Arbitrage and Sampling Uncertainty in Financial Stochastic Programming Models*

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Abstract

Asset Liability Management (ALM) is an important and challenging problem for institutional investors and financial intermediaries. The requirement to fulfill its liabilities constrains the institutional investor in its asset allocation possibilities. We formulate an ALM model for pension funds as a multi-stage stochastic programming model. Relevant variables such as future inflation rates, stock returns, and bond yields are unknown. This is incorporated in the ALM model by means of an event tree, which represents the expected development of the economic variables as well as the corresponding uncertainty. The event tree is constructed by sampling from a time series model for the variables, and is therefore subject to sampling uncertainty. Moreover, for the event tree to be realistic, it is required not to exhibit arbitrage opportunities. In this paper we examine the effect of sampling uncertainty and the structure of the event tree on the optimal policies. Furthermore, we consider the effect of random sampling and the tree structure on the probability of arbitrage-free trees. We also compare the optimal solutions to the ALM problem for trees with and without arbitrage. For these purposes, we consider data from a Dutch pension fund.

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1 Introduction

Risk plays an important role in economic life. Financial intermediaries aim at providing insurance against a broad spectrum of risks. Pension funds provide an income after retirement. Insurance companies provide financial compensation for calamities such as fire, theft, or car accidents. Banks provide loans and OTC contracts. Housing associations provide real property, e.g. houses, buildings, and shops, which are liable to aging and need to be renovated over time. In order to meet their liabilities the financial intermediary has to decide on an investment strategy and a premium-strategy. Often such decisions are complicated by solvency requirements formulated by regulating authorities. Moreover, the clients of a financial intermediary might potentially display conflicting objectives. The cohesive and integral environment in which this problem is embedded is known as Asset Liability Management. In this paper we are primarily concerned with integral pension management.[1]

A pension fund provides old-age benefits to its members. In a defined benefit pension scheme the benefit promises are fixed and do not depend on past investment performance. The risk associated with future investment performance is borne by the plan sponsor by means of contributions. Each year a pension fund has to decide upon the appropriate level of contributions and determine a trading strategy to invest the capital administered. These decisions are being complicated by the different interests of the participants of the fund, the sponsor of the fund, and regulating authorities. The key objective of a pension fund is to create a balance between these opposite interests by means of an indexation policy, a contribution policy, and an investment strategy, taking into account its agreed liabilities.

Specific to pension funds is the long time horizon: it should not only be able to fulfill its liabilities today, but also in 30 years. Clearly, the uncertainty involved with such a long horizon is immense. The usual way to deal with this uncertainty is by scenarios, in which relevant variables, like inflation, bond yields and stock returns, are projected into the future. The scenarios are collected into an event tree. The aim of this event tree is twofold: first, it should represent expected future developments, and second, it should also reflect the uncertainty involving these developments.

Conditional on an event tree, the ALM problem can be regarded as a multi-stage stochastic programming problem and be solved by appropriate techniques. Clearly, the solution of the problem depends on the tree. We investigate the sensitivity of the results with respect to the construction of the tree in two ways. First, we examine sampling uncertainty by repeatedly sampling scenarios. Next, we examine the effect of the structure of the tree. Also, attention is paid to the role of arbitrage. In general, arbitrage opportunities can not be exploited by pension funds, and therefore the event tree should not exhibit this opportunities. We investigate how the structure of the tree affects the occurrence of arbitrage opportunities and whether the solution to the ALM problem is sensitive to the (non)existence of arbitrage.

The paper is organized as follows. In Section 2 we discuss a generic asset liability management model. In particular, we set up a mathematical framework and introduce relevant concepts,

[1] For ALM for insurance companies see, *e.g.*, Carinõ et al.[3], for banks Klaassen [8] and for housing associations Holmer [7].

such as arbitrage. The generation of scenarios is discussed in Section 3. We use Vector AutoRegressive (VAR) models to sample scenarios for the relevant economic variables. The possibility of arbitrage opportunities in the event tree is also discussed. Simulation experiments are set up in Section 4. Considering data from a Dutch pension fund, we investigate sampling uncertainty as well as sensitivity of the results with respect to the construction of the event tree. Also, attention is paid to the role and impact of arbitrage on the optimal dynamic policies. Section 5 concludes.

2 Asset Liability Management Model

2.1 Introduction

Many authors have considered ALM models for pension funds in the spirit of the traditional mean-variance model of Markowitz [13]. The most notable of these are the static, i.e. one-period, models considered by Sharpe and Tint [18], Leibowitz, Kogelman, and Bader [12], and Boender and Heemskerk [2]. Sharpe and Tint [18] introduce the so-called liability hedging credit (LHC) in a mean-variance framework. The LHC quantifies the degree to which a particular asset (or class of assets) provides a benefit for an investor with a particular set of liabilities. An approach based on the funding-ratio (assets/liabilities), was proposed by Leibowitz, Kogelman, and Bader [12]. Their model makes a trade-off between the mean funding ratio return (FRR) and the volatility of the FRR. Boender and Heemskerk [2] assume a fixed funding ratio, the so-called clean-funding model. Their model makes a trade-off between the mean contribution-level and the volatility of the contributions.

Others have considered dynamic, i.e. multi-stage, stochastic programming approaches to Asset Liability Management. For an overview of the state-of-the-art in Asset Liability Management, we refer to the recent book by Mulvey and Ziemba [15]. Some of the relevant models include Carino et al. [3], Consigli and Dempster [4], Dert [6], Kouwenberg [10], and Mulvey [14]. In the next subsections we describe our asset liability management model, which also falls into the class of multi-stage stochastic programming models. Before describing the ALM model in Subsection 2.3, we review some relevant concepts from the field of mathematical finance in Subsection 2.2.

2.2 Some financial background

We consider a securities market economy, where economic activity (e.g. trading) takes place at discrete dates $0, 1, \dots, T$. The set of possible states of the world is denoted by Ω, with generic elements ω. The set Ω belongs to a probability space $(\Omega, \mathbb{F}, P)$, where $\mathbb{F} = \{\mathcal{F}_t;\ t = 0, 1, \dots, T\}$ is the filtration of subsets of Ω and P a probability measure. The set Ω with σ-field $\mathbb{F}$, $(\Omega, \mathbb{F})$ is called a measurable space. The support of $(\Omega, \mathcal{F}, P)$ is the smallest closed subset of Ω with probability 1. If the support is defined through a countable union of points, the probability distribution is called discrete. Throughout this paper we assume that Ω has a finite support.

We assume that $K+1$ long-lived securities are available for trading in the securities market (i.e. each security is available for trading at all dates). A *trading strategy* (or *portfolio*) $\theta(t)$ in the economy described above is a $K+1$ dimensional stochastic process defined on Ω, indicating the amount of wealth, in terms of the underlying currency in the economy, invested in each of the $K+1$ securities at date t. A portfolio entered with at time t should be determined by information available up to time $t-1$ only. This enforces an informational constraint on the investors[2]. To model this informational constraint we assume that the information structure in the economy consists of a given family of increasingly finer partitions of Ω, denoted by $\{\mathcal{F}_t,\ t=0,1,\dots,T\}$, i.e. $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$. For every t, the partition $\mathcal{F}_t$ contains all available information to investors at time t. We assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \{\{\omega\} : \omega \in \Omega\}$. These assumptions imply that:

- no information has arrived at time 0;
- as time progresses, investors come to know about gradually decreasing subsets of Ω which contain the true state of the world;
- the true state of the world is fully revealed at the horizon T.

Non-anticipativity of the trading strategy means that $\theta(t)$ is $\mathcal{F}_{t-1}$ measurable. A random variable X is said to be *measurable* with respect to the sigma-algebra $\mathcal{F}$ if, for every real number x, the subset $\{\omega \in \Omega : X(\omega) = x\}$ is an element of $\mathcal{F}$. A stochastic process $x = \{x(t);\ t = 0,\dots,T\}$ is said to be *adapted* to the filtration $\mathbb{F}$ if the random variable $x(t)$ is measurable w.r.t. $\mathcal{F}_t$, $t=0,\dots,T$. A stochastic process $x = \{x(t);\ t=0,\dots,T\}$ is said to be *predictable* w.r.t. the filtration $\mathbb{F}$ if the random variable $x(t)$ is measurable w.r.t. $\mathcal{F}_{t-1}$, $t=1,\dots,T$. Hence, all predictable stochastic processes are adapted. A non-anticipative stochastic process is a predictable stochastic process.

In this section we first establish some well-known principles in financial economics (we refer to Pliska [16] for an excellent textbook on mathematical finance). A securities market is said to be *perfect (or frictionless)* if there are no constraints on trading and no transaction costs. A crucial concept in financial economics is the notion of arbitrage-free security prices. To this end we define the value process V as follows:

$$V_t = V_{t-1} + \sum_{k=0}^{K} r_k(t)\theta_k(t-1),$$

where $r_k(t)$ denotes the net return on security k from period $t-1$ to t. The gains process G is defined by:

$$G_t = \sum_{k=0}^{K}\sum_{u=1}^{t} r_k(t)\theta_k(t-1)$$

A trading strategy is called *self-financing* if

$$V_t = V_0 + G_t, \quad t = 1,\dots,T.$$

[2] In the stochastic programming literature this informational constraint is called *non-anticipativity.*

The discounted return process $r_k^*(t)$ is defined by

$$r_k^*(t) = \frac{r_k(t) - r_0(t)}{1 + r_0(t)}, \quad k = 0, \dots, K$$

where security 0 is the numeraire asset in the economy. The discounted value and gains process are defined in terms of $r_k^*(t)$.

Definition 1 *An* arbitrage opportunity *is a self-financing trading strategy that generates a strictly positive cash flow between 0 and T in at least one state and does not require an outflow of funds at any date. Formally, an arbitrage opportunity is a self-financing trading strategy θ such that:*

$$V_0 = 0, \quad V_T \geq 0, \quad \mathbb{E}[V_T] > 0,$$

Using standard duality results from linear programming, the following well-known result can easily be proven.

Theorem 1 *The following statements are equivalent in the economy described above:*

(i) *There exist no arbitrage opportunities in the securities market.*

(ii) *There exists an equivalent martingale measure (also called risk-neutral probability measure) Q such that the discounted value process V_t^* corresponding to θ is a martingale under Q.*

It is not difficult to generalize the above theorem to a market with frictions such as e.g. bid-ask spreads, transaction costs, and markets where some securities can only be held short or long.

A securities market is said to be *complete* if, at $t = 0$, investors in the market can trade in state contingent claims (also referred to as Arrow-Debreu securities) for every time and event pair that can occur. A *state contingent claim* is a security which pays one unit of the consumption good when one particular state of the world occurs and nothing otherwise. Other existing securities can be viewed as complex bundles of these state contingent claims. We say that a securities market is *dynamically complete* if the cash flow on every Arrow-Debreu security can be replicated by some trading strategy θ. If at every node of the event tree, there are as many securities with linearly independent payoffs available as the number of successor nodes, we can find the strategy that replicates the payoff on any Arrow-Debreu security. There is an important characterization of dynamically complete markets (see, e.g. Pliska [16]).

Theorem 2 *Suppose that there exist no arbitrage opportunities in a securities market model. Then, the risk-neutral probability measure in such a model is unique if and only if the market is dynamically complete.*

When the market in incomplete there is a infinite number of risk-neutral probability measures. Throughout this paper we are mainly concerned with dynamically incomplete markets.

2.3 The model

In this section we consider a generic Asset Liability Management model for pension funds. As stated in Section 1, the pension fund has to decide on an indexation policy, a contribution policy and an investment strategy, taking into account its agreed liabilities. Throughout the paper we concentrate on the contribution policy and investment strategy, taking the indexation-policy as fixed.

The pension fund receives contributions $c(t)$ at time t from the sponsor of the fund (we denote these contributions as percentage of the wages $w(t)$). The received contribution-payments plus the available capital is invested in $K+1$ different asset-categories. We denote this investment from $t-1$ to t by $\theta_k(t-1)$. These investments earn an uncertain return $r_k(t)$ from period $t-1$ to t. At time t the pension fund pays benefit-payments $p(t)$. The pension fund aims at minimizing the downside variance of the funding level and the mean-absolute deviation of the contributions, and maximizing the expected terminal wealth. Throughout, we denote $x^+ = \max(x, 0)$. The generic Asset Liability Management model for pension funds can be stated as follows:

$$\begin{array}{ll} \min & \mu_u \mathbb{E}[(c-c_0)^+] + \mu_d \mathbb{E}[(c_0-c)^+] + \mathbb{E}[(\hat{X}-X)^+]^2 - \lambda \mathbb{E}[W_T] \\ \text{s.t.} & \end{array} \tag{1}$$

$$W_t \quad = \quad W_{t-1} + \textstyle\sum_{k=0}^{K} r_k(t)\theta_k(t-1) + w(t)c(t) - p(t),$$

where

$$\theta_k = \{\theta_k(t) \geq 0; t = 0, \ldots, T-1\},$$

and

$$c = \{c_l \leq c(t) \leq c_u; t = 0, \ldots, T\},$$

are predictable stochastic processes, with $c(0) = c_0$, and

$$\theta_k(0) = X(0)\ell(0)u_k; \quad k = 0, \ldots, K, \qquad \sum_{k=0}^{K} u_k = 1.$$

The stochastic process for the funding-ratio $X(t)$ is defined by

$$X = \left\{ X(t) = \frac{\sum_{k=0}^{K} \theta_k(t)}{\ell(t)}; t = 1, \ldots, T \right\}$$

The filtration $\mathbb{F} = \{\mathcal{F}_t;\ t = 0, 1, \ldots, T\}$ can be represented by means of an event tree. This enables us to recast the model in this section into a multi-stage stochastic programming model with a convex quadratic objective. The reader is referred to Appendix A for the multi-stage stochastic programming formulation of the model. In the next section we discuss the construction of an event tree by scenario generation.

3 Scenario Generation for Asset Liability Management

In the previous section the asset liability management problem has been described as a multi-stage stochastic programming problem. The input to this problem is an event tree, representing the uncertainty involved with the future development of the relevant economic variables. For the generated values of the variables in the event tree to be realistic, they should satisfy the following requirements:

- they should be consistent with historical data, including any interaction amongst the variables.
- they should not exhibit arbitrage opportunities.

In order to fulfill the first requirement, the variables are modeled according to a Vector AutoRegressive (VAR) Model. This is discussed in the next subsection. The role of arbitrage opportunities is discussed in Subsection 3.3.

3.1 VAR models

In order to generate an event tree which is consistent with the observed economic time series we consider a Vector AutoRegressive (VAR) model. Following, *e.g.*, Sims [19], VAR models have become extremely popular forecasting tools. In general, the model is given by

$$y_t = \mu + A_1 y_{t-1} + \ldots + A_p y_{t-p} + \varepsilon_t, \tag{2}$$

where μ, y_t and ε_t are $K \times 1$ vectors, A_i is a $K \times K$ coefficient matrix, and ε_t is independently identically multivariate Normally distributed with mean vector 0 and covariance matrix Σ, $\varepsilon_t \sim N(0, \Sigma), t = 1, \ldots, T$. We can write this model more compactly as

$$\begin{pmatrix} Y^1 \\ Y^2 \\ \vdots \\ Y^K \end{pmatrix} = \begin{pmatrix} X^1 & 0 & \ldots & 0 \\ 0 & X^2 & 0 & \cdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & X^K \end{pmatrix} \begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^K \end{pmatrix} + \begin{pmatrix} E^1 \\ E^2 \\ \vdots \\ E^K \end{pmatrix} \tag{3}$$

where

$$
\begin{aligned}
Y^k &:= \begin{pmatrix} y_{1,k} \\ \vdots \\ y_{T,k} \end{pmatrix}, \\
X &:= (\iota_T, Y_{-1}, \ldots, Y_{-p}), \\
B &:= (\mu, A_1, \ldots, A_p)', \\
E &:= (E^1, \ldots, E^K), \\
E^k &:= \begin{pmatrix} \varepsilon_{1,k} \\ \vdots \\ \varepsilon_{T,k} \end{pmatrix},
\end{aligned}
$$

with the matrix Y_{-i} defined as the matrix Y lagged by i periods. The matrices X^k in (3) correspond to the matrix X with (possibly) one or more columns removed, and β^k is the k^{th} column of B, with the appropriate elements removed.

In case all X^i are identical, this model can be estimated by Ordinary Least Squares (OLS). In the general case, where the X^i may be different due to restrictions, the model corresponds to Zellner's Seemingly Unrelated Regression (SUR) model (see Zellner [20]) and can be estimated by (iterative) Generalized Least Squares (GLS). To obtain the "best" model, we use the Bayesian Information Criterion (Schwarz [17]), given by

$$
\mathrm{BIC} = \ln|\hat{\Sigma}| + \frac{k \ln T}{T} \tag{4}
$$

where k is the number of location parameters of the model under consideration and $\hat{\Sigma}$ is an estimate of the covariance matrix. Estimating all models, the model is chosen for which the BIC is minimal.

3.2 Generation of economic scenarios

Given an estimated VAR model, forecasts can be generated by sampling from the distribution of the innovation vector ε. For example, consider a VAR(1) model and assume Y_T is known. A drawing of the vector of future observations Y_{T+1} may be obtained as

$$
Y_{T+1} = \hat{\mu} + \hat{A}_1 Y_T + \bar{\varepsilon}, \tag{5}
$$

where $\bar{\varepsilon}$ is a drawing from the multivariate Normal distribution $N(0, \hat{\Sigma})$. Clearly, given a value for Y_{T+1}, a value for Y_{T+2} may be obtained in a similar way. In this way, it is also possible to sample k-step ahead forecasts, Y_{T+k}.

An event tree now can be constructed as follows. First, the initial values of the economic variables, or *state* of the economy, are known, with corresponding vector of economic variables Y_0. This can be represented by a single node at time 0. Next, we sample n_0 times 1-step ahead forecasts, Y_{T+1}, where each forecast describes a possible state of the world at time 1. Each state has probability $\frac{1}{n_0}$. For each node at time 1, this process can be repeated, to obtain possible states of the world at time 2. Repeating this T times, where T is the number of decision moments, results in an event tree.

Note that an event tree can be regarded as a single random drawing from an underlying distribution. Clearly, the solution of the ALM problem depends on the event tree and can therefore also be regarded as random. This kind of uncertainty, known as sampling uncertainty, can be investigated by sampling many trees and solving each corresponding ALM problem. In this way, a sampling distribution of the decision variables is obtained, which may be presented by, *e.g.*, boxplots.

Now a tree is constructed with desirable time series properties, we turn to the issue of arbitrage.

3.3 Arbitrage

Several choices have to be made in constructing an event tree, like

- planning horizon
- the number of stages (decision moments)
- the number of successors per node (children) at each stage

An important argument in making these choices is the issue of arbitrage. In reality, arbitrage opportunities do exist, but they are in general to short-lived to be profitable to pension funds. Therefore, it is desirable that also the event tree does not display arbitrage opportunities. In principal, the VAR model does not generate arbitrage opportunities, i.e., infinite sampling will abolish all arbitrage opportunities. In practice however, all trees are finite, and arbitrage opportunities may exist. Also, the density of the tree, which is likely to effect the number of arbitrage opportunities, is limited by computational restrictions (solution time).

The issue of arbitrage opportunities is often neglected in stochastic programming models for financial applications. There are only a few contributions in the financial stochastic programming literature which reckon with arbitrage opportunities. Klaassen [8] tried to analyze the effect of arbitrage in case of market imperfections for a generic stochastic programming model based on Asset Liability Management for banks. However, since he uses internal sampling to analyze the effect of arbitrage, it is not clear whether he truly analyzes the effect of arbitrage or merely analyzes the effect of inconsistency of the security prices w.r.t. their statistical properties in the pruned event tree obtained after performing internal sampling. In order to obtain an arbitrage free event tree Klaassen [9] uses an arbitrage-free pricing model from the literature resulting in a very large event tree. Then he performs state and time aggregation to condense the event tree while preserving consistency with current market prices and keeping the event tree arbitrage-free. Dert [5] discusses an approach to obtain arbitrage-free prices a priori. Starting with an event tree, Dert [5] adds an additional state of nature in such a way as to eliminate arbitrage opportunities prevalent in the event tree. The disadvantage of this approach is that the resulting event tree no longer admits the statistical properties of the underlying distribution prior to adding the new state of the world. Kouwenberg and Vorst [11] take into account both arbitrage opportunities and the statistical properties of the event tree. Using a fitting procedure the generated event tree is both arbitrage-free and the first few moments of the underlying distribution are matched.

In our paper we do not a priori restrict the event tree to exclude arbitrage opportunities. Instead, we examine the influence of the structure of the tree, i.e. the number of stages and children at each stage, on the existence of arbitrage opportunities. Also, we investigate whether the existence of arbitrage opportunities seriously affects the optimal solution to the ALM problem. To study these issues, we conduct a large simulation experiment.

4 Simulation experiments

In this section we present the results of an extensive simulation experiment. In Subsection 4.1 we first operationalize the ALM model and the VAR model. Subsection 4.2 presents the actual results and their discussion.

4.1 Operationalization

In order to operationalize the ALM model and the VAR model, we have to choose numerical values for some of the model's parameters. We first discuss the ALM model. Table 1 contains most of the relevant parameter choices.

Table 1: Values of parameters in the ALM model, given in equation (1)

μ_u	μ_d	c_0	$X(0)$	μ	$\hat{X}$	λ	c_l	c_u
1.5	0.5	0.08	1.1	0.01	1	0.01	0	0.2

The base contribution level c_0 is set to 8%, while an increase of the contribution is penalized more heavily in the objective function than an decrease of the contribution, $\mu_u > \mu_d$. Contributions and final wealth receive a similar weight in the objective function, $\lambda = \mu$. Note that the minimum and maximum contribution level are set to 0% and 20%, respectively. This means we do not allow for refunds. The initial funding ratio is 1.1, which is 10% in excess of the benchmark $\hat{X} = 1$. This benchmark is common in the context of pension funds. Most supervisors check the funding position of pension funds on a regular, *e.g.*, annual basis.

For the liabilities, we take empirical data[3] from a medium-sized Dutch pension fund, comprising the liability level, the benefit payments, and wage sum over a period of 26 years. These amounts are computed using a push-pull Markov chain for the life and career developments of all present and future fund participants, see, *e.g.*, Boender [1]. The computations are all based on a 0% inflation rate. For simplicity, we incorporate inflation effects in the actual event tree by scaling

[3]These data are obtained from ORTEC Consultants.

up the liability levels, benefit payments, and wage sums by the appropriate price levels. These price levels are dictated by the simulated inflation rates, see also further below.

The VAR model is also operationalized using empirical data. We consider annual Dutch data[3] over the period 1956–1997 on short term interest rates, inflation rates, and annual returns on a diversified stock and bond portfolio, respectively. We construct continuously compounded versions of each of these four series and fit a VAR model. Using the BIC as mentioned in Section 3.1, we select the following model:

$$\begin{aligned}
cash(t) &= 0.013 + 0.769 * cash(t-1) + \varepsilon_1(t), \\
bond(t) &= -0.002 + 1.366 * cash(t-1) + \varepsilon_2(t), \\
stock(t) &= 0.095 + \varepsilon_3(t), \\
infl(t) &= 0.020 + 0.645 * infl(t-1) + \varepsilon_4(t),
\end{aligned}$$

where the covariance matrix of the errors is given by

$$10^{-5} * \begin{pmatrix} 28.466 & -12.702 & -150.027 & 13.972 \\ -12.702 & 393.538 & 134.710 & 32.563 \\ -150.027 & 134.710 & 2430.607 & -166.369 \\ 13.972 & 32.563 & -166.369 & 92.465 \end{pmatrix},$$

and where $cash(t)$ is the short term interest rate, $infl(t)$ is the inflation rate, and $bond(t)$ and $stock(t)$ are the returns on a diversified bond and stock portfolio, respectively. We use the 1997 values of the time series as the starting point for the ALM decision process.

4.2 Simulations and results

In this section we investigate the effect of arbitrage on ALM policies in a stochastic programming framework. Throughout, we use the ALM model as laid out in Subsection 2.3 with the parameter values from Table 1.

Given the general framework, we can build a scenario tree as described in Section 3.2. This requires the further choice of certain parameters. In particular, choices must be made regarding the length of the planning period, the number of scenarios, the number of decision times, and the number of successor nodes per time period and/or node in the event tree.

First, we fix the planning period to 8 years. Of course, this is unrealistically short for pension fund ALM. This case serves as a benchmark however. In future experiments we intend to check the sensitivity of our results to the length of the planning horizon. The main advantage of fixing the length of the planning period is that we can investigate the effect of the number of decision times. For example, we can check whether there are significant differences between a model with one period of 8 years versus 8 periods of 1 year. In particular, we study 1, 2, 4, and 8 period models for the 8 year planning horizon. The higher the number of periods, the more flexible the fund manager is in adapting the asset mix and contribution level.

Regarding the number of scenarios, we study two different effects. First, for a given number of periods, we consider how fund policies and characteristics change if we increase the number of successors per node. In particular, we check whether arbitrage opportunities evaporate and if so, how fast. Second, we isolate the effect of the number of decision times by fixing the length of the planning horizon as well as the total number of scenarios. For example, we compare a 1-period model with 1296 scenarios with a 2-period model in which each node has 36 successors.

In the present article we concentrate on sampling uncertainty and arbitrage. The set-up is as follows. First, we generate an event tree using the estimated VAR model from Subsection 4.1. Second, we solve the ALM model. This process is repeated 2,500 times. Note that it is far from straightforward how the results of these experiments should be reported. A solution to the ALM model consists of optimal asset allocations and contribution levels in each period and state of the world, i.e., node in the event tree. For example, for the 2-period model with 16 successors per node, we have an initial asset mix, 16 asset mixes at time 1, and 16 and 256 contribution levels at time 1 and 2, respectively. Given that we have 3 asset categories, this amounts to a total of 323 numbers. Apart from contribution levels and asset mixes, we are also interested in switches in the contribution level and the asset mix, as well as in the funding ratio, the downside deviation of the funding ratio, and the probability of underfunding. This leads to an overwhelmingly large number of interesting quantities for event trees of a realistic size.

We compress the number of reporting measures to reasonable proportions by taking averages. In particular, per time period we only look at the averaged quantities, where the average is taken over the different nodes at that time. In this way, we obtain for example an average contribution level at each time. Similarly, we obtain the average absolute change in the contribution level between time 0 and time 1.

The effect of sampling uncertainty is now revealed by plotting the above averages per time period in a boxplot. The boxplot displays the distribution of a particular fund characteristic or decision variable over the 2,500 replications. In order to distinguish solutions based on event trees with and without arbitrage, we make separate boxplots for each of these two cases.

Table 2 displays some of the results. The table contains the percentage of replications that exhibits arbitrage.

For 1-period models, the number of scenarios can be set quite high for the solution time to remain manageable. The probability of generating an event tree that exhibits arbitrage clearly decreases with the number of successor nodes per period. For 81, 625, or 1296 possible scenarios, there are almost no arbitrage opportunities.

In the 2-period model, the situation is much worse. Note that in a 2-period model with 25 successor nodes per period, we already have 625 scenarios. The probability of arbitrage in such a tree, however, is almost 30%. This contrasts with a probability of only 3% and 0% for a 1-period model with 25 and 625 successor nodes, respectively. Note that the arbitrage percentage is below the figure one would expect by decomposing the 2-period tree into 1-period problems. By doing so, one can construct 26 1-period trees with 25 successor nodes. From Table 2 one could expect an arbitrage probability of around $1 - 0.971^{26} \approx 53\% > 30\%$. The difference is due, i.a., to the different lengths of the periods. Whereas the 1-period model considers periods of 8 years, the 2-period model consists of two adjacent 4 year periods. Given the results for the 2-period model, it is not surprising that for all 4-period models under consideration every

Table 2: Arbitrage Frequencies

# periods	# successors	% arbitrage	% no arbitrage
1	2	100.0	0.0
1	3	88.7	11.3
1	6	57.8	42.2
1	10	32.3	67.7
1	25	2.9	97.1
1	81	0.0	100.0
1	625	0.0	100.0
1	1296	0.0	100.0
2	2	100.0	0.0
2	3	99.9	0.1
2	6	98.7	1.3
2	10	92.8	7.2
2	25	29.2	70.8
2	36	10.3	89.7
1	16	12.5	87.5
2	4	97.0	3.0
4	2	100.0	0.0
1	81	0.0	100.0
2	9	94.5	5.5
4	3	100.0	0.0
1	256	0.0	100.0
2	16	68.6	31.4
4	4	100.0	0.0
1	625	0.0	100.0
2	25	29.2	70.8
4	5	100.0	0.0
1	1296	0.0	100.0
2	36	10.3	89.7
4	6	100.0	0.0

generated event tree exhibits arbitrage opportunities. This result also holds for the 4-period model with 8 successors per node (not reported in Table 2).

Table 3: Tree Structures Used in Literature[1]

Authors	# Assets	# Stages	Tree Structure	AF[2]	EMF[3]
Carino et al. [3]	7	6	$8^1\ 4^2\ 2^1\ 1^1$	No	Yes
Consigli and Dempster [4]	5	10	$2^k\ 1^{9-k},\ a^1\ 3^1\ 2^7$	No	No
Dert [6]	4	10	$100^2\ 1^7$	Yes	No
Kouwenberg [10]	3	5	$10^1 6^2 4^2$	Yes	Yes
Mulvey [14]	7	9	$(8^1\ 4^2\ 2^5)$	No	No

[1] This table presents an overview of the types of event trees regularly used in stochastic programming models for financial applications.
[2] AF: Arbitrage-Free; Is the event tree arbitrage-free a priori or not?
[3] EMF: Exact Moment Fit; Are the moments of the underlying distribution fitted in the event tree or not?

In Table 4.2 we present an overview of the types of event trees used in the literature. As we discussed in Section 2.2 markets are complete if the number of assets with linearly independent payoffs equals the number of successors. If the number of assets (with linearly independent payoffs) exceeds the number of successors there exists arbitrage opportunities for sure (see Section 2.2). From Table 4.2 we conclude that this is the case for the models of Carinõ et al [3], Consigli and Dempster [4], and Mulvey [14]. The event trees generated by Dert and Kouwenberg are arbitrage-free a priori (see Section 3.3). Our experiments show that even if the number of successors exceeds the number of assets the probability of generating (by random sampling) an event tree which exhibits arbitrage opportunities is quite high. These findings are not surprising. However, enlarging the number of successors naturally diminishes the probability of randomly sampling an event tree which exhibits arbitrage opportunities.

We now turn to the effect of arbitrage possibilities on the actual solutions. Some selected results are given in Figures 4.2 through 4.2. The figures contain plots of the contribution level, the absolute change in the contribution level, the upward deviation in the contribution level, the asset mixes and the switches in these mixes over time, the funding ratio, the probability of underfunding, and the downside deviation of the funding ratio. The solid and dashed boxplots represent the solutions for trees without and with arbitrage, respectively. The widths of the boxes reflect the probability of (no) arbitrage. The boxes contain the interquantile range, with the median marked by a horizontal line. The whiskers of the boxes have horizontal markings at the 1st, 5th, 95th, and 99th percentiles.

The contribution policies and asset allocations are relatively similar between trees with and without arbitrage, with the exception of the 1-period model with 10 successors. For this setting the asset mix is evidently more aggressive, containing more stock and less bonds. For all figures shown, the funding ratios at the planning horizon are generally higher for the event trees with arbitrage. The difference is more pronounced if the number of successors is smaller, and especially if the 1-period rather than the 2-period model is considered. The higher funding ratios under the arbitrage scenarios suggest that one may be overly optimistic about future funding prospects if one bases the ALM policy on event trees with arbitrage.

If we compare the (1,625) experiment with the (2,25) experiment, we notice the much higher sampling variability in the outcomes of the latter. Though the median policies and characteristics match quite closely, the spread in possible outcomes is much smaller for the 1-period model. A median allocation of 70% stocks and 30% bonds appears optimal. This results in a median funding ratio after 8 years of about 1.64 for the 1-period model, and 1.7 for the 2-period model. This result is intuitively clear, as the 2-period allows for more flexibility from the part of the decision maker. The upper quantiles of the funding ratio outcomes clearly lie much higher for the 2-period model compared to the 1-period model, while the lower quantiles lie about 10 percentage points lower. This clearly suggests dynamic contribution and asset allocation strategies may enhance future pension fund solvency positions significantly. This enhancement, however, is not without risk. In particular, the displayed sampling variability suggests that situations are conceivable in which the dynamic policy results in a lower funding ratio than a static policy. This conclusion is also illustrated by the probabilities of underfunding.

Concluding, arbitrage possibilities appear very common in realistically sized multi-period ALM problems. The effect on asset allocation and contribution policies appears limited if the number of successor nodes is sufficiently large. If this number is too limited, the effects are more pronounced. This warrants caution for the interpretation of the results of ALM optimization models found in the literature, at least if a limited number of scenarios is used and dynamic policies are allowed. For event trees of (too) limited complexity, we moreover found that projected funding ratios tend to be higher for trees with arbitrage. Finally, we note that due to market imperfections in our ALM model, (*e.g.*, we do not allow for short-selling), arbitrage opportunities prevalent in the event tree cannot be fully exploited. As a result, the evidence that the optimal ALM solutions are driven by arbitrage opportunities appears even stronger.

5 Conclusion

The problem of Asset Liability Management can be formulated as a stochastic programming model. To solve this problem an event tree, describing future developments of relevant economic variables, is generated by sampling the economic variables from a Vector AutoRegressive Model. As a result, optimal solutions to the ALM model are subject to sampling uncertainty. Also, generated event trees may exhibit arbitrage opportunities.

In this paper we have investigated the effect of sampling uncertainty and the structure of the event tree on the probability of arbitrage-free trees. Moreover, the effect on the optimal ALM policies has been studied, both for event trees with and without arbitrage.

Considering data from a Dutch pension fund, we obtain the following conclusions:

- In realistically sized multi-period models, arbitrage opportunities appear frequently. The frequency increases with the number of periods (holding the planning horizon and total number of scenarios fixed).

- These arbitrage opportunities may drive the optimal ALM solutions, in particular if the number of successor nodes in a multi-period model is small.
- More flexibility, i.e. more decision stages, results in higher funding ratios, but also to more sampling variability, resulting in a higher probability of underfunding.

As a consequence, ALM results obtained from scenario analysis must be interpreted with caution, in particular for moderately sized trees.

Several issues remains to be investigated. For example, we plan to examine the effect of the planning horizon on the optimal ALM strategy. Also, the sensitivity of results with respect to model assumptions, both on the ALM and the VAR model, are subject to further research.

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A The Deterministic Equivalent

The filtration $\mathbb{F} = \{\mathcal{F}_t;\ t = 0, 1, \ldots, T\}$ can be represented by means of an event tree. We index the nodes of this event tree (which correspond to the atoms ω) by $n = 0, \ldots, \mathcal{N}_t$ for each $t = 0, \ldots, T$, where $\mathcal{N}_t = |\mathcal{F}_t|$. In the formulation of the model below the predictability (non-anticipativity) of θ and c is automatically satisfied due to the nodal description. We denote the predecessor of node n by n^-.

The balance equation of a pension fund can be rewritten as:

$$\sum_{k=0}^{K}(1 + r_{kn}(t))\theta_{kn^-}(t) + w_n(t)c_n(t) = \sum_{k=0}^{K}\theta_{kn}(t) + p_n(t),\ t = 2, \ldots, T-1,\ n = 1, \ldots, \mathcal{N}_t,$$

At $t = 1$ and $t = T$ we respectively have:

$$\begin{aligned} X(0)\ell(0)\sum_{k=0}^{K}(1 + r_{kn}(1))u_k + w_n(1)c_n(1) &= \sum_{k=0}^{K}\theta_{kn}(1) + p_n(1),\quad n = 1, \ldots, \mathcal{N}_1, \\ \sum_k u_k &= 1, \end{aligned}$$

and

$$\sum_{k=0}^{K}(1 + r_{kn}(T))\theta_{kn^-}(T-1) + w_n(T)c_n(T) = W_n + p_n(T),\ n = 1, \ldots, \mathcal{N}_T,$$

The funding-ratio $X_n(t)$ of the pension fund measures the ability of the fund to fulfill its obligations:

$$\begin{aligned} X_n(t) &= \frac{\sum_{k=0}^{K}\theta_{kn}(t)}{\ell_n(t)}, t = 1, \ldots, T-1,\ n = 1, \ldots, N_t \\ X_n(T) &= \frac{W_n}{\ell_n(T)},\ n = 1, \ldots, \mathcal{N}_T \end{aligned}$$

where $\ell_n(t)$ denote the liabilities at time t and node $n = 1, \ldots, N_t$.

Short selling is not allowed, hence

$$\theta_{kn}(t) \geq 0,\ \forall k,\ \forall t,\ n = 1, \ldots, \mathcal{N}_t.$$

and the contribution-rate has to stay between certain bounds:

$$c_l \leq c_n(t) \leq c_u,\ t = 1, \ldots, T,\ n = 1, \ldots, \mathcal{N}_t$$

The objective of the ALM model consists of three parts. The first part measures the mean-absolute deviation of the contributions from a basic contribution level:

$$\sum_{t=1}^{T}\sum_{n=1}^{\mathcal{N}_t} p_n(t)\left|c_n(t) - c_0\right|, \tag{6}$$

where c_0 denotes the base contribution-rate and $p_n(t)$ the probability of reaching node $n = 1, \dots, N_t$ at time t from node n^- at time $t-1$. The mean-absolute deviation of the contributions (6) can be incorporated linearly in the objective by introducing the following constraints

$$\chi_n(t) + \mu_d(c_n(t) - c_0) \geq 0, \; t = 1, \dots, T, \; n = 1, \dots, \mathcal{N}_t,$$

$$\chi_n(t) - \mu_u(c_n(t) - c_0) \geq 0, \; t = 1, \dots, T, \; n = 1, \dots, \mathcal{N}_t.$$

Now, $\chi_n(t)$ measures the deviation of the contributions from the basic contribution level c_0 at time-period t in node n. Moreover, we penalyze upside and downside deviations differently. The second part of the objective measures the downside variance of the funding-level below a threshold $\widehat{X}$

$$\sum_{t=1}^{T} \sum_{n=1}^{\mathcal{N}_t} p_n(t) \min\{0, X_n(t) - \widehat{X}\}^2.$$

The downside variance can be incorporated as a convex quadratic part in th objective by introducing auxiliary variables $y_n(t)$ and $z_n(t)$ and the following constraints:

$$X_n(t) - y_n(t) + z_n(t) = \widehat{X}, \; t = 1, \dots, T, \; n = 1, \dots, N_t,$$

and enforcing $y_n(t) \geq 0$, $z_n(t) \geq 0$. Now, $z_n(t)$ measures the shortfall of the funding-level below the target $\widehat{X}$ at time-period t in node n. The final part of the objective aims at maximizing the expected terminal wealth. With the notation introduced above, we can write the objective of the model as follows:

$$\mu \sum_{t=1}^{T} \sum_{n=1}^{\mathcal{N}_t} p_n(t) \chi_n(t) + \sum_{t=1}^{T} \sum_{n=1}^{\mathcal{N}_t} p_n(t) (z_n(t))^2 - \lambda \sum_{n=1}^{\mathcal{N}_T} p_n(T) W_n$$

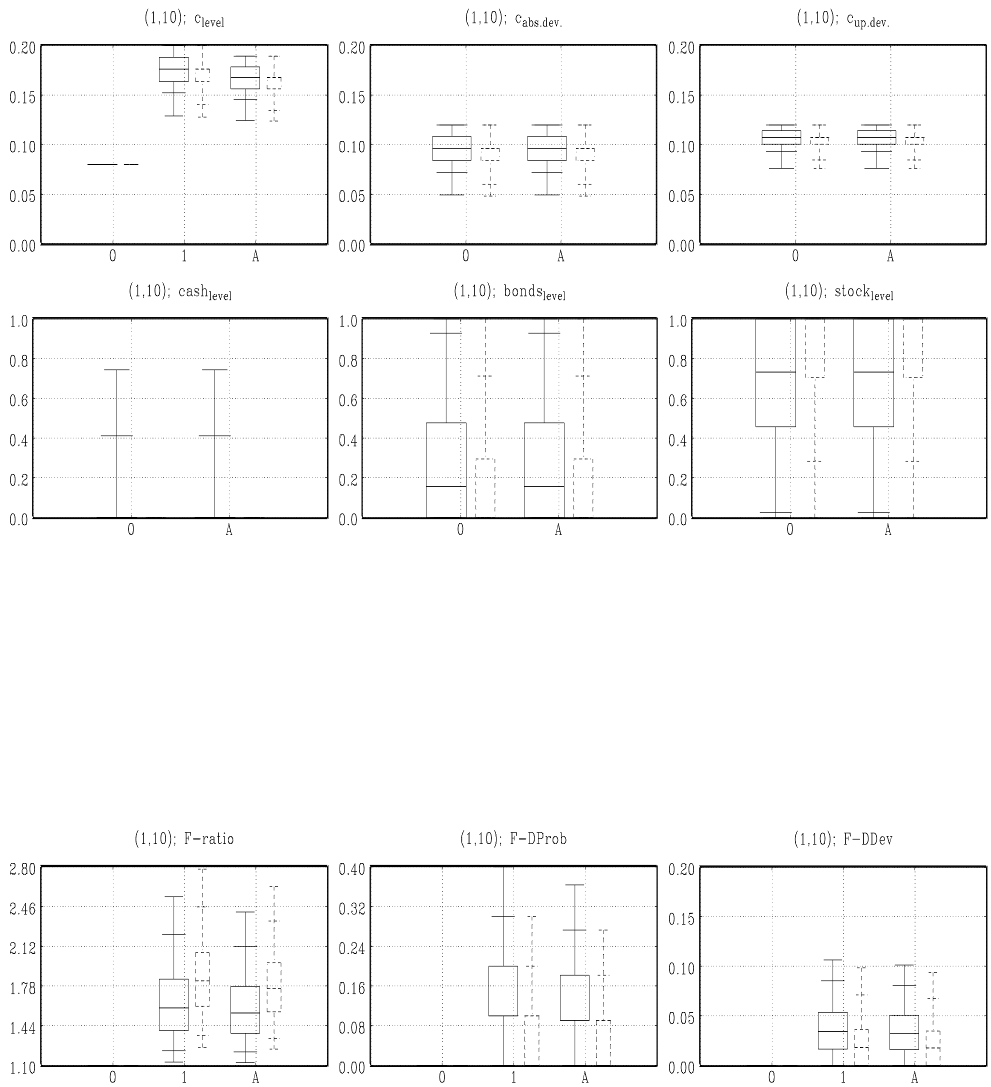

Figure 1: Results 1-period, 10 successors

Figures display respectively the contribution level, absolute changes in contribution level, upward deviation of contribution level, asset mixes, switches in mixes over time (not for the 1-period models), the funding ratio, probability of underfunding and downside deviation of funding ratio at different stages. "A" denotes average. Whiskers corresponds to respectively the 1st, 5th, 95th and 99th percentile. The width of the box represents the probability of (no) arbitrage. Results for the no arbitrage trees are given by the solid boxes, for trees with arbitrage by the dashed boxes.

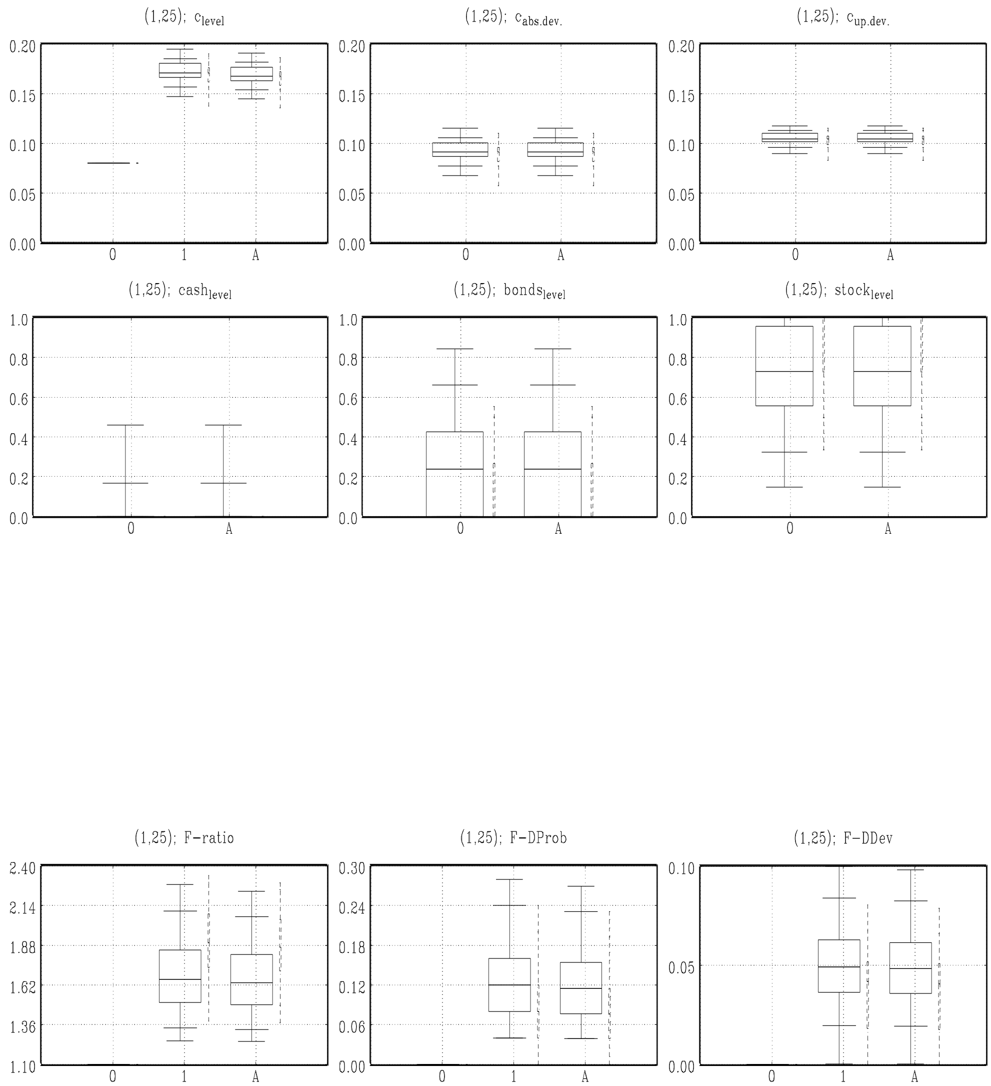

Figure 2: Results 1-period, 25 successors

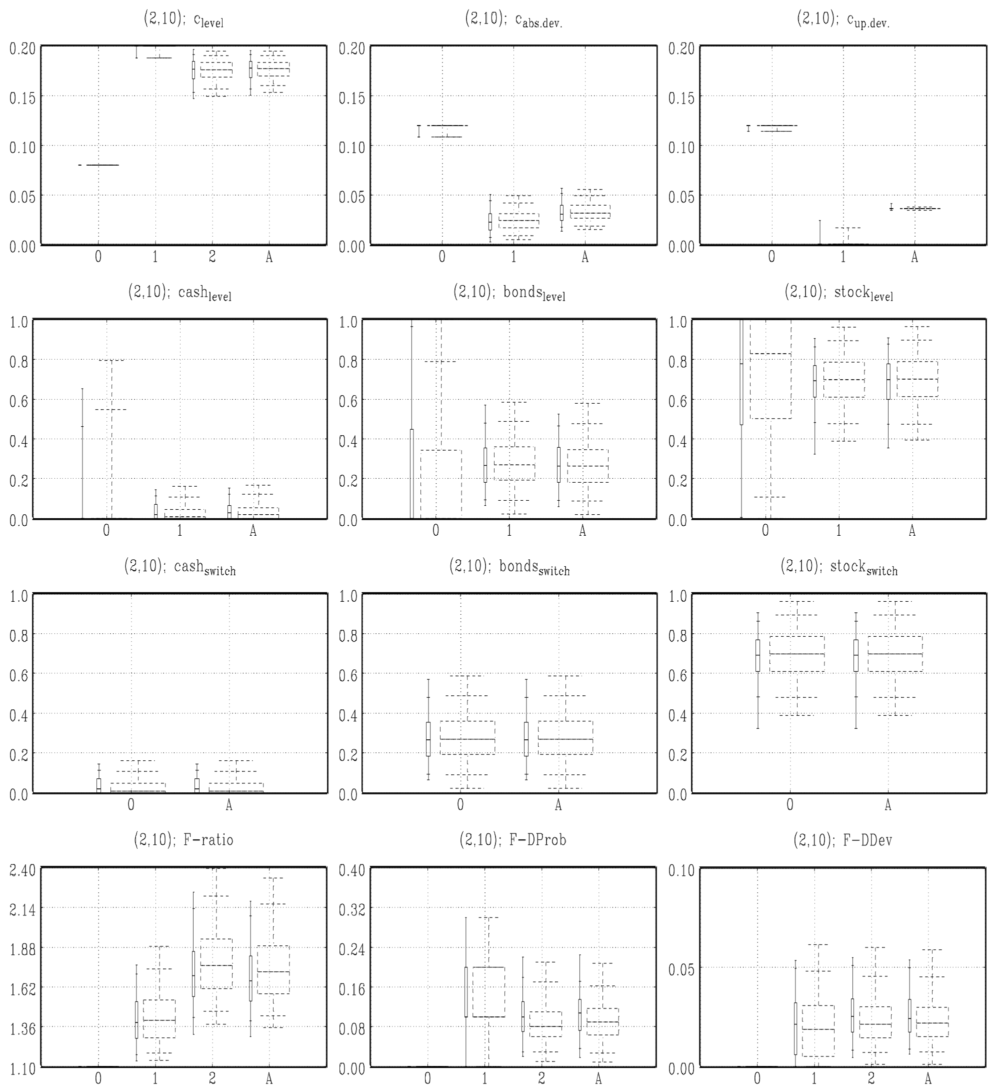

Figure 3: Results 2-period, 10 successors

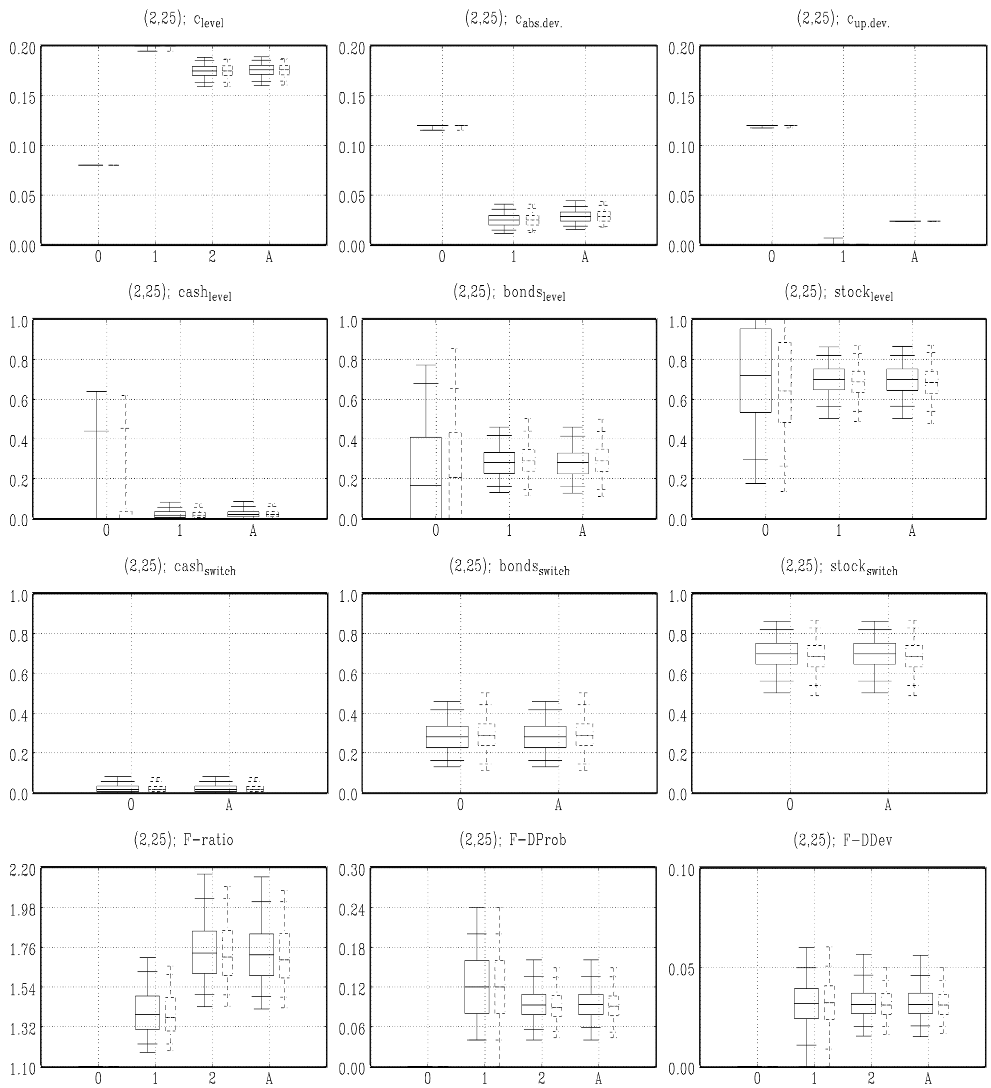

Figure 4: Results 2-period, 25 successors

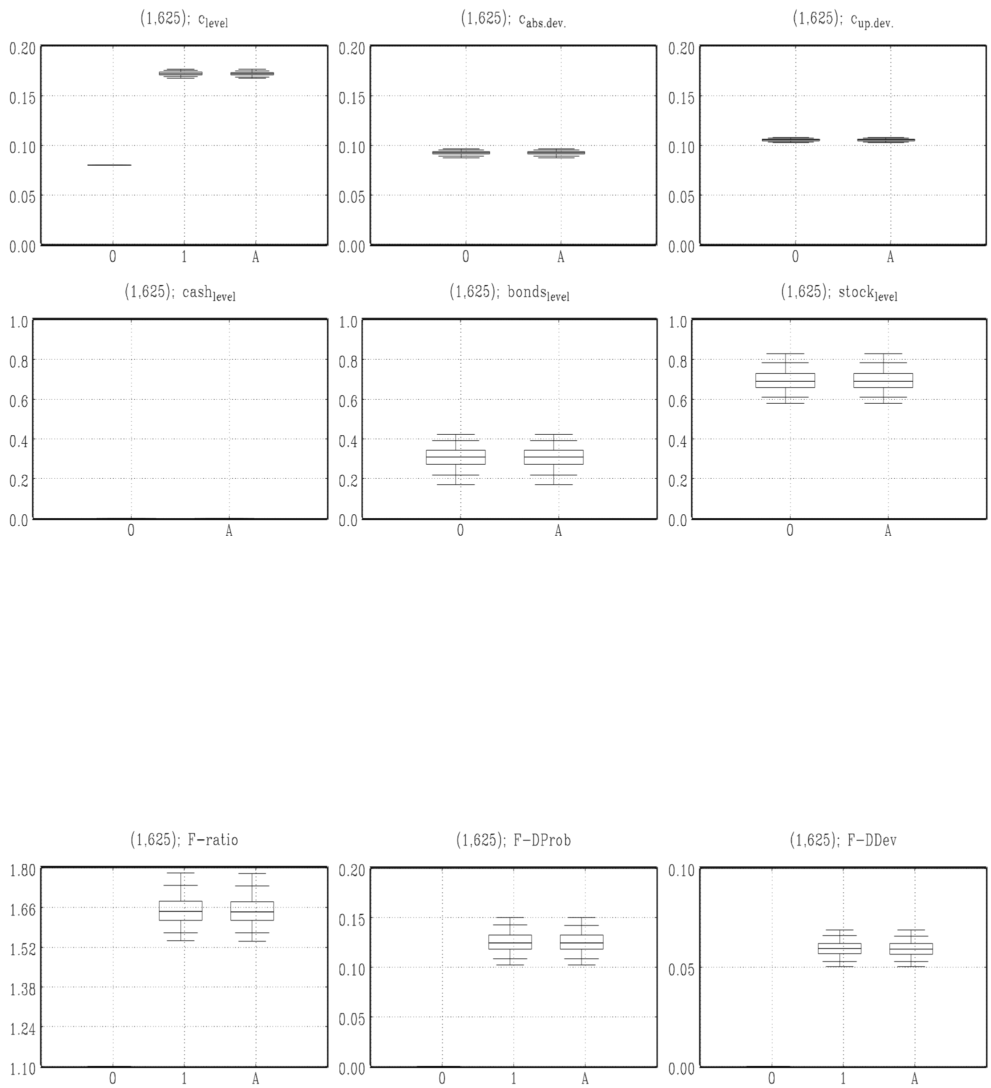

Figure 5: Results 1-period, 625 successors