CHAPTER IV

THE LOGICAL STRUCTURE OF THE NORMAL QUANTITATIVE POLICY PROBLEM (TARGETS AND INSTRUMENTS IN EQUAL NUMBERS); DIRECTIVES

1. Having summed up the various groups of variables and constants playing a rôle in our problems and the relations they have to satisfy we are now able to discuss the logical structure of these problems. Can they, generally speaking, be solved? Are they just determinate or under-determined, or are they perhaps over-determined and therefore insoluble? For well-defined economic models the number N' of structural relations will be equal to the total number of unknowns in the explanatory economic problem, i.e. the number of economic variables $N + n$ (irrelevant + target variables):

$$N' = N + n \tag{37}$$

If now, in addition, the *number of targets is equal to the number of instruments*, or $n' = n$, we have

$$N' = N + n' \tag{38}$$

enabling us to solve for all the unknowns of the policy problem, where the unknowns are the values of the instrument variables and the irrelevant variables. The values z_l^0 satisfying the equations will be functions of the target values y_k^0 that maximise Ω and of the data:

$$z_l^0 = \sum_1^n{}^k \zeta_{lk}^y y_k^0 + \sum_1^{N'}{}^i \zeta_{li}^u u_i. \tag{39}$$

In matrix form:

$$z^0 = Z^y y^0 + Z^u u. \tag{40}$$

Only on certain conditions to be specified in § 2 will these instrument values be determined and unique. If these conditions are satisfied the following general conclusions are valid:

(i) the conclusion of the *interdependency of economic policies*: the values of the instrument variables are dependent, generally speaking, on all the targets set and cannot be considered in isolation;

(ii) the values of the instrument variables are *dependent on those of the data*, i.e. they must vary with the data. In fact, equations (39) or (40) indicate how they have to vary with the data. As soon as certain fixed targets have been chosen the y-terms in (39) or (40) are constant numbers, but the u-terms are not. In this form we shall call these equations "*directives for economic policy*" since they indicate how the political parameters have to be varied in relation to the changing data. It is also interesting to note that

(iii) the coefficients Z^u are independent of the y's.

2. The coefficients ζ^v_{lk} and ζ^u_{li} depend, of course, on the coefficients in the structural equations. We assume that these equations are known to us in the "completely simplified" version:

$$\textstyle\sum^k \delta_{jk}\, y_k = \sum^l \varepsilon_{jl}\, z_l + \sum^i \varphi_{ji}\, u_i \tag{41}$$

$$(j, k, l = 1 \ldots n) \text{ or in matrix form: } \varDelta y = Ez + \varPhi u \tag{42}$$

This means that the irrelevant variables x have already been eliminated. Solution of (41) or (42) for z_l yields:

$$\zeta^v_{lk} = \frac{\sum^m E_{ml}\, \delta_{mk}}{|E|} \tag{43}$$

or

$$Z^v = E^{-1} \varDelta \tag{44}$$

Similar expressions may be found for ζ^u_{li} and Z^u.

If follows that z is unique if and only if $\varrho(E) = n'$ where $\varrho(E)$ is the rank of E [1]).

It is useful to mention a few special cases that may arise.

(1) If $|E| \sim 0$ (almost equal to zero) without the numerators of (43) being small, *very large values* of z_l will only satisfy the equations. This case will be dealt with in chapter VII

(2) If $|E| = 0$ and all n expressions

$$\sum^k \sum^m E_{ml}\, \delta_{mk}\, y_k = 0 \tag{45}$$

the values of z will be *indeterminate* and the equations (41) dependent. Evidently the occurrence of this situation is dependent on the numerical values of the y's too.

(3) The matrices Δ and E may show, each of them or both, the properties of being *consecutive* or *partitionable*.

We call a (square) matrix *triangular* or *consecutive* if, by an appropriate rearrangement of columns, all elements northeast of the principal diagonal vanish, i.e.

$$\varepsilon_{jl} = 0 \text{ if } l > j. \tag{46}$$

If E is consecutive and if we have rearranged the numbers j, the value of z_1 can be found from equation (41, $j = 1$) without the aid of any other equation; z_2 from equations (41, $j = 1, 2$) without the further equations, etc. This means that only coefficients ε_{1l}, δ_{1k} and φ_{1i} and data and targets occurring in equation (41, $j = 1$) are relevant for z_1, etc. The values of z_j, $j = 1, 2 \ldots n$ can be found one after the

[1]) The following rigorous proof was formulated by Prof. TJ. C. KOOPMANS in a conversation: If $\varrho(E) < n'$, we can add to z any solution z' of $Ez' = 0$, $z' \neq 0$, and obtain the same target y.

Note that these solutions z' can all be written, and only such solutions can be written, as $z' = Fw$, where F is a right orthogonal complement of E, i.e. a matrix satisfying $\varrho(F) = c(E) - \varrho(E) > 0$ $EF = 0$. (c = number of columns of a matrix). Hence, if $\varrho(E) < n'$ and z achieves y, then $z + Fw$ also achieves y.

other and the process of solution easily reproduced verbally.

If, by the same rearrangement, both E and Δ appear to be consecutive, an additional peculiarity presents itself: z_1 is only dependent on one target, viz. y_1, z_2 on the two targets y_1 and y_2, etc. The policy whose instrument is z_1 may then be said to be the necessary and sufficient policy in order to reach target y_1; target y_2 can only be reached by either "z_1" or "z_2-policy" etc. We will call this situation one of *corresponding consecutivity*.

There may, of course, also be consecutivity for both E and Δ, but non-corresponding; and there may be consecutivity for Δ without its occurence for E. In the latter case y_1 will depend only on a restricted number of coefficients and, perhaps, instruments and data.

We call a (square) matrix *partitionable* if, after a suitable rearrangement of the columns, all elements outside certain subsquares vanish. These sub-squares are such that their principal diagonals coincide with the principal diagonal of the complete matrix; each of the elements of the principal diagonal belonging to one and only one sub-square. If E is partitionable, the z's can be grouped in subsets which each of them only depend on the corresponding ε's, δ's and φ's. The values of one subset do not depend on the coefficients characteristic of any other.

Also here there may be *corresponding partition*, i.e. that the groups of the y's show a correspondence to those of the z's.

An algebraic example of partition of E is:

$$\begin{aligned} \delta_{11}y_1 + \delta_{12}y_2 + \delta_{13}y_3 &= \varepsilon_{11}z_1 \\ \delta_{21}y_1 + \delta_{22}y_2 + \delta_{23}y_3 &= \qquad\quad \varepsilon_{22}z_2 + \varepsilon_{23}z_3 \\ \delta_{31}y_1 + \delta_{32}y_2 + \delta_{33}y_3 &= \qquad\quad \varepsilon_{32}z_2 + \varepsilon_{33}z_3 \end{aligned}$$

An example of corresponding partition of E and Δ is:

$$\begin{aligned} \delta_{11}y_1 \qquad\qquad\qquad\quad &= \varepsilon_{11}z_1 \\ \delta_{22}y_2 + \delta_{23}y_3 &= \qquad\quad \varepsilon_{22}z_2 + \varepsilon_{23}z_3 \\ \delta_{32}y_2 + \delta_{33}y_3 &= \qquad\quad \varepsilon_{32}z_2 + \varepsilon_{33}z_3 \end{aligned}$$

In the latter case z_1-policy is necessary and sufficient for target y_1, whereas z_2- and z_3-policy is necessary and sufficient for targets y_2 and y_3. The most extreme case is a corresponding partition into groups of one variable each. In that case each target y_k can only be reached by z_k-policy. [1]

A few economic examples may be added in order to illustrate the concepts just introduced. In static systems all quantities and relative prices are determined by technical equations, demand and supply equations and certain balance equations, in which neither the absolute price level P nor the quantity of money B appear; and P is dependent on B as a consequence of one equation about the desired cash-balances. This is an example of a partition of the traditional economic variables (i.e., in our case, the target variables).

As a second, more specific example, let us take the following system, using the notation of our basic model: [2]

Income definition.	$Y = X - D$
Balance of payments deficit definition . .	$D = M - E$
Import value definition	$M = mp^m$
Export value definition	$E = ep$
Demand for imports	$m = \mu y$
Demand for exports	$e = e_0 - \bar{e}\varepsilon^e\, p$
Credit creation equation	$K = X - Y$

The new symbol K represents the extent of credit creation; the last equation is based on the assumption that there are no idle cash balances and that hence the excess of expenditure X over income Y has to be financed by credit creation. Import prices p^m are considered as a datum.

We consider as targets the value of D and that of y

[1] Both Δ and E are diagonal matrices in that case. Economists or economic politicians holding the opinion that there is such a one-by-one correspondence between targets and instruments evidently assume a very special structure.

[2] The symbols indicate, however, absolute values, not deviations.

(employment policy, say); as instruments K, representing monetary and financial policy and p, representing price-and-wage policy. The other variables will then be, in our terminology, irrelevant variables and the "simplified version" of the system of equations is

$$
\begin{aligned}
D &= K \\
D &= \mu y\, p^m - (e_0 - \bar{e}^e p)\, p.
\end{aligned}
$$

The system is a consecutive system for the targets and a partitioned system for the instruments: the first equation only contains one target (D) and one instrument (K); the second equation both targets and only the second instrument (p). This implies that credit policy is only governed by the balance of payments target and, in its turn, this latter target can only be obtained by the correct credit policy. Price policy has to obey the two targets together; and the second target depends on both instruments, but, once that credit policy has been fixed it can only be taken care of by price-and-wage policy.

3. These general statements on the logical structure of our problem may now be illustrated with regard to our two examples. Starting with *example* (1) in its simplified version (ch. III, § 5) we re-write equations (1 I) to (1 IV) with the unknowns l', ξ_0, p and π_0 on the left-hand side:

$$
\begin{aligned}
\xi_2 p - \xi_3 l' - \xi_0 & &= -\xi_1 y - D &= 0.04 \\
-\delta p & &= \mu y - D &= 0.04 \\
p - \pi_1 l' & - \pi_0 &= +\pi_2 y &= 0 \\
p - \lambda l' & &= 0 &
\end{aligned}
$$

It is a happy circumstance for pedestrian methods that these equations show several open places, which facilitate their solution very much. The results are, in general form:

$$p = \frac{-\mu y + D}{\delta} \left(\text{i. e. } \zeta_{pv}^{\text{targ}} = -\frac{\mu}{\delta};\ \zeta_{pD}^{\text{targ}} = \frac{1}{\delta}\right)$$

$$l' = \frac{-\mu y + D}{\delta\lambda}$$

$$\xi_0 = \frac{-\lambda\xi_2\mu + \xi_3\mu + \xi_1\delta\lambda}{\delta\lambda} y + \frac{\lambda\xi_2 - \xi_3 + \delta\lambda}{\delta\lambda} D$$

$$\pi_0 = \frac{-\mu\lambda + \pi_1\mu - \pi_2\delta\lambda}{\delta\lambda} y + \frac{\lambda - \pi_1}{\delta\lambda} D.$$

It may be noted that even so the expressions for ξ_0 and π_0 are already fairly complicated if a verbal translation should be required.

For $y = 0$ and $D = -0.04$ this yields:

$$p = -0.076;\ l' = -0.11;\ \xi_0 = -0.04;\ \pi_0 = -0.04.$$

From the practical point of view it is interesting that, according to this model, a devaluation of 11 % with respect to the rest of the world would have been sufficient to restore equilibrium in the balance of payments, provided, however, that national expenditure be sufficiently curtailed in order to make the necessary export goods available. [1])

Because of the very simple structure of our equations in this special case it is possible to obtain the result by verbal methods too. The second equation may in the following way be translated into plain reasoning. An improvement of the balance of payments by 4 % of the national income is desired. No change in production should be envisaged. The only way is, therefore, to lower prices. A price reduction by 1 % would decrease imports by 0.3 % and increase exports by 2 %, hence the value of exports by 1 %. Since imports were 110 % of exports the total improvement in the balance of payments to be obtained from a 1 % fall in prices is (0.33 + 1) % of exports and since exports are 40 % of

[1]) A further calculation shows that if instead of y the value of x should be kept constant, a devaluation of 20 % would have been necessary.

national income, $0.4 \times 1.33\ \% = 0.53\ \%$ of national income. Hence, a price decline by 4/0.53 or 7.6 % is necessary. If this decline is to be obtained by devaluation, the reduction must be obtained from home costs only, which are 69 % of total price. These have to decline by 7.6/0.69 or 11 %. This reasoning could, without great difficulties, be continued for the other variables, also for all the variables of the basic model, by using the complete set of relations (1) — (19). Their structure is simple enough to permit the verbal interpretation, once y, D, l' and p are known. The following values are found for all the "irrelevant" variables:

$a = 0$		
$e = 0.06$	$E = 0.03$	
$l = -0.11$	$L = -0.06$	
$l^R = -0.03$	$L^R = -0.02$	
$m = -0.01$	$M = -0.01$	
$p^x = -0.08$		
$x = -0.06$	$X = -0.16$	$X^F = -0.14$
	$Y = -0.10$	$Z^R = -0.01$
	$Z = -0.04$	

4. In the case of example (2) the equations are, after substitution of the numerical values of our target variables:

$$
\begin{array}{llllll}
(\xi_6 - \xi_4)\,h & -(\xi_5+\xi_6)\,l & & +\ (1-\sigma+\sigma\overline{L}+\xi_5)\tau & = & \xi_7 p^m \\
\mu h & -\ \delta l & & +\ \delta\tau & = & 0.02 - \delta^m p^m - e_0 \\
(\pi_1-\pi_2)\,h & +\ (1-\pi_1)\,l & -\ \pi_0 & -\ \tau & = & \pi_3 p^m \\
h & +\ \eta_1 l & & -\ \eta_1\tau & = & 0.02 + \eta_2 p^m
\end{array}
$$

Here the structure of the equations is less simple; π_0 only occurs in one equation, but the other instruments occur in all equations. We have to solve h, l and τ from three simultaneous equations. It is, moreover, interesting to note that the data in three of the four equations show proportional movements. The solutions are:

$$\begin{aligned}
h &= 0.04 + 0.60\, p^m - 0.66\, e_0 \\
l &= -0.05 - 1.13\, p^m + 2.52\, e_0 \\
\pi_0 &= -0.05 - 0.94\, p^m - 1.56\, e_0 \\
\tau &= 0.02 - 0.04\, p^m + 0.09\, e_0
\end{aligned}$$

Supposing, first, that the *data do not show any change* in comparison with the initial situation (i.e. $p^m = e_0 = 0$) we find that the targets can be fulfilled only if:

(i) productivity shows an additional increase by 4 %;
(ii) entrepreneurial margins decline by 0.05, i.e. 13 %;
(iii) nominal wage rates and hence prices by 5 % and
(iv) indirect taxes increase by 2 % of national income.

The realisation of these values of the instrument variables would not seem an easy job: both the increase in productivity and a decline in nominal wage rates would seem very difficult. The discussion of this result will be continued in chapter VI where the consequences of boundary conditions are studied.

If, however, the *data should change*, the various problems would become different. Consider, first, an improvement in export conditions. It follows from the formulae that the necessary wage decreases would be smaller and profit decreases larger, or even that a change would not be necessary. Also, the required rise in productivity would be smaller.

p^m	e_0	h	l	π_0	τ
0.05	0.02	0.06	−0.06	−0.13	0.02
0.05	0.04	0.04	−0.01	−0.16	0.02
0.05	0.08	0.02	0.09	−0.22	0.03
0.10	0.02	0.09	−0.11	−0.18	0.02
0.10	0.04	0.07	−0.06	−0.21	0.02
0.10	0.08	0.05	−0.04	−0.27	0.02
0.20	0.02	0.15	−0.23	−0.27	0.01
0.20	0.04	0.13	−0.18	−0.30	0.01
0.20	0.08	0.11	−0.08	−0.36	0.01

required increase in taxes would be smaller. The chance of realisation for each of the instrument variables would improve.

Almost completely the reverse is true for a rise in import prices. Only in the case of taxes both price rises work into the same diıection. Some selected values of the instrument variables are given in the foregoing table.