TYPES OF EQUILIBRIUM AND BUSINESS-CYCLE MOVEMENTS*

1. In this article will be presented some remarks about the different types of positions of equilibrium between which we must distinguish in economic theory. The motive of these remarks can be found in the fact that business-cycle theory contains certain controversies, which —although actually existing—are seldom found clearly expressed. These controversies are caused by the important role that the position of equilibrium plays in business-cycle explanation; two viewpoints are in direct contradiction:

a. that business-cycle movement is a movement around a position of equilibrium;
b. that it is a movement between two consecutive positions of equilibrium.

The former opinion is implicit in many econometric business-cycle models;[1] the latter view is defended for instance, by GOUDRIAAN, although he makes only a casual reference to these problems.[2] On closer examination it appears that the different types of equilibrium must be analysed before a choice between these two viewpoints or a synthesis is possible.

2. Let us start with *the concept of equilibrium*. By a position of equilibrium we shall mean *an economic situation which remains unchanged, unless interference from outside takes place*. A priori *it is far from certain that any positions of equilibrium actually do exist*. Postulation of their existence assumes that all the characteristic magnitudes of the economy *can* actually be kept constant. Economic models can be

* Ligevægtstyper og Konjunkturbevægelse, Nordisk Tidsskrift for Teknisk Økonomi, 1944, pp. 45-63.

[1] I refer here to Kalecki's work and to my own contributions; there are many other examples as well.

[2] J. GOUDRIAAN, "De berekening van de omvang der werkloosheid als gevolg van prijsdaling in een gesloten economisch milieu", *De Economist* 83 (1934), p. 849.

constructed in which equilibrium appears to be impossible. An economy with a positive net investment can, in the most limited sense, never be in equilibrium. For, if the net investment is constant through time, the total stock of capital cannot also remain constant. These two conditions are incompatible with each other. We can take the liberty of assuming the existence of a position of equilibrium only if either the total stock of capital can be looked upon as irrelevant to the other economic magnitudes—and consequently need not be included in the model as a variable—, or if we are content with an approximation. In the following we shall assume the existence of a position of equilibrium.

The determination of the position of equilibrium of an economic system is based on certain *static* relationships between the variables that describe the system, *e.g.* static supply and demand functions. For example, if we put demand equal to supply in a single market, we obtain specific values for the price, p, and for the quantity sold, u. Once having assumed certain values, these magnitudes will remain constant as long as no interference from outside takes place.

3. What will happen, however, if a disturbance occurs? It may have different results, and it is by means of these results that we distinguish between different *types of position of equilibrium*. The distinction between *stable* and *unstable* equilibrium is generally known. However, there are other types which should be added to the list. We talk about a stable equilibrium in cases where, after a disturbance, there occurs a movement back towards the original position of equilibrium, and about an unstable equilibrium if the result is a further movement away from this position. Thus to determine the type of equilibrium we must know the movements of the system, *i.e.* we must know its dynamic characteristics. Static characteristics are not sufficient in this case. However, we often meet the opposite opinion in text books, where certain *assumptions* regarding the dynamics of the system are easily made. An example is the well-known sentence, that there will be stable equilibrium in the single market when and only when the slope of the supply curve with the p-axis is denoted by a *algebraically* bigger number than the slope of the demand curve. This statement is wrong, if we describe the dynamics of the market by

means of the well-known cob-web theorem. For in this case it is essential, in order to obtain a stable equilibrium, that the absolute value of the slope of the supply curve is smaller than the absolute value of the slope of the demand curve.

Thus we must study the dynamics of the system more carefully in order to reach a better understanding of the types of equilibrium. By doing so it will become clear to us that even the simplest hypothesis regarding the movement will lead to an abundance of new possibilities. But as long as it cannot be settled empirically which among the possible assumptions corresponds best to the reality of a given problem, it is rather aimless to lose oneself in the consequences of the more complicated hypotheses.

4. Any movement in an economic system in a state of equilibrium arises by a departure from the original position of equilibrium. For it follows from our definition of the concept of equilibrium that no movement could possibly begin without such a departure. The best-known examples of such disturbances are those caused by abnormal agricultural crops, changes in economic policy, new technical inventions, *etc.* The further development of the movement until a new external disturbance takes place—the so-called *endogenous* part of the movement—is now dependent on the way in which the economic system will react. Thus a sequence of interdependent situations develops, every new situation being dependent upon the preceding. Dynamics is characterized in this way; without this interdependency the dynamic analysis would be superfluous and static theory would be sufficient to describe the phenomena.

5. The different forms of dynamic development depend upon the complexity of the assumptions. The simplest model is obtained, when every situation is dependent only on the proceding one. The next group of models are those where every situation is dependent on the two previous ones, and so on. This presupposes that we work with unit periods of a finite length, throughout which the economic magnitudes are assumed to be constant.

As an example we can choose the following simple model of a closed economy, where only two variables are of importance, *viz.* real income, y, and real expenditure (for consumption and net investment), u. We find that the income of the unit period of time t is equal to the expenditure; that is

(1) $$y_t = u_t,$$

while the expenditure u_t is directly dependent on the income of the proceding period, y_{t-1}:

(2) $$u_t = ay_{t-1} + c,$$

By analogy with the Keynesian concepts, we could now call a the marginal "propensity to spend". It is easy to express the position of equilibrium and the dynamic qualities of this model graphically (Fig. 1). If, for a moment, we disregard differences in time, we can measure y along the horizontal, u along the vertical axis and then draw both functions as straight lines. The straight line OR, which forms an angle of 45° with the axes, shows "the income-formation function" (1), while the straight line ST forms "the expenditure function" (2). As regards the latter function we can distinguish between several cases, according to whether $a \gtrless 1$ (with the limiting case $a = 1$) and according to whether the point of intersection S has got positive or negative co-ordinates (with the limiting case when the co-ordinates of S are 0,0). (In Fig. 1 only the two cases with positive co-ordinates are shown.) There exists only one point of equilibrium *viz.* the point of intersection S. As soon as another situation occurs, for instance if $y = y_1$, a movement will set in. The value of u_2 is obtained by substituting y_1 in equation (2), whereafter the value found for u_2 is used to obtain y_2 from equation (1); we then get u_3 from (2) by insertion of y_2 and so on.

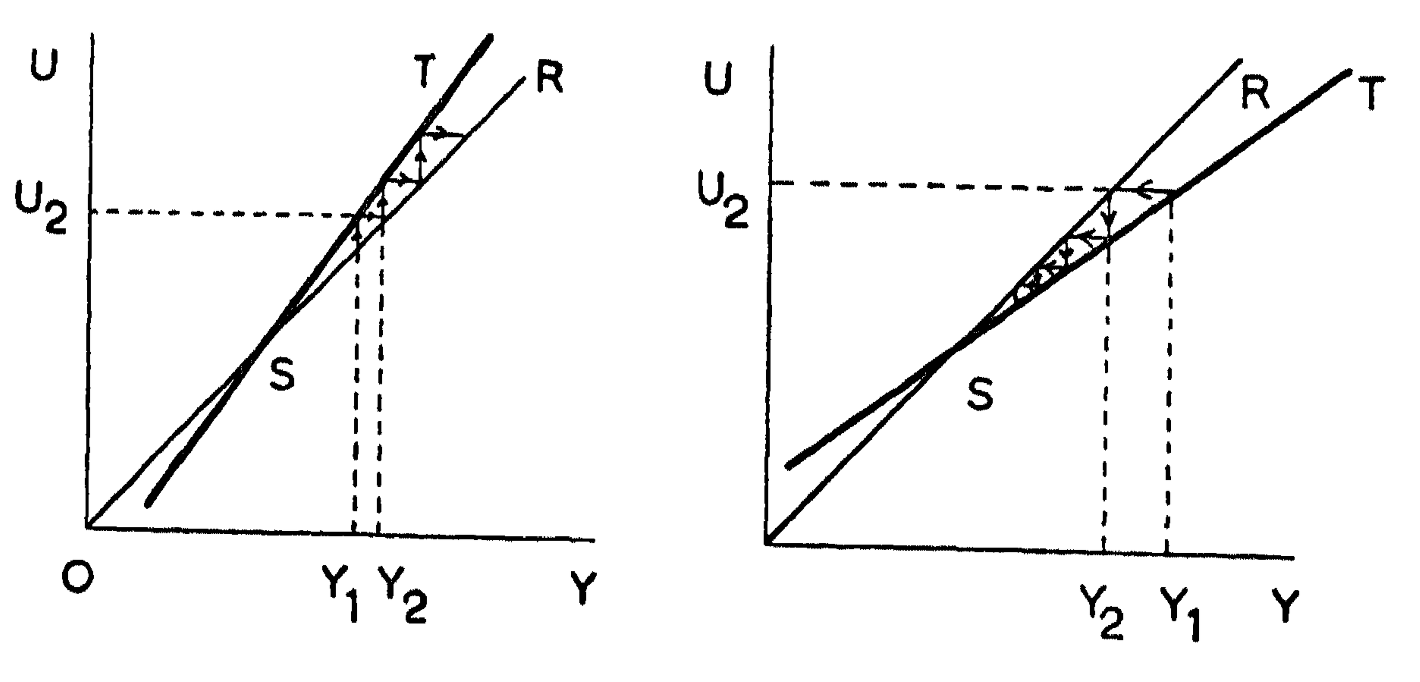

Fig. 1

If $a > 1$ the system will work itself away from the initial position; if, however, $a < 1$ it will gradually approach the point S. This can be easily shown algebraically, using the transformation:

$$y' = y - y_s \qquad \text{and} \qquad u' = u - u_s$$

when y_s and u_s are the co-ordinates of the point S. In this way the equations (1) and (2) are changed into

(1′) $$y'_t = u'_t$$
and

(2′) $$u'_t = ay'_{t-1}$$
so that

(3′) $$y'_t = ay'_{t-1}.$$

This equation, in which only one of the two variables occurs, we will term *the final equation*. It is, as we shall see, of decisive importance for determining how the system will react.

If we let y'_0 denote the initial situation then:

(4) $$y'_1 = ay'_0\,; \quad y'_2 = a^2 y'_0 \,\ldots.. \, y'_t = a^t y'_0.$$

We thus see that an approximation to the equilibrium takes place for $a < 1$; the equilibrium is in this case *a stable* one. On the other hand a continued movement away from the initial position will take place if $a > 1$ (*unstable* equilibrium). In the intermediate case, when $a = 1$, we must distinguish between $c = 0$ and $c \neq 0$. For $c = 0$ the two functions (1) and (2) are identical; there are infinitely many points of intersection and each one of these constitutes a position of equilibrium. This is an indifferent equilibrium.

The formulae derived above have economic meaning only if they lead to positive values of the original co-ordinates (*i.e.* the co-ordinates before the transformation occurred). We shall not enter here into a discussion regarding the complications we should have to introduce if negative values appeared.

Furthermore, we must stress that the position of equilibrium S is characterized by the equivalence of income and expenditure, *i.e.* that there is neither positive or negative hoarding. This equilibrium we shall call *the Keynes equilibrium*.

6. The number of possibilities is increased considerably when we pass on to the second group of equations, *i.e.* when we assume that every situation is dependent on the *two* preceding situations. The final equation then assumes the form:

(5) $$y'_t = ay'_{t-1} - by'_{t-2}.$$

Several different economic interpretations of this equation are possible. The best one seems to us to be the following: expenditure is dependent on income and on the rate of increase of income, but with a certain time lag, that is

$$y'_t = a'y'_{t-1} + b(y'_{t-1} - y'_{t-2}).$$

This expression is identical with (5) if we put:

$$a = a' + b.$$

A dependence on the increase of income can occur not only directly, but also indirectly through the fact that expenditure is directly dependent on price increases (gains from speculation), while prices move parallel to income.[3]

It is easy to show that all sorts of movements now can take place. The solution of this difference equation is reached by putting:

(6) $$y'_t = Cx^t,$$

in which way (S) is changed into:

(7) $$Cx^t - a\,Cx^{t-1} + b\,Cx^{t-2} = 0.$$

We can thus choose the value of C arbitrarily; y' will be a solution only if x satisfies the "characteristic equation"

(8) $$x^2 - ax + b = 0.$$

This equation has two roots x_1 and x_2:

(9) $$x_1 = \frac{a}{2} + \sqrt{\frac{a^2}{4} - b} \quad \text{and} \quad x_2 = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}$$

which satisfy the original equation (5), but also every sum

(10) $$y'_t = C_1 x_1^t + C_2 x_2^t.$$

The reader can convince himself of this by carrying out the com-

[3] See also my analysis: *Business Cycles in the U.S.A.* 1919–1932, League of Nations, Geneva 1939.

putations. Furthermore, it can be proved that an expression with two arbitrary constants must be the most general solution of equation (5). For in this way we can freely choose two values of y' (*i.e.* by appropriate choice of the constants). A free choice of more y'-values is excluded, however. From the original equation we can immediately see that on the basis of any two values of y', for instance y'_0 and y'_1, all further values of y' are determined. For that reason (10) is the most general solution of (5).

Those types of movements that can be produced from the latter equation are dependent on the numerical values of x_1 and x_2, which are either both real numbers or both complex numbers; in the latter case they are conjugate complex numbers.

This occurs if

$$b > \frac{a^2}{4}.$$

In order that we can then obtain real y'-values, C_1 and C_2 must also have conjugate complex values; then we can write y' in the form:

$$y'_t = C\sqrt{b^t}\sin\frac{2\pi(t-t_0)}{T}. \tag{11}$$

C and t_0 are now the arbitrary constants. T is dependent on the constants of the given equation in the following way:

$$T = \frac{2\pi}{\operatorname{arc\,tg}\dfrac{\sqrt{\dfrac{a^2}{4}-b}}{\dfrac{a}{2}}}, \tag{12}$$

T is the period of the sinus function contained in (11). It can assume any value between 0 and ∞, depending on the pair of values a and b. The movements represented by (11) are strictly periodical fluctuations for $b = 1$, with a period T. For $b < 1$ they are damped periodical and for $b > 1$ they are negatively damped or "explosive".

If $b < a^2/4$, both x_1 and x_2 are real. It is now of interest whether these roots of the characteristic equation are negative, positive < 1 or > 1. A negative value corresponds to a zigzag movement, which can be either undamped, damped, or explosive, depending on whether

$x_1 \lesseqgtr 1$. However, negative values of either of the two roots never correspond to positive values of a and b. If one of the roots were negative, b would have to be negative, and if they were both negative, a would have to be negative. A positive value smaller than one corresponds to a unilateral movement towards an equilibrium, and a value bigger than one corresponds to a movement away from the equilibrium. In general, the values of x_1 and x_2 can fall in any one of these three intervals; thus any combination of the three types of movement mentioned above can occur, *e.g.* a damped zigzag movement and a movement away from the equilibrium, if

$$-1 < x_1 < 0 \qquad \text{and} \qquad x_2 > 1.$$

Here we must add that a combination of two movements of the latter type cannot have a single turning-point. In our case, however, (when a and b are positive) the zigzag components cannot occur, as we said before.

Finally we shall refer also to the case when

$$C = \frac{a^2}{4}.$$

The two roots x_1 and x_2 are equal in this case, and the movement of y_t' is described by

$$y_t' = C_1 x_1^t + C_2 t x_1^t. \tag{13}$$

By the abundance of possible movements we can again classify the equilibria corresponding with them. Where only damped or equilibrium-directed components of movement occur the equilibrium is evidently stable; but if any one of the components shows a unilateral movement away from the equilibrium, the equilibrium is unstable, for eventually this component will lead to a movement away from equilibrium. The same holds true in the presence of an explosive periodical component. But here we get a new type with undamped periodical movement or oscillation. In this case neither a constant removal from nor any approach to the equilibrium occurs. We call this form a *semi-stable* equilibrium.

7. Till now our reflections regarded only cases with one single

equilibrium position. By introducing the possibility of several positions of equilibrium, we reach important new insight.[4]

This complication is indispensable for the discussion of our subject since our problem is whether the business cycle is a movement between two points of equilibrium, or movement about a single point of equilibrium as mentioned at the beginning. The possibility of several equilibrium positions is excluded as long as we work only with linear relations between the variables. If the straight lines in Fig. 1 had been curved, more points of intersection would have been possible, but

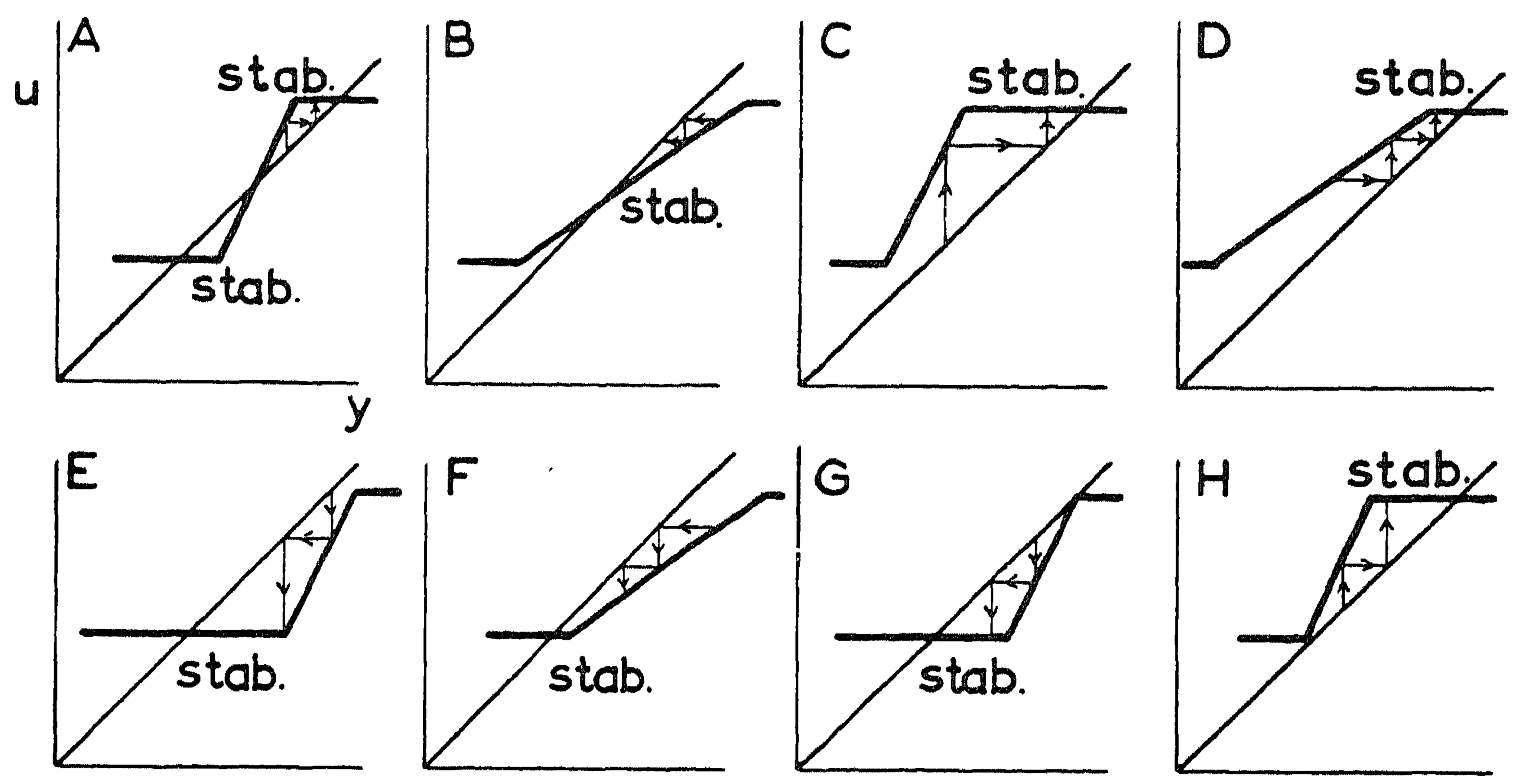

Fig. 2

not al all certain. But the main thing which interests us is whether such a plurality of equilibria actually occurs. Certain hints of such a supposition we find in GOUDRIAAN.[5]

His thinking might be summarized in the following way: real expenditure has both a certain maximum value and a certain minimum

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[4] J. G. KOOPMANS has already indicated this possibility in his treatise: "De mogelijkheid van meervoudig economisch evenwicht", *De Economist* par. 1, (1932), p. 679, 766 and par. 41. A. WALD has contributed essentially to the solution of this problem in: "Über einige Gleichungssysteme der mathematischen Ökonomie", *Zeitschrift für Nationalök.* VII (1936), p. 637.

[5] *Op. cit.*

value. The maximum value is determined by the capacity limit of production, the minimum value by the subsistence level of consumption and the necessary capital replacement. Even if real income, for some reason or other, should exceed this maximum or fall under this minimum, real expenditure would still remain within these limits. If income does not cover the subsistence level, borrowing will be the result. In consequence the straight expenditure line ST will, in principle, change into a broken line with two horizontal parts (Fig. 2). We shall now investigate the consequences of this change of form for the movements of the system. This can be done by simple mathematical operations if we use the discontinuous form of the curve. In reality the curve is likely to be continuous, because not all firms reach their capacity limit, and not all families reach their minimum level of subsistence at the very same moment. But if we had to take this into account the mathematical treatment would become much more complicated, without changing the basic result.

8. Let us first see how the first model, shown in Fig. 1, now behaves. All possible cases are shown in Fig. 2. The direction of the movement is again denoted by arrows. We see that in case A three positions of equilibrium appear, two of which, the extreme ones, are stable, whereas the one in the middle is unstable. We shall call the stable positions the Goudriaan-equilibria. This case corresponds to the case of the unstable equilibrium in Fig. 1. The introduction of a maximum and a minimum has thus had important consequences.

Just as before we find only one position of equilibrium in case B, and that a stable one. In both these cases the Keynes-point lies between the Goudriaan-points. However it is also conceivable that it lies outside both of these. In that case the examples C–F will appear. Here we still get only one position of equilibrium, a stable one. In addition there are the two limiting cases G and H, where the Keynes-point coincides with one of the two Goudriaan-points. These are especially interesting because of the nature of these points of coincidence. For deviations to one side they are stable positions of equilibrium, for deviations to the other side unstable, and in such a way that the positions are unstable for disturbances in the direction of the other equilibrium, stable for disturbances in the opposite direction. This sort of equilibria, whose importance for business-cycle explanation has already been stressed by KALDOR,[6] we shall name *mixed* equilibria.

In spite of the enrichment of the picture of movements, which is obtained by the development of the theory of the Goudriaan-points, it is not possible to use the models of the first class for the explanation of business cycles, because in these theories every situation is dependent only on the one preceding situation. Thus is illustrated by Fig. 3, where the path of the several possible movements is shown. The letters correspond to those of Fig. 2. Thus not all cases of Fig. 2 have been taken into consideration; the reader might add the rest for himself. The dotted lines denote the position of the Goudriaan-points and the Keynes-point. In case A the Goudriaan-points are stable equilibria. The endogenous movements are thus steadily directed towards this equilibrium. Once the system is in equilibrium it can leave it again only if disturbed by outside forces, and if the disturbance is a small one the system will always return to the same equilibrium. A movement from one Goudriaan-point to the other will take place only if there occurs a disturbance strong enough to carry the system across the Keynes-point. We must think of a business cycle as a movement from the upper equilibrium to the lower one, and back again; we have

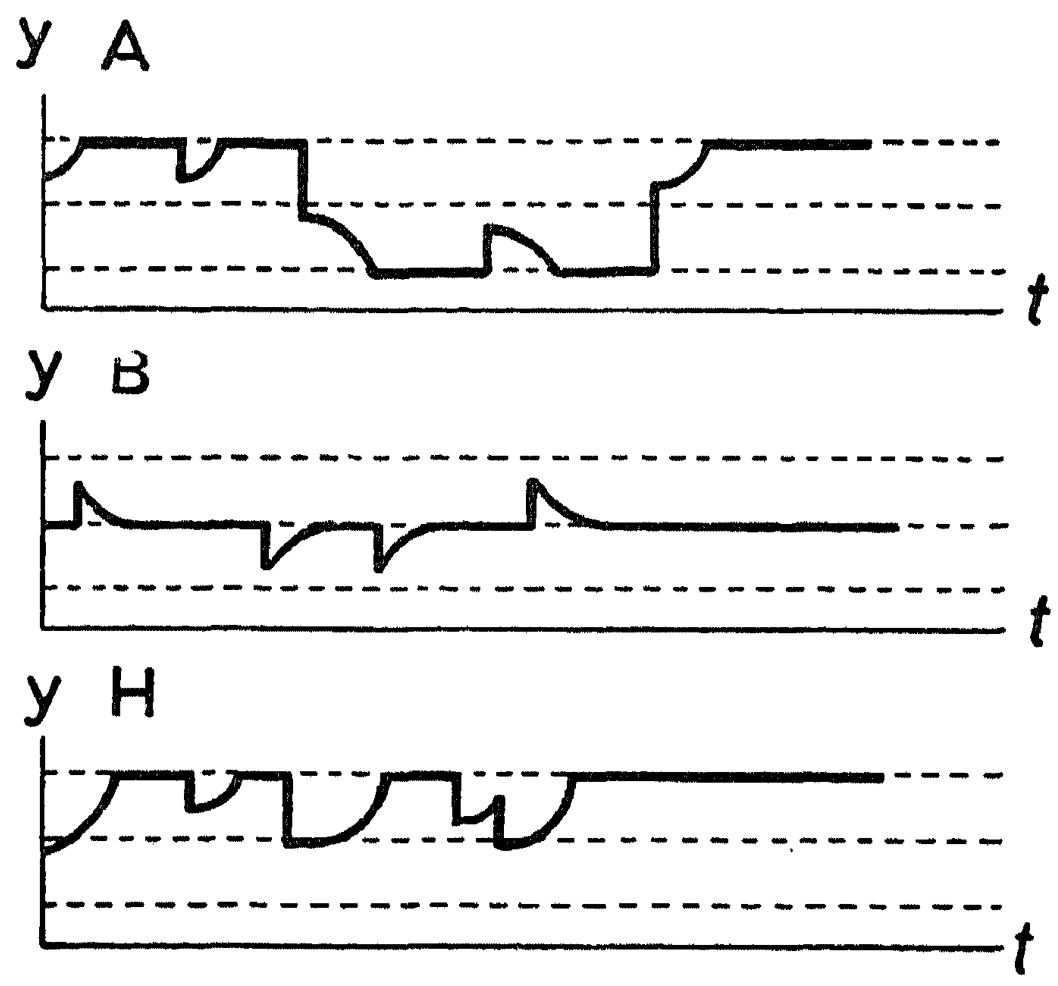


Fig. 3

[6] N. KALDOR, "A Model of the Trade Cycle", *Economic Journal*, L (1940), p. 79.

just seen that this only can happen if a sudden fall or rise occurs carrying the system over the Keynes-point. This sort of a movement is, however, unrealistic.

In case B matters are quite different, but also unrealistic. Only movements towards the Keynes-point can occur. After any disturbance from outside, the movement whether it is up or down, proceeds toward the dotted line in the middle. Thus this sort of model can not be used either as an explanation of business-cycle movements.

In the same way the reader can convince himself that none of the equations of the first group, shown in Fig. 2, contribute to the explanation of business cycles.

9. Consequently we must direct our attention to *the models of the second group, i.e.* those in which we assume that every situation is dependent on two preceding situations. We assume the existence of a Goudriaan-maximum and a Goudriaan-minimum and further it is assumed that *the maximum lies above and the minimum below the Keynes-point.* This means that income y_t is determined by equation (5):

$$y_t = ay_{t-1} - by_{t-2}, \tag{14}$$

as long as we do not get values above y^0 or below $y^{0'}$, where y^0 is the Goudriaan maximum and $y^{0'}$ the Goudriaan minimum. If we obtain such extreme values they must be replaced by y^0 and $y^{0'}$ instead of

$$ay_{t-1} - by_{t-2}.$$

In equation (14) we have removed the prime after y_{t-1}. That means that we measure income from the Keynes-equilibrium, or in other words, that the Keynes-equilibrium lies at the income level of 0. This is just a matter of measurement. For the moment we put $y^0 >$ and $y^{0'} < 0$.

We shall now systematically study the movements of our model, in relation to the values of the two coefficients a and b.

As mentioned above it is necessary to solve the characteristic equation (8); the roots are given by (9). These roots are apparently complex if, and only if, $b > a^2/4$ or $a < 2\sqrt{b}$; if we illustrate all possible pairs of values a, b graphically in the plane of Fig. 4, those pairs of values when $a < 2\sqrt{b}$ will be represented by all the points below the parabola OPN. Thus above this parabola we find only the a-b-values that lead to real roots. According to our previous assumption, for positive values of a and b the roots are also positive. It is also of importance whether the roots are $\lesseqgtr 1$; let us investigate in which sectors of the a-b-plane this is the case.

$$x \gtrless 1, \text{ when } \sqrt{\frac{a^2}{4} - b} \gtrless 1 - \frac{a}{2}. \tag{15}$$

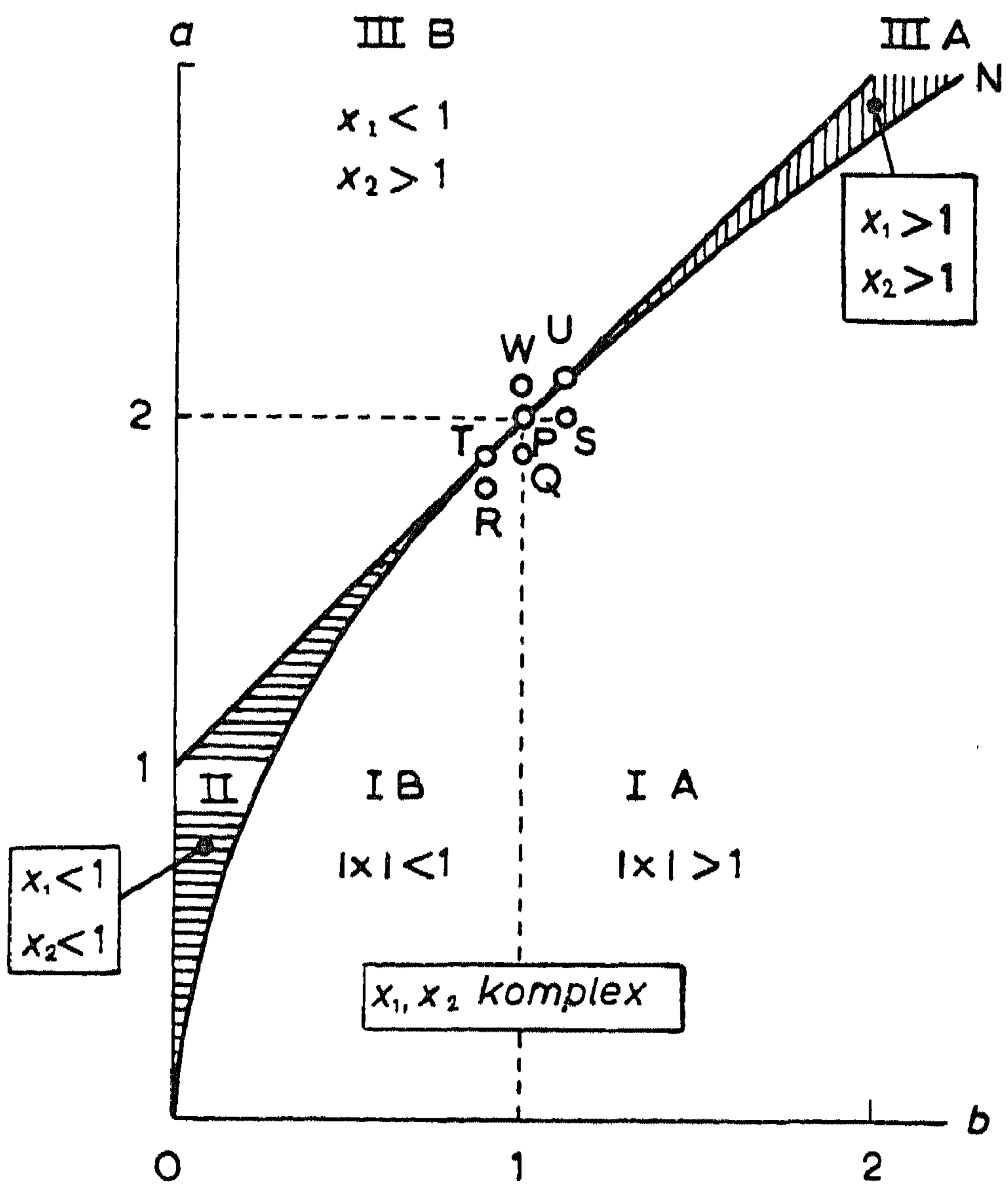


Fig. 4

Here we must distinguish between two cases, *viz.* $a > 2$ and $a < 2$. For $a > 2$ we have $1 - \frac{a}{2} < 0$ and thus always $\sqrt{\frac{a^2}{4} - b} > 1 - \frac{a}{2}$ or $x_1 > 1$. For $a < 2$, however, we have $x_1 \gtrless 1$ depending on whether $\frac{a^2}{4} - b \gtrless 1 - a + \frac{a^2}{4}$ or $a \lesseqgtr b + 1$.

Thus all points above the line $a = b + 1$ represent, for a-values smaller than 2, pairs of values (a, b) with $x_1 > 1$; and all points below this line represent pairs of values with $x_1 < 1$; the latter is true, however, only as long as the points are lying above the parabola. The distribution of the points over the plane is shown by Fig. 4. In the same way we find the distribution of the x_2-values; all results are given in Fig. 4. In this we see that in sector II both roots are real and < 1 and thus both components of movements are directed towards the Keynes-equilibrium: this equilibrium is here a *stable* one. But in sector III at least one of the roots is > 1 (in sector IIIA even both), and consequently the movements are directed away from this equilibrium: the Keynes-equilibrium is *unstable*.

10. The next question is: are the Goudriaan-points also equilibrium positions?

In order to answer this question we must investigate whether the system, once having reached such a point, will stay there, *e.g.*, if first we have $y_{t-1} = y_{t-2} = y^0$ will we also get $y_t = y^0$.

This requires that either

$$ay^0 - by^0 = y^0 \tag{16}$$

or

$$ay^0 - by^0 > y^0 \quad \text{if} \quad y^0 > 0 \tag{17}$$

or finally

$$ay^0 - by^0 < y^0 \quad \text{if} \quad y^0 < 0. \tag{18}$$

These conditions are equivalent to

$$a - b \geqq 1 \quad \text{or} \quad a \geqq b + 1.$$

The equality holds good for the straight line; the inequality for all points above this line. *We thus find that the Goudriaan-points are equilibrium positions if the Keynes-equilibrium is unstable.* But this conclusion only holds good in the cases where the Keynes-point is lying between the two Goudriaan-points.

11. Eventually we shall investigate different kinds of the Goudriaan-equilibrium. Here we must first answer the following question:

Which movement sets in if the system makes a small departure δ from the point in question? The initial values of this movement are $y_0 = y^0$, $y_1 = y^0 - \delta$; which means that the constants A and B must satisfy the condition

$$y^0 = Ax_1^0 + Bx_2^0 = A + B \tag{19}$$

and

$$(20) \qquad y^{0} - \delta = Ax_1^{1} + Bx_2^{1} = Ax_1 + Bx_2.$$

In principle the nature of the movement is the same for the Keynes- and the Goudriaan-equilibrium and only dependent on x_1 and x_2. In the cases when the Goudriaan-point represents a position of equilibrium, which thus means if $a \geqq b + 1$, we always get $x_1 > 1, x_2 < 1$ (except for the case $a = b + 1$ to which we shall return later on) and as a rule the components corresponding to x_1 make the equilibria unstable. But the matter is actually changed for the Goudriaan-points, because certain movements are "prohibited", *viz.* movements upwards at the upper point and downwards movements at the lower point. If, for certain initial values of y_1 the components directed away from the equilibrium take the prohibited direction, they will fall away and they will—*for these initial values*—change the equilibrium into a stable one.

This state of affairs is characterized by A's assuming positive values in the foregoing formulae. But if A has a negative value the equilibrium remains stable. From (19) and (20) we find that:

$$(21) \qquad A = \frac{y^0 (1 - x_2) - \delta}{x_1 - x_2}.$$

In order to get $A \gtrless 0$, we must have either $y^0(1 - x_2) - \delta \gtrless 0$

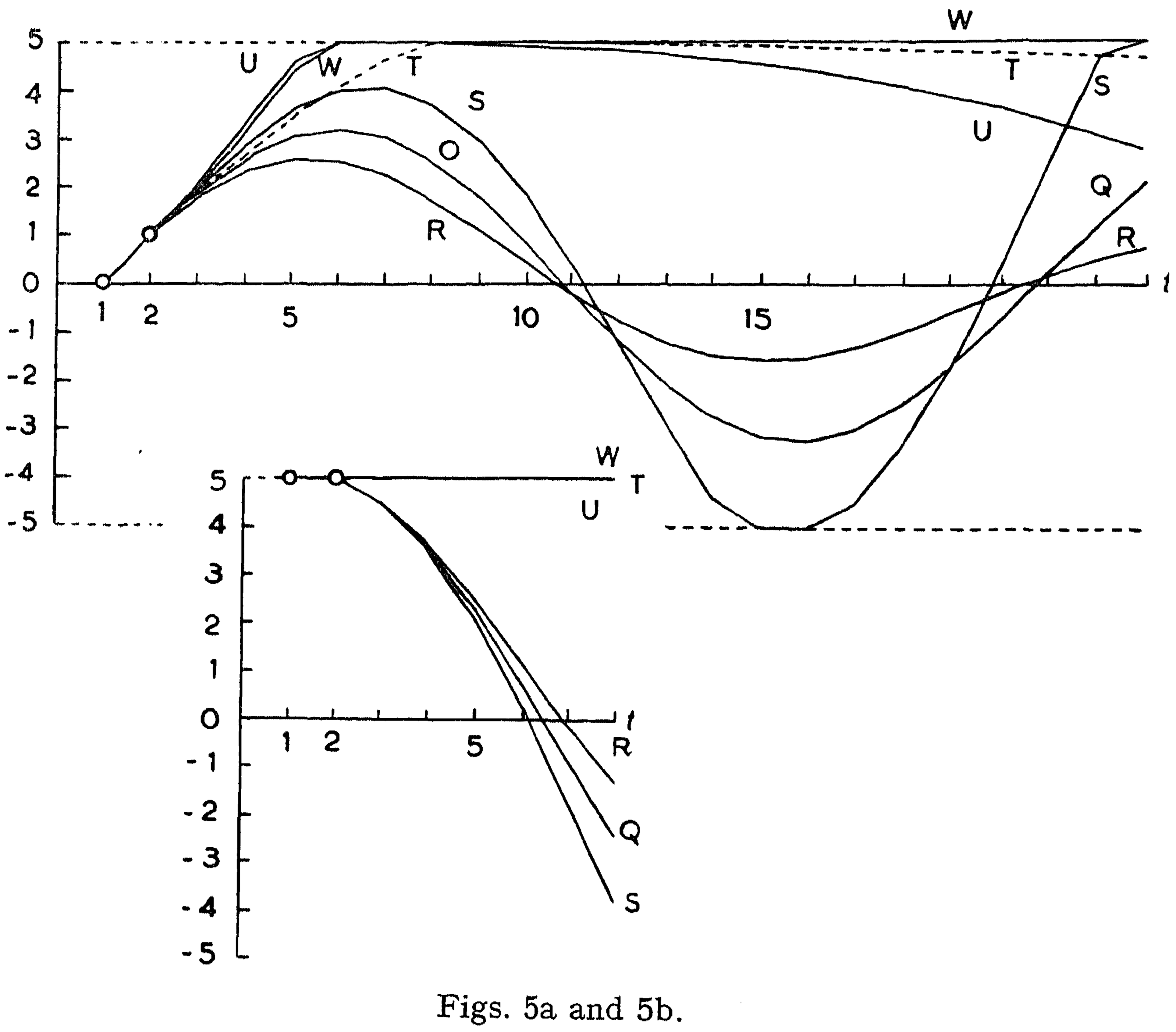
or

$$(22) \qquad \frac{\delta}{y_0} \lesseqgtr 1 - x_2.$$

If thus the system gets a small deflection from the Goudriaan-equilibrium, a movement starts towards the equilibrium position and the original equilibrium is a stable one; for greater departures it is unstable, however. The critical value of this departure lies at $(1 - x_2)y^0$, and is thus proportional to the distance from the Goudriaan-point to the Keynes-point. The factor of proportionality $1 - x_2$ can assume all possible values between 1 and 0, and it assumes the smallest of these if $x_2 \sim 1$, which is in the neighbourhood of the line $a = b + 1$. Such an equilibrium—the nature of which is dependent on the size of the deflections—we shall call a *meta-stable* equilibrium, by analogy with the physical equilibrium of the oversaturated solution: this situation survives small departures but in case of greater departures a sudden crystallization takes place.

Further it is seen from (22) that no critical value exists for $x_2 > 1$, that means in sector IIIA; thus all movements are "unstable" in that sector.

12. Our results may be illustrated by the computation of a few examples, the graphical representation of which is to be seen in the Figs. 5 and 6. Fig. 5 refers to the movements of the different models the *a*-*b*-pairs of values of which are denoted by the letters Q, R, S, T, U, and W in Fig. 4.



Figs. 5a and 5b.

The initial values are 0 and 1 for the curves of the upper half of Fig. 5, and 5, 5 for the curves of the lower half. For both halves the numerical value of 5 has been chosen for y^0 and $y_1^{0\prime}$ thus $y^0 = 5$ and $y^{0\prime} = -5$.

The a-b-values are :

Model	Q	R	S	T	U	W
a	1.9	1.8	2.0	1.899	2.098	2.04
b	1.0	0.9	1.1	0.9	1.1	1.0

The choice has been made in such a way that Q, R, and S lie in the first section, and also in such a way that Q shows an undamped, R a damped, and S an explosive movement; further T lies in section II, U in IIIA and W in IIIB. Accordingly, the curve W, for which the Goudriaan-point is a position of stable equilibrium, will lie in this level (since we assume that no shocks from outside occur). But the curves T and U, for which the point is not a position of stable equilibrium, will slowly return. Without the existence of the Goudriaan-maximum these curves would by degrees have moved away from the zero-point (the Keynes-point). But the curve S would have shown a periodical movement without the presence of the Goudriaan-points, although with a steadily growing amplitude. This latter, of course remains ≤ 5.

Fig. 6 shows the behaviour of model W after shocks of different sizes. For initial departures of the sizes 0.5 and 0.8, the system returns to its starting point. This is not the case for departures of the sizes 1.0 and 1.5.

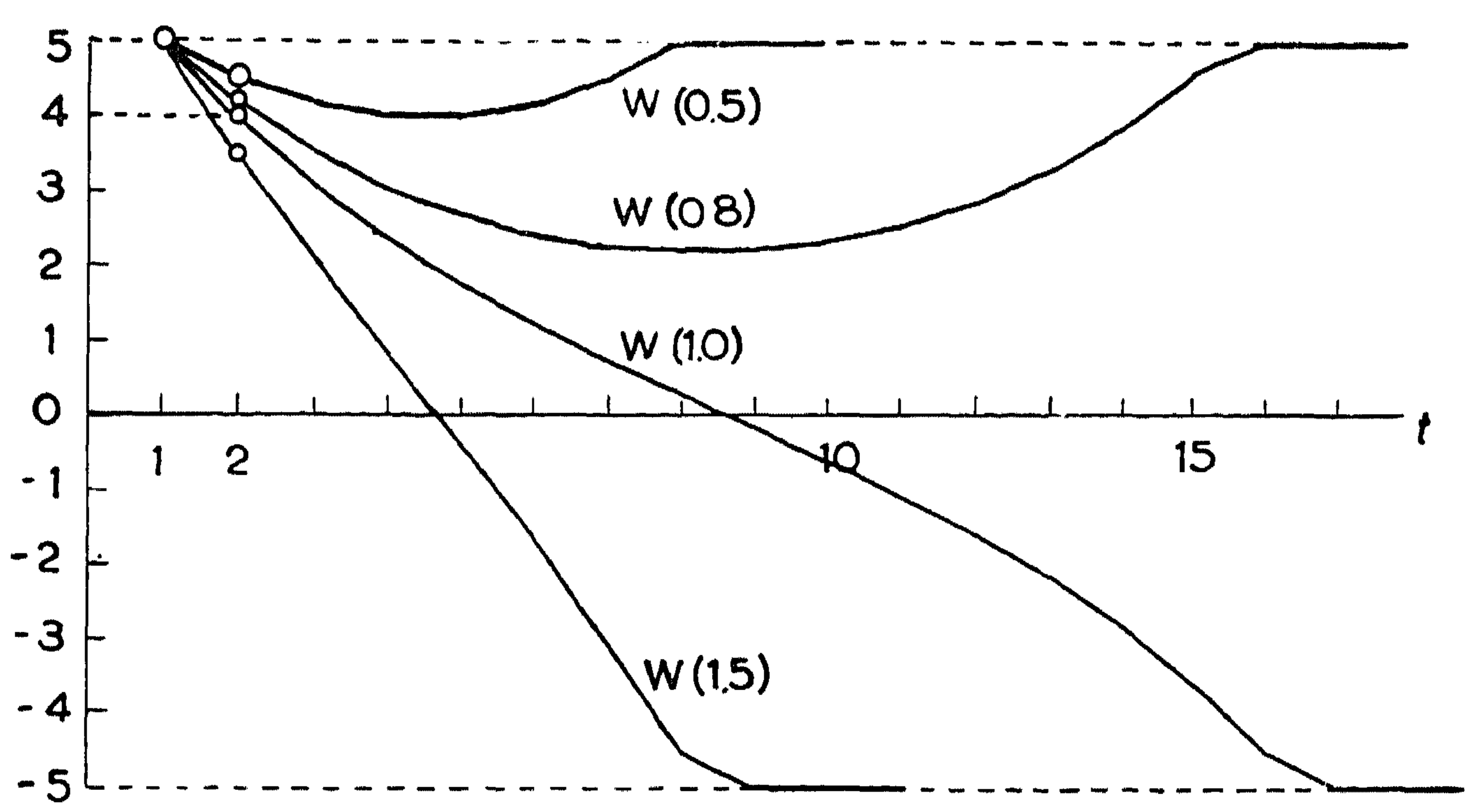


Fig. 6.

13. We shall now look at some *special cases*, the first of which is the case when $a = b + 1$. According to our results in section 10 any position is now a position of equilibrium; any position is stable as long as new shocks do not occur. The characteristic equation now runs: $x^2 - (b + 1)\, x + b = 0$ and we have:

$$\begin{array}{llll} x_1 = b & \text{and} \quad x_2 = 1 & \text{for} & b > 1 \\ x_1 = 1 & \text{and} \quad x_2 = b & \text{for} & b < 1 \end{array}$$

for $b > 1$ we get $\delta/y^0 = A\ (1 - b) < 0$.
Thus no critical value exists here, *i.e.* all equilibria are *unstable.*

Whereas for $b < 1$ the general form of the movements is given by

$$y = A + Bb^t \tag{23}$$

and any equilibrium is thus stable.

Next we shall study the still more particular case $a = 2$, $b = 1$. Here the two roots coincide and both are equal to one. Using equation (13) we now have $y = (A + Bt)x_1^t$, which in this case may be simplified to

$$y_t = A + Bt. \tag{24}$$

This result is quite evident, for we can now write the difference equation:

$$y_t = y_{t-1} + (y_{t-1} - y_{t-2}), \tag{25}$$

that is the increase $y_t - y_{t-1}$ is equal to the preceding increase $y_{t-1} - y_{t-2}$; it thus describes a straight line. This movement is of course broken only if a Goudriaan-point is reached. The movement is then changed into a horizontal one, until a new departure from the Goudriaan-point occurs. Then a straight-line movement to the other side continues until the next shock from outside occurs.

14. Till now we have assumed that the Keynes-point was lying between the two Goudriaan-points, corresponding to the sections A and B in Fig. 2. We shall now assume that *the Keynes-point is lying outside both the Goudriaan-points*. This is equivalent to the supposition that the signs of y^0 and $y^{0\prime}$ are equal. The condition that a point with a positive $y^{0\prime}$ is a position of equilibrium is now:

$$ay^{0\prime} - by^{0\prime} \leqq y^{0\prime} \quad \text{or} \quad a \leqq b + 1. \tag{26}$$

We get the same condition that a point with a negative y^0 can be a position of equilibrium. This condition is thus not fulfilled if the condition of equilibrium is satisfied for the other Goudriaan-point, and *vice versa.* We thus find that in these models without any Keynes-point *only one of the two Goudriaan-points can be a position of equilibrium.*

Let us now see what sort of equilibrium this is. As we have seen earlier, the condition (26) is satisfied in the sectors I, II, and IIIA. These require a special treatment. In sector I any movement is periodic. If thus the system gets a shock from outside away from the position y^0, it will theoretically start a movement, which automatically returns to the lower Goudriaan-point, if the upper Goudriaan-point is not reached. If the upper point is reached the movement will also return, because the upper point is not a point of equilibrium. Since all movements setting out from the lower point will automatically return, it is a stable equilibrium.

For sector III somewhat the same holds true; here the movement returns immediately after the shock has been given, since we find here only movements directed towards the Keynes-equilibrium. Here the upper Goudriaan-point can only be reached if the shock carries the system directly to it.

But in sector IIIA we can also get movements directed away from the equilibrium. Here again we must analyse the movements more closely. If the system gets a shock δ away from the position $y_t = y^{0\prime}$ the initial values are $y_0 = y^{0\prime}$ and $y_1 = y^{0\prime} + \delta$ and we have

$$y^{0\prime} = A + B$$

$$y^{0\prime} + \delta = Ax_1 + Bx_2$$

which yields us:

$$A = \frac{-y^{0\prime}(x_2 - 1) + \delta}{x_1 - x_2} \quad \text{and} \quad B = \frac{y^{0\prime}(x_1 - 1) - \delta}{x_1 - x_2} \tag{27}$$

Here $x_1 - x_2$, $x_1 - 1$, and $x_2 - 1$ all are positive values, as well as $y^{0\prime}$ and δ. We must now distinguish between the three following cases:

a) $\delta < y^{0\prime}(x_2 - 1) < y^{0\prime}(x_1 - 1)$, which involves $A < 0$, $B > 0$:
b) $y^{0\prime}(x_2 - 1) < \delta < y^{0\prime}(x_1 - 1)$, which involves $A > 0$, $B > 0$ and
c) $y^{0\prime}(x_2 - 1) < y^{0\prime}(x_1 - 1) < \delta$ which involves $A > 0$, $B < 0$.

In case b) the movement is directed away from the equilibrium. A movement with a turning point can occur only if the negative component dominates in the long run; this is possible only for the component Ax_1^t, because $x_1 > x_2$ and thus in the long run we always get $|A| x_1^t > Bx_2^t$; a movement of this sort is thus possible only in case a). This means that in reality we have found another critical value of δ, *viz.*

$$\delta = y^{0\prime} (x_2 - 1). \tag{28}$$

If the shock assumes values smaller than this, it will have as a consequence a movement automatically returning towards the equilibrium; values bigger than (28) result in a steady movement away from the equilibrium.

In the limiting case when $y^{0\prime} = 0$, *i.e.* where the lower Goudriaan-point coincides with the Keynes-point, the abovementioned results are changed as follows: the lower Goudriaan-point is now a general position of equilibrium, in sector IIIB, however, it is unstable. In the sectors IA and II the equilibrium is always stable, and in sector IB the movements are certainly returning, but they are not damped. In sector IIIA, at length, the formula (27) is changed in such a way (y^0 falls away) that of the three possibilities *a*, *b*, and *c* only the last one is left, which means that we get an unstable equilibrium here also.

15. We are now able to take a survey of the abundance of possible movements that are derived from the theories of the second group, under the assumption of Goudriaan-points. We shall thus have to distinguish between models with:

I. KEYNES-POINT BETWEEN THE GOUDRIAAN-POINTS and
II. KEYNES-POINT OUTSIDE THE GOUDRIAAN-POINTS.

I. Here we have models with:

a) *a stable Keynes-point.* In that case the Goudriaan-points are not equilibrium positions. These systems always have a tendency to reverse the direction of the movement as soon as they have reached a Goudriaan-point. If they belong to sector II, the Goudriaan-points can be reached only by shocks that take the systems all the way to these positions, because all endogenous movements are directed towards the

Keynes-point. If the system belongs to sector I then automatic turning-points can also occur before the Goudriaan-points are reached; in sector II this can only take place because of shocks from outside which make the movement turn. In addition we have:

b) *models with an unstable Keynes-point.* Here the Goudriaan-points are meta-stable equilibria. These models thus have a tendency to remain in the Goudriaan-points. A lasting movement out from these points can occur only as a result of bigger shocks which carry the system over to the other equilibrium. There is no automatic return from the boom or the slump.

The models in I all have a similarity between boom and depression; this is lacking in the models in II. Here only one of the two positions is stable or meta-stable; the other is not an equilibrium at all. If both Goudriaan-points lie above the Keynes-point the position of equilibrium is the upper point in sector IIIB and the lower point in the other sector. In sector IIIB this position is meta-stable, stable in the sectors I and II, and again meta-stable in IIIA. If, however both Goudriaan-points are located under the Keynes-point, the upper and the lower Goudriaan-points change roles.

If we look at the historical data of business cycles, it is difficult to assume the existence of an evident asymmetry between the upper and the lower turning points. We cannot assert that there is a general tendency to remain in the upper points and no such tendency for the lower points; neither can we assert the opposite. In my opinion it is better to conclude that a significant tendency to remain in equilibrium does not appear at all. This leads us to the models in I, *i.e.* the models in which the presence of the Goudriaan-points is only of secondary importance. Here their only function is to set a limit to the amplitude of the movement in the cases in which—due to abnormally strong shocks or antidamping—there is a tendency to excessive deviation.

We think, that this opinion is also stressed by attempts to base models on statistically discovered relationships. However, it would lead us too far afield to discuss this point of view in this article. In conclusion we must add some final remarks about *the explanation of the turning-points*. Summarizing our reasoning, there are three types of turning-points, *viz.*:

1. *the perfectly endogenous:* those that occur without any shocks from outside and without the presence of a Goudriaan-point;
2. *the extended endogenous:* those that occur without any shocks from outside as a consequence of the presence of a Goudriaan-point;
3. *the exogenous:* those that are caused by shocks from outside.

Formulated mathematically the perfectly endogenous turning-points can occur if all relations are linear, and no shocks take place; the extended endogenous turning-points can be obtained if certain relations are non-linear, and the exogenous ones only if shocks occur. For laymen the exogenous turning-points are the easiest to "explain"; the shock is "the cause". After these the extended endogenous types are the simplest for laymen and non-mathematical economists; in explanation we can always refer to the economic phenomena which lead to the Goudriaan-points, for instance the existence of a limited capacity and so on. The most difficult ones to "explain" verbally are the perfectly endogenous turning-points. This is perhaps the reason why they have claimed so little attention in literary theory, in spite of the fact that they are likely to be of the utmost importance for the understanding of economic dynamics.