# Do the US and Canada Have a Common Nonlinear Cycle in Unemployment?

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**Abstract**

To enable answering the question in the title, we introduce a bivariate censored latent effects autoregression, and discuss representation, parameter estimation, diagnostics and inference. We show that this bivariate nonlinear model is very useful for examining common nonlinearity. We apply the model to the monthly unemployment rate in the US and Canada to examine if these variables have common cyclical properties conditional on lagged explanatory variables such as industrial production, the oil price and interest spread. We find that US variables have explanatory value for Canadian unemployment, but that Canadian variables do not predict cyclical patterns in the US. Also, we find that recessionary shocks in Canada are more persistent than similar sized shocks in the US in the same period. Finally, we obtain some evidence for a common nonlinear business cycle.

**Keywords**: Nonlinear time series, censored regression, persistence, common nonlinearity

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# 1 Introduction

Aggregate unemployment in industrialized countries, when measured over a few decades, displays a pattern which at first sight seems to be common across these countries. Generally, the unemployment rate has a tendency to converge to some natural rate of unemployment, which may differ across countries, while the level of the unemployment rate seem to be lifted upward due to important events such as oil price shocks and economic crises, see for example Blanchard and Summers (1987) and Bianchi and Zoega (1998). Typically, during recessions, the increase in the unemployment rate can be quite large, and it often takes lengthy expansions to bring the unemployment rate back at the level of before the recession. In other words, unemployment displays asymmetric patterns, where it tends to rise sharply in recessions, while it decreases only slowly in expansions.

In order to describe such asymmetric variables, one often has to consider nonlinear time series models. Typical examples of such models, which are regularly considered for unemployment data, are the switching regime models with observed and exogenous variables determining the switches, see Granger and Teräsvirta (1993) and Teräsvirta (1998), and the Markov switching models with an unobserved switching variable, see Hamilton (1989) and Diebold *et al.* (1994). Noticing that unemployment tends only to be lifted upward in recessions, Franses and Paap (1998) introduce another nonlinear time series model, assuming that linear combinations of exogenous variables generate shocks, which are only observed and allowed to predict unemployment if they are positive. The application in Franses and Paap (1998) of this so-called censored latent effect autoregressive [CLEAR] model to monthly US unemployment shows that it can adequately describe the data with only a small number of parameters. Also, their model is found to outperform related models in terms of fit and out-of-sample forecasting performance.

Given the substantial linkage of economies of several industrialized countries, it is of interest to study if macroeconomic aggregates in two or more countries have common properties. In terms of unemployment, for example, one may wonder if two closely linked countries show similar asymmetric patterns across recessions and expansions. A related question concerns the lead-lag relationship between two countries. For example, do explanatory variables that have predictive value for an upcoming recession in one country also have such predictive power for a second country? Another issue concerns the persistence of shocks. It may be of interest to examine if oil price shocks have a different long run impact across countries. In the present paper, we aim to examine these issues for monthly unemployment in the US and in Canada.

In order to empirically examine common features in two unemployment series, we need a multivariate nonlinear time series model. There are a few requirements for such a model for our purposes. First of all, the model should be reasonably parsimonious. Secondly, the model should allow for a simple analysis of common features. For example, the results in Boswijk and Franses (1997) and Vahid and Anderson (1998) show that it is not very straightforward to define common nonlinearity and common asymmetry in the smooth transition regression framework. Thirdly, parameter estimation in our multivariate model should be computationally tractable. Therefore, we consider a bivariate extension of the univariate CLEAR model proposed in Franses and Paap (1998) which is a parsimonious model, where straightforward methods apply for inference in the parameters, and where common asymmetries are naturally defined.

The outline of our paper is as follows. In Section 2 we discuss the representation of a bivariate CLEAR model. We illustrate that this parsimonious model can be used to investigate common cyclical (nonlinear) features. Next, in Section 3, we discuss parameter estimation and inference issues. In Section 4, we apply our bivariate model to describe cyclical patterns in the monthly unemployment rate of the US and of Canada. In Section 5, we give some final remarks.

## 2 A Bivariate CLEAR Model

In this section we first give a general outline of the representation of the model. Next, we discuss which restrictions on the parameters and variables in the model correspond with common nonlinearity. Finally, we give some details on how the model can be used for probabilistic inference.

## 2.1 Representation

A bivariate censored latent effects vector autoregression [CLEAR] of order $p$ for $y_t = (y_{1,t}, y_{2,t})'$, $t = 1, \ldots, T$ (while having in mind an application to unemployment rates) can be represented by

$$y_t = \mu + \sum_{i=1}^{p} \Psi_i y_{t-i} + v_t + \varepsilon_t, \tag{1}$$

where $\varepsilon_t \sim \text{NID}(\mathbf{0}, \Sigma_\varepsilon)$ with $\Sigma_\varepsilon$ a $(2\times 2)$ covariance matrix, $\mu$ is a $(2\times 1)$ parameter vector and $\Psi_i$, $i = 1, \ldots, p$, are $(2 \times 2)$ parameter matrices. This amounts to a straightforward extension of the univariate CLEAR model in Franses and Paap (1998). The $(2\times 1)$ vector $v_t = (v_{1,t}, v_{2,t})'$ contains positive innovation outliers, which are assumed to be generated by

$$v_{1,t} = \begin{cases} x'_{1,t}\beta_1 + u_{1,t} & \text{if } x'_{1,t}\beta_1 + u_{1,t} \geq 0 \\ 0 & \text{if } x'_{1,t}\beta_1 + u_{1,t} < 0 \end{cases} \tag{2}$$

and

$$v_{2,t} = \begin{cases} x'_{2,t}\beta_2 + u_{2,t} & \text{if } x'_{2,t}\beta_2 + u_{2,t} \geq 0 \\ 0 & \text{if } x'_{2,t}\beta_2 + u_{2,t} < 0 \end{cases} \tag{3}$$

where $u_t = (u_{1,t}, u_{2,t})' \sim \text{NID}(\mathbf{0}, \Sigma_u)$, with $\Sigma_u$ a $(2 \times 2)$ covariance matrix. The $x_{i,t}$, $i = 1, 2$, are $(k_i \times 1)$ vectors containing explanatory variables, and the corresponding $\beta_i$ are $(k_i \times 1)$ parameter vectors. The first element of $x_{i,t}$ equals 1.

The expressions (2) and (3) denote two censored regression models. The unobserved random variable $v_{1,t}$ is zero unless $x'_{1,t}\beta_1$ exceeds the stochastic and time varying threshold level $-u_{1,t}$. In that case a positive shock $v_{1,t}$ is added to the equation for $y_{1,t}$ in (1). Similarly, a positive shock $v_{2,t}$ is added to the equation for $y_{2,t}$ if $x'_{2,t}\beta_2$ exceeds the stochastic threshold $-u_{2,t}$. Given their inclusion in (1), equations (2) and (3) constitute a multivariate innovation outlier generating mechanism, where it is assumed that these outliers only take positive values. We restrict attention here to such positive outliers, as we assume that such positive shifts in the level mimic the typical patterns in unemployment where the first few months of recessionary periods show an explosive behavior in unemployment, see also Bianchi and Zoega (1998) and Blanchard and Summers (1987).

In Franses and Paap (1998) we find that a univariate CLEAR model can describe and forecast the US unemployment rate data rather well. Of course, for applications other than unemployment, one may want to modify (2) and (3) accordingly.

Positive shocks $v_{1,t}$ and $v_{2,t}$ may enter the equations of $y_{1,t}$ and $y_{2,t}$ at the same time, although this is not imposed. The correlation between the $u_{1,t}$ and $u_{2,t}$ variables, reflected by the covariance matrix $\Sigma_u$, indicates that the stochastic threshold levels are allowed to be correlated. It is easy to understand that, given certain values of $x'_{1,t}\beta_1$ and $x'_{2,t}\beta_2$, a high positive correlation between $u_{1,t}$ and $u_{2,t}$ increases the probability that a positive shock occurs in both equations. When this correlation is very close to 1, it becomes more likely that the two variables $y_{1,t}$ and $y_{2,t}$ display common nonlinear cyclical behavior. The precise nature of such a common property depends on the explanatory variables $x_{1,t}$ and $x_{2,t}$ and their corresponding parameters, as we will illustrate next.

## 2.2 Common Nonlinearity

The key feature of our bivariate CLEAR model, which makes it distinct from alternative multivariate nonlinear models see for example Philips (1991), Krolzig (1997), Diebold and Rudebusch (1996) and Kim and Nelson (1998), is that we introduce nonlinearity in $y_{1,t}$ and $y_{2,t}$ through an innovation outlier generating mechanism. When this mechanism is absent, the model is linear.

There are several interesting restricted versions of (1)–(3), which somehow imply common properties across the two time series $y_{1,t}$ and $y_{2,t}$. The first case is that $y_{1,t}$ and $y_{2,t}$ have absolutely no nonlinearity in common. This would mean that the correlation between $u_{1,t}$ and $u_{2,t}$ equals zero, and that the off-diagonal elements of the $\Psi_i$ matrices are all zero. In all other cases the series have some degree of common nonlinearity.

A second case of interest concerns perfect common nonlinearity. We define perfect common nonlinearity by the existence of a linear combination of $y_{1,t}$ and $y_{2,t}$ that is linear, while the individual series are nonlinear. From (1) it is easy to see that the process $y_{2,t} - \alpha y_{1,t}$ with $\alpha \neq 0$ is a linear process, if $v_{2,t} = \alpha v_{1,t}$. Hence, under perfect common nonlinearity the model in (1) reduces to

$$y_t = \mu + \sum_{i=1}^{p} \Psi_i y_{t-i} + \begin{pmatrix} 1 \\ \alpha \end{pmatrix} v_{1,t} + \varepsilon_t, \tag{4}$$

with $v_{1,t}$ given in (2). A special case amounts to $\alpha = 1$, where the difference between $y_{1,t}$ and $y_{2,t}$ is a linear process. Note that model (4) with (2) and with imposed common nonlinearity is nested in the general model (1) to (3) only if $x_{1,t} = x_{2,t}$. If however $x_{1,t} \neq x_{2,t}$ it is unlikely that the relation $v_{2,t} = \alpha v_{1,t}$ will hold for every observation $t$. As we only have one censored regression in (4), estimation and inference in this model proceeds in the same way as in a univariate CLEAR model, see Franses and Paap (1998) for details.

Other interesting cases concern the correlations between $u_{1,t}$ and $u_{2,t}$ and the explanatory variables $x_{1,t}$ and $x_{2,t}$. In case $x_{1,t} = x_{2,t}$, then the variables that have predictive value for a recession for a variable $y_{1,t}$ are the same as those for $y_{2,t}$. If $u_{1,t}$ and $u_{2,t}$ have high positive correlation, common recession periods become more likely. An interesting bivariate CLEAR model, which is slightly different from (1) to (3) imposes that shocks, enter the equation for $y_{1,t}$ and $y_{2,t}$ simultaneously. This model is given by (1) with

$$\begin{pmatrix} v_{1,t} \\ v_{2,t} \end{pmatrix} = \begin{cases} \begin{pmatrix} x'_{1,t}\beta_1 \\ x'_{2,t}\beta_2 \end{pmatrix} + \begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} & \text{if } x'_{1,t}\beta_1 + u_{1,t} \geq 0 \wedge x'_{2,t}\beta_2 + u_{2,t} \geq 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{elsewhere.} \end{cases} \tag{5}$$

In this specification it is not possible that $v_{1,t}$ enters the first equation of (1) while $v_{2,t}$ does not enter the second equation and *vice versa*. The size of the shocks may differ, of course, and therefore we do not label this model as one with common nonlinearity, but we refer to it as a model with a common cycle. Note that this model specification is not nested in (2) and (3) and hence it is not possible to use a standard likelihood ratio test to compare the models. In the empirical section below, we will therefore propose an encompassing test to decide between (2)–(3) and (5). In the remainder of this paper we focus on estimation and inference for model specification (1) with (2) and (3), as the analysis of specification (1) with (5) can be done in a similar way. In our empirical section we will of course compare both models.

## 2.3 Unconditional Inference

The bivariate variable $v_t$ is unobserved, and therefore we can only have probabilistic inference on its value. We now discuss the probabilities that the $v_{i,t}$ $i = 1, 2$ variables are zero or positive. We can distinguish four cases, summarized in the following identity:

$$\begin{aligned}\Pr[v_{1,t} = 0 \wedge v_{2,t} = 0|x_t] + \Pr[v_{1,t} = 0 \wedge v_{2,t} \neq 0|x_t] \\ + \Pr[v_{1,t} \neq 0 \wedge v_{2,t} = 0|x_t] + \Pr[v_{1,t} \neq 0 \wedge v_{2,t} \neq 0|x_t] = 1, \quad (6)\end{aligned}$$

where $x_t = (x_{1,t} \vee x_{2,t})$. The probabilities $\Pr[v_{1,t} = 0 \wedge v_{2,t} \neq 0|x_t]$ and $\Pr[v_{1,t} \neq 0 \wedge v_{2,t} = 0|x_t]$ indicate whether the series $y_{1,t}$ and $y_{2,t}$ have different nonlinear cycles, since a common nonlinear cycle requires that the probabilities $\Pr[v_{1,t} = 0 \wedge v_{2,t} \neq 0|x_t]$ and $\Pr[v_{1,t} \neq 0 \wedge v_{2,t} = 0|x_t]$ are zero, see (5).

The probability that $v_{1,t} = v_{2,t} = 0$ given $x_t$ is given by

$$\Pr[v_{1,t} = 0 \wedge v_{2,t} = 0|x_t] = \int_{-\infty}^{-x'_{1,t}\beta_1} \int_{-\infty}^{-x'_{2,t}\beta_2} |\Sigma_u|^{-\frac{1}{2}} \phi(\Sigma_u^{-\frac{1}{2}} u_t)\, du_{1,t}\, du_{2,t}, \quad (7)$$

where $\phi(\cdot)$ is the probability density function [pdf] of a standard bivariate normal distribution with zero mean and an identity as covariance matrix. Likewise,

$$\begin{aligned}\Pr[v_{1,t} = 0 \wedge v_{2,t} \neq 0|x_t] &= \int_{-\infty}^{-x'_{1,t}\beta_1} \int_{-x'_{2,t}\beta_2}^{\infty} |\Sigma_u|^{-\frac{1}{2}} \phi(\Sigma_u^{-\frac{1}{2}} u_t)\, du_{1,t}\, du_{2,t} \\ \Pr[v_{1,t} \neq 0 \wedge v_{2,t} = 0|x_t] &= \int_{-x'_{1,t}\beta_1}^{\infty} \int_{-\infty}^{-x'_{2,t}\beta_2} |\Sigma_u|^{-\frac{1}{2}} \phi(\Sigma_u^{-\frac{1}{2}} u_t)\, du_{1,t}\, du_{2,t} \quad (8) \\ \Pr[v_{1,t} \neq 0 \wedge v_{2,t} \neq 0|x_t] &= \int_{-x'_{1,t}\beta_1}^{\infty} \int_{-x'_{2,t}\beta_2}^{\infty} |\Sigma_u|^{-\frac{1}{2}} \phi(\Sigma_u^{-\frac{1}{2}} u_t)\, du_{1,t}\, du_{2,t}.\end{aligned}$$

The marginal probabilities that $v_{1,t} = 0$ and $v_{2,t} = 0$ are simply

$$\begin{aligned}\Pr[v_{1,t} = 0|x_t] &= \Pr[v_{1,t} = 0 \wedge v_{2,t} = 0|x_t] + \Pr[v_{1,t} = 0 \wedge v_{2,t} \neq 0|x_t] \\ \Pr[v_{2,t} = 0|x_t] &= \Pr[v_{1,t} = 0 \wedge v_{2,t} = 0|x_t] + \Pr[v_{1,t} \neq 0 \wedge v_{2,t} = 0|x_t].\end{aligned} \quad (9)$$

The probabilities in (7), (8) and (9) can be used to calculate the probability that there is a positive outlier $v_{i,t}$ in (one of) the series at time $t$ given the value of $x_t$. A forecast for the magnitude of $v_t$ given $x_t$ may be obtained from the expectation of $v_{1,t}$ and $v_{2,t}$ given

$x_t$. The expectation of $v_{1,t}$ equals

$$\begin{aligned}\mathrm{E}[v_{1,t}|x_t] &= \mathrm{E}[v_{1,t}|v_{1,t}=0,x_t]\Pr[v_{1,t}=0|x_t] + \mathrm{E}[v_{1,t}|v_{1,t}\neq 0,x_t]\Pr[v_{1,t}\neq 0|x_t] \\ &= \mathrm{E}[v_{1,t}|v_{1,t}\neq 0 \wedge v_{2,t}=0,x_t]\Pr[v_{1,t}\neq 0 \wedge v_{2,t}=0|x_t] \\ &\quad + \mathrm{E}[v_{1,t}|v_{1,t}\neq 0 \wedge v_{2,t}\neq 0,x_t]\Pr[v_{1,t}\neq 0 \wedge v_{2,t}\neq 0|x_t].\end{aligned} \tag{10}$$

Hence, the expectation of $v_{2,t}$ equals

$$\begin{aligned}\mathrm{E}[v_{2,t}|x_t] &= \mathrm{E}[v_{2,t}|v_{1,t}=0 \wedge v_{2,t}\neq 0,x_t]\Pr[v_{1,t}=0 \wedge v_{2,t}\neq 0|x_t] \\ &\quad + \mathrm{E}[v_{2,t}|v_{1,t}\neq 0 \wedge v_{2,t}\neq 0,x_t]\Pr[v_{1,t}\neq 0 \wedge v_{2,t}\neq 0|x_t].\end{aligned} \tag{11}$$

A closed form expression in terms of the pdf and CDF of a univariate normal distribution for the expectations of a censored bivariate normal distribution can be found in Rosenbaum (1961), see also Maddala (1983, p. 368).

# 3 Estimation

In this section we discuss maximum likelihood estimation of the parameters and conditional inference on the bivariate innovation outlier mechanism.

## 3.1 Maximum Likelihood Estimation

The model parameters of the bivariate CLEAR model (1) to (3) are given by $\theta = (\mu, \psi_1, \ldots, \psi_p, \Sigma_\varepsilon, \beta_1, \beta_2, \Sigma_u)$. The estimation of these parameters can be done by maximum likelihood. To derive the likelihood function we first consider the conditional pdf of $y_t$ given its past $Y_{t-1} = \{y_{t-1}, \ldots, y_1\}$ and given $v_t$. This function is given by

$$f(y_t|Y_{t-1}, v_t; \theta) = |\Sigma_\varepsilon|^{-\frac{1}{2}} \phi(\Sigma_\varepsilon^{-\frac{1}{2}}(e_t - v_t)) \tag{12}$$

with $e_t = y_t - \mu - \sum_{i=1}^{p} \Psi_i y_{t-i}$ and where we write $\varepsilon_t$ as $e_t - v_t$ to denote that it is conditional on $v_t$. To obtain the pdf of $y_t$ conditional on $Y_{t-1}$ but unconditional on $v_t$, we have to integrate over the unknown error process $u_t$ in the censored regressions. Hence the pdf of $y_t$ given its past is given by

$$
\begin{aligned}
f(y_t|Y_{t-1},x_t;\theta) &= \Pr[v_{1,t}=0 \wedge v_{2,t}=0|x_t] f(y_t|Y_{t-1},v_t;\theta)|_{v_{1,t}=0,v_{2,t}=0} \\
&+ \int_{-\infty}^{-x'_{1,t}\beta_1} \int_{-x'_{2,t}\beta_2}^{\infty} |\Sigma_u|^{-\frac{1}{2}} \phi(\Sigma_u^{-\frac{1}{2}} u_t) f(y_t|Y_{t-1},v_t;\theta)|_{v_{1,t}=0,v_{2,t}=x'_{2,t}\beta_2+u_{2,t}}\ du_{1,t}\ du_{2,t} \\
&+ \int_{-x'_{1,t}\beta_1}^{\infty} \int_{-\infty}^{-x'_{2,t}\beta_2} |\Sigma_u|^{-\frac{1}{2}} \phi(\Sigma_u^{-\frac{1}{2}} u_t) f(y_t|Y_{t-1},v_t;\theta)|_{v_{1,t}=x'_{1,t}\beta_1+u_{1,t},v_{2,t}=0}\ du_{1,t}\ du_{2,t} \\
&+ \int_{-x'_{1,t}\beta_1}^{\infty} \int_{-x'_{2,t}\beta_2}^{\infty} |\Sigma_u|^{-\frac{1}{2}} \phi(\Sigma_u^{-\frac{1}{2}} u_t) f(y_t|Y_{t-1},v_t;\theta)|_{v_{1,t}=x'_{1,t}\beta_1+u_{1,t},v_{2,t}=x_{2,t}\beta_2+u_{2,t}}\ du_{1,t}\ du_{2,t}
\end{aligned}
\tag{13}
$$

which consists of four parts corresponding to whether the $v_{i,t}$ terms are zero or not, see also (6). The log likelihood function is now simply the sum of the log of the unconditional pdfs, that is

$$
\mathcal{L}(Y_T|x_T;\theta) = \sum_{t=1}^{T} \ln f(y_t|Y_{t-1},x_t;\theta). \tag{14}
$$

To maximize this log likelihood we can use standard optimization algorithms, for instance the BHHH algorithm of Berndt *et al.* (1974) with a numerical gradient per observation. To ensure that the covariance matrix $\Sigma_\varepsilon$ is positive definite we write $\Sigma_\varepsilon$ as $Q'Q$ where $Q$ is a matrix with one of the off-diagonals equal to zero. The same applies to the $\Sigma_u$ covariance matrix. Standard errors of the parameter estimates can be estimated from the outerproduct of the vector of gradients per observation, evaluated in the maximum likelihood estimates.

To simplify the computation of the three integrals in the log likelihood we use the following result

$$
\begin{aligned}
&u'\Sigma_u^{-1}u + (Su-a)'\Sigma_\varepsilon^{-1}(Su-a) \\
&\qquad = u'\Sigma_u^{-1}u' + u'S'\Sigma_\varepsilon^{-1}Su - a'\Sigma_\varepsilon^{-1}Su - u'S'\Sigma_\varepsilon^{-1}a + a'\Sigma_\varepsilon^{-1}a \\
&\qquad = u'(\Sigma_u^{-1} + S'\Sigma_\varepsilon^{-1}S)u + a'\Sigma_\varepsilon^{-1}Su - u'S'\Sigma_\varepsilon^{-1}a + a'\Sigma_\varepsilon^{-1}a \\
&\qquad = (u-b)'(\Sigma_u^{-1} + S'\Sigma_\varepsilon^{-1}S)(u-b) - b'(\Sigma_u^{-1} + S'\Sigma_\varepsilon^{-1}S)b + a'\Sigma_\varepsilon^{-1}a
\end{aligned}
\tag{15}
$$

with $b = (\Sigma_u^{-1} + S'\Sigma_\varepsilon^{-1}S)^{-1}S'\Sigma_\varepsilon^{-1}a$ and where $S$ is a 2-dimensional diagonal matrix with on the diagonal $(0,1)$, $(1,0)$ and $(1,1)$ for the first, second and third integral, respectively.

This result allows us to write the product of the two normal pdfs in the three integrals which both depend on $u_t$ as the product of a normal pdf which depends on $u_t$ and a remainder that does not depend on $u_t$. Hence, each integral can be expressed in terms of a CDF of a bivariate normal distribution.

## 3.2 Conditional Inference

In Section 2.3 we already discussed inference on the unobserved variables in the model. This inference was unconditional on the value of $y_t$ and can therefore be used in forecast exercises. In this section we consider probabilistic inference conditional on the observed values of the time series $y_t$. First we consider the conditional probabilities that the elements of the unobserved vector $v_t$ equal zero. For instance, the conditional probability that $v_{1,t} = v_{2,t} = 0$ given $Y_t$ and $x_t$ equals

$$\Pr[v_{1,t} = 0 \wedge v_{2,t} = 0 | Y_t, x_t] = \frac{\Pr[v_{1,t} = 0 \wedge v_{2,t} = 0 | x_t] f(y_t | Y_{t-1}, v_t; \theta)|_{v_{1,t}=0, v_{2,t}=0}}{f(y_t | Y_{t-1}, x_t; \theta)}. \quad (16)$$

The other conditional probabilities are simply

$$\begin{aligned}
&\Pr[v_{1,t} = 0 \wedge v_{2,t} \neq 0 | Y_t, x_t] = \\
&\quad \frac{\int_{-\infty}^{-x'_{1,t}\beta_1} \int_{-x'_{2,t}\beta_2}^{\infty} |\Sigma_u|^{-\frac{1}{2}} \phi(\Sigma_u^{-\frac{1}{2}} u_t) f(y_t | Y_{t-1}, v_t; \theta)|_{v_{1,t}=0, v_{2,t}=x'_{2,t}\beta_2 + u_{2,t}} \, du_{1,t} \, du_{2,t}}{f(y_t | Y_{t-1}, x_t; \theta)} \\
&\Pr[v_{1,t} \neq 0 \wedge v_{2,t} = 0 | Y_t, x_t] = \\
&\quad \frac{\int_{-x'_{1,t}\beta_1}^{\infty} \int_{-\infty}^{-x'_{2,t}\beta_2} |\Sigma_u|^{-\frac{1}{2}} \phi(\Sigma_u^{-\frac{1}{2}} u_t) f(y_t | Y_{t-1}, v_t; \theta)|_{v_{1,t}=x'_{1,t}\beta_1 + u_{1,t}, v_{2,t}=0} \, du_{1,t} \, du_{2,t}}{f(y_t | Y_{t-1}, x_t; \theta)} \\
&\Pr[v_{1,t} \neq 0 \wedge v_{2,t} \neq 0 | Y_t, x_t] = \\
&\quad \frac{\int_{-x'_{1,t}\beta_1}^{\infty} \int_{-x'_{2,t}\beta_2}^{\infty} |\Sigma_u|^{-\frac{1}{2}} \phi(\Sigma_u^{-\frac{1}{2}} u_t) f(y_t | Y_{t-1}, v_t; \theta)|_{v_{1,t}=x'_{1,t}\beta_1 + u_{1,t}, v_{2,t}=x'_{2,t}\beta_2 + u_{2,t}} \, du_{1,t} \, du_{2,t}}{f(y_t | Y_{t-1}, x_t; \theta)}.
\end{aligned} \quad (17)$$

Hence, the marginal probabilities that $v_{1,t} = 0$ and $v_{2,t} = 0$ given $Y_t$ and $x_t$ equal

$$\begin{aligned}
\Pr[v_{1,t} = 0 | Y_t, x_t] &= \Pr[v_{1,t} = 0 \wedge v_{2,t} = 0 | Y_t, x_t] + \Pr[v_{1,t} = 0 \wedge v_{2,t} \neq 0 | Y_t, x_t] \\
\Pr[v_{2,t} = 0 | Y_t, x_t] &= \Pr[v_{1,t} = 0 \wedge v_{2,t} = 0 | Y_t, x_t] + \Pr[v_{1,t} \neq 0 \wedge v_{2,t} = 0 | Y_t, x_t].
\end{aligned} \quad (18)$$

These conditional probabilities indicate whether it is likely that a positive shock affects (one of) the series at time $t$. They can be used to give a business cycle chronology, see Franses and Paap (1998) for an example.

An estimate of the magnitude of the shock at time $t$ follows from the conditional expectation of $v_t$ given $Y_t$ and $x_t$, that is,

$$\begin{aligned}\mathrm{E}[v_{1,t}|Y_t,x_t] &= \mathrm{E}[v_{1,t}|v_{1,t}=0,Y_t,x_t]\,\mathrm{Pr}[v_{1,t}=0|Y_t,x_t]\\ &\quad + \mathrm{E}[v_{1,t}|v_{1,t}\neq 0,Y_t,x_t]\,\mathrm{Pr}[v_{1,t}\neq 0|Y_t,x_t]\\ &= \mathrm{E}[v_{1,t}|v_{1,t}\neq 0\wedge v_{2,t}=0,Y_t,x_t]\,\mathrm{Pr}[v_{1,t}\neq 0\wedge v_{2,t}=0|x_t]\\ &\quad + \mathrm{E}[v_{1,t}|v_{1,t}\neq 0\wedge v_{2,t}\neq 0,Y_t,x_t]\,\mathrm{Pr}[v_{1,t}\neq 0\wedge v_{2,t}\neq 0|Y_t,x_t],\end{aligned} \tag{19}$$

and

$$\begin{aligned}\mathrm{E}[v_{2,t}|Y_t,x_t] &= \mathrm{E}[v_{2,t}|v_{1,t}=0\wedge v_{2,t}\neq 0,Y_t,x_t]\,\mathrm{Pr}[v_{1,t}=0\wedge v_{2,t}\neq 0|x_t]\\ &\quad + \mathrm{E}[v_{2,t}|v_{1,t}\neq 0\wedge v_{2,t}\neq 0,Y_t,x_t]\,\mathrm{Pr}[v_{1,t}\neq 0\wedge v_{2,t}\neq 0|Y_t,x_t].\end{aligned} \tag{20}$$

A typical element in the expression for these conditional expectations is for instance

$$\begin{aligned}&\mathrm{E}[v_{1,t}|v_{1,t}\neq 0\wedge v_{2,t}\neq 0,Y_t,x_t]\,\mathrm{Pr}[v_{1,t}\neq 0\wedge v_{2,t}\neq 0|Y_t,x_t]\\ &\quad = [f(y_t|Y_{t-1},x_t;\theta)]^{-1}\int_{-x'_{1,t}\beta_1}^{\infty}\int_{-x'_{2,t}\beta_2}^{\infty}(x'_{1,t}\beta_1+u_{1,t})|\Sigma_u|^{-\frac{1}{2}}\phi(\Sigma_u^{-\frac{1}{2}}u_t)\\ &\qquad\qquad f(y_t|Y_{t-1},v_t;\theta)|_{v_{1,t}=x'_{1,t}\beta_1+u_{1,t},v_{2,t}=x_{2,t}\beta_2+u_{2,t}}du_{1,t}du_{2,t}.\end{aligned} \tag{21}$$

This expression can be evaluated using the simplification in (15) and the results in Rosenbaum (1961) or Maddala (1983, p. 368) concerning the expectation of a truncated bivariate normal random variable. The same holds for the other expectations in (19) and (20).

As we do not observe $v_t$, evaluating the model in the estimated parameters does not automatically result in estimated fitted values and residuals. We consider two estimators for fitted values based on an unconditional and on a conditional fit. The unconditional fit of the bivariate CLEAR model for $y_t$ is

$$\hat{\mu}+\sum_{i=1}^{p}\hat{\Psi}_i y_{t-i}+\mathrm{E}[v_t|x_t], \tag{22}$$

where we replace the unknown $v_t$'s by the unconditional expectations given in (10) and (11). For the conditional fit we replace the unknown $v_t$'s by the conditional expectations given in (19) and (20), giving

$$\hat{\mu} + \sum_{i=1}^{p} \hat{\Psi}_i y_{t-i} + \mathrm{E}[v_t | Y_t, x_t]. \tag{23}$$

The residuals are simply the differences between the fitted values and the true values of the time series $y_t$. We use these residuals to test for residual autocorrelation and normality using the standard diagnostic measures.

# 4 Application

In this section we will show what a bivariate CLEAR model can do in practice. First, we discuss the relevant data and variables. Next, we estimate (1) with (2) and (3), and we test for parameter restrictions which show us which variables from which economy help to predict unemployment rates. We then turn to a discussion of common nonlinearity and a common cycle. Finally, we give the impulse response functions.

## 4.1 Data and Variables

We consider the log of the seasonally adjusted monthly unemployment rate of the United Stated and Canada for the period 1970.01–1997.12. Figure 1 shows a graph of both series. For both series we notice short periods characterized by large increases in unemployment, which can be called recessions, and longer periods with slow decline in the unemployment rate, the expansions. The recessions periods in both series seem to occur roughly in the same period, except for the first oil crisis halfway the seventies where the Canadian unemployment rate displays a somewhat different pattern than the US unemployment rate in the sense that this recession seems to last longer.

As explanatory variables in the censored regressions (2) and (3) we use the same kind of variables as in Franses and Paap (1998) for their univariate CLEAR model for US unemployment. They consider (lagged values of) monthly seasonally adjusted US industrial production ($i_t^{US}$), the log of the oil price in dollars deflated by seasonally adjusted US CPI ($o_t$), the log of the Dow Jones index ($s_t^{US}$) and the difference between the 10 year treasury bill rate with constant maturity and a 3-month treasury bill rate of the United

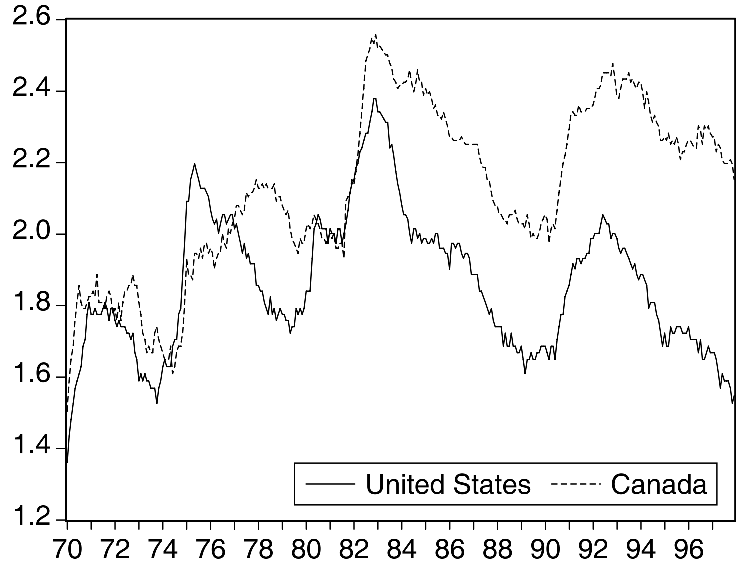


Figure 1: The logarithm of monthly US and Canadian unemployment rate, 1970.01–1997.12

States ($r_t^{US}$). The inclusion of the oil price is based on the results in Hamilton (1983), Tatum (1988) and Mork (1989), while the results in for instance Harvey (1988), Estrella and Hardouvelis (1991) and Estrella and Mishkin (1997) suggest that the term structure of interest may be a good predictor for turning points. Notice that these turning points concern the first and last observations of recessions, and therefore may come close to observations with positive values of $v_{1,t}$ or $v_{2,t}$. In a similar spirit, we use for Canada monthly seasonally adjusted Canadian industrial production ($i_t^{CA}$), the log of the Toronto 300 SE index ($s_t^{CA}$) and the difference between yield on government bonds with life over 10 years and a 3-month treasury bill rate of Canada ($r_t^{CA}$). To remove possible (stochastic) trends in the explanatory variables we take first differences of the industrial production series, the oil price and the stock indices.

## 4.2 A Bivariate CLEAR Model

First, we estimate the bivariate CLEAR model (1)–(3) with the US and Canadian explanatory variables in both censored regressions. To determine the lag structure of the explanatory variables, we first construct two univariate CLEAR models for US as well as Canadian unemployment, where we either include US or Canadian explanatory variables for both series. It turns out that the order $p$ of the univariate model equals 1. For each of the four univariate models we determine the lag structure as in Franses and Paap (1998), that is, the explanatory variables are added one by one to the censored regression and in each step the optimal lag structure of the added variable is based on the maximum value of the likelihood over all possible lag structures. At the end of this procedure, we check whether a change in the lag structure of one of the explanatory variables gives a higher value of the likelihood. Finally, the lag structure of the bivariate model simply follows from the lag structure of the univariate models.

With the above specification procedure, we arrive at the following bivariate CLEAR model with estimated standard errors in parentheses

$$y_t = \begin{pmatrix} \underset{(0.0177)}{0.0240} \\ \underset{(0.0158)}{0.0004} \end{pmatrix} + \begin{pmatrix} \underset{(0.009)}{0.978} & \underset{(0.009)}{0.004} \\ \underset{(0.009)}{0.011} & \underset{(0.007)}{0.988} \end{pmatrix} y_{t-1} + \begin{pmatrix} v_{1,t}\mathbb{I}[v_{1,t} \geq 0] \\ v_{2,t}\mathbb{I}[v_{2,t} \geq 0] \end{pmatrix} + \varepsilon_t, \tag{24}$$

with

$$\begin{aligned} v_{1,t} = \; & \underset{(0.007)}{0.020} - \underset{(0.007)}{0.013}\; \Delta i^{US}_{t-2} - \underset{(0.004)}{0.016}\; r^{US}_{t-10} + \underset{(0.045)}{0.060}\; \Delta o_{t-12} - \underset{(0.120)}{0.096}\; \Delta s^{US}_{t-7} \\ & - \underset{(0.004)}{0.005}\; \Delta i^{CA}_{t-6} - \underset{(0.003)}{0.001}\; \Delta r^{CA}_{t-9} - \underset{(0.080)}{0.128}\; \Delta s^{CA}_{t-7} + u_{1,t} \end{aligned} \tag{25}$$

and

$$\begin{aligned} v_{2,t} = & -\underset{(0.020)}{0.024} + \underset{(0.013)}{0.010}\; \Delta i^{US}_{t-4} - \underset{(0.008)}{0.007}\; r^{US}_{t-10} + \underset{(0.110)}{0.086}\; \Delta o_{t-12} - \underset{(0.169)}{0.215}\; \Delta s^{US}_{t-8} \\ & - \underset{(0.008)}{0.016}\; \Delta i^{CA}_{t-1} - \underset{(0.005)}{0.010}\; \Delta r^{CA}_{t-9} - \underset{(0.159)}{0.280}\; \Delta s^{CA}_{t-9} + u_{2,t} \end{aligned} \tag{26}$$

and

$$\Sigma_\varepsilon = 10^{-4} \begin{pmatrix} 5.16 & 0.77 \\ 0.77 & 5.62 \end{pmatrix}, \; \Sigma_u = 10^{-3} \begin{pmatrix} 0.47 & 0.78 \\ 0.78 & 1.30 \end{pmatrix}, \; \mathcal{L}_{\max} = 1526.39, \tag{27}$$

where $\mathbb{I}[\cdot]$ is an indicator function which is 1 if the argument holds and zero otherwise. To analyze possible misspecification in the lag order of the VAR model (24), we regress estimated residuals on its lagged values and test for the significance of the lagged residuals using a likelihood ratio [LR] test. The residuals here are taken as the differences between the series $y_t$ and the unconditional fit, as in (22). The LR test statistic equals 6.00 with a $p$-value 0.20, and hence a VAR model of order 1 seems sufficiently capturing the dynamics in $y_t$. A normality test on the residuals of the US and Canadian equation equals 1.62 (0.44) and 5.68 (0.06) respectively with $p$-values in parentheses. Hence we cannot reject normality at the 5 % level.

Before we discuss the parameters estimates, we first examine whether the Canadian explanatory variables have predictive power in the innovation generating mechanism for the US unemployment rate and *vice versa*. A LR test for the absence of the Canadian variables in the $v_{1,t}$-equation equals 6.23, which, when compared with fractiles from the $\chi^2(3)$ distribution, is not significant at the 5% level. Since the off-diagonal elements of the $\Psi_1$ matrix in (24) are not significantly different from zero, we may conclude that the Canadian variables do not seem to have explanatory power for positive shocks in US unemployment.

A joint test for the absence of Canadian variables in the $v_{1,t}$-equation and the absence of US variables in the $v_{2,t}$-equation leads to a LR statistic of 15.67, which is significant at the 5% level. Hence, the US variables seem to have explanatory power for forecasting positive shocks in Canadian unemployment. Finally, the LR statistic for the joint test of the absence of the Canadian variables in both censored regressions equals 29.92, which is again not significant at the 5% level. This means that Canadian variables do have information value for predicting positive shocks for Canadian unemployment over that contained in US variables. In the sum, these test results suggest that the condition of the US economy has predictive value for that of Canada, but not *vice versa*.

The above test results suggest that the Canadian variables can be removed from (26). The ML estimates of the resulting model are not much different from the estimates of (24)–(26). The main difference is found for the parameter estimate of the Dow Jones returns in the $v_{1,t}$-equation. This value decreases from $-0.096$ to $-0.223$. The discussion on estimation results in the remainder of this subsection is based on this restricted model.

To save space we do not give the full results and a detailed outline of the estimation results can be obtained from the authors.

The estimated autoregressive parameter matrix $\Psi_1$ in (24) corresponds with two characteristic roots, which are close to but smaller than 1. This indicates that if $v_t = 0$ there is slow convergence to an equilibrium. We may of course impose two unit roots in the VAR model, but for our analysis this is not necessary. The equilibrium can be interpreted as the natural unemployment rate in both countries. Since we have log-transformed the data, this natural unemployment rate is approximately equal to $\exp((\mathbf{I}_2 - \Psi_1)^{-1}\mu)$ which results in 3.20 % for the US and 3.27% for Canada. Interestingly, these natural rates are almost equal. Note that these values may never be reached as future shocks may move the unemployment rates away from the equilibrium values. It is also of interest that our bivariate CLEAR models allows for a straightforward calculation of equilibrium values, which is in contrast to many other nonlinear models.

**Model Interpretation**

Most coefficients of the explanatory variables in the censored regressions (25)–(26) have the expected sign. Negative growth in industrial production, negative stock returns and a negative difference between long and short term interest rates increase the probability of a positive $v_t$ in (24) and hence of a sudden increase in the level of unemployment. The same applies for an increase in the real oil price. For the US, the interest spread and the stock market returns seem to be the most important variables, while for Canada the interest spread is very important. The estimated correlation between the $u_{1,t}$ and $u_{2,t}$ variable is very close to 1 and given the value of the explanatory variables, it is likely that $v_{1,t}$ enters the equation for $y_{1,t}$ at the same time as $v_{2,t}$ does for $y_{2,t}$.

Figure 2 shows the four joint conditional probabilities given in (16) and (17) for the estimated model. The upper left cell shows the conditional probability that both series are unaffected by a positive shock. The lower right cell contains the conditional probability that both series experience a positive shock. The off-diagonal cells show the conditional probability that a positive shock only affects one of the series. Note that the

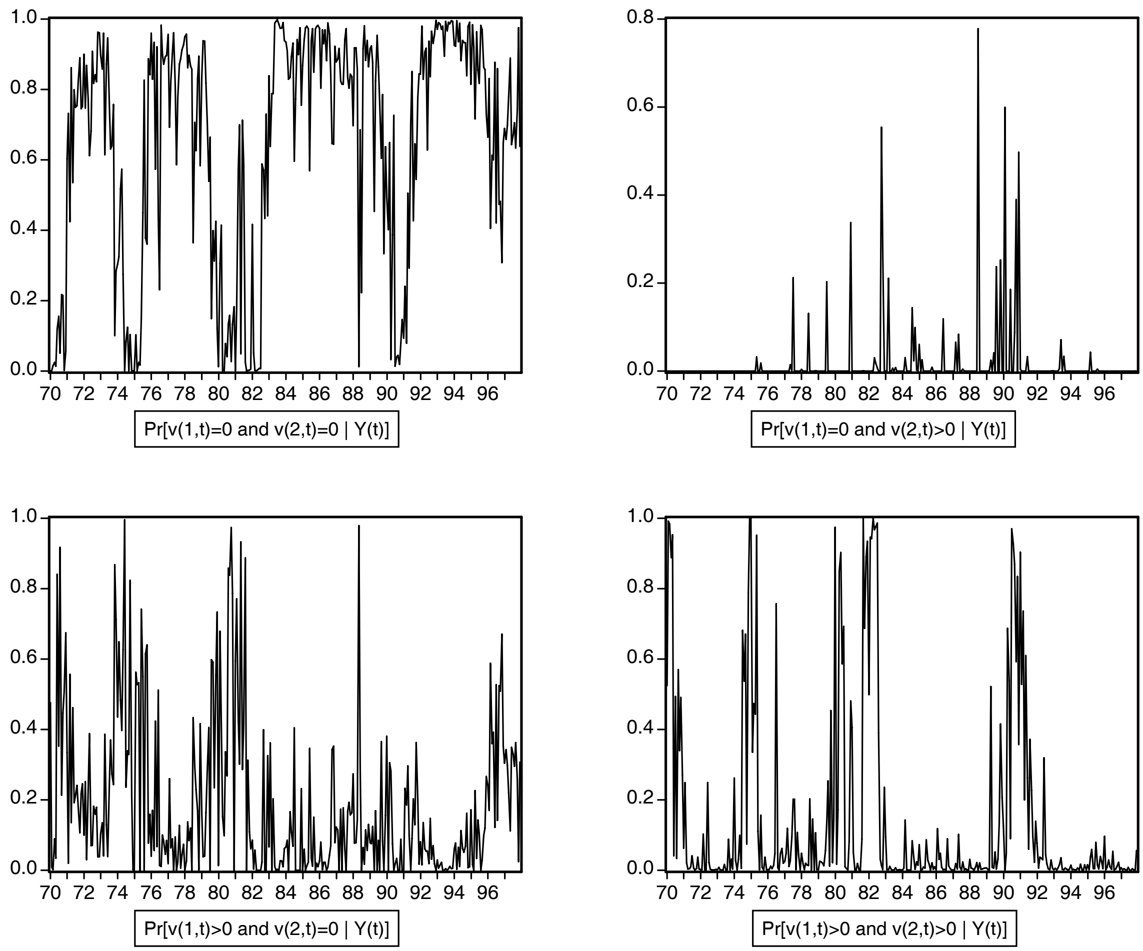


Figure 2: Estimated conditional probabilities (16) and (17) for model (24)–(26)

Table 1: Peaks and troughs for US and Canadian unemployment based on estimated conditional probabilities[1].

| US | | Canada | | common | |
|---|---|---|---|---|---|
| peak | trough | peak | trough | peak | trough |
| – | 70.12 | – | 70.05 | – | 70.12 |
| 74.04 | 75.07 | 74.06 | 75.01 | 74.05 | 75.05 |
| 79.07 | 81.02 | | | | |
| 81.08 | 82.07 | 81.08 | 82.07 | 81.08 | 82.12 |
| 90.03 | 91.05 | 90.02 | 91.05 | 90.03 | 91.08 |

[1] A recession is defined by 6 consecutive months for which $\Pr[v_{i,t} \neq 0 \mid Y_t, x_t] > 0.5$. A peak corresponds with the last expansion observation before a recession and a trough with the last observation in a recession.

four conditional probabilities sum to one.

The probabilities in the upper right cell are almost always zero and this can be interpreted as that Canadian unemployment is almost never in a recession when the US is not. This matches the above findings that US variables are important for both unemployment rates. The lower left cell, which shows the conditional probability that shocks affect US unemployment but not Canadian unemployment, contains more peaks. This occurs mostly in the first half of the sample. Very large and sudden increases in unemployment during the first oil crises are more pronounced for the US series than for the Canadian series, see also Figure 1. Furthermore, the increase in unemployment around 1979 is larger in the US than in Canada.

The marginal conditional probabilities (18) are show in Figures 3 and 4. These probabilities can be used to determine turning points in the business cycle of the individual series, see Franses and Paap (1998) for an example. We may define a recession by 6 consecutive months for which $\Pr[v_{i,t} \neq 0|Y_t, x_t] > 0.5$. A peak is defined by the last expansion observation before a recession. A trough is defined by the last observation in a recession.

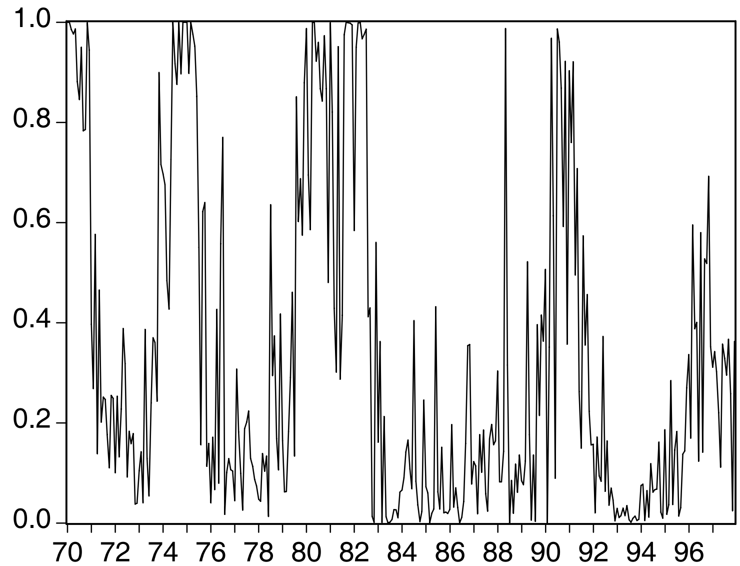


Figure 3: The marginal conditional probabilities $\Pr[v_{1,t} \neq 0|Y_t, x_t]$ for the estimated model (24)–(26) (US)

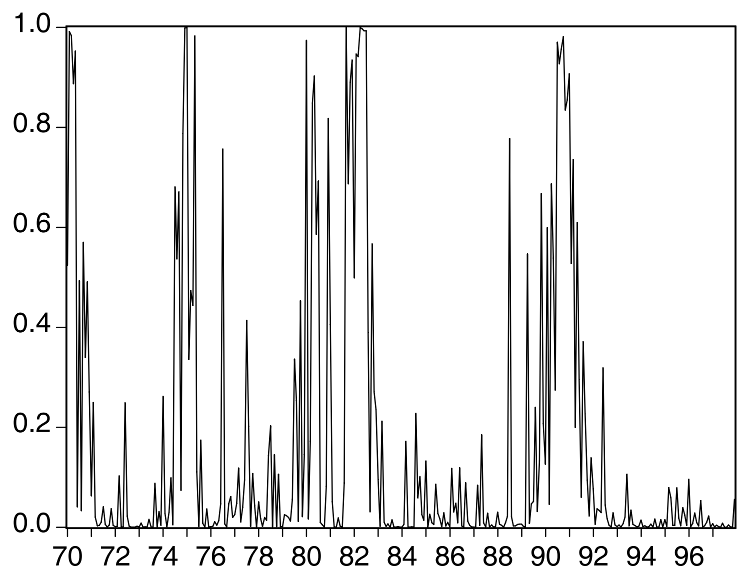


Figure 4: The marginal conditional probabilities $\Pr[v_{2,t} \neq 0|Y_t, x_t]$ for the estimated model (24)–(26) (Canada)

The first four columns of Table 1 show the estimates for the peaks and troughs. Since the definition of a recession may be too restrictive, we allow that the marginal conditional probability may be smaller than 0.5 for one month in a row during a recession. The peaks and troughs of the US unemployment based on the multivariate CLEAR model are quite similar to the estimated peaks and troughs based on a univariate CLEAR model, see Franses and Paap (1998). If we compare the peaks and troughs of US and Canadian unemployment, we notice clear differences for the first three recessions. For the last two recessions, the dates of the peaks and troughs are almost exactly the same, suggesting potential common features. In the next subsection we formally analyze if the US and Canada have common nonlinear features in their unemployment series.

## 4.3 Investigating Common Features

Figure 5 and 6 show the conditional expectations of the value of the shocks $v_t$ defined in (19) and (20). As implied by the model, the shocks are positive during the recessionary periods. In case of perfect common nonlinearity in the two unemployment series, these values of $v_{1,t}$ and $v_{2,t}$ would be the same.

To analyze perfect common nonlinearity we estimate (4) with (2). The estimation results of this model are

$$y_t = \begin{pmatrix} \underset{(0.0149)}{0.0497} \\ \underset{(0.0168)}{-0.0276} \end{pmatrix} + \begin{pmatrix} \underset{(0.009)}{0.977} & \underset{(0.008)}{-0.006} \\ \underset{(0.009)}{0.015} & \underset{(0.007)}{0.996} \end{pmatrix} y_{t-1} + \begin{pmatrix} 0 \\ \underset{(0.73)}{2.96} \end{pmatrix} v_{1,t}\mathbb{I}[v_{1,t} \geq 0] + \varepsilon_t, \tag{28}$$

with

$$\begin{aligned} v_{1,t} = \underset{(0.004)}{0.017} - \underset{(0.004)}{0.005}\ \Delta i^{US}_{t-2} - \underset{(0.002)}{0.010}\ r^{US}_{t-10} + \underset{(0.015)}{0.059}\ \Delta o_{t-12} + \underset{(0.067)}{0.046}\ \Delta s^{US}_{t-7} \\ - \underset{(0.002)}{0.004}\ \Delta i^{CA}_{t-6} - \underset{(0.002)}{0.002}\ \Delta r^{CA}_{t-9} - \underset{(0.051)}{0.212}\ \Delta s^{CA}_{t-7} + u_{1,t} \end{aligned} \tag{29}$$

and

$$\Sigma_\varepsilon = 10^{-4}\begin{pmatrix} 5.99 & 0.36 \\ 0.36 & 4.35 \end{pmatrix},\ \Sigma_u = 0.12 \times 10^{-3},\ \mathcal{L}_{\max} = 1509.64, \tag{30}$$

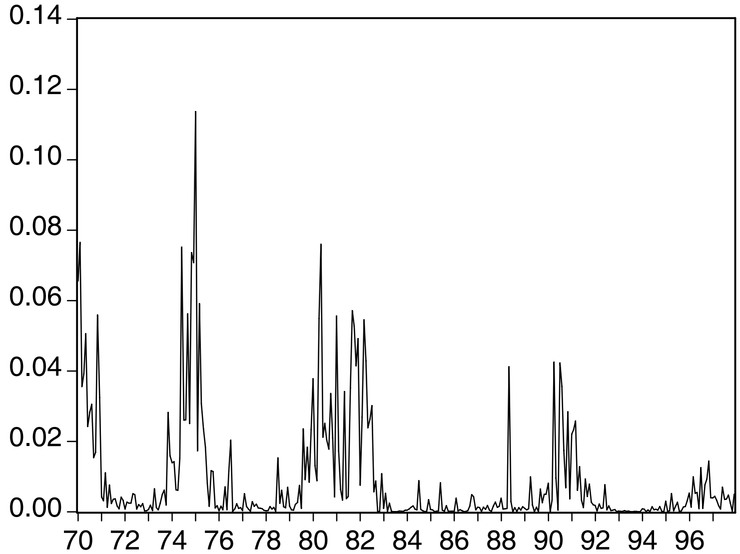


Figure 5: The conditional expectation of $v_{1,t}$ $\mathrm{E}[v_{1,t}|Y_t, x_t]$ for the estimated model (24)–(26) (US)

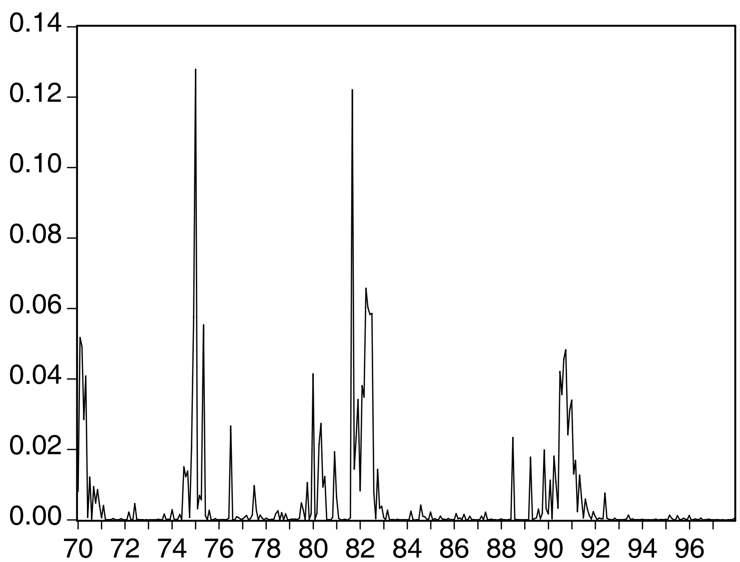


Figure 6: The conditional expectation of $v_{2,t}$ $\mathrm{E}[v_{2,t}|Y_t, x_t]$ for the estimated model (24)–(26) (Canada)

Note again that this model is not nested in (24)–(26).

To compare both models we follow an encompassing test strategy, see Davidson and MacKinnon (1993, p.381–388). We add the unconditional fit of (24)–(26) as explanatory variables to the VAR equation (28) and re-estimate the model. A LR test for the absence of the unconditional fit variables equals 31.62, which, when compared to the fractiles from the $\chi^2(4)$ distribution, is clearly not significant at the 1% level. Hence, perfect common nonlinearity for the US and Canadian unemployment rate gets rejected convincingly.

Although US and Canada do not display perfect common nonlinearity, they still may have a common nonlinear cycle. To analyze the presence of such a cycle we consider the model (1) with (5). The estimated model is

$$y_t = \begin{pmatrix} \underset{(0.0155)}{0.0330} \\ \underset{(0.0151)}{0.0015} \end{pmatrix} + \begin{pmatrix} \underset{(0.009)}{0.980} & \underset{(0.008)}{-0.002} \\ \underset{(0.009)}{0.009} & \underset{(0.007)}{0.989} \end{pmatrix} y_{t-1} + \begin{pmatrix} v_{1,t} \\ v_{2,t} \end{pmatrix} \mathbf{I}[v_{1,t} \geq 0 \wedge v_{2,t} \geq 0] + \varepsilon_t, \tag{31}$$

with

$$\begin{aligned} v_{1,t} = \; & \underset{(0.006)}{0.039} - \underset{(0.009)}{0.018}\, \Delta i^{US}_{t-2} - \underset{(0.004)}{0.011}\, r^{US}_{t-10} + \underset{(0.079)}{0.052}\, \Delta o_{t-12} - \underset{(0.154)}{0.021}\, \Delta s^{US}_{t-7} \\ & - \underset{(0.004)}{0.003}\, \Delta i^{CA}_{t-6} - \underset{(0.003)}{0.003}\, \Delta r^{CA}_{t-9} - \underset{(0.104)}{0.196}\, \Delta s^{CA}_{t-7} + u_{1,t} \end{aligned} \tag{32}$$

and

$$\begin{aligned} v_{2,t} = & - \underset{(0.007)}{0.006} + \underset{(0.010)}{0.012}\, \Delta i^{US}_{t-4} - \underset{(0.004)}{0.007}\, r^{US}_{t-10} + \underset{(0.073)}{0.064}\, \Delta o_{t-12} - \underset{(0.098)}{0.219}\, \Delta s^{US}_{t-8} \\ & - \underset{(0.005)}{0.008}\, \Delta i^{CA}_{t-1} - \underset{(0.003)}{0.009}\, \Delta r^{CA}_{t-9} - \underset{(0.091)}{0.233}\, \Delta s^{CA}_{t-9} + u_{2,t} \end{aligned} \tag{33}$$

and

$$\Sigma_\varepsilon = 10^{-4} \begin{pmatrix} 5.09 & 0.66 \\ 0.66 & 5.68 \end{pmatrix}, \; \Sigma_u = 10^{-3} \begin{pmatrix} 0.51 & 0.19 \\ 0.19 & 0.73 \end{pmatrix}, \; \mathcal{L}_{\max} = 1527.36. \tag{34}$$

The maximum likelihood value of this model is somewhat higher than the maximum likelihood of model (24)–(26), which equals 1526.39. As the models are not nested, we construct again an encompassing test by adding the unconditional fitted values of the estimated model (24)–(26) as explanatory variables to equation (31). The LR statistic for the absence of these unconditional fitted values equals 11.36 with a $p$-value of 0.02. Hence, there is some evidence for a common nonlinear cycle, but this evidence is not very strong. On the other hand, if we compare the conditional fitted values of both models there is not much difference, see Figures 7 and 8. The off-diagonal graphs in Figure 2, however, together with the estimated turning points in Table 1 suggest that there seems to be a common nonlinear cycle only after 1982. Apparently, this explains that the first fitted model has some explanatory power over the model (31)–(33).

A LR statistic to test for the absence of the Canadian explanatory variables in the $v_{1,t}$ equation (32) equals 7.04. Hence, this hypothesis cannot be rejected at the 5% level. Again, parameter estimates for the model without the Canadian explanatory variables in the $v_{1,t}$ equation do not differ much from the estimated model in (31)–(33), except for the parameter concerning Dow Jones returns in the $v_{1,t}$ equation which decreases in from $-0.021$ to $-0.232$. The discussion in the remainder of this section is based on the estimation results of the restricted model. A detailed outline of the estimation results can be obtained from the authors.

Figure 9 shows the joint conditional probability that a positive shock appears simultaneously in both series for the estimated 'common cycle' model (31)–(33). Since the model does not allow that only one of the series is affected by a positive shock, we may interpret these conditional probabilities in terms of a common business cycle in US and Canadian unemployment. Again, we may define a recession by 6 consecutive months for which $\Pr[v_{1,t} \neq 0 \wedge v_{2,t} \neq 0 | Y_t, x_t] > 0.5$. The final two columns of Table 1 show the estimated peaks and troughs based on this strategy. We do not find a recessionary period at the end of the seventies/beginning of the eighties. There is no sequence of six months in a row in this period for which $\Pr[v_{1,t} \neq 0 \wedge v_{2,t} \neq 0 | Y_t, x_t] > 0.5$, although the conditional probabilities suggest that the series are affected by quite a number of positive shocks in this period, see Figure 9.

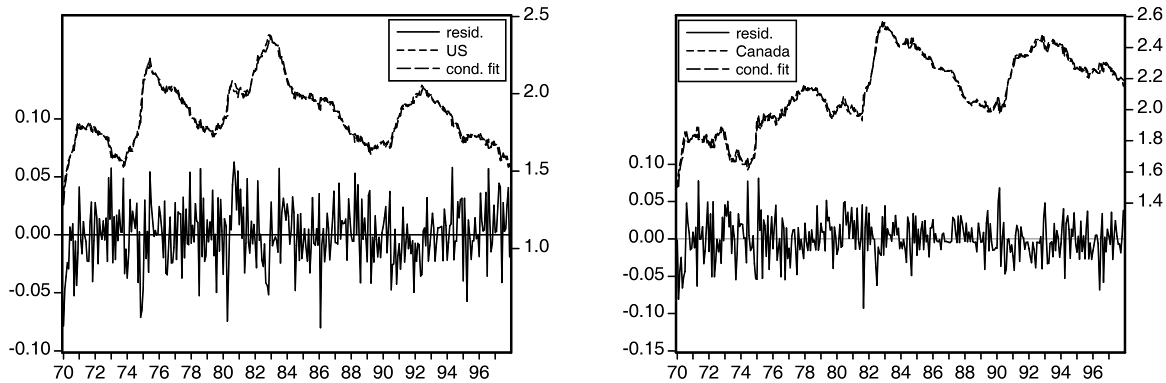


Figure 7: Unconditional fit and residuals of the estimated model (24)–(26) (unrestricted nonlinearity and cycle)

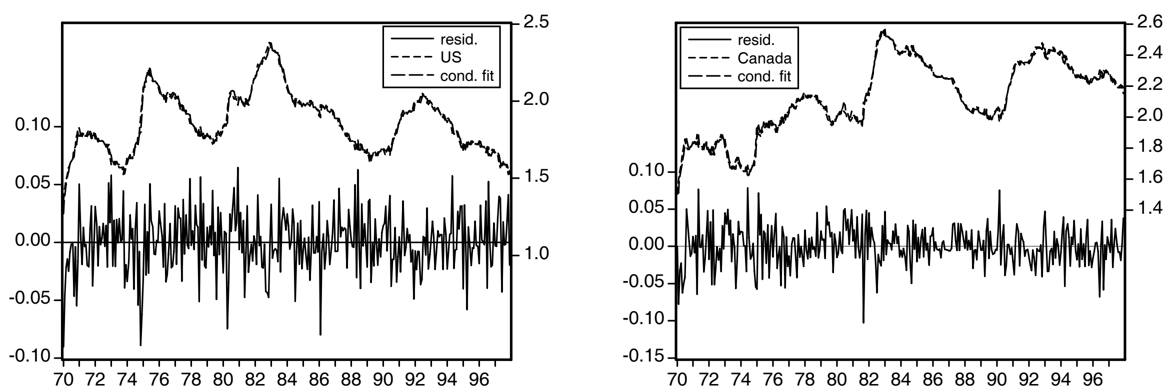


Figure 8: Unconditional fit and residuals of the estimated model (31)–(33) (common nonlinear cycle)

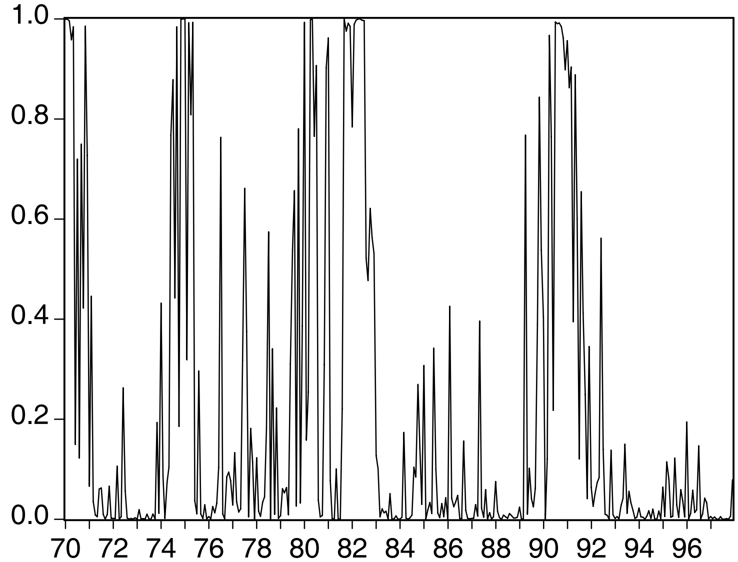


Figure 9: The conditional probability $\Pr[v_{1,t} \neq 0 \wedge v_{2,t} \neq 0 | Y_t, x_t]$ for the estimated model (31)-(33)

## 4.4 Impulse Response Function

The conditional expectations of the components of $v_t$ for the estimated model (31)–(33) are displayed in Figures 10 and 11. These conditional expectations indicate the magnitude of a shock during a recession. Note that the total impact of the shocks on future values of the unemployment rates also depends on the value of the autoregressive parameters in $\Phi_1$. For instance, the impact of the shock at time $t$ on unemployment at period $t+1$ is $\Psi_1 v_t$. To analyze the impact of the recessions on future values of the series we calculate the total impact of the recessions for both series.

Figure 12 shows the total impact of the four recessions identified by the turning point analysis above. The graphs show the impact of the estimated innovations $\hat{v}_t = \mathrm{E}[v_t | Y_t, x_t]$ on future values of the time series. In fact, if $t$ denotes the first month of the recession, we display in each graph $\hat{v}_t$, $\hat{v}_{t+1} + \Psi_1 \hat{v}_t$, $\hat{v}_{t+2} + \Psi_1 \hat{v}_{t+1} + \Psi_1 \Psi_1 \hat{v}_t$, and so on. To compare the impact of the different recessions directly, we put $\hat{v}_t$ equal to zero after each recession. The recessionary periods are based on the final two columns of Table 1. The two graphs in the first row of Figure 12 show similar patterns, but are different from the two graphs in the

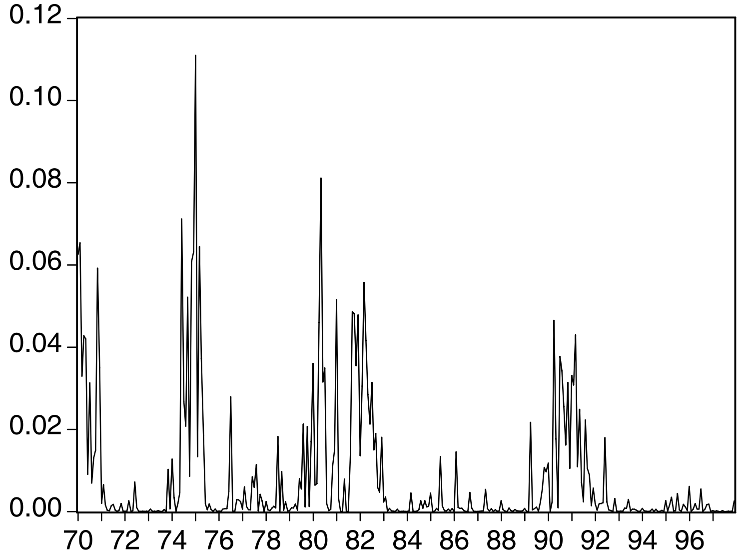


Figure 10: The conditional expectation of $v_{1,t}$ $\mathrm{E}[v_{1,t}|Y_t, x_t]$ for the estimated model (31)–(33) (US)

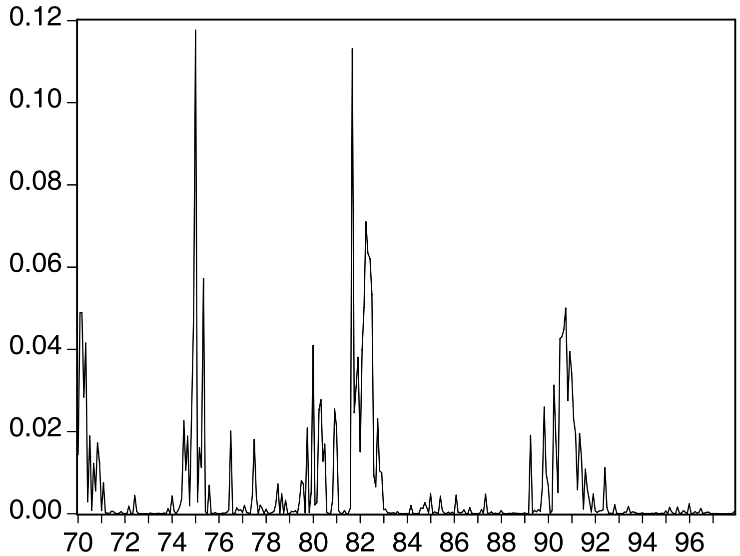


Figure 11: The conditional expectation of $v_{2,t}$ $\mathrm{E}[v_{2,t}|Y_t, x_t]$ for the estimated model (31)–(33) (Canada)

bottom row, which display again a similar pattern. For the recession in the beginning and middle of the seventies we see that the total impact of the shocks during the recession has been larger for the US than for Canada but that due to a larger autoregressive parameter for the Canada the impact of the recession is longer for Canada than for the US. However, for the recessions in the beginning of the eighties and the nineties, the total impact of shocks during the recessionary periods has been larger for Canada than for the US.

To summarize this empirical section, the bivariate CLEAR models presented in this paper seem to be a useful tool in analyzing possible common nonlinear patterns in the US and Canadian unemployment series. US explanatory variables have explanatory power for Canadian recession, while this is not the case for the reverse relation. There is no perfect common nonlinearity across the two series, but there is some evidence for a common nonlinear cycle, especially for the second half of the sample period. Finally, the impact of a recession is quite different for Canadian and US unemployment. For the recessions in the beginning and middle of the seventies, the total impact of the shocks during recessions is larger for the US than for Canada, but the effect of the impact lasts longer for Canada, and lasts even beyond the recession period. However, for the recession in the beginning of the eighties and the nineties the total impact of the shocks during the recessionary periods is larger for Canada than for the US.

# 5 Concluding Remarks

In this paper we have proposed a bivariate time series model to analyze common nonlinear patterns in time series. The model is a standard vector autoregression which is sometimes affected by positive innovation shocks. The mechanism that generates and explains the innovation shocks consists of one or more censored regression models. Positive shocks enter the autoregression if a linear combination of explanatory variables exceeds a stochastic threshold level. We used this bivariate censored latent effects autoregressive model to describe the nonlinear cycles in the US and Canadian unemployment rate.

The proposed model appeared to describe the typical characteristics of the two un-

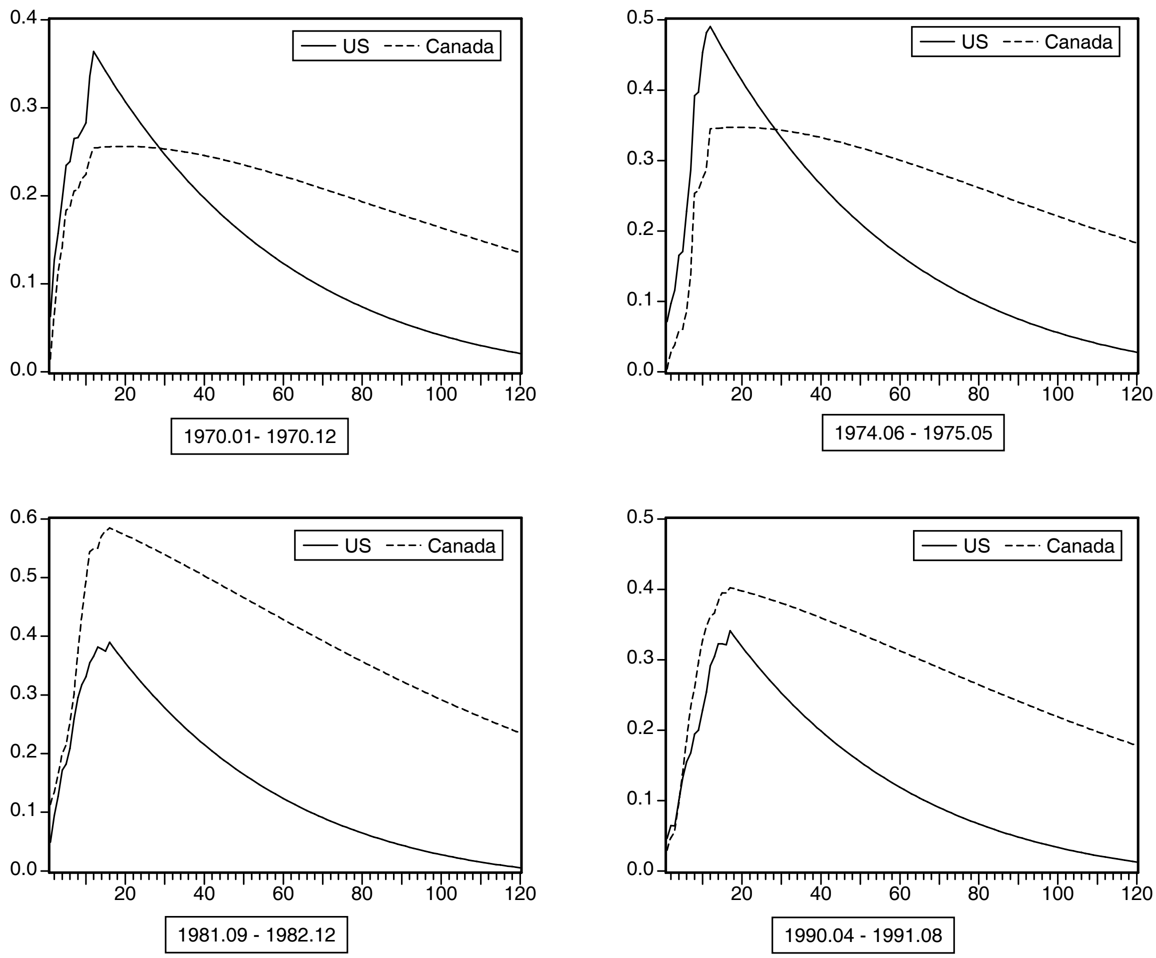


Figure 12: Total effect of recessions on future values of the US and Canadian unemployment rate.

employment series rather well, while using only a relatively small number of parameters. The estimation results showed that US variables, like lagged Dow Jones returns, lagged US interest spread and the growth in industrial production, have explanatory power for predicting recessions in the Canadian unemployment rate, while the same variables for Canada have no explanatory power for predicting recessions in the US unemployment rate. Although there is no perfect common nonlinearity in the US and Canadian unemployment series, there is some evidence for a common nonlinear cycle in the US and Canadian unemployment rate, especially after 1981.

The bivariate CLEAR model can easily be extended to analyze three or more time series. Especially, the specification which imposes that the series are effected by shocks at the same time, seems to be appropriate to estimate a business cycles based on more than two variables. These multivariate nonlinear models can be used to investigate and estimate common cycles in unemployment rates or other macroeconomic time series of several industrialized countries. Further research should shed light on the usefulness of the multivariate CLEAR model in other occasions, where it can occur that these applications requires straightforward modifications of the model.

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