# Bemerkungen

## Income Distribution: A Correction and a Generalization

By

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In an article published in this journal by the second author of the present note in 1956[1] the problem he proposed to tackle was how earnings must depend on the frequency distributions' parameters of the supply of and the demand for labour of different quality. The quality of the labour supplied as well as the quality required in order properly to perform the jobs available were supposed to be *two-dimensional* and the frequency distributions of both to be the normal or Gaussian distribution. The parameters appearing in the distribution of qualities required, hence the demand side were: $\bar{s}_1$, the average intensity or degree of capability 1, $\bar{s}_2$ that for capability 2, $\sigma_1$ the standard deviation of the $s_1$, $\sigma_2$ that of the $s_2$ and $r_s$ the correlation coefficient between $s_1$ and $s_2$. Similarly, for the capabilities of the workers supplying their labour the parameters $\bar{t}_1$, $\bar{t}_2$, $\tau_1$, $\tau_2$ and $r_t$ were assumed to apply.

The way in which workers choose their job was specified by *assuming their maximization of a utility function*[2]:

$$(5.1) \quad \omega = \omega_0 \log l - \frac{1}{2}\omega_1(s_1-t_1)^2 - \frac{1}{2}\omega_2(s_2-t_2)^2$$

The main finding of the 1956 article was that an income scale for log l of a *quadratic form* was sufficient to establish equilibrium:

$$(5.5) \quad \log l = \lambda\,(s_1, s_2) = \lambda_{00} + \lambda_{10}\, s_1 + \lambda_{01}\, s_2 + \frac{1}{2}\lambda_{20}\, s_1^2 + \lambda_{11}\, s_1\, s_2 + \frac{1}{2}\lambda_{02}\, s_2^2$$

Expressions for the λs, specifying the income scale were given in the 1956 article. They must be functions of the parameters characterizing the demand and supply frequency distributions and show some interesting features of equilibrium income scales.

The first author of this note discovered some errors in the solutions proposed by the second author. He was also able to give the general solution, whereas the second author had only presented solutions for particular values of the correlation coefficients $r_s$ and $r_t$.

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[1] Tinbergen, Jan, "On the Theory of Income Distribution". *Weltwirtschaftliches Archiv, Vol. 77,* 1956, pp. 155–175.

[2] Formulae numbers refer to the numbering in Tinbergen, *ibid.*

Since in the last few years more statistical data on at least one capability required have become available it seemed worthwhile to present, in this brief note, the corrected and generalized solutions found by the first author. Starting point for the mathematical part of Tinbergen's article are the formulae (6.1) and (6.2) on page 161 in which the bivariate normal densities of ($s_1$, $s_2$) and ($t_1$, $t_2$), respectively, are defined:

$$(6.1)\quad \frac{m_0}{2\pi\sigma_1\sigma_2\sqrt{1-r_s^2}} \exp \frac{-{}^1/_2}{1-r_s^2}\left[\frac{(s_1-\bar{s}_1)^2}{\sigma_1^2} - 2r_s \frac{(s_1-\bar{s}_1)(s_2-\bar{s}_2)}{\sigma_1\sigma_2} + \frac{(s_2-\bar{s}_2)^2}{\sigma_2^2}\right]$$

$$(6.2)\quad \frac{n_0}{2\pi\tau_1\tau_2\sqrt{1-r_t^2}} \exp \frac{-{}^1/_2}{1-r_t^2}\left[\frac{(t_1-\bar{t}_1)^2}{\tau_1^2} - 2r_t \frac{(t_1-\bar{t}_1)(t_2-\bar{t}_2)}{\tau_1\tau_2} + \frac{(t_2-\bar{t}_2)^2}{\tau_2^2}\right]$$

Out of the maximization of the utility function (5.1),together with the income scale (5.5), equations between $s_i$ and $t_i$ were derived: (5.6) and (5.7), with (5.8): $\omega_3' = \omega_0/\omega_1$ and $\omega_3'' = \omega_0/\omega_2$ (compare 5.8 of the original article).

Using $t_i' = t_i - \bar{t}_i$ and $s_i' = s_i - \bar{s}_i$ and the reformulations (6.4) and (6.5) the links between the two distributions can be written as (6.6) and (6.7). The exponent of the density n ($t_1$, $t_2$), with $\tau_1 = \tau_2 = 1$, can now be written as:

$$\frac{-{}^1/_2}{1-r_t^2}\,[t_1'^2 - 2r_t t_1' t_2' + t_2'^2]\text{, or}$$

$$\frac{-{}^1/_2}{1-r_t^2}\,[s_1'^2\,(x^2 + 2\alpha r_t xy + \alpha^2 y^2) + s_1' s_2'\,(-2xy - 2r_t xz - 2r_t \alpha y^2 - 2\alpha yz) + s_2'^2\,(y^2 + 2r_t yz + z^2)]$$

(compare 6.8)

This expression has to be identical to the exponent of m ($s_1$, $s_2$) which can be written as

$$\frac{-{}^1/_2}{1-r_s^2}\left[\frac{s_1'^2}{\sigma_1^2} - 2r_s \frac{s_1' s_2'}{\sigma_1\sigma_2} + \frac{s_2'^2}{\sigma_2^2}\right] = \frac{-{}^1/_2}{1-r_t^2}\left[\rho_1^2 s_1'^2 - 2r_s\rho_1\rho_2 s_1' s_2' + \rho_2^2 s_2'^2\right]$$

(see 6.9)

with $\rho_1$ and $\rho_2$ defined in (6.10).

A comparison of the coefficients of $s_1'^2$, $s_1'$ $s_2'$ and $s_2'^2$, respectively, yields the model (6.11) - (6.13):

$$(6.11)\quad x^2 + 2\alpha r_t xy + \alpha^2 y^2 = \rho_1^2$$

$$(6.12)\quad \alpha r_t y^2 + (x+\alpha z)\, y + r_t xz = r_s \rho_1 \rho_2$$

$$(6.13)\quad z^2 + 2r_t yz + y^2 = \rho_2^2$$

which can be solved for three different choices of ($r_s$, $r_t$):

(i) $r_s = 0$, $r_t = 0$. This is the situation in which the results are interpreted by Tinbergen in his Section VII.

From (6.12) it is easy to see that $y = 0$, yielding $x = \rho_1 = \sigma_1^{-1}$ and $z = \rho_2 = \sigma_2^{-1}$, is the only feasible solution, provided that x and z should be nonnegative.

(ii) $r_s \neq 0$, $r_t = 0$. This is the situation for which Tinbergen presents an – erroneous – solution.

(6.16) $x^2 + \alpha^2 y^2 = \rho_1^2$ (compare 6.16 of the original article)

(6.17) $(x + \alpha z)y = r_s \rho_1 \rho_2$

(6.18) $y^2 + z^2 = \rho_2^2$

$$\left.\begin{array}{ll} \text{From (6.17)}: & x + \alpha z = \dfrac{r_s\rho_1\rho_2}{y} \\ \text{(6.16)}: & x^2 + \alpha^2 y^2 = \rho_1^2 \\ \text{(6.18)}: & \alpha^2 z^2 + \alpha^2 y^2 = \alpha^2 \rho_2^2 \\ \hline & (x+\alpha z)(x - \alpha z) = \rho_1^2 - \alpha^2\rho_2^2 \end{array}\right\} \begin{array}{l} x - \alpha z = \dfrac{[\rho_1^2 - \alpha^2\rho_2^2]}{r_s\rho_1\rho_2} y \\ x + \alpha z = \dfrac{r_s\rho_1\rho_2}{y} \\ \hline \qquad\qquad\qquad + \\ x = \dfrac{(\rho_1^2 - \alpha^2\rho_2^2)\, y}{2 r_s \rho_1\rho_2} + \dfrac{r_s\rho_1\rho_2}{2y} \end{array} \quad (6.22)$$

$$\left.\begin{array}{l} \text{or } x^2 = \left[\dfrac{\rho_1^2\alpha^2\rho_2^2}{2r_s\rho_1\rho_2}\right]^2 y^2 + \dfrac{[r_s\rho_1\rho_2]^2}{4y^2} + \dfrac{\rho_1^2 - \alpha^2\rho_2^2}{2} \\ \text{whereas (6.16) gives } x^2 = \rho_1^2 - \alpha^2 y^2 \end{array}\right\} \rightarrow$$

which gives us a correct formulation of (6.24), using the transformations from squares to capital letters (see 6.23):

$$Y^2\,[(P_1 - AP_2)^2 + 4AR_sP_1P_2] + Y\,[-2(P_1 + AP_2)R_sP_1P_2] + R_s^2P_1^2P_2^2 = 0$$

The solution for Y should have read:

$$(6.25)\quad Y = \frac{R_sP_1P_2[(P_1 + AP_2) + 2\sqrt{AP_1P_2\,(1 - R_s)}]}{(P_1 - AP_2)^2 + 4AR_sP_1P_2}$$

This would have yielded

$$(6.26)\quad y^2 = \frac{r_s^2}{1 - r_s^2} K$$

$$(6.27)\quad x^2 = \frac{1}{1 - r_s^2}\left[\frac{1}{\sigma_1^2} - \frac{\omega_1^2}{\omega_2^2} r_s^2 K\right] \text{ and}$$

$$(6.28)\quad z^2 = \frac{1}{1 - r_s^2}\left[\frac{1}{\sigma_2^2} - r_s^2 K\right]$$

$$\text{with } K = \omega_2^2 \cdot \frac{\omega_1^2\sigma_1^2 + 2\omega_1\omega_2\sqrt{1-r_s^2}\;\sigma_1\sigma_2 + \omega_2^2\sigma_2^2}{\omega_1^4\sigma_1^4 + 2\omega_1^2\omega_2^2(2r_s^2-1)\sigma_1^2\sigma_2^2 + \omega_2^4\sigma_2^4}$$

(iii) $r_s \neq 0$, $r_t \neq 0$, the generalized solution.

From (6.11) it follows that $x = -\alpha r_t y \pm \sqrt{\rho_1^2 - \alpha^2(1-r_t^2)y^2}$

From (6.13) it follows that $z = -r_t y \pm \sqrt{\rho_2^2 - (1-r_t^2)y^2}$

In (6.12) the "solutions" for x and z can be filled in. This yields as equation:

$$a_1y^2 + a_2y + a_3 = 0$$

with $a_1 = -\alpha r_t(1-r_t^2)$

$a_2 = (1-r_t^2)\,[\pm\sqrt{\rho_1^2 - \alpha^2(1-r_t^2)y^2} \pm \sqrt{\alpha^2\rho_2^2 - \alpha^2(1-r_t^2)y^2}]$

$a_3 = \alpha^{-1}r_t\,[\pm\sqrt{(\rho_1^2 - \alpha^2(1-r_t^2)y^2)(\alpha^2\rho_2^2 - \alpha^2(1-r_t^2)y^2)}\,] - r_s\rho_1\rho_2$

This equation can be rewritten in a way that the left-hand side contains all terms with square roots and the right-hand side contains all other terms. Squaring both sides, rearranging in the same manner and squaring again eventually leads to an equation $b_3y^4 + b_4y^2 + b_5 = 0$

with $b_3 = (1-r_t^2)^2\,[\rho_1^4 + 4\alpha r_s r_t \rho_1^3\rho_2 + 2\alpha^2(2r_s^2 + 2r_t^2 - 1)\rho_1^2\rho_2^2 + 4\alpha^3 r_s r_t \rho_1\rho_2^3 + \alpha^4\rho_2^4]$

$b_4 = (1-r_t^2)\;[\rho_1^4\rho_2^2\{-2\{r_s^2(1-r_t^2) + r_t^2(1-r_s^2)\}\} + \rho_1^3\rho_2^3 4\alpha r_s r_t(r_s^2 + r_t^2 - 2)$
$+ \alpha^4\rho_1^2\rho_2^4\{-2\,\{r_s^2(1-r_t^2) + r_t^2(1-r_s^2)\}\}\,]$

$b_5 = \rho_1^4\rho_2^4(r_s^2 - r_t^2)^2$

This quadratic form in $y^2$ has a solution:

$$y^2 = \frac{c_1\rho_1^4\rho_2^2 + c_2\rho_1^3\rho_2^3 + c_3\rho_1^2\rho_2^4}{c_4\rho_1^4 + c_5\rho_1^3\rho_2 + c_6\rho_1^2\rho_2^2 + c_7\rho_1\rho_2^3 + c_8\rho_2^4}$$

In order to save space, the coefficients c will not be shown. Eliminating $\rho_1$ and $\rho_2$ (using 6.10) gives an – almost similar – solution for $y^2$:

$$y^2 = \frac{a_1\sigma_1^2 + a_2\sigma_1\sigma_2 + a_3\sigma_2^2}{b_1\sigma_1^4 + b_2\sigma_1^3\sigma_2 + b_3\sigma_1^2\sigma_2^2 + b_4\sigma_1\sigma_2^3 + b_5\sigma_2^4}$$

with $a_1 = \alpha^2[r_s\sqrt{1-r_t^2} \pm r_t\sqrt{1-r_s^2}]$

$a_2 = 2\alpha\,[r_s r_t\,\{(1-r_s^2) + (1-r_t^2)\} + (r_s^2 + r_t^2)\{\pm\sqrt{(1-r_s^2)(1-r_t^2)}\}]$

$a_3 = r_s\sqrt{1-r_t^2} \pm r_t\sqrt{1-r_s^2}$

$b_1 = (1-r_s^2)\alpha^4$

$b_2 = (1-r_s^2)4\alpha^3 r_s r_t$

$$b_3 = (1-r_s^2)2\alpha^2(2r_s^2+2r_t^2-1)$$
$$b_4 = (1-r_s^2)4\alpha r_s r_t$$
$$b_5 = (1-r_s^2)$$

Solutions for $x^2$ and $z^2$ can now be deduced, but their formulation is not very elegant. Moreover, their solutions are not as interesting as the solution for $\lambda_{11}$ which can be deduced from:

$$\lambda_{11} = \frac{\omega_1}{\omega_0} y$$

There is, however, a way to express $\lambda_{20}$ and $\lambda_{02}$ as functions of $\lambda_{11}$, which makes solutions for x and z superfluous. By rewriting (6.11) and (6.13), not in x, y and z but in λ's, solutions for these λ's can be given:

$$\lambda_{20} = \frac{\omega_2}{\omega_0} + \frac{\omega_1}{\omega_2} r_t\lambda_{11} \pm \frac{\omega_1}{\omega_0}\sqrt{1-r_t^2}\sqrt{\frac{1}{\sigma_1^2}\frac{1}{1-r_s^2} - \frac{\omega_0^2}{\omega_2^2}\lambda_{11}^2}$$

and

$$\lambda_{02} = \frac{\omega_2}{\omega_0} + \frac{\omega_2}{\omega_1} r_t\lambda_{11} \pm \frac{\omega_2}{\omega_0}\sqrt{1-r_t^2}\sqrt{\frac{1}{\sigma_2^2}\frac{1}{1-r_s^2} - \frac{\omega_0^2}{\omega_1^2}\lambda_{11}^2}$$