THE EQUALIZATION OF FACTOR PRICES BETWEEN FREE-TRADE AREAS

1. INTRODUCTORY

Recently, Professor SAMUELSON has shown[1] that, under certain conditions, free trade in final products is sufficient to equalize, in the areas concerned, not only the prices of these products, but also the prices of productive agents or factors. It is the purpose of the present article to give a mathematical proof of his statement for a number of possible constellations, to go into a number of complications that arise for other constellations, and to discuss the limits of its applicability. When speaking of constellations I have in mind:

(I) the number of products, areas and factors considered, and

(II) the degree to which the production function for the various products differ.

2. TWO COUNTRIES, TWO PRODUCTS AND TWO FACTORS

First of all we may consider the case where the number of products, as well as the number of factors and that of countries, equals two. Let the countries be indicated by subscripts 1 and 2 respectively, the number of workers in the countries by a_1 and a_2, the quantities of the other factor—say land—by n_1 and n_2, and denote by one and two primes respectively the quantities of these factors devoted to the production of food and clothing, which are the two products being considered. This gives rise to the following set of symbols, as shown on the next page.

The quantities produced of both products will depend on the quantities of both factors used in their production and may be indicated by the functions φ for food and ψ for clothing; $\varphi(a_1', n_1')$ representing the quantity of food produced in country 1 and $\varphi(a_2', n_2')$ the quantity

[1] P. A. SAMUELSON: "International Trade and the Equalization of Factor Prices", *The Economic Journal*, LVIII, 1948, page 163.

QUANTITIES OF LABOUR AND LAND

Devoted to production of	Labour		Land	
	Country: 1	2	1	2
Food	a_1'	a_2'	n_1'	n_2'
Clothing	a_1''	a_2''	n_1''	n_2''
Total	a_1	a_2	n_1	n_2

of food produced in country 2. *The same production function will be assumed to exist for the same product* in both countries, but for the other product another production function is supposed to exist.

Food will be taken as the standard of value or numéraire. The following prices will then be relevant:

p, the price of one unit of clothing, in terms of food;

l_1 and l_2, the prices of labour in country 1 and 2 respectively and r_1 and r_2, the prices of the use of land (rent) in these countries, all of them in terms of food.

Since we assume *free trade*, prices of products are the same for both countries; prices of factors may be different but it will be proved that, under certain conditions, they will also be equal.

It will further be assumed that there is:

(I) *full utilization of productive resources* and

(II) *free competition between entrepreneurs.*

Condition (I) leads to the following equations:

(1) $$a_1' + a_1'' = a_1$$

(2) $$a_2' + a_2'' = a_2$$

(3) $$n_1' + n_1'' = n_1$$

(4) $$n_1' + n_2'' = n_2$$

Condition (II) yields the following equations:

(5) (6) $$l_1 = \frac{\partial \varphi}{\partial a_1'} = \frac{\partial \psi}{\partial a_1''} p$$

(7) (8) $$l_2 = \frac{\partial \varphi}{\partial a_2'} = \frac{\partial \psi}{\partial a_2''} p$$

(9) (10) $$r_1 = \frac{\partial \varphi}{\partial n_1'} = \frac{\partial \psi}{\partial n_1''} p$$

(11) (12) $$r_2 = \frac{\partial \varphi}{\partial n_2'} = \frac{\partial \psi}{\partial n_2''} p.$$

Finally there will be an equation expressing that p equals the ratio of marginal utilities of the two products, which itself depends on the quantities produced and the incomes. This equation, of which the details need not concern us, will be of this type:

(13) $$p = f(\varphi_1 + \varphi_2, \psi_1 + \psi_2, l_1, l_2, r_1, r_2).$$

These 13 equations will, as a rule, enable us to determine the 13 unknowns introduced, *viz.* 4 a's, 4 n's, 2 l's, 2 r's and 1 p. Implicitly, however, some further assumptions have been made, *viz.* that *both countries do produce both products*; or, in other words, that all a's and all n's are positive. If one or more of them should prove to be negative, some of the equations would have to be replaced by other ones. We will discuss this later.

3. HOMOGENEOUS LINEAR PRODUCTION FUNCTIONS

In addition to the assumptions already enumerated one further hypothesis will be made, *viz.* that *the production functions are homogeneous linear functions*, *i.e.* that a proportionate increase in the use of both labour and land in the industry concerned will lead to an increase in the same proportion of the quantity produced. This will be the case if the optimum size of the firm is small in comparison with the quantity produced. It can easily be shown that this implies that the marginal productivities in equations (5) to (12) are dependent only on the ratios of the quantities of the factors used, *i.e.*

$$\frac{\partial \varphi}{\partial a_1'} = \varphi_a \left(\frac{a_1'}{n_1'}\right), \frac{\partial \varphi}{\partial a_2'} = \varphi_a \left(\frac{a_2'}{n_2'}\right) \text{ etc.}$$

We may now introduce, as new variables, the ratios:

$$\frac{a_1'}{n_1'} = x_1' \qquad \frac{a_2'}{n_2'} = x_2'$$

$$\frac{a_1''}{n_1''} = x_1'' \qquad \frac{a_2''}{n_2''} = x_2''$$

and rewrite our equations (1) to (12):

$$\text{I} \begin{cases} x_1' n_1' + x_1'' n_1'' = a_1 & (1') \\ x_2' n_2' + x_2'' n_2'' = a_2 & (2') \\ n_1' + n_1'' = n_1 & (3') \\ n_2' + n_2'' = n_2 & (4') \end{cases}$$

$$\text{II} \begin{cases} \varphi_a (x_1') = p\psi_a (x_1'') & (6') \\ \varphi_a (x_2') = p\psi_a (x_2'') & (8') \\ \varphi_n (x_1') = p\psi_n (x_1'') & (10') \\ \varphi_n (x_2') = p\psi_n (x_2'') & (12') \end{cases}$$

$$\text{III} \begin{cases} l_1 = \varphi_a (x_1') & (5') \\ l_2 = \varphi_a (x_2') & (7') \\ r_1 = \varphi_n (x_1') & (9') \\ r_2 = \varphi_n (x_2') & (11') \end{cases}$$

For any value of p chosen provisionally, this system of equations may be used in the following way: equations II may be used to determine the four x's, equations I to find the values of the four n's and equations III will yield the l's and r's. By means of the definitions of the x's we may calculate the a's and from (13) the corresponding p. If this p does not equal the value taken provisionally, we make another choice, until the two coincide. If no such coincidence can be obtained, this is an indication that one of the cases of "specialization" has to be tried. This is to be discussed later (section 5).

4. A PROOF OF PROFESSOR SAMUELSON'S STATEMENT

Equations (6'), (8'), (10') and (12') at once make it clear that $x_1' = x_2'$ and $x_1'' = x_2''$. The equations by means of which x_1' and x_2'' have to be determined, *viz.* (6') and (10'), are identical with (8') and (12') used for calculating x_2' and x_2''. It follows from (5') and (7') that $l_1 = l_2$ and from (9') and (11') that $r_1 = r_2$: *under the conditions enumerated, both wage rate and rent rate are equal in both countries. No transfer of factors is needed to obtain this equality if positive values for all a's are compatible with the data of our problem, i.e.* the given values of total labour and total land supply in both countries and the nature of the production functions. We will now investigate in somewhat more detail the nature of these "boundary conditions".

5. BOUNDARY CONDITIONS; SPECIALIZATION

For each of the countries these may be written in this way:

(14) (15) $$n' \geqq 0 \qquad n'' \geqq 0$$

(16) (17) $$x' \geqq 0 \qquad x'' \geqq 0.$$

It will be necessary to express all of these conditions in terms of x' and x'', in order to find out as quickly as possible whether a solution is or is not acceptable. Conditions (16) and (17) have a simple meaning, which can also easily be translated into geometrical terms: the point $(x'; x'')$ shall be situated above the horizontal and to the right of the vertical axis. Conditions (14) and (15) may be expressed as follows: since $n'' = n - n'$ we have $x'n' + x''(n - n') = a$, the total labour supply.

Solving for n' we have:

(18) $$n' = \frac{a - nx''}{x' - x''} \geqslant 0.$$

Similarly we have

(19) $$n'' = \frac{a - nx'}{x'' - x'} \geqslant 0.$$

These conditions may be interpreted in the following way.

$$(18') \qquad \begin{cases} \text{If } x' > x'', \text{ we must have } x'' \leqslant \dfrac{a}{n} \\ \text{if } x' < x'', \text{ we must have } x'' \geqslant \dfrac{a}{n}. \end{cases}$$

$$(19') \qquad \begin{cases} \text{Similarly, if } x'' > x', \quad x' \leqslant \dfrac{a}{n} \\ \qquad\qquad \text{if } x'' < x', \quad x' \geqslant \dfrac{a}{n}. \end{cases}$$

These conditions may be summarized by saying that one of the x must be smaller and the other greater than a/n, *i.e.*

$$(20) \qquad x' \geqq \frac{a}{n} \geqq x'' \quad \text{or} \quad x' \leqq \frac{a}{n} \leqq x''.$$

In geometrical terms, the point (x', x'') is not permitted to enter the areas I and II (*cf.* Fig. 1). It may be shown that the borderlines L and M correspond to the cases of *specialization*, *i.e.* the cases where one

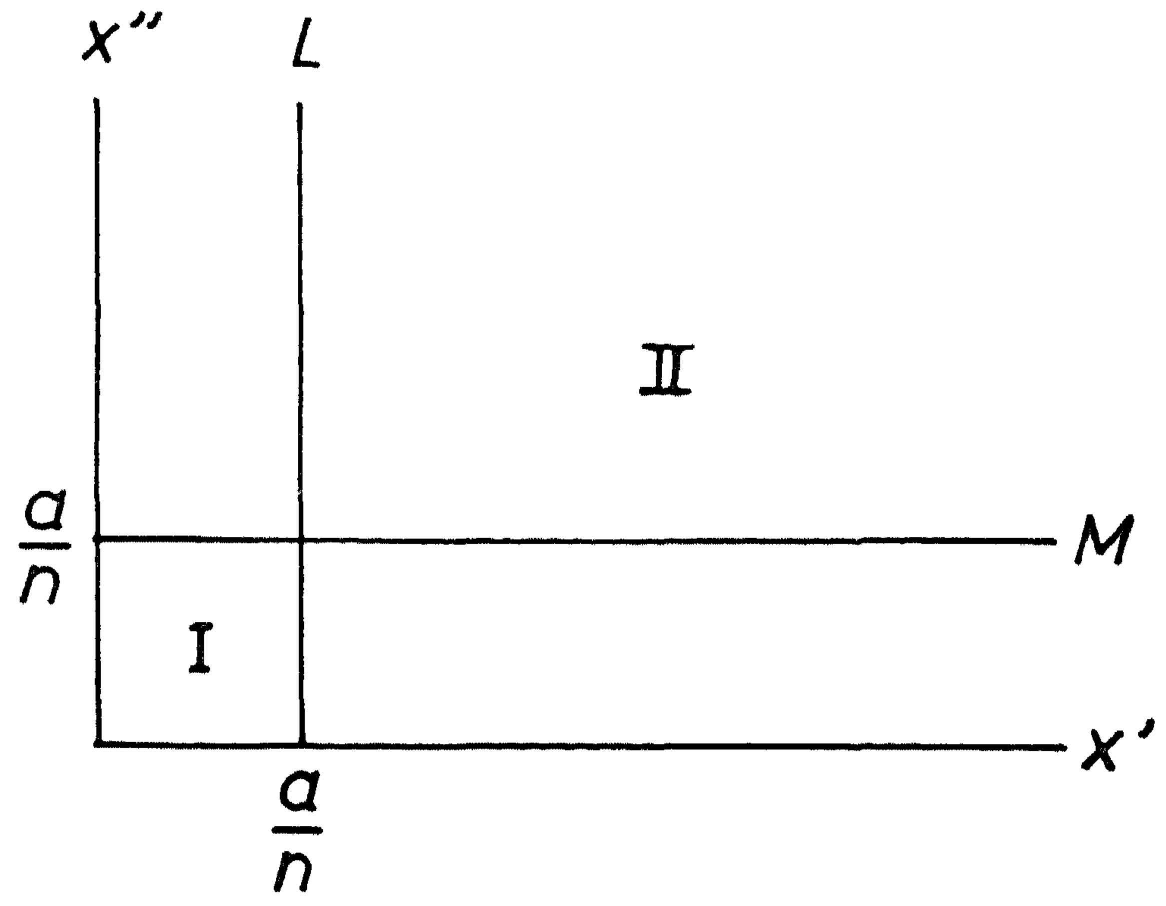

Fig. 1.

of the countries or both are producing only one of the two commodities.

Suppose that the solutions of our equations (1) to (13) inclusive do not obey these conditions and that both x's are either $< a/n$ or both are $> a/n$. In such cases the number of workers and the area of land devoted to one of the industries will be found to be negative. Since this is an impossible proposition, we have to choose these quantities equal to zero; and these equations have to be substituted for the corresponding equations among (6′), (8′), (10′) and (12′). Three cases may be distinguished:

(1) only country 1 specializes;
(2) only country 2 specializes; and
(3) both countries specialize.

It may, of course, be proved that specialization in the industry requiring comparatively more labour will occur in the country where a/n is larger than in the other country, and specialization on the "land-intensive" industry in the other country. If the values of a_1' and n_1' are negative and those for a_2'' and n_2'' are not, only country 1 will specialize and it will specialize on clothing. *It will be clear that as soon as specialization enters into the picture, the equality of wage rates is no longer necessary.*

6. DENSITY OF POPULATION AND WAGE LEVEL

In the foregoing analysis we have considered the total labour and land supplies of the countries as given, and inquired into the influence of varying price conditions, and varying production functions. In fact, the results will be different according to the technical properties of the industries considered. With a great difference between the degrees of "labour intensity" (no matter in what way defined) of the two industries there will be a greater probability of non-specialization than with a small difference. We will now consider another aspect of the mechanism described, by asking *what influence on a country's wage level will be exerted* (given a "world market" price ratio between the two products) *by a changing ratio between the supplies of labour and land, i.e.* a changing density of population. Fresh interest in this way of formulating the population problem has recently been aroused by the stimulating investigations by E. WAGEMANN.[2] Our analysis will make it clear that the curve representing (*cf.* Fig. 2) the functional relationship between a/n as the independent variable and l as the

[2] E. WAGEMANN: *Menschenzahl und Völkerschicksal*, Hamburg, 1948.

dependent one will consist of three parts. For low values of a/n there will be specialization on the land-intensive industry (say food); and l will be the higher, the smaller a/n; for high values, on the other hand, there will be specialization on the labour-intensive industry (clothing), l being lower, the larger a/n. In between there will be co-existence of

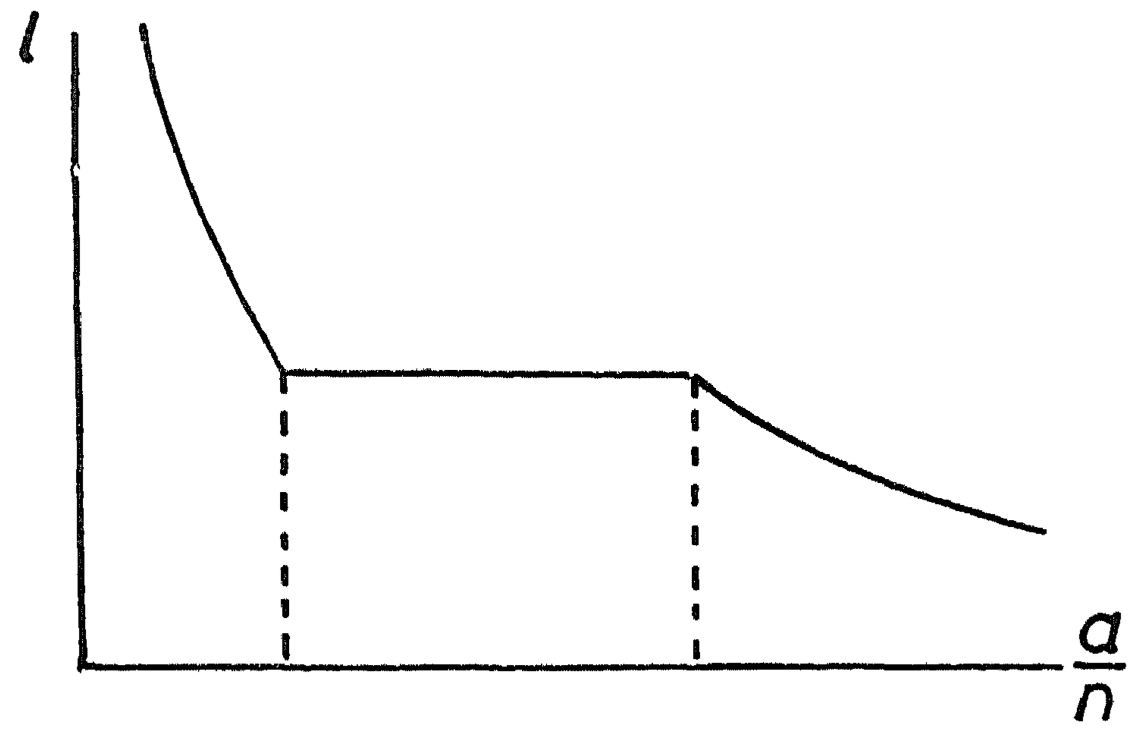

Fig. 2.

the two industries and for this whole range l will be constant, since, according to equations (4′) to 12′) inclusive, it does not depend on a and n as long as there is co-existence of the two industries. This middle part of the curve will therefore be horizontal.[3]

7. INCREASING THE NUMBER OF COUNTRIES

We may summarize our findings in the previous sections by saying that specialization is possible, but not necessary in the case of two countries, two products and two factors. It will be worth while studying other constellations with regard to these numbers; and we will find that for some of them specialization is always a necessity.

Let us therefore ask what happens if any of these numbers increases. *Increasing the number of countries* is a simple matter. Our system of equations already consisted of an equal number of equations for each

[3] The interesting analogy of this curve with the graphical representation of Van der Waals' Law in physics may be pointed out. In this law the connection between pressure (vertical axis) and density (horizontal axis) for a substance is expressed; the horizontal part representing the range of the co-existence of the fluid and the gas phase.

of the two countries together with one equation for the price ratio p: the equations for country 1 *e.g.* being characterized by the subscripts 1. Similar groups of equations may be added for any number of new countries and their discussion would then proceed along the same lines as given above. Evidently the equalization of factor prices will exist only as long as none of the countries is forced—by its data in connection with those of the production functions and the price ratio—to specialize. Of course this becomes increasingly improbable as the number of countries increases and in particular as the range of their population density widens.

8. INCREASING THE NUMBER OF FACTORS

Next we consider an *increase in the number of factors*. We shall assume three: labour, land and capital, the quantities of the latter factor applied in the two industries being k', and k'', respectively. Omitting the subscript for the country we have the following equations for each country (where m represents the interest rate, $u' = k'/n'$ and $u'' = k''/n''$:

$$(l =)\ \varphi_a(x', u') = p\psi_a(x'', u'') \tag{21}$$

$$(r =)\ \varphi_n(x', u') = p\psi_n(x'', u'') \tag{22}$$

$$(m =)\ \varphi_k(x', u') = p\psi_k(x'', u'') \tag{23}$$

$$x'n' + x''n'' = a \tag{24}$$

$$n' + n'' = n \tag{25}$$

$$u'n' + u''n'' = k \tag{26}$$

These six equations must be satisfied by the values of the six unknowns x', x'', u', u'' and n', n''. The interesting difference with the case of two factors is that no longer the ratios x and u, now four in number, can be determined by means of the equations (21), (22) and (23). *Our proof on the equalization of factor prices is no longer valid*: the equations by which x', x'', u' and u'' are found are no longer identical for all countries. And it may easily be seen that, if a certain set of values x', u', x'', u'' satisfies these six equations, it will no longer satisfy them if the value of one of the data a, n and k is changed. Only a proportionate change in these data will not affect the values of x',

u', x'' and u''. Hence our conclusion: *Samuelson's statement is not correct if the number of factors is three while the number of products is two.*[4]

9. INCREASING THE NUMBER OF PRODUCTS

Let us now increase the number of products to three while leaving the number of factors equal to two. Indicating by x''' the ratio a'''/n''' for the third industry by χ the corresponding production function and by q the price ratio between the third and the first product, our system of equations now becomes, for each country:

$$(27)\ (28) \qquad [l =]\ \varphi_a(x') = p\psi_a\ (x'') = q\chi_a\ (x''')$$

$$(29)\ (30) \qquad [r =]\ \varphi_n(x') = p\psi_n\ (x'') = q\chi_n\ (x''')$$

$$(31) \qquad x'\,n' + x''\,n'' + x'''\,n''' = a$$

$$(32) \qquad n' + n'' + n''' = n$$

Disregarding the expressions in square brackets, we have six equations for six unknowns (three x's and three n's). Upon closer inspection, however, we see that in the first four equations (27) to (30) inclusive only three unknowns appear, whereas in the last two equations three more unknowns are introduced. This means that generally the system is partly overdetermined and partly underdetermined. Generally speaking, therefore, there will be no solution; a closer study reveals that in this case *as a rule specialization will be necessary.* This closer study may be based upon the introduction of hypotheses on the dynamics of the system from which the development of production in the three industries may be derived. It may be shown, under rather general assumptions, that this development ends with the elimination of one of the industries—except for the trivial case that the production functions of two industries are the same.

10. AN EQUAL NUMBER OF FACTORS AND PRODUCTS

The difficulties dealt with will not present themselves if we increase at the same time the number of commodities and the number of factors in such a way that their numbers remain equal; say n. Indicating by

[4] It can easily be proved that the same is true whatever be the numbers of factors and products if only there are more factors than products.

superscripts the industries and by subscripts the factors, our unknowns are now a_k^i being the quantity of factor k applied in the production of commodity i. Indicating by x_k^i the proportion a_k^i/a_1^i, we have $n(n-1)$ unknowns x; in addition we are left with the unknowns a_1^i $(i = 1 \ldots n)$. Using a corresponding system of notation for the production functions and their derivatives, we indicate by φ_k^i the derivative with respect to factor k of the production function for commodity i. Finally the prices of the factors may be written as l_k and the price ratio of commodity i with respect to 1 as p_i. Our equations are:

$$[l_k =]\ \varphi_k^1 (x_2^1 \ldots x_n^1) = p^2 \varphi_k^2 (x_2^2 \ldots x_n^2) = \ldots p^n \varphi_k^n (x_2^n \ldots x_n^n) \quad (k = 1 \ldots n) \tag{33}$$

$$\Sigma_i\, x_k^i a_1^i = a_k \quad (k = 2 \ldots n) \tag{34}$$

$$\Sigma_i\, a_n^i = a_1. \tag{35}$$

This system of n^2 equations (omitting the expressions in square brackets) determines the n^2 unknowns, *viz.*, $n(n-1)$ values x and n values a_1^i). It again appears that the equations (33), whose number amounts to $n(n-1)$ are identical for all countries. They are sufficient to determine the $n(n-1)$ quantities $(i = 1 \ldots n, k = 2 \ldots n)$. It follows that *in this case too there is an equalization of factor prices.*

11. Summary

We will briefly summarize our findings. It appears that in an international economy, under certain conditions, *viz.*:

(I) free trade;
(II) free competition between entrepreneurs;
(III) full utilization of productive resources;
(IV) identical production functions in all countries for the same product;
(V) optimum units small in comparison with markets and
(VI) certain numerical conditions are fulfilled by the data, making specialization undesirable,

there will be an equalization of factor prices, even if factors cannot move from one country to another.

The proposition holds good for an equal number of factors and products, independently of the number of countries. It may be proved by showing that factor prices within any country are independent of the relative abundance of the various factors, being the only data differing from one country to another. It is not correct, however, except in borderline cases, if the number of factors is unequal to the number of products. If the number of factors is smaller than the number of products, specialization is always necessary and this no longer warrants the equality of factor prices. If, on the other hand, the number of factors exceeds the number of products, factor prices will no longer be independent of the relative abundance of the factors for each individual country and hence will not be equal for all countries. Finally, specialization may also occur if the number of factors equals the number of products. This depends on the divergencies in the relative abundance of factors in the various countries, together with the differences in production functions or, in simple but somewhat vague words, the differences in "land intensity" or "labour intensity" between industries.