Second order regular variation and the domain of attraction of stable distributions

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Abstract

We characterize second order regular variation of the tail sum of F together with a balance condition on the tails in terms of the behaviour of the characteristic function near zero.

Keywords: Domain of attraction, (generalised) second order regular variation.

1 Introduction and main results

Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution function F. It is well known that F is in the domain of attraction of a stable law with index $\alpha \in (0,2)$ if and only if the function $1-F(x)$ is regularly varying with index $-\alpha$, $\alpha > 0$ (written $1-F \in RV(-\alpha)$) if

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha}, x > 0. \tag{1}$$

and the tail balance condition

$$\lim_{t\to\infty} \frac{1-F(t)}{1-F(t)+F(-t)} \to p \in [0,1], \tag{2}$$

holds. In that case there exist sequences $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} P(\sum_{i=1}^{n} X_i/a_n - b_n \leq x) = G_\alpha(x), x \in \mathbb{R} \tag{3}$$

where G_α is the distribution function of a stable distribution.

The rate of convergence in (3) is studied by many authors. Cramèr [2] considers the case where $1-F(t)+F(-t)$ is a power function and a remainder term. His work is followed by Ibragimov and Linnik [15], Hall [13, 14], Christoph and Wolf [3] and others. Instead of power functions with a remainder term it is a natural extension of (1) to consider second order regular variation.

The function $1-F$ is second order regularly varying with first order parameter $-\alpha$ and auxiliary function a if there exists a function $a(t) \to 0, t \to \infty$ which ultimately has constant sign such that

$$\lim_{t\to\infty} \frac{\frac{1-F(tx)}{1-F(t)} - x^{-\alpha}}{a(t)} = M(x) := cx^{-\alpha}\int_1^x u^{\rho-1}du, x > 0 \text{ for } c \neq 0. \tag{4}$$

Notation: $1-F \in 2RV(-\alpha, \rho, a)$.

Observe that for $x > 0$

$$M(x) = \begin{cases} cx^{-\alpha}\log x, & \text{if } \rho = 0, \\ cx^{-\alpha}\frac{x^\rho - 1}{\rho}, & \text{if } \rho < 0. \end{cases}$$

It is well known ([5], Theorem 1.9) that if (4) holds with $M(x)$ not a multiple of $x^{-\alpha}$, then M satisfies the above representation, $|a| \in RV(\rho)$ with $\rho \leq 0$ and no other choices of ρ are consistent with $a(t) \to 0$. Moreover convergence in (4) is uniform in x on compact intervals of $(0, \infty)$.

Under the condition of second order regular variation a natural refinement of (2) is the second order tail balance condition

$$\lim_{t\to\infty} \frac{\frac{1-F(t)}{1-F(t)+F(-t)} - p}{a(t)} = r \tag{5}$$

for some constants $p \in [0,1]$ and $r \in \mathbb{R}$.

Recently in [9] the asymptotic behavior of the sequence

$$q_n := \sup_{x \in \mathbb{R}} |P(\sum_{i=1}^{n} X_i/a_n - b_n \leq x) - G_\alpha(x)|$$

under conditions (4) and (5) is given in terms of the function a and the sequence a_n.

In this paper we consider the behavior of the characteristic function of distribution functions satisfying (4) and (5).

Instead of condition (4) on the right tail it is more convenient to use the second order regular variation condition on the tail sum $H(t) = 1 - F(t) + F(-t)$. It is easy to see (Lemma 2 below) that under the tail balance condition (5) both conditions are equivalent if $p \neq 0$. The main results in this paper are extensions to second order regular variation of results by Pitman [16].

As in the first order case, which is studied in [4], the result for $\alpha = 2$ (given in theorem 2) is different from the $\alpha < 2$ case.

Before we formulate the theorems we define the following constants

$$s_\alpha = \int_0^\infty x^{-\alpha} \sin x dx$$

$$c_\alpha = \int_1^\infty x^{-\alpha} \cos x dx + \int_0^1 x^{-\alpha}(\cos x - 1) dx$$

$$S_{\alpha,\rho} = \int_0^\infty x^{-\alpha} \frac{x^\rho - 1}{\rho} \sin x dx$$

$$C_{\alpha,\rho}(p,r) = \int_1^\infty \{(2p-1)x^{-\alpha}\frac{x^\rho - 1}{\rho} + 2rx^{\rho-\alpha}\} \cos x dx + \int_0^1 \{(2p-1)x^{-\alpha}\frac{x^\rho - 1}{\rho} + 2rx^{\rho-\alpha}\}(\cos x - 1) dx.$$

Suppose the random variable X has distribution function F and expectation EX. We use the following notation:

$$H(t) = 1 - F(t) + F(-t)$$

$$K(t) = 1 - F(t) - F(-t)$$

$$\lambda(t) = Ee^{itX} = U(t) + iV(t)$$

Theorem 1 *Suppose $1 < \alpha < 2, \rho \le 0, \alpha - \rho < 2$. Consider the following statements.*

(i) $H(t) = 1 - F(t) + F(-t) \in 2RV(-\alpha, \rho, a_1)$ *and*

$$\lim_{t\to\infty} \frac{\frac{1-F(t)}{H(t)} - p}{a_1(t)} \to r \tag{6}$$

as $t \to \infty$ for some constants $p \in [0,1]$ and $r \in \mathbb{R}$.

(ii) $1 - U(1/t) \in 2RV(-\alpha, \rho, a_1)$ *and there exist $c_\alpha^* \in [-(1+(1-\alpha)c_\alpha)/s_\alpha, (1+(1-\alpha)c_\alpha)/s_\alpha]$, $d_\alpha^*(x)$ such that for $x > 0$*

$$\frac{\frac{txV(1/tx) - tV(1/t)}{t(1-U(1/t))} - c_\alpha^* \frac{x^{1-\alpha}-1}{1-\alpha}}{a_1(t)} \to d_\alpha^*(x) \text{ as } t \to \infty. \tag{7}$$

Then (i) implies (ii). Conversely if $t^\alpha(1 - F(t))$ and $t^\alpha H(t)$ are monotone for t sufficiently large and $c_\alpha^* \neq 0$, then (ii) implies (i).
If (i) holds, then as $t \to \infty$

$$1 - U(1/t) - s_\alpha H(t) \sim S_{\alpha,\rho} a_1(t) H(t), \tag{8}$$

$$c_\alpha^* = (2p-1)[1 + (1-\alpha)c_\alpha]/s_\alpha.$$

and

$$\begin{aligned} d_\alpha^*(x) &= (2p-1)\int_1^x s^{-\alpha}\frac{s^\rho - 1}{\rho} ds + (2p-1)^2 c_\alpha x^{1-\alpha}\frac{x^\rho - 1}{\rho} + \\ &+ \{C_{\alpha,\rho}(p,r) + \frac{2r}{\rho - \alpha + 1}\}(x^{\rho+1-\alpha} - 1) - S_{\alpha,\rho} c_\alpha^* \frac{x^{1-\alpha} - 1}{1-\alpha}. \end{aligned}$$

Remarks

1. The statement $1 - U(1/t) \in 2RV(-\alpha, \rho, a_1)$ follows from (7) in case $c_\alpha^* \neq 0$.
2. The monotonicity conditions on $t^\alpha(1 - F(t))$ and $t^\alpha H(t)$ can be replaced with other monotonicity conditions on linear combinations of $t^\alpha(1 - F(t))$ and $t^\alpha F(-t)$ by adapting the proof.
3. In case $0 < \alpha < 1$ application of dominated convergence is no longer possible, but similar results can be obtained using suitable monotonicity conditions. We leave the details to the reader.

Theorem 2 *Suppose $EX = 0$ and $-1 < \rho \leq 0$. The following statements are equivalent.*

(i)

$$H \in RV(\rho - 2),$$

and

$$\frac{1 - F(t)}{H(t)} \to p \in [0, 1].$$

(ii)

$$1 - U(1/t) \in 2RV(-2, \rho, a_1)$$

and as $t \to \infty$

$$\frac{V(1/t)}{(1 - U(1/t))a_1(t)} \to h_\rho \in \left[\frac{\Gamma(1+\rho)\sin\frac{\rho\pi}{2}}{\rho(\rho-1)}, -\frac{\Gamma(1+\rho)\sin\frac{\rho\pi}{2}}{\rho(\rho-1)}\right]. \tag{9}$$

Moreover, given (i) (or (ii)), we have

$$\frac{\frac{1-U(1/tx)}{1-U(1/t)} - x^{-2}}{\frac{t^2 H(t)}{H_1(t)}} \to (1 + \rho g_\rho)x^{-2}\frac{x^\rho - 1}{\rho},\ (t \to \infty), \tag{10}$$

where

$$g_\rho = \frac{1}{\rho(\rho-1)}\Gamma(1+\rho)\cos\frac{\rho\pi}{2} + \frac{1}{\rho} \tag{11}$$

and the constant h_ρ in (9) satisfies

$$h_\rho = \frac{2p-1}{\rho(\rho-1)}\Gamma(1+\rho)\sin\frac{\rho\pi}{2}, \tag{12}$$

2 Proofs and auxiliary results

In the sequel we use the following integrals of the functions H and K.

$$H_1(x) = \int_0^x u(1 - F(u) + F(-u))du, \tag{13}$$

$$H_2(x) = \int_0^x H_1(u)du, H_3(x) = \int_0^x H_2(u)du \tag{14}$$

and

$$K_1(y) = \int_0^y K(x)dx. \tag{15}$$

In the proof of the main theorem we make use of another generalisation of the class RV, namely the class of functions ϕ for which there exist constants $c, c^*, \alpha \in \mathbb{R}$ and functions a, a_1 with $a_1(t) \to 0$ $(t \to \infty)$ such that

$$\frac{\frac{\phi(tx)-\phi(t)}{a(t)} - \frac{x^{1-\alpha}-1}{1-\alpha}}{a_1(t)} \to c\int_1^x s^{-\alpha}\frac{s^\rho - 1}{\rho}ds + c^*\int_1^x s^{\rho-\alpha}ds, \tag{16}$$

where $a_1(t) \to 0$. It is well known that if (16) holds, then as $t \to \infty$

$$\frac{\frac{a(tx)}{a(t)} - x^{1-\alpha}}{a_1(t)} \to cx^{1-\alpha}\int_1^x u^{\rho-1}du \text{ and } \frac{a_1(tx)}{a_1(t)} \to x^\rho \quad (x > 0). \tag{17}$$

See de Haan and Stadtmüller [12].

In the sequel we also make use of the inversion relations

$$H(x) = \frac{2}{\pi}\int_0^\infty \frac{1-U(t)}{t}\sin txdt \tag{18}$$

and

$$K(x) = \frac{2}{\pi}\int_0^\infty \frac{V(t)}{t}\cos txdt. \tag{19}$$

In order to prove the main result we need some Lemmas.

Lemma 1 *Suppose there exist $c, c^*, k \in \mathbb{R}$ and functions a, a_1 with $a_1(t) \to 0$ $(t \to \infty)$ such that as $t \to \infty$*

$$\frac{\frac{\Psi_1(tx)-\Psi_1(t)}{a(t)} - k\frac{x^{1-\alpha}-1}{1-\alpha}}{a_1(t)} \to L(x) + L_2(x) \text{ for } x > 0, \tag{20}$$

where $L(x) = c\int_1^x s^{-\alpha}\frac{s^\rho-1}{\rho}ds$, $L_2(x) = c^*\int_1^x s^{\rho-\alpha}ds$ and $a_1(t) \to 0$. Assume the function Ψ satisfies

$$\frac{\frac{\Psi(t)-\Psi_1(t)}{a(t)} - c_0}{a_1(t)} \to d_0 \ (t \to \infty) \tag{21}$$

for some constants $c_0, d_0 \in \mathbb{R}$. Then as $t \to \infty$

$$\frac{\frac{\Psi(tx)-\Psi(t)}{a(t)} - c_\alpha^+\frac{x^{1-\alpha}-1}{1-\alpha}}{a_1(t)} \to L(x) + L_2(x) + d_0(x^{\rho+1-\alpha} - 1) + c_0cx^{1-\alpha}\frac{x^\rho - 1}{\rho} \tag{22}$$

for $x > 0$ where $c_\alpha^+ = k + c_0(1-\alpha)$.

Proof Note that (22) follows from (20),(21) and the equations (17).

Lemma 2 *Suppose $\phi \in 2RV(-\alpha, \rho, a)$ and*

$$\frac{\frac{\phi_1(t)}{\phi(t)} - p}{a(t)} \to r \quad (t \to \infty),$$

where p, r are constants.

Then $\phi_1 \in 2RV(-\alpha, \rho, a)$ if $p \neq 0$. In particular we have as $t \to \infty$

$$\frac{\frac{\phi_1(tx)}{\phi_1(t)} - x^{-\alpha}}{a(t)} \to (\frac{r\rho}{cp} + 1)M(x) \text{ for } x > 0 \text{ in case } p \neq 0 \tag{23}$$

and

$$\frac{\frac{\phi_1(tx)}{\phi(t)} - px^{-\alpha}}{a(t)} \to rx^{-\alpha+\rho} + pM(x) \tag{24}$$

Proof The left-hand side in (23) equals

$$\frac{\frac{\phi(tx)}{\phi(t)}\{p + (r + o(1))a(tx)\}/\frac{\phi_1(t)}{\phi(t)} - x^{-\alpha}}{a(t)} = \frac{\{x^{-\alpha} + (M(x) + o(1))a(t)\}Q - x^{-\alpha}}{a(t)},$$

where

$$\begin{aligned} Q &= \frac{p + (r + o(1))a(tx)}{p + (r + o(1))a(t)} = 1 + \frac{(r + o(1))(a(tx) - a(t))}{p + (r + o(1))a(t)} = \\ &= 1 + \frac{r}{p}(x^\rho - 1)(1 + o(1))a(t). \end{aligned}$$

Substitution of the last expression for Q gives (23). We omit the proof of (24) which is similar.

Lemma 3

(i) Suppose $\int_0^t u^{\delta-1}\phi(u)du \in 2RV(\delta, \rho, a)$, $\delta > 0$ with ϕ eventually monotone. Then $t^\delta \phi(t) \in 2RV(\delta, \rho, a)$ and

$$\frac{\frac{t^\delta \phi(t)}{\int_0^t u^{\delta-1}\phi(u)du} - \delta}{a(t)} \to c \text{ as } t \to \infty. \tag{25}$$

(ii) Suppose $\int_t^\infty u^{\delta-1}\phi(u)du \in 2RV(\delta, \rho, a)$, $\delta < 0$ with ϕ eventually monotone. Then $t^\delta \phi(t) \in 2RV(\delta, \rho, a)$.

Proof The proof is based on a well known monotonicity argument. We give the proof of (i) in case ϕ is non-decreasing. Define the function γ by

$$\gamma(t) = t^{-\delta} \int_0^t u^{\delta-1}\phi(u)du.$$

Note that the assumption of the lemma is equivalent to the limit relation

$$\frac{\gamma(tx)-\gamma(t)}{a(t)\gamma(t)} \to c\frac{x^\rho - 1}{\rho} \text{ as } t \to \infty \text{ for } x > 0 \text{ where } c \neq 0. \tag{26}$$

Take $x > 1$ fixed. Since ϕ is non-decreasing we have

$$x^\delta \gamma(tx) - \gamma(t) = t^{-\delta} \int_t^{tx} u^{\delta-1}\phi(u)du \geq \frac{x^\delta - 1}{\delta}\phi(t),$$

hence

$$\frac{\phi(t) - \delta\gamma(t)}{a(t)\gamma(t)} \leq \frac{\delta x^\delta}{x^\delta - 1} \frac{\gamma(tx)-\gamma(t)}{a(t)\gamma(t)}. \tag{27}$$

Combination with (26) gives

$$\overline{\lim_{t\to\infty}} \frac{\phi(t) - \delta\gamma(t)}{a(t)\gamma(t)} \leq \frac{\delta x^\delta}{x^\delta - 1} c \frac{x^\rho - 1}{\rho} \tag{28}$$

and a similar lower inequality. Now (25) follows if we let $x \to 1$ from above and below respectively. Combination with Lemma 2 then completes the proof.

The following Lemma combines Theorem 1.9 and 1.10 in [5] with an inequality from [11].

Lemma 4 *Suppose f is measurable and for some positive function a_1*

$$\psi(x) := \lim_{t\to\infty} \frac{f(tx) - f(t)}{a_1(t)}$$

exists for all $x > 0$.

Then there exist constants c, γ such that $\psi(x) = c(x^\gamma - 1)/\gamma$.(For $\gamma = 0$ read $\psi(x) = c \log x$.)

If $c, \gamma > 0$, then $\frac{c}{\gamma} a_1(t) \sim f(t) \in RV(\gamma)$.

If $c > 0, \gamma < 0$, then $f(\infty) := \lim_{t\to\infty} f(t)$ exists and $-\frac{c}{\gamma}a_1(t) \sim f(\infty) - f(t) \in RV(\gamma)$.

Moreover there exists a positive function a with $a(t) \sim a_1(t)$ $(t \to \infty)$ with the property that for every $\varepsilon, \varepsilon' > 0$ there exists a t_0 such that for $t \geq t_0, tx \geq t_0$

$$x^{-\gamma} e^{-\varepsilon|\log x|} \left| \frac{f(tx) - f(t)}{a(t)} - c\frac{x^\gamma - 1}{\gamma} \right| < \varepsilon'. \tag{29}$$

Using (29) and a standard argument in order to prove general kernel theorems (see e.g. [1]) we obtain the following result.

Lemma 5 *Suppose f is integrable on finite intervals of $\mathbb{R}^+$ and satisfies the conditions of Lemma 4.*

(i) If $k : \mathbb{R}^+ \to \mathbb{R}$ is bounded on $(0,1)$ and $\gamma > -1$, then as $t \to \infty$

$$\int_0^1 k(s)\frac{f(ts)-f(t)}{a(t)}ds \to c\int_0^1 k(s)\frac{s^\gamma - 1}{\gamma}ds.$$

(ii) If $t^{\gamma+\varepsilon}k(t)$ is integrable on $(1,\infty)$ for some $\varepsilon > 0$ and $\gamma \geq 0$, then $\int_1^\infty k(s)f(ts)ds < \infty$ for $t > 0$ and as $t \to \infty$

$$\int_1^\infty k(s)\frac{f(ts)-f(t)}{a(t)}ds \to c\int_1^\infty k(s)\frac{s^\gamma - 1}{\gamma}ds.$$

Proof of theorem 1 (i)→ (ii) Using (18) and (19) we have

$$\frac{\frac{\lambda(1/t)-1-\frac{i}{t}\int_0^t K(s)ds}{H(t)} + s_\alpha - i(2p-1)c_\alpha}{a_1(t)} =$$

$$-\int_0^\infty \frac{(tx)^\alpha H(tx) - t^\alpha H(t)}{a_1(t)t^\alpha H(t)}\ \frac{\sin x}{x^\alpha}dx + i\int_1^\infty \frac{(tx)^\alpha K(tx) - (2p-1)t^\alpha H(t)}{a_1(t)t^\alpha H(t)}\ \frac{\cos x}{x^\alpha}dx +$$

$$+i\int_0^1 \frac{(tx)^\alpha K(tx) - (2p-1)t^\alpha H(t)}{a_1(t)t^\alpha H(t)}\ \frac{\cos x - 1}{x^\alpha}\ dx \to -S_{\alpha,\rho} + iC_{\alpha,\rho}(p,r) \quad (30)$$

as $t \to \infty$. Convergence to the limit for the first integral is justified by Lemma 5. Note that we have absolute convergence since $\alpha > 1$. The first factor in the second integral on the right hand side can be written as

$$\frac{K(tx)-(2p-1)H(tx)}{a_1(tx)H(tx)}\ \frac{(tx)^\alpha a_1(tx)H(tx)}{t^\alpha a_1(t)H(t)} + (2p-1)\frac{(tx)^\alpha H(tx) - t^\alpha H(t)}{a_1(t)t^\alpha H(t)}.$$

In view of (5) we have

$$\frac{K(t)-(2p-1)H(t)}{a_1(t)H(t)} \to 2r \text{ as } t \to \infty,$$

hence the first part of the integral can be treated using a general kernel theorem for RV functions. See e.g. [5], Theorem 1.8. The rest of the integral and the third integral can be treated similarly. Taking the real part on both sides in the above limit relation now gives (8).

Application of Lemma 2 (first part) with $\phi(t) = H(t)$ shows that $\phi_1(t) := 1 - U(1/t)$ satisfies (23) with right hand side

$$(\frac{\rho S_{\alpha,\rho}}{s_\alpha} + 1)x^{-\alpha}\frac{x^\rho - 1}{\rho} \neq 0 \text{ for } x \neq 1,$$

hence $1 - U(1/t) \in 2RV(-\alpha, \rho, a_1)$.

Now apply Lemma 2 (second statement) with $\phi(t) = H(t)$ and $\phi_1(t) = K(t)$ to find

$$\frac{\frac{K(tx)}{H(t)} - (2p-1)x^{-\alpha}}{a_1(t)} \to 2rx^{-\alpha+\rho} + (2p-1)x^{-\alpha}\frac{x^\rho - 1}{\rho}$$

as $t \to \infty$ for $x > 0$. Integrating this limit relation over $x \in (1, x')$, we find (20) with $\Psi_1(t) = \int_0^t K(s)ds$, $a(t) = tH(t)$, $k = c = 2p - 1$ and $c^* = 2r$. Taking imaginary parts in (30), it follows that (21) is satisfied with $\Psi(t) = tV(1/t)$, $c_0 = (2p-1)c_\alpha$ and $d_0 = C_{\alpha,\rho}(p, r)$.

Application of Lemma 1 shows that (22) is satisfied with the above specifications. Now (7) follows from (8) and (22).

Proof of theorem 1 (ii)→ (i) Suppose U satisfies

$$\lim_{t\to\infty} \frac{\frac{1-U(1/tx)}{1-U(1/t)} - x^{-\alpha}}{a_1(t)} = cx^{-\alpha}\int_1^x u^{\rho-1}du, x > 0 \text{ for } c \neq 0.$$

With the function H_2 as defined in (14) we have as in Pitman ([16], proof of theorem 5)

$$H_2(x) = \frac{2}{\pi}x^3\int_0^\infty (1 - U(t/x))\frac{2(1-\cos t) - t\sin t}{t^4}dt,$$

hence

$$\frac{\frac{H_2(x)}{x^3(1-U(1/x))} - \gamma_\alpha}{a_1(x)} = \frac{2}{\pi}\int_0^\infty \frac{\frac{1-U(t/x)}{1-U(1/x)} - t^\alpha}{a_1(x)} \frac{2(1-\cos t) - t\sin t}{t^4}dt$$

$$\to \frac{2}{\pi}\int_0^\infty M(1/t)\frac{2(1-\cos t) - t\sin t}{t^4}dt, \tag{31}$$

where

$$\gamma_\alpha = \frac{2}{\pi}\int_0^\infty \frac{2(1-\cos t) - t\sin t}{t^{4-\alpha}}dt.$$

Convergence in (31) follows from Lemma 5.

Lemma 2 now shows that $H_2(x)/x^3 \in 2RV(-\alpha, \rho, a_1)$, i.e. $H_2 \in 2RV(3 - \alpha, \rho, a_1)$. Application of Lemma 3(i) with $\delta = 3 - \alpha, \phi(u) = u^{\alpha-2}H_1(u)$ (which is monotone due to our assumption) gives $H_1 \in 2RV(2 - \alpha, \rho, a_1)$. A second application of Lemma 3(i) with $\delta = 2 - \alpha, \phi(u) = u^\alpha H(u)$ then gives $H \in 2RV(-\alpha, \rho, a_1)$.

Note that

$$\frac{txV(1/tx) - \frac{c_\alpha^*}{1-\alpha}tx(1 - U(1/tx)) - [tV(1/t) - \frac{c_\alpha^*}{1-\alpha}t(1 - U(1/t))]}{a_1(t)t(1 - U(1/t))} =$$

$$\frac{\frac{txV(1/tx)-tV(1/t)}{t(1-U(1/t))}-c_\alpha^*\frac{x^{1-\alpha}-1}{1-\alpha}}{a_1(t)}-\frac{c_\alpha^*}{1-\alpha}\frac{tx(1-U(1/tx))-x^{1-\alpha}t(1-U(1/t))}{a_1(t)t(1-U(1/t))}. \quad (32)$$

By assumption both terms on the right hand side converge. Hence (since $a_1(t)t(1-U(1/t)) \in RV(1-\alpha+\rho)$) in view of Lemma 4 the limit is $d_0\frac{x^{1-\alpha+\rho}-1}{1-\alpha+\rho}$ for some constant d_0. Moreover since $1-\alpha+\rho<0$ the limits

$$d_1 := \lim_{t\to\infty} \{tV(1/t) - \frac{c_\alpha^*}{1-\alpha}t(1-U(1/t))\}$$

and

$$\lim_{t\to\infty}\frac{d_1-tV(1/t)+\frac{c_\alpha^*}{1-\alpha}t(1-U(1/t))}{a_1(t)t(1-U(1/t))}=\lim_{t\to\infty}\frac{\frac{d_1-tV(1/t)}{t(1-U(1/t))}+\frac{c_\alpha^*}{1-\alpha}}{a_1(t)}=\frac{d_0}{1-\alpha+\rho} \quad (33)$$

exist. By Lemma 2 and (33)

$$\lim_{t\to\infty}\frac{\frac{d_1-txV(1/tx)}{t(1-U(1/t))}+\frac{c_\alpha^*}{1-\alpha}x^{1-\alpha}}{a_1(t)} =: d_2(x),\ x>0 \quad (34)$$

exists, where $d_2(x) = \frac{d_0}{1-\alpha+\rho}x^{1-\alpha+\rho}+\frac{c_\alpha^*}{1-\alpha}M(x)$. With the function $K_1(y) = \int_0^y K(x)dx$ we have

$$K_1(y)=\frac{2}{\pi}\int_0^\infty \frac{V(t)}{t}\frac{\sin ty}{t}dt \quad (35)$$

as in [6]. Define the function K^{i2} by

$$K^{i2}(t) := \int_0^t (d_1-K_1(y))dy = \frac{2t}{\pi}\int_0^\infty \{d_1-\frac{t}{s}V(\frac{s}{t})\}\frac{1-\cos s}{s^2}ds. \quad (36)$$

In view of (34) this implies

$$\begin{aligned}\frac{\frac{K^{i2}(t)}{t^2(1-U(1/t))}-c'}{a_1(t)} &= \frac{2}{\pi}\int_0^\infty \frac{\frac{d_1-\frac{t}{s}V(\frac{s}{t})}{t(1-U(1/t))}+\frac{c_{\alpha^*}}{1-\alpha}s^{\alpha-1}}{a_1(t)}\frac{1-\cos s}{s^2}ds \\ &\to \int_0^\infty d_2(\frac{1}{s})\frac{1-\cos s}{s^2}ds =: c'' \text{ as } t\to\infty, \quad (37)\end{aligned}$$

where

$$c' = \frac{-2c_\alpha^*}{\pi(1-\alpha)}\int_0^\infty \frac{1-\cos s}{s^{3-\alpha}}ds.$$

Note that $c_\alpha^* \neq 0$ by assumption. Taking the limit under the integral in (37) is justified by Lebesgue's theorem on dominated convergence using the dominating function from Lemma 6. Application of Lemma 2 (note that $c' \neq 0$) shows that

$$K^{i2} \in 2RV(2-\alpha,\rho,a_1). \quad (38)$$

Now we show that

$$d_1 - K_1(t) = \int_t^\infty K(x)dx, \quad t > 0, \tag{39}$$

hence

$$K^{i2}(t) = \int_0^t \int_s^\infty K(x)dx. \tag{40}$$

Since $|K(t)| = |1 - F(t) - F(-t)| \leq H(t)$ and $H \in 2RV(-\alpha, \rho, a_1) \subset RV(-\alpha)$ with $1 < \alpha < 2$, it follows that $\lim_{y\to\infty} K_1(y) = \lim_{y\to\infty} \int_0^y K(x)dx$ exists as a finite limit. Moreover $\lim_{y\to\infty} K_1(y) = d_1$, since a different limit contradicts $K^{i2} \in RV(2-\alpha)$. Hence (39) follows.

Since $H \in 2RV(-\alpha, \rho, a_1)$ with $1 < \alpha < 2$, we have similarly $H^*(t) := \int_0^t H(x)dx \to \xi$ $(t \to \infty)$ and

$$\xi - H^*(t) = \int_t^\infty H(x)dx. \tag{41}$$

A standard argument shows that $H \in 2RV(-\alpha, \rho, a_1)$ implies that the function H^{i2} defined by

$$H^{i2}(t) = \int_0^t \int_s^\infty H(x)dx = \int_0^t (\xi - H^*(s))ds \in 2RV(2-\alpha, \rho, a_1). \tag{42}$$

From (38),(40) and (42) it follows that

$$H^{i2}(t) + K^{i2}(t) = \int_0^t \int_s^\infty (H(x) + K(x))dxds \in 2RV(2-\alpha, \rho, a_1).$$

(It is easy to see that if $\phi_1, \phi_2 \in 2RV(\beta, \rho, a)$, then $\phi_1 + \phi_2 \in 2RV(\beta, \rho, a)$.) Application of Lemma 3(i) now gives

$$\int_t^\infty (H(x) + K(x))dx = 2\int_t^\infty (1 - F(x))dx \in 2RV(1-\alpha, \rho, a_1)$$

and

$$\frac{\frac{t\int_t^\infty (1-F(x))dx}{\int_0^t \int_u^\infty (1-F(x))dxdu} - (2-\alpha)}{a_1(t)} \to c_1 \text{ as } t \to \infty, \tag{43}$$

where c_1 is a constant. Note that the monotonicity assumption is satisfied since $\phi(t) = 2t^{\alpha-1}\int_t^\infty (1 - F(x))dx = 2\int_1^\infty y^{-\alpha}(ty)^\alpha(1 - F(ty))dy$ is monotone due to our assumption that $t^\alpha(1 - F(t))$ is eventually monotone. An application of Lemma 3(ii) then gives $1 - F \in 2RV(-\alpha, \rho, a_1)$ and

$$\frac{\frac{t(1-F(t))}{\int_t^\infty (1-F(x))dx} - (\alpha - 1)}{a_1(t)} \to c_2^* \text{ as } t \to \infty, \tag{44}$$

where c_2^* is a constant.

We introduce constants d_i, d_i' $(i = 3, \dots, 6)$. As before (8) holds, hence

$$t^2(1 - U(1/t)) = d_3 t^2 H(t) + (d_3' + o(1)) t^2 H(t) a_1(t)$$

and combination with (37) gives

$$K^{i2}(t) = d_4 t^2 H(t) + (d_4' + o(1)) t^2 H(t) a_1(t) \ (t \to \infty). \tag{45}$$

It follows from $H \in 2RV(-\alpha, \rho, a_1)$ that

$$H^{i2}(t) = d_5 t^2 H(t) + (d_5' + o(1)) t^2 H(t) a_1(t) \ (t \to \infty). \tag{46}$$

Addition of (45) and (46) gives

$$K^{i2}(t) + H^{i2}(t) = d_6 t^2 (K(t) + H(t)) + (d_6' + o(1)) t^2 (K(t) + H(t)) a_1(t) \ (t \to \infty). \tag{47}$$

Now (5) follows from (43),(44) and (47).

In order to prove theorem 2 we need the following Lemma which can be found in the appendix of [8].

Lemma 6 *The following statements are equivalent*

(i) $H_1 \in 2RV(0, \rho, a_1)$.

(ii) $H \in RV(\rho - 2)$ *with* $\rho \leq 0$.

If (i) or (ii) holds, then

$$a_1(t) \sim t^2 H(t)/H_1(t) \ (t \to \infty). \tag{48}$$

Proof of theorem 2 (i)→ (ii) From equation (2.8) of [10] we have as $t \to \infty$

$$\frac{1 - U(1/t) - t^{-2} H_1(t)}{H(t)} \to g_\rho \tag{49}$$

$$\frac{V(1/t)}{H(t)} \to h_\rho, \tag{50}$$

where g_ρ and h_ρ satisfy (11) and (12) respectively. Since $t^{-2} H_1(t) \in 2RV(-2, \rho, a_1)$ (by Lemma 6), application of Lemma 2 gives $1 - U(1/t) \in 2RV(-2, \rho, a_1)$ and (10). Note that (49) implies

$$\frac{1 - U(1/t)}{H(t)} \sim \frac{1}{a_1(t)},$$

hence (9) follows from (50).

Proof of theorem 2 (ii)→ (i) With H_2 as defined by (14) we have

$$H_2(x) = \frac{2}{\pi}\int_0^\infty (1-U(t))\frac{2(1-\cos tx) - tx\sin tx}{t^4}dt.$$

Define the function H_3 by $H_3(x) = \int_0^x H_2(u)du$. It follows that

$$\begin{aligned} H_3(x) &= \frac{2}{\pi}\int_0^\infty \frac{1-U(t)}{t^4}\int_0^x 2(1-\cos ut) - ut\sin ut dt \\ &= \frac{2}{\pi}\int_0^\infty \frac{1-U(t)}{t^4}(2x - \frac{3\sin xt}{t} + x\cos xt)dt \\ &= \frac{2}{\pi}x^4\int_0^\infty \{1-U(t/x)\}\frac{t(2+\cos t) - 3\sin t}{t^5}dt. \end{aligned}$$

Introducing the constant γ^* by

$$\gamma^* = \frac{2}{\pi}\int_0^\infty \frac{t(2+\cos t) - 3\sin t}{t^5}dt$$

we have

$$\frac{\frac{H_3(x)}{x^4(1-U(1/x))} - \gamma^*}{a_1(x)} = \frac{2}{\pi}\int_0^\infty \frac{\frac{1-U(t/x)}{1-U(1/x)} - t^2}{a_1(x)}\frac{t(2+\cos t) - 3\sin t}{t^5}dt$$

$$\to \frac{2}{\pi}\int_0^\infty t^2\frac{t^{-\rho}-1}{\rho}\frac{t(2+\cos t) - 3\sin t}{t^5}dt,\ x\to\infty. \tag{51}$$

Note that convergence follows from Lemma 5 since $-1 < \rho \le 0$. Lemma 2 (note that $\gamma^* \neq 0$) now shows that $H_3(x)/x^4 \in 2RV(-2,\rho,a_1)$, equivalently $H_3 \in 2RV(2,\rho,a_1)$. Repeated application of Lemma 3(i) gives respectively $H_2 \in 2RV(1,\rho,a_1)$ and $H_1 \in 2RV(0,\rho,a_1)$.

Using Lemma 6 we find $H \in RV(\rho-2)$ with $\rho \le 0$ and (48). As a consequence H_1 is slowly varying, hence repeated application of Karamata's theorem (see e.g. [1], Ch. 1) gives $H_3(x) \sim \frac{1}{2}x^2H_1(x)$. Combination with (48) then gives $H_3(x)/(x^4H(x)) \sim 1/(2a_1(x))$ as $x\to\infty$. Since (51) implies that $H_3(x) \sim \gamma^*x^4(1-U(1/x))$, we have $(1-U(1/x))/H(x) \sim 1/(2\gamma^*a_1(x))$, hence (by (9))

$$V(1/x)/H(x) \to h_\rho/(2\gamma^*) \text{ as } x\to\infty. \tag{52}$$

For the function $K_2(y) = \int_0^y uK_1(u)du$, we have by (35)

$$K_2(y) = \frac{2}{\pi}\int_0^\infty \frac{V(t)}{t^2}\frac{\sin ty - ty\cos ty}{t^2}dt.$$

It follows that the function $K_3(y) = \int_0^y K_2(u)du$ satisfies

$$
\begin{aligned}
K_3(y) &= \frac{2}{\pi}\int_0^\infty \frac{V(t)}{t^4}\int_0^y (\sin ut - ut\cos ut)dudt \\
&= \frac{2}{\pi}\int_0^\infty V(t)\frac{2(1-\cos ty) - ty\sin ty}{t^5}dt \\
&= \frac{2}{\pi}y^4\int_0^\infty V(t/y)\frac{2(1-\cos t) - t\sin t}{t^5}dt.
\end{aligned}
$$

Since $H \in RV(\rho - 2)$ we have by (52) as $y \to \infty$

$$
\begin{aligned}
\frac{K_3(y)}{y^4 H(y)} &= \frac{2}{\pi}\int_0^\infty \frac{V(t/y)}{H(y/t)}\frac{H(y/t)}{H(y)}\frac{2(1-\cos t) - t\sin t}{t^5}dt \\
&\to \frac{2}{\pi}\int_0^\infty \frac{h_\rho}{2\gamma^*}t^{2-\rho}\frac{2(1-\cos t) - t\sin t}{t^5}dt. \qquad (53)
\end{aligned}
$$

In view of Lemma 7 (which is proved independently below) this implies

$$
\begin{aligned}
\frac{2\int_0^y\int_0^v u\int_u^\infty F(-x)dxdudv}{y^4H(y)} &\to \frac{2}{\pi}\int_0^\infty \frac{h_\rho}{2\gamma^*}t^{2-\rho}\frac{2(1-\cos t) - t\sin t}{t^5}dt \\
+\frac{1}{2}\int_0^1 x^{-2+\rho}x^2(1-x)dx &+ \frac{1}{6}\int_0^1 x^{-2+\rho}x^3dx + \frac{1}{6}\int_1^\infty x^{-2+\rho}dx \qquad (54)
\end{aligned}
$$

as $y \to \infty$. Denote the constant on the right hand side of (54) by k_ρ. If $k_\rho \neq 0$, then (54) implies $\int_0^y\int_0^v u\int_u^\infty F(-x)dxdudv \in RV(\rho+2)$, hence repeated apllication of the monotone density theorem (see e.g. [1], Ch. 3) gives $F(-x) \in RV(\rho-2)$. Repeated apllication of Karamata's theorem then gives $(1-\rho^2)(2+\rho)\int_0^y\int_0^v u\int_u^\infty F(-x)dxdudv \sim y^4F(-y)$, which in combination with (54) shows that $F(-y)/H(y)$ converges as $y \to \infty$. In case $k_\rho = 0$ we can use a similar argument if we express the function $K_3(x)$ in Lemma 7 in terms of $1 - F(x)$ aand $H(x)$.

Lemma 7 *Under the conditions of Theorem 2*

$$
\begin{aligned}
K_3(y) &= 2\int_0^y\int_0^v u\int_u^\infty F(-x)dxdudv - \frac{1}{2}y^4\int_0^1 H(xy)x^2(1-x)dx \\
&- \frac{1}{6}y^4\int_0^1 H(xy)x^3dx - \frac{y^4}{6}\int_1^\infty H(xy)dx,
\end{aligned}
$$

where the function K_3 is as defined in the proof of theorem 2(ii)→(i).

Proof Since $EX = 0$ it follows that

$$
K_1(y) = K_1(y) - \int_0^\infty K(u)du = -\int_y^\infty K(u)du = 2\int_y^\infty F(-u)du - \int_y^\infty H(u)du.
$$

(The function K_1 is given in (15)).

Substitution and simplification then gives

$$\begin{aligned} K_2(y) &= \int_0^y uK_1(u)du = 2\int_0^y u\int_u^\infty F(-x)dxdu - \frac{1}{2}\int_0^y x^2H(x)dx \\ &- \frac{y^2}{2}\int_y^\infty H(x)dx. \end{aligned}$$

Substituting this gives

$$\begin{aligned} K_3(y) &= \int_0^y K_2(v)dv = 2\int_0^y\int_0^v u\int_u^\infty F(-x)dxdudv - \frac{1}{2}\int_0^y x^2H(x)\int_x^y dvdx \\ &- \frac{1}{2}\int_0^y H(x)\int_0^x v^2dvdx - \frac{1}{2}\int_y^\infty H(x)\int_0^y v^2dvdx, \end{aligned}$$

which is equivalent to the statement of the Lemma.

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