ON THE THEORY OF INCOME DISTRIBUTION*

CONTENTS

1. THE STATISTICAL FACTS OF INCOME DISTRIBUTION

The frequency distribution of incomes has aroused the interest of statisticians for a long time. Pareto's Law is one of the forms in which the regularities of income distribution have been expressed. Both VAN DER WIJK[1] and GIBRAT[2] have proposed another formula, known as the logarithmic normal distribution. Their formula says that the logarithms of incomes are normally distributed, *i.e.* according to Gaussian law. The results of the two approaches are not very different and it would be difficult, from a statistical point of view, to decide which of the two formulae deserves preference. Most of the statistical research on comparisons between countries and periods has been done

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* Published in Weltwirtschaftliches Archiv 77 (1956), pp. 155-175 Some errors have been corrected.

[1] J. VAN DER WIJK, *Inkomens- en vermogensverdeling*, Nederlandsch Economisch Instituut Nr. 26, Haarlem, 1939.

[2] R. GIBRAT, *Les inégalités économiques, Applications: Aux inégalités des richesses, A la concentration des entreprises, Aux populations des villes, Aux statistiques des familles, etc. d'une Loi nouvelle, La loi de l'effet proportionnel*, Paris, 1931.

with the aid of Pareto's Law. The experience so far gained may be very briefly summarized as follows. Except for cyclical fluctuations, Pareto's constant α for incomes before taxes has moved upward in most countries, indicating a gradually decreasing inequality in incomes. From a figure of around 1.3 in the early phase of industrial revolution (and this figure applies also to under-developed countries in our times) the level has risen to about 2 in the socially advanced countries. In several western countries incomes after taxes show an α some 0.2 higher, pointing to a further decrease in inequality.[3]

2. THE NEED FOR A THEORETICAL INTERPRETATION

The fairly satisfactory state of affairs with respect to the statistical description of income distribution contrasts with an unsatisfactory state in the area of economic interpretation. No generally accepted interpretation of the statistical regularities seems to exist, and most economic text books do not even deal with such an interpretation.[4] This is the more remarkable since the inequality in income distribution is at the bottom of some of the most important problems of economic policy.

To be sure some attempts, and even a few very interesting ones, have been made to throw light on the forces behind the distribution of income. The distribution of wealth, as well as that of certain physical and psychological properties of man, has rightly been mentioned as basic for the understanding of income distribution.

Some interesting partial investigations into the distribution of incomes within homogeneous groups have tried to shed some light on the details of the mechanism.

Recently, TUCK[5] has made a very interesting attempt to explain not only the distribution of incomes, but also the frequency distribution of enterprises with regard to size, with the aid of a rather complex property of man, the ability to guide other people.

What seems to be lacking in all these attempts is, in the author's

[3] *Cf.* JAN TINBERGEN, *Redelijke inkomstenverdeling*, Haarlem, 1953, p. 15 *sq.*

[4] The interpretation given by CHAMPERNOWNE also does not seem to aim at an economic analysis of the process. *Cf.* D.G. CHAMPERNOWNE, "A Model of Income Distribution", *The Economic Journal*, Vol. LXIII, London, 1953, p. 318 *sq.*

[5] R. H. TUCK, *An Essay on the Economic Theory of Rank*, Oxford, 1954.

opinion, a precise description of the mechanism of income formation in terms of the usual instruments of economic analysis. The present article, while elaborating some ideas put forward earlier, tries to make a contribution in this field. This attempt is very provisional, since an essential ingredient for building an acceptable theory is still lacking: *i.e.* sufficient statistical and factual information.[6] Therefore, any theory now presented cannot have been tested; it can only serve as a basis for collecting the necessary data.

A theory of income distribution is needed not only for purely scientific reasons; it is also needed if a policy is to be devised to influence income distribution in the future.

3. DEMAND FOR AND SUPPLY OF PRODUCTIVE CONTRIBUTIONS

The usual instruments of economic analysis just referred to consist of the concepts of demand and supply applied to the system of markets which determine income distribution. Since incomes derive from production, we have to consider the markets for productive contributions. By this term, not only the labour market, but the markets of the other factors of production as well are indicated, although labour, in the widest sense of this word, will play a predominant role. For this reason we will sometimes speak of the demand for and supply of labour in our examples; it should be kept in mind, however, that the demand for and supply of capital and land are implied as well.

It is essential for our problem that this market for productive contributions consists of a very large number of compartments. Demand, supply and price have to be distinguished for each of these compartments but, at the same time, there will be a considerable degree of interdependence. We might deal with this situation by introducing an index for each compartment, say j, and separate symbols for the supply, the demand and the price in each of them, bearing in mind that supply in compartment $j = 767$ depends not only on the corresponding price but also on a number of other prices. The system of equations describing the equilibrium conditions for all of these compartments would be very large and difficult to handle. We will use another instrument which will considerably simplify this description.

[6] *Cf.* section 10.

We will not use a discontinuous index but a set of continuously changing variables characterizing the many compartments; and we will assume that the price system can be described by one single function of these variables. Instead of saying that supply in a given compartment depends on all the prices in the various compartments we will say that that supply depends on the function just mentioned, *i.e.* on the parameters of that function. This mathematical set-up appears to be much more efficient for the treatment of our problem. In order to clarify the method, we will first specify what variables we are going to use for the description of the market for productive contributions.

4. CHARACTERIZING PRODUCTIVE CONTRIBUTIONS BY THE "DEGREES" OF THEIR ATTRIBUTES OR QUALITIES

The variables to be used will be called the "degrees" to which certain attributes or qualities are needed for the contribution to be made. These attributes or qualities may be intelligence, ability to work under unfavourable physical conditions, ability to deal with other people, and so on. These attributes are partly those known from job evaluation. In addition, there will be others needed for jobs not yet evaluated —especially the most qualified—and there may be such attributes as do not necessarily refer to the personal abilities of a man, but to his wealth, relations *etc.* In contradistinction to the usual approach in job evaluation we will not only use these concepts for the description of "jobs", which are indicative of the demand side of our markets, but also for the description of individuals which represent the supply side.

As already observed, the variables which we are going to use to characterize the various market compartments are the *degrees to which* certain attributes are required or present in that compartment. We will indicate by s_i the degree to which an attribute i is required, and by t_i the degree to which it is present in those compartments supplying a contribution. If we used, say, 25 attributes for the characterization of our market, each compartment would be described by 25 figures $s_1, s_2 \ldots . s_{25}$ on the demand side. Because of the use of continuous variables we are free ourselves to choose the size of the compartment; this size will be indicated, on the demand side, by the intervals $ds_1, ds_2 \ldots . ds_{25}$ admitted. In a space with 25 dimensions this represents a little hypercube if all ds_i chosen are of equal size. Small values of ds_i

will describe a sharply defined "occupation", larger values a whole group of neighbouring occupations; neighbouring, that is, in the sense of requiring almost the same abilities.

In order to simplify our illustrations we will base them on a much more restricted number of attributes, namely 2. This may be justified by the fairly high correlation which appears to exist between several of the "degrees" needed for a number of jobs; such "degrees" may then be represented by one single s_1 and a few others by s_2. The existence of such correlation is illustrated by the fact that for a long time it was assumed that intelligence alone is the relevant criterion, or by Mr. Tuck's approach of using the ability to guide (or command) other people as his sole variable.

The demand in our market system may be said to be exerted by the "organizers of production", who are supposed collectively to specify a frequency distribution $m(s_1, s_2 \ldots)$ of the number of contributions they want in each region of the variables s_i.

On the supply side it is assumed that a similar frequency distribution describes the availability of attributes on the part of the individuals forming the economy; this distribution will be indicated by $n\ (t_1, t_2 \ldots)$. In principle this frequency distribution can be observed by psychotechnical tests; in a rudimentary form these are available as army tests or as sets of school reports for young people.[7]

5. THE INCOME SCALE AND ITS INFLUENCE ON THE CHOICE OF A "JOB"

If the two frequency distributions just introduced were identical, *i.e.* the number of contributions of each type wanted were equal to the number of contributors available, it would be possible to organize pro-

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[7] For such tests to be conducted correctly, the circumstances under which the "degrees" are measured have to be specified. When put under pressure, as *e.g.* in competitions, most people will be able to deliver higher performances than without such pressure. For the economic application of such measurements one of the important "circumstances" is the income that may be derived from a certain performance. Strictly speaking we should specify our definition $n(t_i)$ by saying that it represents the supply distribution over the various jobs when all remunerations are equal. Alternatively we may specify any given system of remunerations or "income scale". *Cf.* also U.S. Employment Services, *Guide to the Use of General Aptitude Test Battery B*–1002, Washington.

duction so that each "job" would be done by a man just fit for it. Apart from the availability of the necessary information, there would be no problem of choosing a job. In reality the two frequency distributions do not coincide and this introduces an element of tension between attributes required and attributes available. In order to get every "vacancy" filled the organizers will have to offer remunerations attracting people to the jobs for which supply is insufficient and repelling them from jobs with oversupply. This system of tensions seems to represent the essence of our problem since the function of income distribution is to induce people to choose the proper contribution to production.

We will assume, as already announced, that the incomes paid in the various compartments may be represented by a system, to be called the *"income scale"*. This scale will be represented mathematically by assuming that income l in each compartment $s_1, s_2 \ldots$ can be determined from the values $s_1, s_2 \ldots$ by a certain function $\Lambda\ (s_1, s_2 \ldots)$. This scale, then, is the unknown of our problem; and we will have to fix it in such a way as to equilibrate demand and supply in each compartment. It is this scale, at the same time, which influences both the supply and the demand side of the market so as to bring them together. Fort he time being we will assume that the demand side is not influenced, *i.e.* that demand is inelastic; later on, however, this assumption will be removed.[8] We will first consider the way in which supply adapts itself by studying in more detail the choice the individual makes as to his contribution. Also here we will use a traditional instrument of economic analysis, namely the utility or indifference function. It will be assumed that each individual tries to maximize his utility function ω, supposed to depend on (I) the income l received for the "job" he chooses, (II) the degrees of the attributes of the job s_i representing its difficulty and (III) the degrees of the attributes t_i characterizing the individual, *i.e.* his "abilities". More specifically we assume that the two latter groups of variables—the s_i and the t_i—affect the individual's well-being only to the extent that there are "tensions" between them. Still more specifically we assume ω to be of the form

$$\omega = \omega_3 \log l - \tfrac{1}{2}\omega_1(s_1 - t_1)^2 - \tfrac{1}{2}\omega_2\,(s_2 - t_2)^2. \tag{5.1}$$

[8] *Cf.* section 8.

One way of interpreting this formula is that utility is measurable, that equal percentage increases in income cause equal additions to utility and that equal increases in "tensions" between s_i and t_i—whether positive or negative—cause parabolic increases in disutility with a disutility $\frac{1}{2}\omega_i$ for the first unit of tension. It is by no means necessary, however, to adhere to the interpretation of measurability. Formula (5.1) may simply be said to define the shape of indifference curves. As long as only small "tensions" $s_i - t_i$ appear, the form of (5.1) may be considered to be an approximation which is sufficiently general; the really restrictive hypothesis is that the compensation wanted by different individuals for a unit tension is a given percentage increase in income. Evidently, the final decision about such assumptions can be made only on the basis of empirical evidence.[9]

When confronted now with an income scale

$$\log l = \lambda\,(s_1, s_2 \ldots), \tag{5.2}$$

the individual will choose his "job" s_1, s_2 ... so as to maximize ω, leading to:

$$\frac{\partial \omega}{\partial s_1} \equiv \omega_3 \frac{\partial \lambda}{\partial s_1} - \omega_1 (s_1 - t_1) = 0 \tag{5.3}$$

$$\frac{\partial \omega}{\partial s_2} \equiv \omega_3 \frac{\partial \lambda}{\partial s_2} - \omega_2\,(s_2 - t_2) = 0 \ etc. \tag{5.4}$$

In the special case where λ is of the shape:

$$\lambda \equiv \lambda_{00} + \lambda_{10}s_1 + \lambda_{01}s_2 + \tfrac{1}{2}\lambda_{20}s_1^2 + \lambda_{11}s_1s_2 + \tfrac{1}{2}\lambda_{02}s_2^2, \tag{5.5}$$

these equations take the form:

$$t_1 = (1 - \omega_3'\lambda_{20})\,s_1 - \omega_3'\lambda_{11}s_2 - \omega_3'\lambda_{10} \tag{5.6}$$

$$t_2 = -\omega_3''\lambda_{11}s_1 + (1 - \omega_3''\lambda_{02})\,s_2 - \omega_3''\lambda_{01} \tag{5.7}$$

where

$$\omega_3' = \frac{\omega_3}{\omega_1}; \quad \omega_3'' = \frac{\omega_3}{\omega_2}. \tag{5.8}$$

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[9] There is no lack of generality in our assumption that one and the same function ω can be used to describe the behaviour of different individuals, since the parameters t_i may always be used in a number sufficient to bring out all relevant differences between individuals.

In our illustration[10] we will use this particular assumption about λ in order to solve the problem.

6. ILLUSTRATION OF THE PROCESS OF INCOME FORMATION

The process of income formation now may be further illustrated by an attempt to solve the problem posed, *i.e.* to determine the unknown function λ, in the case of only two attributes where (a) the demand frequency distribution is inelastic and (b) both the demand and the supply distribution are normal distributions in their variables. In formulae this implies that

$$m(s_1, s_2) = \frac{m_0}{2\,\pi\sigma_1\sigma_2\sqrt{1-r_s^2}} \exp \frac{1}{1-r_s^2}\Big[-\frac{1}{2\,\sigma_1^2}(s_1-\bar{s}_1)^2 + \frac{r_s}{\sigma_1\sigma_2}(s_1-\bar{s}_1)(s_2-\bar{s}_2) - \frac{1}{2\,\sigma_2^2}(s_2-\bar{s}_2)^2\Big] \tag{6.1}$$

$$n(t_1, t_2) = \frac{n_0}{2\,\pi\tau_1\tau_2\sqrt{1-r_t^2}} \exp \frac{1}{1-r_t^2}\Big[-\frac{1}{2\,\tau_1^2}(t_1-\bar{t}_1)^2 + \frac{r_t}{\tau_1\tau_2}(t_1-\bar{t}_1)(t_2-\bar{t}_2) - \frac{1}{2\,\tau_2^2}(t_2-\bar{t}_2)^2\Big]\,. \tag{6.2}$$

These assumptions with regard to the frequency distributions do not imply very serious restrictions to generality since the scales in which the attributes are measured are still at our discretion, meaning that these scales may be so adapted as to make at least either the s- or the t-distribution normal. The assumptions are, moreover, to some extent supported by the general experience that many human characteristics show distributions similar, at least, to the Gauss' distribution. Nevertheless, it should be fully recognized that this is one of the additional points about which empirical evidence will have to say the last word. As will be understood from (6.1) and (6.2), both frequency distributions are characterized by:

(I) the average values of the degrees of the attributes ($\bar{s}_1$, $\bar{s}_2$ and $\bar{t}_1$, $\bar{t}_2$);

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[10] *Cf.* section 6.

(II) their standard deviations (σ_1, σ_2 and τ_1, τ_2)
and

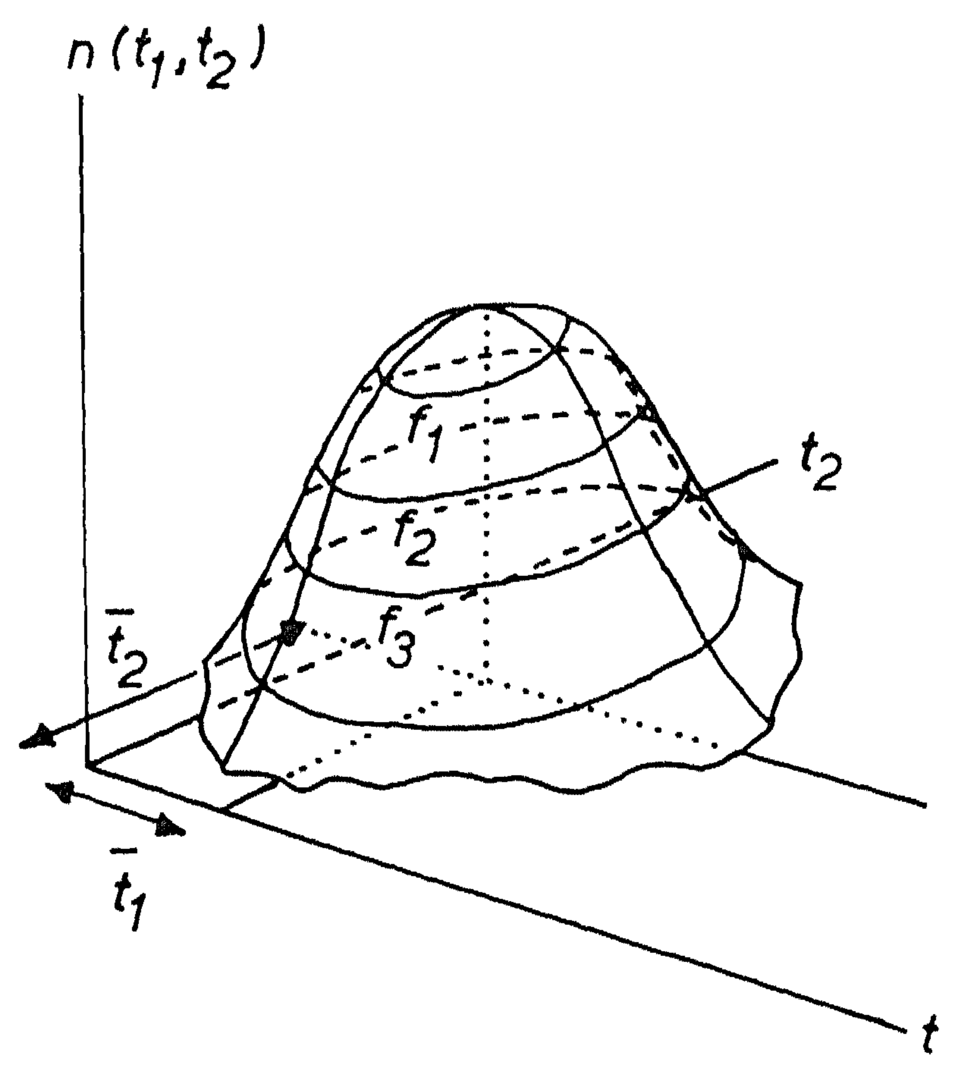
(III) the coefficients of correlation between the "degrees" in the demand and the supply distribution (r_s and r_t).

This means that three forms of divergence between the distributions are admitted. In plain language, and by way of example, this means that there may be differences in average intelligence between requirements and availabilities, and differences in the dispersion of intelligence (and the same for, say, the ability to guide others). In addition, the way in which these abilities are combined on the demand side may differ from the supply side. There may be a relatively strong demand for the combination of the two, whereas the combination may be less frequent among the people who are available.

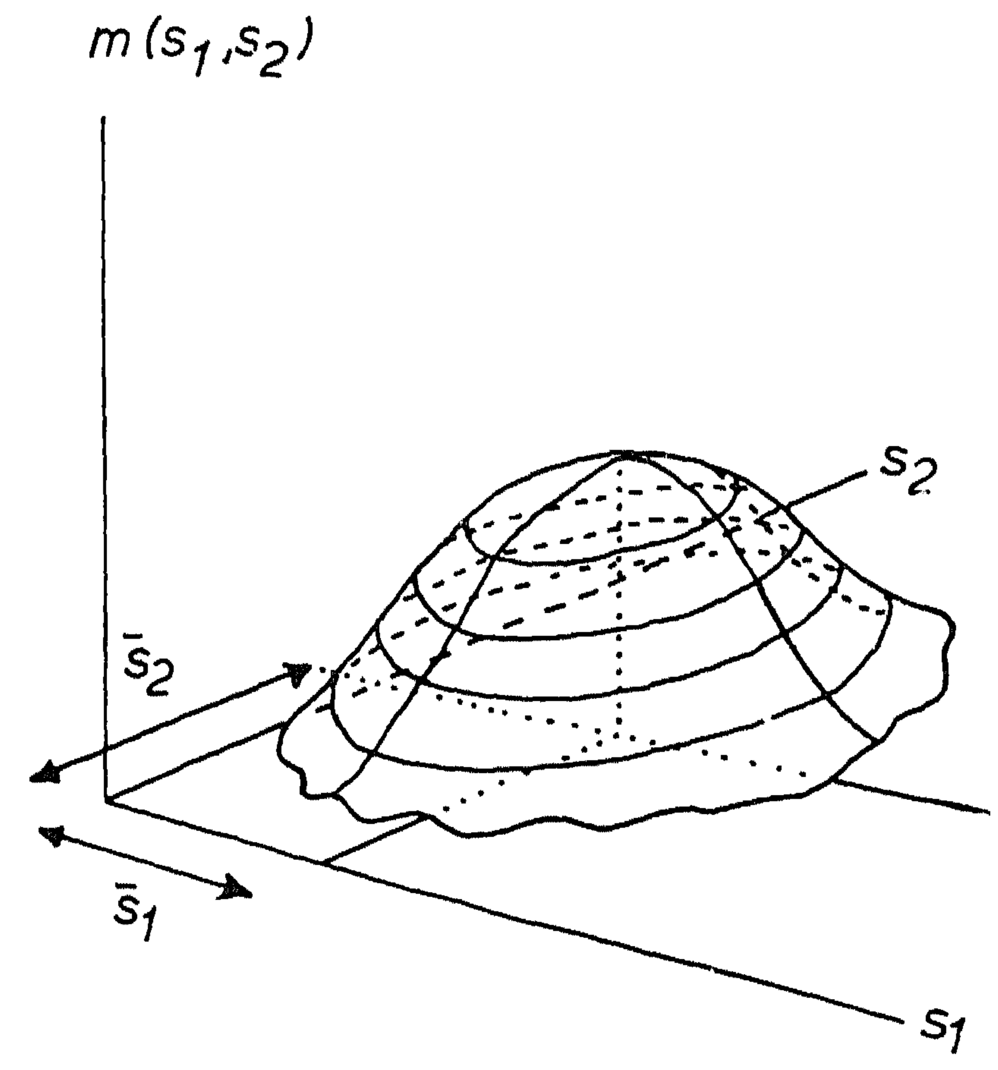
In order to simplify our calculations we choose the units for t_1 and t_2 so as to make $\tau_1 = \tau_2 = 1$. Since s_1 and s_2 are expressed in the same units as t_1 and t_2, this may also be interpreted by saying that our symbols s_1, s_2, $\bar{s}_1$, $\bar{s}_2$, t_1, t_2, $\bar{t}_1$, $\bar{t}_2$ and σ_1, σ_2 represent the ratios of each of these to the corresponding τ_i.

The essence of the process of income formation may now be reformulated by saying that an income scale brings about a choice by each individual with regard to his "job", resulting in a correspondence between s-sets and t-sets illustrated by (5.3) and (5.4), and that this correspondence will have to be such as to transform the t-distribution into an s-distribution identical with the given demand distribution. This identity is another expression for equality, in each compartment, of demand and supply. The procedure may be illustrated geometrically. Graph 1 shows the frequency distribution of t_1 and t_2, *i.e.* the supply side. On the frequency surface some curves of constant frequency densities f_1, f_2, f_3 have been drawn. Graph 2 shows the corresponding distribution of s_1 and s_2. Our income scale has to be one that will make the individuals around t_1^*, t_2^* choose a job around s_1^*, s_2^*, and so on for each point in the t_1-t_2-plane, in such a way that corresponding points in the s- and the t-plane have the same frequencies. This does not mean that the frequency densities—*i.e.* the distances of the points t_1^* t_2^* and s_1^* s_2^* to the horizontal plane—are equal, but that the product of this distance and the surface elements are equal, that is:

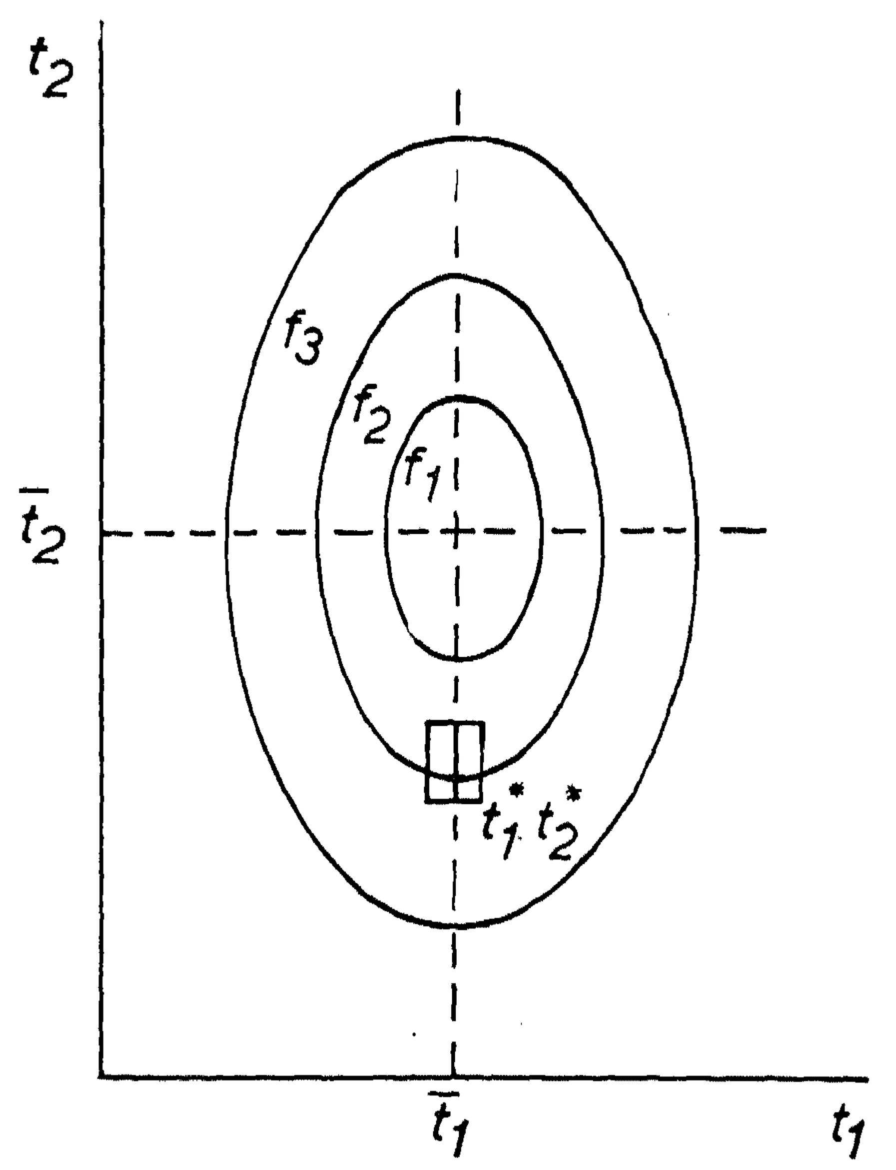
$$n(\overset{*}{t}_1\overset{*}{t}_2)\,d\overset{*}{t}_1 d\overset{*}{t}_2 = m(\overset{*}{s}_1\overset{*}{s}_2)\,d\overset{*}{s}_1 d\overset{*}{s}_2.$$



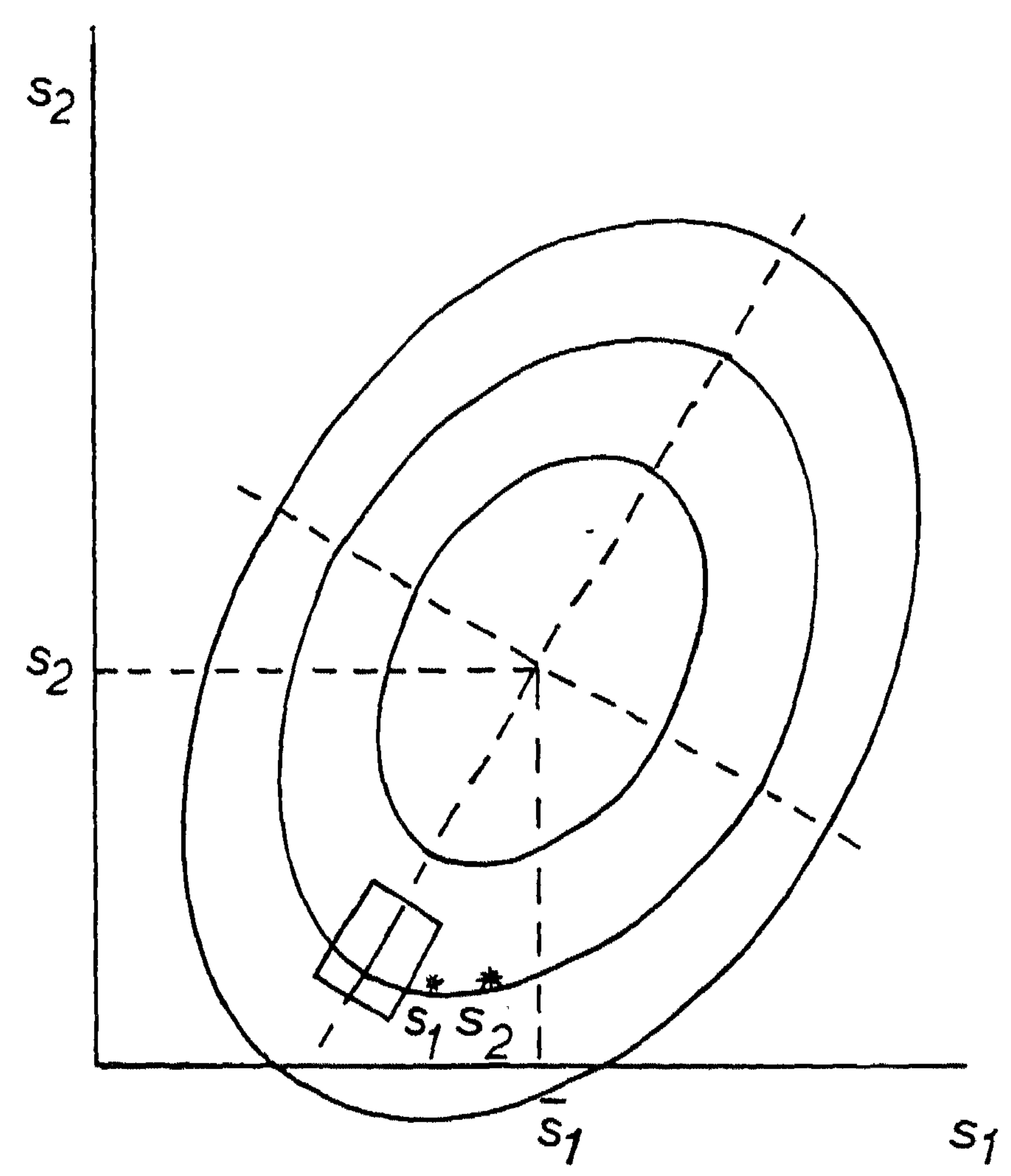
Graph 1



Graph 2



Graph 3



Graph 4

The surface elements have been represented by the small rectangles drawn in graphs 3 and 4, where the projections of the curves of constant density on the horizontal plane have been drawn. These are ellipses, with the axes parallel to the coordinate axes when there is no correlation between the attributes, and an arbitrary position of the axes when there is correlation. The case represented by the graphs is characterized by $r_t = 0$ and $r_s > 0$.

The t-surface has to be "deformed" so as to coincide with the s-surface, otherwise there will not be equilibrium in all compartments of the market.

The proces may now be shown algebraically. As is usual in mathematical problems of this type, we start by making an assumption about the shape of the function λ. We assume it to be of the form (5.5), where still the numerical values of the coefficients $\lambda_{00} \ldots \lambda_{02}$ are open. We try to find values for these coefficients which "do the job". If we are not able to find such values, our assumption (5.5) has to be abandoned. If it can be shown that there is only one solution, and our assumption appears to be a solution, then it will be the solution.

Trying, then, our solution (5.5), the transformation between t and s will be given by (5.6) and (5.7). Substituting these values for t_1 and t_2 in (6.2) must then yield a frequency distribution in s_1, s_2 identical with (6.1). Indicating by t_1' and t_2' the deviations of t_1 and t_2 from their averages:

$$(6.3) \qquad t_1' = t_1 - \bar{t}_1 \qquad t_2' = t_2 - \bar{t}_2$$

using similar symbols s_1' and s_2', and writing for the sake of simplicity,

$$(6.4) \qquad 1 - \omega_3' \lambda_{20} = x \qquad 1 - \omega_3'' \lambda_{02} = z$$

$$(6.5) \qquad \omega_3' \lambda_{11} = y \qquad \frac{\omega_3''}{\omega_3'} = \frac{\omega_1}{\omega_2} = \alpha,$$

we get the transformation

$$t_1' = x s_1' - y s_2'$$

$$(6.7) \qquad t_2' = - \alpha y s_1' + z s_2'.$$

and the result of its application on $n(t_1, t_2)$:

$$-2(1-r_t^2) \times \text{exponent of } n(t_1, t_2) = t_1'^2 + 2r_t t_1' t_2' + t_2'^2$$

$$= s_1'^2(x^2 - 2axyr_t + a^2y^2) + s_1' s_2'(-2xy + 2r_t xz + 2r_t ay^2 - 2ayz)$$

$$+ s_2'^2(y^2 - 2yzr_t + z^2). \tag{6.8}$$

This expression has to be identical to $-2(1-r_s^2) \times$ the exponent of m (s_1, s_2) which, for simplicity's sake, we write as

$$\varrho_1^2 s_1'^2 + 2r_s\varrho_1\varrho_2 s_1' s_2' + \varrho_2^2 s_2'^2 \tag{6.9}$$

$$\text{where } \varrho_1 = \frac{1}{\sigma_1}\sqrt{\frac{1-r_t^2}{1-r_s^2}} \quad \text{and} \quad \varrho_2 = \frac{1}{\sigma_2}\sqrt{\frac{1-r_t^2}{1-r_s^2}}. \tag{6.10}$$

Our unknown coefficients x, y and z have to satisfy the conditions following from a comparison between (6.8) and (6.9):

$$x^2 + 2ar_t xy + a^2y^2 = \varrho_1^2 \tag{6.11}$$

$$-2xy - 2r_t xz - 2r_t ay^2 - 2ayz = -2r_s\varrho_1\varrho_2 \tag{6.12}$$

$$y^2 + 2r_t yz + z^2 = \varrho_2^2. \tag{6.13}$$

In addition to these conditions to be satisfied by x, y and z, two other conditions will have to be satisfied by the coefficients λ_{10} and λ_{01}; these follow from applying (5.6) and (5.7) to the average values of $\bar{s}_1$ and $\bar{s}_2$. Such an application must yield, as its result, the average values of $\bar{t}_1$ and $\bar{t}_2$, since otherwise the exponent of $n(t_1, t_2)$ after transformation would show non-quadratic terms in s_1 and s_2 which are not present in the exponent of $m(s_1, s_2)$. Hence:

$$\bar{t}_1 = (1 - \omega_3' \lambda_{20})\,\bar{s}_1 - \omega_3' \lambda_{11}\bar{s}_2 - \omega_3' \lambda_{10} \tag{6.14}$$

$$\bar{t}_2 = -\omega_3'' \lambda_{11}\bar{s}_1 + (1 - \omega_3'' \lambda_{02})\,\bar{s}_2 - \omega_3'' \lambda_{01}. \tag{6.15}$$

Since, generally speaking, five unknowns can be solved from five equations, it appears that our assumption (5.5) yields a possible solution.

The solution may be undertaken in two stages, since the three unknowns x, y and z appear in (6.11)–(6.13) only and can therefore be solved first; (6.14) and (6.15) will then yield λ_{10} and λ_{01}. Apparently, λ_{00} is then still underdetermined; this coefficient only changes, by its variation, the general level of incomes, but not its distribution. This general level will follow from the condition that no more income can be paid out than has been produced. It may be said to be determined by the general level of productivity.

In order to understand the nature of our solution, we have to give it an explicit, instead of the implicit form we have before us.

For the sake of still further simplification we will now make the assumption that $r_t = 0$, *i.e.* that the attributes are chosen so as to show no correlation on the supply side. Since it is always possible, by a mathematical transformation of $n(t_1, t_2)$, to obtain this state of affairs, there is again no loss of generality.[11]

Equations (6.11)–(6.13) now reduce to:

$$x^2 + a^2y^2 = \varrho_1^2 \tag{6.16}$$

$$xy + ayz = r_s\varrho_1\varrho_2 \tag{6.17}$$

$$y^2 + z^2 = \varrho_2^2, \tag{6.18}$$

which it is not difficult to solve. Writing (6.17) in the form

$$x + az = \frac{r_s\varrho_1\varrho_2}{y} \tag{6.19}$$

and eliminating y between (6.16) and (6.18):

$$x^2 - a^2z^2 = \varrho_1^2 - a^2\varrho_2^2 \tag{6.20}$$

we derive from (6.19) and (6.20)

$$x - az = \frac{\varrho_1^2 - a^2\varrho_2^2}{r_s\varrho_1\varrho_2}\, y. \tag{6.21}$$

Adding up (6.19) and (6.21) yields:

$$2x = \frac{\varrho_1^2 - a^2\varrho_2^2}{r_s\varrho_1\varrho_2}\, y + \frac{r_s\varrho_1\varrho_2}{y} \tag{6.22}$$

This may be used to eliminate x from (6.16). Using capital letters to indicate squares, *i.e.*

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[11] I owe this remark to Mr. J. A. HARTOG. In terms familiar to the statistician the procedure means that factor analysis is applied to the attributes so far used and a number of factors equal to the number of attributes is introduced in their place.

$$Y = y^2; \quad P_1 = \varrho_1^2; \quad P_2 = \varrho_2^2; \quad A = a^2; \quad R_s = r_s^2, \tag{6.23}$$

we get for Y the equation:

$$Y^2[(P_1 - AP_2)^2 + 4AR_sP_1P_2] - 2(P_1 + AP_2)R_sP_1P_2 + R_s^2P_1^2P_2^2 = 0. \tag{6.24}$$

The solutions are:

$$Y = \frac{R_s P_1 P_2}{P_1 + AP_2 \pm 2\sqrt{AP_1P_2(1 - R_s)}} \tag{6.25}$$

or:

$$y^2 = \frac{\omega_2^2 r_s^2}{\omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 \pm 2\,\omega_1\omega_2\sigma_1\sigma_2\sqrt{1 - r_s^2}} \; \frac{1}{1 - r_s^2} \tag{6.26}$$

It follows that

$$x^2 = \left[\frac{1}{\sigma_1^2} - \frac{\omega_1^2 r_s^2}{\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 \pm 2\,\omega_1\omega_2\sigma_1\sigma_2\sqrt{1 - r_s^2}}\right] \frac{1}{1 - r_s^2} \tag{6.27}$$

and

$$z^2 = \left[\frac{1}{\sigma_2^2} - \frac{\omega_2^2 r_s^2}{\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 \pm 2\,\omega_1\omega_{21}\sigma_2\sqrt{1 - r_s^2}}\right] \frac{1}{1 - r_s^2}. \tag{6.28}$$

We thus have found a set of solutions to our problem.

7. INTERPRETATION OF RESULTS

These results may now be used for a reconnaissance and a verbal interpretation. We start by assuming that also on the demand side there is no correlation between the attributes, *i.e.* $r_s = 0$. Our solution then reduces to:

$$\left\{\begin{array}{llll} x = 1 - \dfrac{\omega_3}{\omega_1}\lambda_{20} = \dfrac{1}{\sigma_1} & \text{or} & \lambda_{20} = \dfrac{\omega_1}{\omega_3}\left(1 - \dfrac{1}{\sigma_1}\right) \\ y = \omega_3' \lambda_{11} \quad = 0 & \text{or} & \lambda_{11} = 0 \\ z = 1 - \dfrac{\omega_3}{\omega_2}\lambda_{02} = \dfrac{1}{\sigma_2} & \text{or} & \lambda_{02} = \dfrac{\omega_2}{\omega_3}\left(1 - \dfrac{1}{\sigma_2}\right) \\ & & \lambda_{10} = \dfrac{\omega_1}{\omega_3}\left(\dfrac{\bar{s}_1}{\sigma_1} - \bar{t}_1\right) \\ & & \lambda_{01} = \dfrac{\omega_2}{\omega_3}\left(\dfrac{\bar{s}_2}{\sigma_2} - \bar{t}_2\right) \end{array}\right. \tag{7.1}$$

Remembering what was said about the units used, we may rewrite these formulae for the case where τ_1 and τ_2 are not chosen equal to 1:

$$(7.1') \quad \left\{ \begin{aligned} \lambda_{20} &= \frac{\omega_1}{\omega_3}\left(1 - \frac{\tau_1}{\sigma_1}\right) \\ \lambda_{02} &= \frac{\omega_2}{\omega_3}\left(1 - \frac{\tau_2}{\sigma_2}\right) \\ \lambda_{10} &= \frac{\omega_1}{\omega_3}\left(\frac{\bar{s}_1}{\sigma_1} - \frac{\bar{t}_1}{\tau_1}\right) \\ \lambda_{01} &= \frac{\omega_2}{\omega_3}\left(\frac{\bar{s}_2}{\sigma_2} - \frac{\bar{t}_2}{\tau_2}\right) \end{aligned} \right.$$

The result is interesting enough. It appears that an income scale would be possible where log l is a quadratic function of the "degrees" s_1 and s_2 but without a "mixed term". This means that, generally, income has to depend on the degrees of the attributes of the contributions. The size of the coefficients, which are indicative of the extra remuneration wanted if a higher degree is required, depends on certain "tensions" again, namely, for the linear terms in (5.5), on the tensions between average required and average available degrees[11a]; and for the quadratic terms on the tensions between the dispersions of the required and the available degrees. If the dispersion in the required degrees is larger than that in the degrees available, there will be a positive quadratic term, meaning that successive equal increases in required degrees will have to be remunerated by increasing additional percentage income increases. If the dispersion in the required degrees is smaller than that in the degrees available, successive equal increases in required degrees can be remunerated with decreasing additional percentage income increases. These scales do not depend on the requirements made for the other degree.

Such interdependence will, however, have to be introduced as soon as $r_s \neq 0$. We may therefore say that the "mixed term" $\lambda_{11}s_1s_2$ is necessary because of possible correlation between the required degrees if such correlation is not available on the supply side. It is, in addition, noteworthy that in this case the solutions are no longer unique but that there are two sets of solutions, corresponding to the + and the — signs in our formulae (6.26), (6.27) and (6.28).

The results just obtained may be given a still further interpretation.

[11a] Both expressed in terms of their own dispersion.

We may ask to what extent or under what circumstances income distribution is equivalent to paying a certain price for each of the attributes used to classify jobs and abilities. At least three cases may be distinguished. In order to speak of a price of the attributes, we must have the situation where everybody's income is the sum total of, for each attribute, the product of quantity and price. Quantity, here, stands for the corresponding degree, and this situation would exist only if our income scale were linear. In this strictest sense we cannot therefore speak of the existence of a price, *i.e.* one price, for each attribute. We may, however, somewhat widen our concept of price and—as a second case—accept a different price for larger and for smaller "quantities". Our quadratic income scale without the mixed term $\lambda_{11}s_1s_2$ represents such a situation. Income may still be considered as the total of two amounts, one for the attribute 1 and one for the attribute 2; but each amount is not proportional any longer to the "quantity" s of the attribute. Finally, in the third case to be considered, there is our income scale with a mixed term; in this case one cannot speak at all of a price of an attribute: it not only depends on the quantity of the attribute considered, but also on the quantity of other attributes.

Finally, we may come back to the basic question of our essay, namely, whether the illustrative results obtained imply any hope of arriving at an explanation of the observed income distribution. There would indeed seem to be such a hope, since the income scale found will, under certain circumstances, lead to a logarithmic normal distribution of incomes. This applies when the quadratic terms in our scale vanish, *i.e.* when the dispersions in degrees required do not differ from those in degrees available. In that case, even with a large number of attributes, the logarithm of income can be represented as the sum total of a number of terms, each normally distributed. Such a sum is itself normally distributed and hence the frequency formula defended by VAN DER WIJK and by GIBRAT would result. When quadratic terms appear, the distribution will be distorted, but the main shape need not be very much affected. Again this calls for empirical studies.

8. SOME GENERALIZATIONS

It is possible to remove some of our over-simplifying assumptions and so to generalize our model. First, a stochastic element may be introduced which, according to the observations published by the U.S. Employment Services,[12] plays a very real role. It may be assumed that an individual bases his occupation on assumed values of his attributes rather than on his real attributes. This element of deviation between the supposed values of the t_i and the real values may be brought into our model by the introduction of variables v_i (with means $\bar{v}_i$ and standard deviations φ_i) that have to be added to the supposed value of t_i in order to arrive at the real value u_i. The process of income formation will be based on the supposed values and nothing needs to be changed in the preceding formulae. In order, however, to establish the statistical relations between the real values u_i of t_i and s_i we will have to add the variable v_i:

$$u_i = t_i + v_i. \tag{8.1}$$

This will account for a stochastic relation between u_i and s_i instead of a functional relation.

Next we will *remove our assumption that demand for productive contributions is inelastic.* Instead we will now assume that the distribution of demand will change if a change in relative remunerations occurs. What will be the consequences of a flatter income scale, that is, a decrease in the income differentials between more and less qualified contributions? Evidently, it will become more advantageous for the organizers of production to employ relatively more qualified and relatively less unqualified "workers". The demand distribution, that is, will shift away from unqualified towards more qualified occupations. The averages $\bar{s}_i$ will increase with decreasing values of the λ's. There may also be a concentration on qualified "workers", meaning that the standard deviations σ_i diminish. Both phenomena happen not only because the same products will now be produced with relatively more qualified contributors, but also because the demand for products will shift towards and concentrate upon quality products which can now be obtained at relatively more attractive prices. An interesting question is whether, with complete equality in incomes, the demand for relatively unqualified workers will completely vanish, meaning that the standard deviations σ_i would tend to zero. This is probably not the case

[12] U.S. Employment Services, *op. cit.*

since there is also a considerable degree of complementarity between more and less qualified work. There will always be a need for a certain quantity of relatively simple jobs too.

From these considerations we deduce that the relation between the two distributions may be a simple one or it may be more complicated. The simplest case would be that σ_i would be independent of the parameters λ of the income scale, while

$$\bar{s}_1 = \bar{s}_{10} - \bar{s}_{11}\lambda_{10} \tag{8.2}$$

$$\bar{s}_2 = \bar{s}_{20} - \bar{s}_{21}\lambda_{01}. \tag{8.3}$$

Thanks to the structure of our solution this would only affect the formulae for λ_{10} and λ_{01}, which would now become:

$$\lambda_{10} = \frac{\omega_1(\bar{s}_{10} - \bar{t}_1\sigma_1)}{\omega_3\sigma_1 + \omega_1\bar{s}_{11}} \tag{8.4}$$

and

$$\lambda_{01} = \frac{\omega_2(\bar{s}_{20} - \bar{t}_2\sigma_2)}{\omega_3\sigma_2 + \omega_2\bar{s}_{21}}. \tag{8.5}$$

Since we must assume that $\bar{s}_{11} > 0$ and $\bar{s}_{21} > 0$, the new values will be lower than the previous ones, illustrating that a certain elasticity of demand facilitates the adaptation of supply to demand.

Very probably, however, also the σ_i will be affected by the income scale and, in addition, not only λ_{10} and λ_{01} but the other λ's as well will exert some influence. In a general way we therefore assume that

$$\begin{aligned} \sigma_1 &= \sigma_{10} + \sigma_{11}\lambda_{10} + \sigma_{12}\lambda_{01} + \sigma'_{13}\lambda_{20} + \sigma'_{14}\lambda_{02} \\ &= \sigma_{100} - \sigma_{13}x - \sigma_{14}z, \end{aligned} \tag{8.6}$$

where

$$\sigma_{100} = \sigma_{10} + \sigma_{11}\lambda_{10} + \sigma_{12}\lambda_{01} + \frac{\sigma'_{13}}{\omega'_3} + \frac{\sigma'_{14}}{\omega''_3} \tag{8.7}$$

and that similar formulae for σ_2 apply.

Our formulae become much more complicated and a general solution will not be attempted. For the case where $r_s = 0$ the solution may be undertaken in two stages; in our formulae (7.1) we may substitute (8.6):

$$x = \frac{1}{\sigma_{100} - \sigma_{13}x - \sigma_{14}z}; \quad z = \frac{1}{\sigma_{200} - \sigma_{23}x - \sigma_{24}z}, \tag{8.8}$$

from which x and z may be solved. It may be observed that the additional relations of the type (8.6) do not invalidate equations (6.11) to (6.13) inclusive and that hence, for $r_s = 0$, it remains true that $x = 1/\sigma_1$ and $z = 1/\sigma_2$, or that for λ_{20} and λ_{02} to be zero $\sigma_1 = \sigma_2 = 1$.

9. DEVELOPMENT OF INCOME DISTRIBUTION AND POSSIBILITIES TO INFLUENCE IT

The theory so far presented is static. It assumes the two distributions m and n to be given. They need not, however, be constant. Their coefficients may show movements; these movements may be exogenous or endogenous. They may also be subject to political influence or they may be independent from such influence. Each of these cases deserves some further illustration.

Exogenous movements may be movements in the physical and psychological conditions of the population and its technical development. The latter will continually lead to higher demands on the skill of at least the supervising categories of jobs; and all types of education will try to catch up with these demands or even to surpass them.

Endogenous movements in our context are movements caused by the situation with regard to income distribution of the immediate past. Perhaps the best example may be the role of wealth in income distribution. To-day's incomes will change to-morrow's wealth distribution and to-morrow's wealth distribution will be partly responsible for to-morrow's income distribution. Interesting attempts have been made to schematize this inderdependence and to single out the influence exerted on income distribution by inheritance.[13] The complete mechanism of income distribution and its development would have to be a combination of the influences of wealth and of personal abilities.

The question of whether, by a conscious policy, we can influence income distribution without disrupting equilibrium in the various compartments of our market is of particular importance in the present phase of social development. That question is the key to further social reform which, in its turn, is the only way of eliminating the all-important "social problem". Our illustrative formulae suggest that a de-

[13] A. W. HAMMING, unpubl. calculations on the effect of inheritance on the distribution of wealth.

crease of inequality will be dependent on the possibilities of reducing the "tensions" between required and available attributes. For the purpose of policy design, a subdivision of attributes into attributes that can or that cannot "easily" be changed recommends itself. "Easy" changes may be either cheap or quick or both; difficult changes are those requiring much energy and time. The process of education itself shows the same components. The interesting corollary of our model is that equality of incomes does not require equality of capacities of all individuals but only the correspondence of the distribution of capacities to the distribution required.[14]

In our simplest model, considered in sections 6 and 7, this means τ_1 and τ_2 would have to be adapted to the given values of σ_1 and σ_2. In the more general model discussed in section 8, σ_1 and σ_2 are not given, but are themselves dependent on the distribution of incomes; any move towards a more equal distribution will reduce their values. The τ's have then to be adapted to the reduced values of the σ's; it may of course prove impossible for the τ's to adapt themselves completely. Again, the last word will have to come from empirical studies on both the demand and the supply side of the market.

10. THE NEED FOR CO-ORDINATION OF STATISTICAL DATA IN THE FIELDS CONCERNED

In the preceding discussion the paucity of statistical information relating to the phenomena discussed has been repeatedly stressed. It is this lack of relevant information which prevented us from offering a theory tested as to its degree of realism. This made the theory only illustrative of possible explanations and at the same time obliges us to go first to the facts. Now, the facts have been investigated in many ways and by a large number of research workers. We already referred to job evaluation and to school and army tests. What is wanted foremost, is a co-ordinated survey of our scattered knowledge; and in the second place, a filling up of the most evident lacunae. Formulated in an over-simplified way, what we want is job evaluation for the economy as a whole, indicating frequencies at the same time; and a test

[14] For a further elaboration of these suggestions *cf.* JAN TINBERGEN, *Economic Policy: Principles and Design*, Amsterdam, 1956, section 6.4.

of the population along the same lines. Finally, we want systematic data, again expressed in the same concepts, about the costs, in energy and time, and results of education. It is such a co-ordinated program of research which the Central Planning Bureau of the Netherlands, with the collaboration of a group of experts in these fields, is going to undertake.