_+(y)$ and $c'_-(\infty) = \lim_{y \uparrow \infty} c'_-(y)$ exist and for every $y > 0$ it follows that

$$c'_-(\infty) \leq c'_+(y) \leq c'_+(0). \tag{A.2}$$

Since it is also well-known that

$$c(x) - c(s) = \int_s^x c'_+(z)dz$$

for every $x > 0$ and $s > 0$ [24] and $0 = c(0) = \lim_{s \downarrow 0} c(s)$, we obtain by relation (A.2) and a limiting argument (take $s \downarrow 0$) that

$$c(x) \geq c'_-(\infty)x \tag{A.3}$$

for every $x > 0$. Introducing now for the concave function $h : \mathbb{R} \to [-\infty, \infty)$, its so-called conjugate function $h^* : \mathbb{R} \to [-\infty, \infty]$ given by

$$h^*(\omega) := \inf\{\omega x - h(x) : x \in \mathbb{R}\},$$

one can show the following dual representation for the function $c(\cdot)$.

Lemma A.1 *If* $c : [0, \infty) \to \mathbb{R}$ *is a concave function satisfying* $0 = c(0) = \lim_{x \downarrow 0} c(x)$, *then it follows for every* $x \geq 0$ *that*

$$c(x) = \inf\{\omega x - c^*(\omega) : c'_-(\infty) \leq \omega \leq c'_+(0)\}$$

with $c^*(\cdot)$ *the conjugate function of* $c(\cdot)$ *reduced to*

$$c^*(\omega) = \inf\{\omega x - c(x) : x \geq 0\}.$$

Proof. Extending the concave function $c : [0, \infty) \to \mathbb{R}$ to $\mathbb{R}$ by introducing $c(x) = -\infty$ for every $x < 0$ the extended function $c : \mathbb{R} \to [-\infty, \infty)$ is also concave on $\mathbb{R}$. Moreover, since c is continuous on $(0, \infty)$ and $c(0) = \lim_{x \downarrow 0} c(x)$, the upper level sets

$$U(c, s) := \{x \in \mathbb{R} : c(x) \geq s\}$$

are closed for every $s \in \mathbb{R}$ and so the extended function $c : \mathbb{R} \to [-\infty, \infty)$ is also upper semicontinuous. Hence we may apply the biconjugate or Fenchel-Moreau theorem [25] and so it follows that

$$c(x) = \inf\{\omega x - c^*(\omega) : \omega \in \mathbb{R}\}. \tag{A.4}$$

Looking now at the conjugate function $c^*(\cdot)$ of $c(\cdot)$ we obtain due to $c(0) = 0$ that $c^*(\omega) \leq 0$ for every ω and this shows using $c(x) = -\infty$ for $x < 0$ and $c^*(\omega) \leq 0$ for every ω, that

$$c^*(\omega) = \inf\{\omega x - c(x) : x \geq 0\}.$$

Also, if $\omega < c'_-(\infty)$, it follows by relations (A.3) and (A.4) that $c^*(\omega) \leq \omega x - c(x) \leq (\omega - c'_-(\infty))x$ for every $x > 0$ and so $c^*(\omega) = -\infty$ for every $\omega < c'_-(\infty)$. Moreover if $\omega > c'_+(0)$ we obtain by the subgradient inequality for convex functions that

$$\omega x - c(x) \geq (\omega - c'_+(0))x$$

for every $x \geq 0$ and so $0 \geq c^*(\omega) \geq \inf\{(\omega - c'_+(0))x : x \geq 0\} = 0$. Substituting this in relation (A.4) and using $c(\cdot)$ is finite on $[0, \infty)$ the desired representation follows. □

B Structure of Problem (3.10)

Lemma B.1 *The function $v : [0, \infty) \times (0, \infty) \to \mathbb{R}$ (see also relation (3.11)) given by*

$$v(x, y) := \min_{T>0} \left\{ \frac{a + y}{T} + \frac{b\sigma T}{2} - \frac{\lambda\sigma(bT)^2}{2\lambda T(h + b + rx) + 2ry} \right\}$$

is a concave function on $[0, \infty) \times (0, \infty)$.

Proof. Let $v_T : [0, \infty) \times (0, \infty) \to \mathbb{R}$ be a real valued function given by

$$v_T(x, y) := \frac{a + y}{T} + \frac{b\sigma T}{2} - \frac{\lambda\sigma(bT)^2}{2\lambda T(h + b + rx) + 2ry}.$$

Clearly for every $T > 0$, the function $(x, y) \to 2\lambda T(h + b + rx) + 2ry$ is linear and positive on the convex set $[0, \infty) \times (0, \infty)$. Hence, by Avriel *et. al.* [20, Corollary 5.18] we obtain that the function

$$(x, y) \to \frac{(bT)^2}{2\lambda T(h + b + rx) + 2ry}$$

is convex on $[0, \infty) \times (0, \infty)$. This implies that the function $v_T(\cdot, \cdot)$ is a concave function on $[0, \infty) \times (0, \infty)$. Since $v(\cdot, \cdot)$ is a minimum of concave functions, it is also a concave function and thus, the desired result follows. □

C Bounding Interval and Lipschitz Constants

To apply Lipschitz optimization, we should first construct an interval $[T_l, T_u]$ that contains the optimal solution T^* of problem (P_n). After this construction we need to compute a Lipschitz constant for the objective function in (4.1) on this interval.

The objective function of the multi-item joint replenishment problem is given by

$$\Theta(T) = \frac{A}{T} + \sum_{i=1}^{n} \{\min_{k_i \in \mathbb{N}} \Phi_i(k_i T)\}. \tag{C.1}$$

Before finding the bounds T_l and T_u, we first select some $T_0 > 0$ and compute $\Theta(T_0)$. This allows us to consider the lower level set

$$L(\Theta(T_0)) := \{T : \Theta(T) \leq \Theta(T_0)\}.$$

Clearly the optimal solution T^* belongs to $L(\Theta(T_0))$. Recall that $T_i(\omega)$ is the optimal solution of problem (4.10) for $\omega \in \Omega_i$, $1 \leq i \leq n$. Using now relations (4.4) and (4.5) leads to

$$\Phi_i(T) = \min_{\omega \in \Omega_i} H_i(\omega, T) \geq \min_{\omega \in \Omega_i} H_i(\omega, T_i(\omega)) \tag{C.2}$$

for all $T > 0$. To compute a lower bound T_l for any element T belonging to $L(\Theta(T_0))$, we observe by relations (C.1) and (C.2) that

$$\frac{A}{T} + \sum_{i=1}^{n} \{\min_{\omega \in \Omega_i} H_i(\omega, T_i(\omega))\} \leq \Theta(T) \leq \Theta(T_0).$$

Therefore, the lower bound can be computed as

$$T_l = \frac{A}{\Theta(T_0) - \sum_{i=1}^{n} \{\min_{\omega \in \Omega_i} H_i(\omega, T_i(\omega))\}},$$

which implies that a larger lower bound can be gained by decreasing the value of $\Theta(T_0)$. Since we have the individual solutions $T_i(\omega)$, one possible way is to set

$$T_0 = \arg\min\{\Theta(T_i(\omega)) : \omega \in \Omega_i, 1 \leq i \leq n\}.$$

To compute an upper bound, T_u, we again use the optimal solutions $T_i(\omega)$. Since the function $T \rightarrow H_i(\omega, T)$ is increasing for $T > T_i(\omega)$, it follows by relation (4.7) that $k_i(\omega, T) = 1$, and hence

$$H_i(\omega, T) = \min_{k_i \in \mathbb{N}} H_i(\omega, k_i T) \tag{C.3}$$

for every $T > T_i(\omega)$. By relations (4.4) and (4.5), we know for every $T > 0$ that

$$\Phi_i(T) = \min_{\omega \in \Omega_i} H_i(\omega, T). \tag{C.4}$$

This implies using (C.3) and (C.4) that

$$\begin{aligned} \min_{k_i \in \mathbb{N}} \Phi_i(k_i T) &= \min_{k_i \in \mathbb{N}} \min_{\omega \in \Omega_i} H_i(\omega, k_i T) \\ &= \min_{\omega \in \Omega_i} \min_{k_i \in \mathbb{N}} H_i(\omega, k_i T) \\ &= \min_{\omega \in \Omega_i} H_i(\omega, T) = \Phi_i(T) \end{aligned} \tag{C.5}$$

for every $T > \hat{T} := \max\{T_i(\omega) : \omega \in \Omega_i, 1 \leq i \leq n\}$. Moreover, since $T \rightarrow H_i(\omega, T)$ is increasing on $[\hat{T}, \infty)$ for every i and $\omega \in \Omega_i$, we also obtain that the functions $\Phi_i(\cdot)$ are increasing on $[\hat{T}, \infty)$. This shows that the function $T \rightarrow \sum_{i=1}^{n} \Phi_i(T)$ is increasing on $[\hat{T}, \infty)$, and since $\lim_{T \rightarrow \infty} \Phi_i(T) = \infty$ for every $1 \leq i \leq n$, we can always find some $T_u > \hat{T}$ satisfying

$$\sum_{i=1}^{n} \Phi_i(T) \geq \sum_{i=1}^{n} \Phi_i(T_u) \geq \Theta(T_0)$$

for every $T \geq T_u$. Using now relation (C.5) shows

$$\Theta(T) \geq \sum_{i=1}^{n} \Phi_i(T) \geq \Theta(T_0)$$

for every $T \geq T_u$. Thus, for every T belonging to $L(\Theta(T_0))$ we obtain $T \leq T_u$. The value of T_u can be found by doing a simple line search on $[\hat{T}, \infty)$.

Next we compute the Lipschitz constant for the objective function (C.1). Suppose that L_0 is the Lipschitz constant for the function $T \rightarrow A/T$ on $[T_l, T_u]$ and L_i are the Lipschitz constants for the functions $T \rightarrow \min_{k_i \in \mathbb{N}} \Phi_i(k_i T)$, $1 \leq i \leq n$ on the same interval. If we denote now the Lipschitz constant of the objective function (C.1) on $[T_l, T_u]$ by L, then it is clear that

$$L = L_0 + \sum_{i=1}^{n} L_i. \tag{C.6}$$

Since the function $T \rightarrow A/T$ is convex and differentiable on $[T_l, T_u]$, the Lipschitz constant L_0 is given by the maximum of its derivative in absolute values on this interval. That is

$$L_0 = \frac{A}{T_l^2}. \tag{C.7}$$

By relations (4.7) and (C.5), the values L_i are also the Lipschitz constants for the functions $T \rightarrow \min_{\omega \in \Omega_i} H_i(\omega, k_i(\omega, T)T)$ on $[T_l, T_u]$. Recall from relations (4.11) and (4.12) that given $T > 0$

$$\left\lfloor \frac{T_i(\omega)}{T} \right\rfloor \leq k_i(\omega, T) \leq \left\lceil \frac{T_i(\omega)}{T} \right\rceil.$$

This implies for $T \in [T_l, T_u]$ that

$$\left\lfloor \frac{T_i(\omega)}{T_u} \right\rfloor \leq \left\lfloor \frac{T_i(\omega)}{T} \right\rfloor \leq k_i(\omega, T) \leq \left\lceil \frac{T_i(\omega)}{T} \right\rceil \leq \left\lceil \frac{T_i(\omega)}{T_l} \right\rceil.$$

Therefore, if we define the finite index set

$$\mathcal{K}_i(\omega) := \left\{ k \in \mathbb{N} : \left\lfloor \frac{T_i(\omega)}{T_u} \right\rfloor \leq k \leq \left\lceil \frac{T_i(\omega)}{T_l} \right\rceil \right\},$$

then for any $T \in [T_l, T_u]$ we have

$$\min_{k_i \in \mathbb{N}} H_i(\omega, k_i T) = \min_{k_i \in \mathcal{K}_i(\omega)} H_i(\omega, k_i T).$$

Consequently, if we denote the Lipschitz constants for the functions $T \rightarrow H_i(\omega, k_i T)$ by $L_i(\omega, k_i)$, then by relation (C.5) we obtain

$$L_i = \max_{\omega \in \Omega_i} \max_{k_i \in \mathcal{K}_i(\omega)} L_i(\omega, k_i). \tag{C.8}$$

To find the Lipschitz constants $L_i(\omega, k_i)$, $1 \leq i \leq n$, we need the following auxiliary result.

Lemma C.1 *If the function* $g : [a, b] \rightarrow \mathbb{R}$, $a > 0$ *is differentiable on* $[a, b]$ *and* $h(x) := g(x^{-1})$ *is a convex function on* $[1/b, 1/a]$*, then it follows that*

$$| g(x) - g(y) | \leq L_g \mid y - x \mid$$

with the Lipschitz constant

$$L_g = \max\{(b/a)^2 \mid g'(b) \mid, \mid g'(a) \mid\}.$$

Proof. Let L_h denote the Lipschitz constant for $h(\cdot)$ on $[1/b, 1/a]$. Since $h(\cdot)$ is a convex function on $[1/b, 1/a]$, the Lipschitz constant is given by

$$L_h = \max\{\mid h'(1/b) \mid, \mid h'(1/a) \mid\} = \max\{\mid b^2 g'(b) \mid, \mid a^2 g'(a) \mid\}.$$

Moreover, for $x, y \in [a, b]$ we have

$$\begin{aligned} \mid g(x) - g(y) \mid \quad &= \mid h(x^{-1}) - h(y^{-1}) \mid \\ &\leq L_h \mid x^{-1} - y^{-1} \mid \\ &\leq \tfrac{L_h}{xy} \mid x - y \mid \\ &\leq \tfrac{L_h}{a^2} \mid x - y \mid . \end{aligned}$$

By setting $L_g := \frac{L_h}{a^2}$, the desired result follows. □

Notice by Lemma 3.1 that the functions $T \to H_i(\omega, (k_i T)^{-1})$ are convex functions. Therefore, by Lemma C.1 we obtain

$$L_i(\omega, k_i) = \max\left\{ \frac{k_i T_u^2}{T_l^2} \mid H_i'(\omega, k_i T_u) \mid, k_i \mid H_i'(\omega, k_i T_l) \mid \right\} \tag{C.9}$$

Combining now relations (C.6), (C.7),(C.8) and (C.9) gives the Lipschitz constant for the objective function of the joint replenishment problem.

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