CHAPTER III

RELATIONS: STRUCTURAL RELATIONS AND BOUNDARY CONDITIONS. EXAMPLES OF MODELS; ALTERNATIVE SETUPS

1. The logic of economic policy is largely determined by the *nature and number of relations* existing between the various types of variables described in chapter II. These relations are in principle the relations now usually called *structural relations* of an economy or an economic model, which are nothing but the "primary" relations described by the pioneers of mathematical economics. Here the word primary is used in contradistinction to derived or reduced forms obtained, by any process of combination or elimination, from the primary relations. As a rule the primary relations represent the direct logical ties between the variables introduced by economic behaviour or by the logic of definition or technique.

Hence the structural equations may be grouped into *demand*, *supply*, *definition* and *technical* equations or relations. Their number evidently depends on the degree of detail used in the model considered; for each separate market introduced into the picture, a supply and demand equation and maybe technical relations are needed. Definition equations may be equations defining such concepts as "total value sold" out of "prices" and "quantities"; or they may define certain incomes or balance items. In almost every model a definition equation for national income will be needed. Technical equations are among others such equations as the one existing between the volume of production and the quantities of productive agents used. It seems desirable to follow some generally accepted system of introducing all these relations in order that a judgment of the set of relations concerning its completeness be easily possible. This might be the beginning of a systematic study of the characteristics of economic models.

Such a systematic setup should consist first of all of a list of the markets considered, indicating product markets and factor markets separately. For each of them demand and supply relations should be present. In those relations prices and quantities handled will be among the variables; moreover, incomes will come in for certain sectors. These incomes have to be defined; among them may be the public sector. Technical relations will have to be added according to the physical side of the model presented.

The set of structural equations just discussed is the same for problems of economic policy and for the traditional problems of economic theory. But the use made of these equations is different since, as follows from chapter II, the given and the unknown quantities are different. In the traditional problems, political instruments are among the given entities and the targets among the unknowns. In the theory of economic policy, these two categories interchange places; in a sense, *the problem is inverted.*

2. Under certain conditions, to be considered somewhat more carefully in chapter VII, the relations may be given the linear form. With this simplification, systematic studies are possible on the structure of models and of problems of economic policy, which are hardly possible without it. We shall devote some special attention to that type of model. After the choice of the variables the system of structural relations may be given the general form:

$$\begin{array}{lr} \Sigma_j \, a_{ij} x_j + \Sigma_k \, \beta_{ik} y_k + \Sigma_l \, \gamma_{il} z_l = u_i & (1) \\ \text{or} \quad Ax + By + \Gamma z = u & (2) \end{array} \left\{ \begin{array}{l} i = 1 \ldots N' \\ j = 1 \ldots N \\ k = 1 \ldots n \\ l = 1 \ldots n' \end{array} \right.$$

Here N' represents the number of structural relations;

α_{ij}, β_{ik} and γ_{il} coefficients with their corresponding matrices A, B and Γ; x, y, z and u are vectors representing the irrelevant variables, the target variables, the instrument variables and the data. The data have here been aggregated already according to their appearance in the structural equations, u_i representing the sum of all terms in equation i containing data.

Typical of the structure of any economic model are the matrices A, B and Γ. If the model is represented in some detail, a great many of the coefficients α, β and γ will be zero (cf. this chapter, § 3).

3. Apart from the structural relations another type of equations may play a rôle, to be labelled "*boundary conditions*". They are a very handsome supplement to the use of linear relations and in a sense represent all the protests of reality against the supposed linearity. They may be used, first of all, to express the condition that negative prices or quantities are technically impossible or the stricter condition that certain prices and quantities cannot exceed certain narrower limits. One may also think of upper limits in the case of bottle necks, important in micro-models. Apart from technical reasons there may be a number of other reasons. Tax rates must not exceed certain limits because otherwise the tendency to fraud or evasion will become too strong. Wage rates cannot, for social or political reasons, be lowered or at least lowered more than a few percent. Boundary conditions may also be used to express certain political devices such as "social equilibrium" i.e. certain proportionalities between the incomes or the sacrifices of different social groups.

An important type of boundary conditions is due to the limits which may be set to real activity by certain financial situations or measures. If, by an increase in activity, the point is reached where a further increase can only be financed out of new money creation and the central bank refuses to do so, an important brake may be put to that activity. An even stronger example presents itself if gold and exchange reserves of a country have been exhausted and foreign payments, as a consequence, are impossible.

It will be clear that, in principle, boundary conditions are *inequalities*, with equalities as their limiting form. As long as they are satisfied they need not to be introduced and do not, so to say, play an active rôle. They only become "active", i.e. have to be added to the other relations, if their fulfilment is threatened and then have to be given their limiting form, i.e. then they are equations.

In principle it may be possible to avoid the introduction of such boundary conditions. Negative prices or quantities will not come into existence if only the shape of the demand and supply functions be such that their geometric representation does not intersect with the axes. It is often easier, however, to use linear approximations with boundary conditions.

The rôle of boundary conditions is not exactly such as to be supplementary to the structural relations; they rather have to be substituted for some of the latter, as soon as a solution should be found contradictory to one or more of the boundary conditions. The problem has then to be reformulated with one or more boundary conditions instead of some of the structural relations (cf. Chapter VI).

4. The two types of relations just discussed will now be exemplified by the description of some simple models used by the author for the study of problems of economic policy. Two examples will be discussed and also be used to illustrate the subject-matter of the following chapters, relating to practically the same model used for incidental policies in the situation in 1949 and in 1950 respectively, but (2) somewhat more elaborated.

Examples (1) *and* (2) are derived from the same *basic model* that will first be set out.

The model represents the economy of a country with international trade. Three markets are distinguished, viz. two commodity markets and one factor market. The commodity markets are those for home products (sold at home as well as abroad) and for import products (sold at home only). The factor considered is labour.

Accordingly, the variables considered may be grouped as follows:

Market		Prices	Quantities	Values (market)
Home products	sold at home . . .	p^x	x	X^M
	exported	p	e	E
Import products		p^m	m	m
Labour		l	a (index)	L

Prices p^x are market prices; the corresponding factor-cost prices are p.

For home-sold home products the relevant incomes are supposed to be labour and non-labour income. Exports are supposed to depend on world incomes and competitive world prices, but world incomes are not introduced as a separate variable; only the "shift in export volume due to changes in world income". Import goods are supposed to be raw materials and semi-finished products only (even "finished" products are only so in the technical sense and hardly ever sold to consumers directly); hence "demand" for imports is supposed to be of a technical nature, determined by the production function. The same applies to the demand for labour. (In a sense we might have considered the import market also as a "factor" market).

Accordingly the following further variables have been considered:

Incomes	Nominal	Real
Labour	L	L^R
Non-Labour.	Z	Z^R
Total National (at factor prices) . . .	Y^F	.
Volume of production	.	y

Shift in export volume due to changes in world income	e^W
Competitive world market prices	p^W

Finally the following variables have been introduced:

Home sales at factor prices	X^F
Autonomous component in national expenditure .	ξ_0
Profit margin	π_0
Level of indirect taxes (as a fraction of factor values)	τ
Labour productivity	h
Labour costs per unit of product	l'
Real wage rate	l^R
Balance of payments deficit	D

The rôle of some of these variables will become clear from the setting of the problems, i.e. the choice of targets and instruments in the examples and their discussion may be postponed for a while.

The *symbols* just mentioned will be used in three different ways. With a double bar ($\bar{\bar{e}}$) they indicate the optimum values, expressed in the units to be indicated, of the variables; with a single bar ($\bar{e}$) they indicate some initial value, i.e. the value before the optimum policy has been applied; without a bar (e) they represent the difference $\bar{\bar{e}} - \bar{e}$. Single-barred values are considered as given for any of the variables. For the data among the variables, also the non-barred values are supposed to be given. Non-barred values are treated as differentials, i.e. as small numbers in comparison to initial values.

Units. Prices are expressed as indexnumbers with initial values as their bases; hence $\bar{p} = \bar{p}^m = \bar{p}^x = \bar{l} = \bar{p}^W = 1$. For

$\bar{p}^x$ this is only possible since, for simplicity's sake, it has been assumed that $\bar{\tau} = 0$. All values are expressed as a percentage of initial national income at factor prices. Initial quantities are therefore equal to the corresponding initial values.
$\bar{p}^x \neq 1$.

The following values, being the values for the Netherlands economy in 1949, have been assumed:

$$\overline{X^F} = \bar{x} = 1.04 \; ; \; \bar{E} = \bar{e} = 0.40 \; ; \; \overline{M} = \bar{m} = 0.44 \; ; \; \overline{D} = 0.04 \; ;$$
$$\overline{L} = \overline{L^R} = 0.55 \; ; \; \overline{Z} = \overline{Z^R} = 0.45 \; ; \; \overline{Y^F} = 1; \quad \overline{X} = 1.04.$$
$$\bar{\tau} = 0.12 \; ; \; \overline{X} = 1.17.$$

The *structural relations* are given below in a systematic way. Some equations have been included which are not real equations, since they only contain data. They have been put in brackets and will not be counted.

Type of relation	No	Formula	Meaning of constants
Home demand . .	(1)	$x=(1-\sigma)\, Z^R+L^R+\xi_0$	$1-\sigma$ marginal propensity to spend
Export demand .	(2)	$e=e^W+\bar{e}\varepsilon^e\,(p^W-p)$	ε^e price elast. of exp.
Home supply	(3)	$p^x=\pi_0+\pi_1 l'+\pi_2 y+\pi_3 p^m+\tau$	π_1 marginal labour quota π_2 supply flexibility π_3 marginal import quota
Export supply . .	(4)	$p=\pi_0+\pi_1 l'+\pi_2 y+\pi_3 p^m$	
Import demand .	(5)	$m=\mu y+\bar{m}\varepsilon^m\,(p-p^m)$	$\mu =$ net marginal imp. quota
Import supply . .		$(p^m=\text{given})$	$\left(\mu = \frac{\pi_3}{1-\pi_3}\right)$
Labour demand . .	(6)	$a=y-h$	$\varepsilon^m =$ price elast. of imp.
Labour supply . .		$(l=\text{given})$	
Physical balance .	(7)	$(1+\mu)\, y=x+e$	
Definition	(8)	$Z^R=Z-\overline{Z}p^x$	
,, ...	(9)	$L^R=L-\overline{L}p^x$	
,, ...	(10)	$Z=Y-L$	

Definition	(11)	$L = \overline{L}\,(y + l')$
,,	(12)	$Y = X^F + E - M$
,,	(13)	$X^F = x + \bar{x}p$
,,	(14)	$X = (1 + \bar{\tau})\, X^F + X^F \tau$
,,	(15)	$M = m + \bar{m}p^m$
,,	(16)	$E = e + \bar{e}p$
,,	(17)	$D = M - E$
,,	(18)	$l' = l - h$
,,	(19)	$l^R = l - p - \tau$

Equations (6), (8), (9), (11), (13) — (16), (18) and (19) are all linearised forms of products or quotients and therefore only approximations. Equations (10), (12) and (17) are value balance equations.

Example (1)

The basic model is simplified by disregarding any changes in e^W, p^W, p^m (the external data), τ and h, i.e. these five variables are taken $= 0$. One *side condition* is added, expressing that changes in nominal wage rates and prices are proportional:

$$\lambda\, l = p \quad \left(\lambda = \frac{\overline{Y}}{\overline{Y}+\overline{M}} = 0.69\right). \tag{20}$$

This equation may be considered either as a condition of "social equilibrium" or as a technical condition if wage decreases and price decreases are supposed to be effectuated by devaluation only. (They have to be measured, then, in foreign currency). In fact by a devaluation all national cost elements diminish in the same proportion whereas the imported goods remain equal in price. That means that the resulting fall in p is only λ times the fall in wage costs and that l' will represent the degree of devaluation.

We now propose to consider as *target variables*: D and y and we choose

$$D = -\,0{,}04 \tag{21}$$

$$y = 0 \tag{22}$$

expressing that equilibrium in the balance of payments is desired under the condition that the volume of production (or of employment which, for $h = 0$, comes to the same) is maintained. Alternatively (22) may be interpreted as a second boundary condition, telling that there was, in the initial situation, full employment and that no extension of production therefore is possible.

As *instruments* we propose l' and ξ_0, i.e. devaluation and a possible change in public expenditure.

The problem of economic policy considered is: what level of the rate of exchange and of public expenditure is necessary in order to equilibrate the balance of payments under conditions of high employment, given the situation (of the Netherlands) mid 1949?

Using the definitions of this and the preceding chapters we may describe our problem as follows:

Target variables:	D, y, i.e.	$n = 2$
Instruments:	l', ξ_0, i.e.	$n' = 2$
Data:	none	
Irrelevant variables:	x, e, p^x, p, π_0, m, a, Z^R, L^R, Z, L, Y, X^F, X, M, E, l, l^R, i.e.	$N = 18$

The number of structural relations $N' = 19$, plus 1 boundary condition.

Example (2)

Here the external data e^W, p^W and p^m are considered variable; the variable ξ_0 is taken constant, i.e. $= 0$, however; to begin with, no boundary condition is added. (At a later stage, however, such conditions will be introduced according to necessity).

As target variables we now introduce D, x, a and l^R, desiring that

$$D = -0{,}02 \tag{23}$$

$$x = a = l^R = 0 \tag{24}$$

ssing that the balance of payments gap should be reduced to half its previous extent and real national expenditure, employment and real wage rate maintained.

As *instruments* we now propose the indirect tax rate τ, the nominal wage rate l, the profit margin π_0 and labour productivity h.

The problem of economic policy discussed in example 2 corresponds to the situation of the Netherlands discussed towards the end of 1950. The deficit on the balance of payments had not diminished, but it was considered feasible to eliminate it during the two remaining years for which E.R.P. aid was expected; the subject under public discussion was whether it would be possible by means of tax, wage and price policy and of an increase in labour productivity to close the balance of payments in two years' time while maintaining real welfare, i.e. real national expenditure, real wage rates and employment. More precisely, the latter two objectives were those of the trade unions and the maintenance of total real expenditure was desired because of the necessity to industrialise and to increase the defence program. One of the questions under discussion was, how far the development in external data would or would not permit the realisation of the desired targets.

Summarising, we have now:

Target variables:	D, x, a, l^R, i.e.	$n = 4$
Instruments:	τ, l, π_0, h, i.e.	$n' = 4$
Data:	e^W, p^W, p^m	
Irrelevant variables:	e, p, p^x, m, y, Z^R, L^R, Z, L, Y, X^F, X, M, E, l' i.e.	$N = 15$
The number of structural relations		$N' = 19$

In these two examples the following numerical values for the structural constants have been used [1]: $\sigma = 0.3$; $\varepsilon^e = 2$; $\pi_1 = 0.3$; $\pi_2 = 0.125$; $\pi_3 = 0.3$; $\mu = 0.44$; $\varepsilon^m = 0.3$.

[1]) For a documentation, cf. J. TINBERGEN, Econometrics, § 45 (The Blakiston Company, Philadelphia 1951).

5. *Alternative setups* will be obtained if some of the targets are not considered as such but as conditions; this may be the case e.g. with the target for D: one may as well say that the equation $D = -0.04$ or $D = -0.02$ is a condition. This diminishes the number of target variables by one, but at the same time increases the number of relations by one. Another example of an alternative setup is the elimination of a certain number of irrelevant variables and, at the same time, of an equal number of equations. The number may even be chosen in different ways. An easy example is the elimination of X as a variable and of equation (14). It is just a matter of taste whether one prefers to have them in or not. In example (1), where $\tau = h = 0$, p^x may be left out together with equation (3) if only p^x is replaced by p whenever it appears in other relations; and l' may be left out together with equation (18), if we replace l' by l; a may also be left out with equation (6) if we substitute y for a. But this process of, in a sense, simplification may be carried much further. We may even eliminate all irrelevant variables if we want to; we shall speak in that case of the "completely simplified" version.

In both our examples it seems particularly attractive to eliminate a large number of variables, the dependence of which on the more "strategic" variables is easily remembered. The following simplified versions are therefore presented:

Example (1), *simplified version:*

Equations: $(N' = 4)$

$$
\begin{array}{ll}
(1\ \mathrm{I}) & \xi_1 y + D + \xi_2 p = \xi_3 l' + \xi_0 \\
(1\ \mathrm{II}) & -\mu y + D - \delta p = 0 \\
(1\ \mathrm{III}) & -\pi_2 y \quad + \quad p = \pi_1 l' + \pi_0 \\
(1\ \mathrm{IV}) & p = \lambda l'
\end{array}
$$

Here:

$$\xi_1 = \sigma (1 - \overline{L}) \qquad = 0{,}135 \qquad (25)$$

$$\xi_2 = 1 - \bar{x} + \sigma\,(\bar{m} + \overline{L} - \mu\varepsilon^m) \quad = 0{,}217 \qquad (26)$$

$$\xi_3 = \sigma\overline{L} \quad = 0{,}165 \qquad (27)$$

$$\delta = \mu\varepsilon^m + \bar{e}\varepsilon^e - \bar{e} \quad = 0{,}53 \qquad (28)$$

Target variables: $D = -\,0.04 \qquad y = 0$ $\qquad (n = 2)$
Instruments: $l',\ \xi_0$ $\qquad (n' = 2)$
Irrelevant variables: $p,\ \pi_0$ $\qquad (N = 2)$

The meaning of the equations — which can be obtained from our previous list by eliminating all the other variables — may be summarised verbally in the following way:

(1 I) represents a "multiplier equation", generated from (12) and expressing national income (decomposed in its constituents y and p) as a function of the balance of payments deficit and the autonomous elements in national expenditure (ξ_0, l') [1].

(1 II) may be considered as an explanatory equation for the balance of payments deficit D, which from equation (17), by the intermediary of (15), (16), (5) and (2) may be easily expressed as a function of y and p: national activity and national (export) price level.

(1 III) is nothing but equation (4), the supply equation for national products.

(1 IV) is the "social equilibrium" or "devaluation" relation.

Example (2), *simplified version.*

Equations ($N' = 4$)

$$\begin{aligned}
&(2\,\mathrm{I}) && -\xi_4 a + (\xi_6 - \xi_4)\,h - (\xi_5 + \xi_6)\,l + \xi_5 l^R + \\
&&& + (1 - \sigma + \sigma\overline{L} + \xi_5)\,\tau + x = \xi_7 p^m \\
&(2\,\mathrm{II}) && -\mu a + D - \mu h - \delta l + \delta l^R + \\
&&& + \delta\tau = -\,\delta^m p^m - e_0 \\
&(2\,\mathrm{III}) && -\pi_2 a + (\pi_1 - \pi_2)\,h + (1 - \pi_1)\,l - l^R - \\
&&& - \pi_0 - \tau = \pi_3 p^m
\end{aligned}$$

[1] For a fuller interpretation cf. J. TINBERGEN, loc. cit. This somewhat complicated relation has, because of its complexity, been decomposed in two equations in example (2).

$$(2\,\mathrm{IV}) \qquad a + D + \qquad h + \qquad \eta_1 l - \eta_1 l^R - \\ - \eta_1 \tau - x = \eta_2\, p^m$$

Here:

$\xi_4 = 1 - \sigma + \sigma\overline{L}$	$=$	0.865	(29)
$\xi_5 = (1 - \sigma)\,(\bar{m} + \overline{L} - \mu\varepsilon^m) - \overline{L}$	$=$	0.05	(30)
$\xi_6 = \sigma\overline{L}$	$=$	0.165	(31)
$\xi_7 = \mu\,(1 - \sigma)\,(\varepsilon^m - 1)$	$=$	$-\,0.22$ [1]	(32)
$\delta^m = \mu\,(\varepsilon^m - 1)$	$=$	$-\,0{,}31$	(33)
$\eta_1 = 1 + \mu - \mu\varepsilon^m - \bar{x}$	$=$	0.27	(34)
$\eta_2 = \mu\,(1 - \varepsilon^m)$	$=$	0.31	(35)
$e_0 = \bar{e}\varepsilon^e p^W + e^W$	$=$	$0.80\, p^W + e^W$	(36)
Target variables:	$D,\ x,\ a,\ l^R$	$n = 4$	
Instruments:	$\tau,\ l,\ \pi_0,\ h$	$n' = 4$	
Irrelevant variables:	none	$N = 0$	

These equations are obtained from equations in the variables $x,\ y,\ D,\ p,\ l',\ \xi_0$ and π_0 running:

$$(2\,\mathrm{I}') \qquad x - \xi_4 y - \xi_5 p = \xi_6 l' - (1 - \sigma + \sigma\overline{L})\,\tau + \xi_7 p^m$$
$$(2\,\mathrm{II}') \qquad -\mu y + D - \delta p = -\delta^m p^m - e_0$$
$$(2\,\mathrm{III}') \qquad -\pi_2 y + p = \pi_1 l' + \pi_0 + \pi_3 p^m$$
$$(2\,\mathrm{IV}) \qquad -x + y + D + \eta_1 p = \eta_2 p^m$$

This set of equations may be briefly interpreted in a similar way as has been done for the simplified version of example (1). In fact, equation (2 II′) is the same as (1 II) and (2 III′) the same as (1 III), except that the terms with the data have now been maintained. Equation (2 I′) is a version of the demand equation for home products, deduced from equation (1) of the basic model. Equation (2 IV′), on the other hand, may be described as a version of the income definition equation (12) of the basic model. It is found by the equalisation of two different transformations of the right-hand side of (12): one by re-writing X^F in terms of x and keeping $E - M$

[1]) Since $\bar{m} = \mu$, μ has been written for $\bar{m}$ in this formula.

together as $-D$; the other by combining X^F and E and expressing it, via (7), in y. This process, although somewhat lengthy, explains the relation between Y and y which is more complicated than those between e.g. M and m, E and e or X and x. Instead of (2 IV′) it would also have been possible to add a version of (7). The remark may be added that equation (1 I), the "multiplier equation" of example (1) may be obtained from (2 I′) and (2 IV′) by the elimination of x, the usual process in Keynesian literature to obtain the multiplier equation.

The set (2 I′) — (2 IV′) has finally been transformed into (2 I) — (2 IV) with the help of equation (6) saying that $y = a + h$, equation (18) telling that $l' = l - h$ and equation (19) giving an expression for p in terms of l and l^R: $p = l - l^R - \tau$. These transformations are essential for the second example in contradistinction to the first; they bring into the simplified set of relations the target variables a and l^R and the instruments l and h.

One could characterize the simplified systems of equations in a mathematical way by saying that the original matrix of coefficients has been condensed so asn ot to leave too many open places and still give a rather simple economic meaning to each of the elements of the matrix.