Forecasting with Periodic Autoregressive Time Series Models*

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Abstract

This chapter is concerned with forecasting univariate seasonal time series data using periodic autoregressive models. We show how one should account for unit roots and deterministic terms when generating out-of-sample forecasts. We illustrate the models for various quarterly UK consumption series.

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1 Introduction

There are various approaches to modeling and forecasting (seasonally unadjusted) seasonal time series, see Franses (1996c) for a recent survey. One approach builds on the work of Box and Jenkins (1970) and relies on moving average models for double differenced time series (so-called seasonal ARIMA [SARIMA] models). Another approach assumes that seasonal time series can be decomposed into trend, cycle, seasonal and irregular components, see Harvey (1989). Reduced forms of the resultant models have many similarities with the aforementioned SARIMA models. A third approach questions the aforementioned adequacy of the double differencing filter in SARIMA models and mainly addresses the issue of how many unit roots should be imposed in autoregressive models, see Hylleberg *et al.* (1990, HEGY). Finally, a fourth approach assumes the seasonal variation is best described by allowing the parameters in an autoregression to vary with the seasons, that is, the so-called periodic autoregression [PAR]. Of course, on may want to consider periodic ARMA models, but is rarely done in practice. Periodic autoregressions have been frequently used in environmental and hydrological studies, see Franses (1996b) for a summary of early references, but it was introduced into the economic literature by Osborn (1988) and Osborn and Smith (1989). The latter study focused on out-of-sample forecasting of quarterly UK consumption series. Since that study the literature on periodic models has developed substantially, and in this chapter we will highlight some of these issues in more detail. Specifically, we will address unit roots and deterministic terms and how they should be incorporated in a PAR model. There have appeared several studies on evaluating forecasts from PAR models, see Novales and Flores de Fruto (1997), Wells (1997), Herwartz (1997, 1999) and Franses and Paap (1996) and they yield mixed results. The novelty of this chapter is that we take explicit account of a proper inclusion of deterministic terms in our PAR models and that we use encompassing tests to formally evaluate forecast performance. Following the seminal study in Osborn and Smith (1989), we will also consider various UK consumption series.

In Section 2, we first discuss several preliminaries on PAR models, like representation, estimation, unit roots and deterministic terms. In Section 3 we discuss out-of-sample forecasting. In Section 4 we consider PAR models for forecasting several quarterly UK consumption series. In Section 5, we conclude this chapter with some remarks.

2 Preliminaries

In this section we give a brief overview of periodic autoregressions. The discussion draws heavily from material covered in detail in Franses (1996a,b), Boswijk and Franses (1996), Boswijk *et al.* (1997), Franses and Paap (1994, 1996) and Paap and Franses (1999). In Section 2.1 we consider representation and estimation. Section 2.2 deals with unit roots and periodic integration. To save notation we consider in these two sections models without intercept and trend. As these are very relevant in practice, we dedicate Section 2.3 to this issue.

2.1 Representation and Parameter Estimation

Consider a univariate time series y_t which is observed quarterly for N years, that is $t = 1, 2, \ldots, n = N/4$. A periodic autoregressive model of order p [PAR(p)] for y_t can be written as

$$y_t = \phi_{1,s} y_{t-1} + \cdots + \phi_{p,s} y_{t-p} + \varepsilon_t, \tag{1}$$

or $\tilde{\phi}_{p,s}(L) y_t = \varepsilon_t$, where L is the usual lag operator, and where $\phi_{1,s}$ through $\phi_{p,s}$ are autoregressive parameters which may take different values across the seasons $s = 1, 2, 3, 4$. The disturbance ε_t is assumed to be a standard white noise process with constant variance σ^2. Of course, this assumption may be relaxed by allowing for different variances σ_s^2 in each season.

The periodic process described by model (1) is nonstationary as the variance and autocovariances are time-varying within the year. For some purposes a more convenient representation of a PAR(p) process is given by rewriting it in a time-invariant form. As the PAR(p) model considers different AR(p) models for different seasons, it seems natural to rewrite it as a model for annual observations, see also Gladyshev (1961), Tiao and Grupe (1980), Osborn (1991) and Lütkepohl (1993). In general, the PAR(p) process in (1) can be rewritten as an AR(P) model for the 4-dimensional vector process $Y_T = (Y_{1,T}, Y_{2,T}, Y_{3,T}, Y_{4,T})'$, $T = 1, 2, \ldots, N$, where $Y_{s,T}$ is the observation in season s in year

T, $s = 1, 2, 3, 4$, and where $P = 1 + [(p-1)/4]$, where $[\cdot]$ denotes the integer function. The corresponding vector autoregressive [VAR] model is given by

$$\Phi_0 Y_T = \Phi_1 Y_{T-1} + \cdots + \Phi_P Y_{T-P} + \varepsilon_T, \tag{2}$$

where $\varepsilon_T = (\varepsilon_{1,T}, \varepsilon_{2,T}, \varepsilon_{3,T}, \varepsilon_{4,T})'$, with $\varepsilon_{s,T}$ is the value of the error process ε_t in season s in year T. The Φ_0, Φ_1 to Φ_P are (4×4) parameter matrices with elements

$$\begin{aligned} \Phi_0(i,j) &= 1 && i = j \\ &= 0 && j > i \\ &= -\phi_{i-j,i} && j < i \\ \Phi_k(i,j) &= \phi_{i+4k-j,i}, \end{aligned} \tag{3}$$

for $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$ and $k = 1, 2, \ldots, P$. The lower triangular of Φ_0 shows that (2) is in fact a recursive set of equations.

As an example, consider the PAR(2) model

$$y_t = \phi_{1,s} y_{t-1} + \phi_{2,s} y_{t-2} + \varepsilon_t, \tag{4}$$

which can be written as

$$\Phi_0 Y_T = \Phi_1 Y_{T-1} + \varepsilon_T, \tag{5}$$

with

$$\Phi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\phi_{1,2} & 1 & 0 & 0 \\ -\phi_{2,3} & -\phi_{1,3} & 1 & 0 \\ 0 & -\phi_{2,4} & -\phi_{1,4} & 1 \end{pmatrix} \quad \text{and} \quad \Phi_1 = \begin{pmatrix} 0 & 0 & \phi_{2,1} & \phi_{1,1} \\ 0 & 0 & 0 & \phi_{2,2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

In order to avoid confusion with multivariate time series models, one often refers to models like (5) as the vector of quarters [VQ] representation. Notice from (5) and (6) that one can also write a nonperiodic AR model in a VQ representation.

There are two useful versions of (2) for the analysis of unit roots and for forecasting. The first is given by simply pre-multiplying (2) with Φ_0^{-1}, that is

$$Y_T = \Phi_0^{-1}\Phi_1 Y_{T-1} + \cdots + \Phi_0^{-1}\Phi_P Y_{T-P} + \Phi_0^{-1}\varepsilon_T. \tag{7}$$

which amounts to a genuine VAR(P) for Y_T. When $\varepsilon_T \sim \mathrm{N}(\mathbf{0}, \sigma^2 I_4)$, it follows that $\Phi_0^{-1}\varepsilon_T \sim \mathrm{N}(\mathbf{0}, \sigma^2 \Phi_0^{-1}(\Phi_0^{-1})')$. It is easy to see that Φ_0^{-1} for any PAR process is also a lower triangular matrix. For example, for the PAR(2) model in (5) it can be found that

$$\Phi_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \phi_{1,2} & 1 & 0 & 0 \\ \phi_{1,2}\phi_{1,3} + \phi_{2,3} & \phi_{1,3} & 1 & 0 \\ \phi_{1,2}\phi_{1,3}\phi_{1,4} + \phi_{1,2}\phi_{2,4} + \phi_{2,3}\phi_{1,4} & \phi_{1,3}\phi_{1,4} + \phi_{2,4} & \phi_{1,4} & 1 \end{pmatrix}. \tag{8}$$

This implies that the first two columns of $\Phi_0^{-1}\Phi_1$ contain only zeros, that is

$$\Phi_0^{-1}\Phi_1 = \begin{pmatrix} 0 & 0 & \phi_{2,1} & \phi_{1,1} \\ 0 & 0 & \phi_{1,2}\phi_{2,1} & \phi_{1,2}\phi_{1,1} + \phi_{2,2} \\ 0 & 0 & (\phi_{1,2}\phi_{1,3} + \phi_{2,3})\phi_{2,1} & \phi_{1,1}\phi_{1,2}\phi_{1,3} + \phi_{1,1}\phi_{2,3} + \phi_{1,3}\phi_{2,2} \\ 0 & 0 & (\phi_{1,2}\phi_{1,3}\phi_{1,4} + \phi_{1,2}\phi_{2,4} + \phi_{2,3}\phi_{1,4})\phi_{2,1} & (\phi_{1,2}\phi_{1,3}\phi_{1,4} + \phi_{1,2}\phi_{2,4} + \phi_{2,3}\phi_{1,4})\phi_{1,1} \end{pmatrix}, \tag{9}$$

displaying that Y_T depends only on the third and fourth quarters in Y_{T-1}.

A second version of (2) is based on the possibility of decomposing a p-th order polynomial $\tilde{\psi}_p(L)$ with at least k real roots as $\tilde{\psi}_{p-k}(L)(1-\psi_k L)\dots(1-\psi_1 L)$. Hence, it can be useful to rewrite (2) as

$$[\tilde{\Psi}_{p-k}(L)\ \Psi_k(L)\dots\Psi_1(L)]\ Y_T = \varepsilon_T, \tag{10}$$

where the $\Psi_i(L)$, $i = 1,\dots,k$ are (4×4) matrices with elements that are first order polynomials in L and $\tilde{\Psi}_{p-k}(L)$ is a matrix polynomial of order $(p-k)$. An example is again given by the PAR(2) process in (5), which can be written as

$$\Psi_2(L)\Psi_1(L)Y_T = \varepsilon_T, \tag{11}$$

with

$$\Psi_2(L) = \begin{pmatrix} 1 & 0 & 0 & -\beta_1 L \\ -\beta_2 & 1 & 0 & 0 \\ 0 & -\beta_3 & 1 & 0 \\ 0 & 0 & -\beta_4 & 1 \end{pmatrix},\ \Psi_1(L) = \begin{pmatrix} 1 & 0 & 0 & -\alpha_1 L \\ -\alpha_2 & 1 & 0 & 0 \\ 0 & -\alpha_3 & 1 & 0 \\ 0 & 0 & -\alpha_4 & 1 \end{pmatrix}, \tag{12}$$

such that (4) becomes

$$(1-\beta_s L)(1-\alpha_s L)y_t = \varepsilon_t. \tag{13}$$

This expression equals

$$y_t - \alpha_s y_{t-1} = \beta_s(y_{t-1} - \alpha_{s-1}y_{t-2}) + \varepsilon_t, \tag{14}$$

as the backward shift operator L also operates on α_s, that is, $L\alpha_s = \alpha_{s-1}$ for all $s = 1,2,3,4$ and with $\alpha_0 = \alpha_4$.

Parameter Estimation

To estimate the parameters in a PAR model, we use seasonal dummy variables $D_{s,t}$ which are equal to 1 if t corresponds to season s, and zero elsewhere. The parameters of the PAR(p) model in (1) can be estimated by considering the regression model

$$y_t = \sum_{s=1}^{4} \phi_{1,s} D_{s,t} y_{t-1} + \cdots + \sum_{s=1}^{4} \phi_{p,s} D_{s,t} y_{t-p} + \varepsilon_t. \tag{15}$$

Under normality of the error process ε_t and with fixed starting values, the maximum likelihood [ML] estimators of the parameters $\phi_{i,s}$, $i = 1, 2, \ldots, p$ and $s = 1, 2, 3, 4$, are obtained from ordinary least squares [OLS] estimation of (15). For alternative estimation methods and asymptotic results, see Pagano (1978) and Troutman (1979). Notice that the available sample for estimating the periodic parameters is in fact $N = n/4$, that is, the number of observations can be small.

Once the parameters in a PAR(p) process have been estimated, an important next step involves testing for periodic variation in the autoregressive parameters. Boswijk and Franses (1996) show that the likelihood ratio test for the null hypothesis

$$H_0 : \phi_{i,s} = \phi_i \text{ for } s = 1, 2, 3, 4 \text{ and } i = 1, 2, \ldots, p, \tag{16}$$

has an asymptotic $\chi^2(3p)$ distribution, whether the y_t series has units root or not. We denote by F_{per} the F-version of this test. An important implication of this result is that (15) can be estimated for the y_t series itself, that is, there is no need to *a priori* difference the y_t series to remove stochastic trends when one wants to test for periodicity. This suggests that, for practical purposes, it seems most convenient to start with estimating the model in (15) and testing the H_0 in (16). In a second step one may then test for unit roots in periodic models or nonperiodic models depending on the outcome of the test. An additional advantage is that this sequence of steps allows the possibility of having a periodic differencing filter, which is useful in case of periodic integration. We address this issue in more detail in the next subsection.

Order Selection

To determine the order p of a periodic autoregression, Franses and Paap (1994) recommend to use the BIC in combination with diagnostic tests on residual autocorrelation. As we are dealing with periodic time series models, it seems sensible to opt for an LM test for periodic serial correlation in the residuals. This test corresponds to a standard F-test for the significance of the ρ_s parameter in the following auxiliary regression

$$\hat{\varepsilon}_t = \gamma_{1,s} y_{t-1} + \cdots + \gamma_{p,s} y_{t-p} + \rho_s \hat{\varepsilon}_{t-1} + \eta_t, \tag{17}$$

where $\hat{\varepsilon}_t$ are the estimated residuals of (15), see Franses (1993). Of course, one may also consider the nonperiodic version, where one imposes in (17) that $\rho_s = \rho$ for all s. Finally, standard tests for normality and ARCH effects can also be applied.

2.2 Unit Roots and Periodic Integration

To analyze the presence of stochastic trends in y_t we consider the solutions to the characteristic equation of (5), that is, the solutions to

$$|\Phi_0 - \Phi_1 z - \cdots - \Phi_P z^P| = 0. \tag{18}$$

When k solutions to (18) are on the unit circle, the Y_T process, and also the y_t process, has k unit roots. Notice that the number of unit roots in y_t equals that in Y_T, and that, for example, no additional unit roots are introduced in the multivariate representation. We illustrate this with several examples.

As a first example, consider the PAR(2) process in (4) for which the characteristic equation is

$$|\Phi_0 - \Phi_1 z| = \begin{vmatrix} 1 & 0 & -\phi_{2,1}z & -\phi_{1,1}z \\ -\phi_{1,2} & 1 & 0 & -\phi_{2,2}z \\ -\phi_{2,3} & -\phi_{1,3} & 1 & 0 \\ 0 & -\phi_{2,4} & -\phi_{4,1}z & 1 \end{vmatrix} = 0, \tag{19}$$

which becomes

$$\begin{aligned} 1 - (&\phi_{2,2}\phi_{1,3}\phi_{1,4} + \phi_{2,2}\phi_{2,4} + \phi_{2,1}\phi_{1,2}\phi_{1,3} + \phi_{2,1}\phi_{2,3} + \phi_{1,1}\phi_{1,2}\phi_{1,3}\phi_{1,4} \\ &+ \phi_{1,1}\phi_{1,2}\phi_{2,4} + \phi_{1,1}\phi_{1,4}\phi_{2,3})z + \phi_{2,1}\phi_{2,2}\phi_{2,3}\phi_{2,4}z^2 = 0. \end{aligned} \tag{20}$$

Hence, when the nonlinear parameter restriction

$$\begin{aligned} &\phi_{2,2}\phi_{1,3}\phi_{1,4} + \phi_{2,2}\phi_{2,4} + \phi_{2,1}\phi_{1,2}\phi_{1,3} + \phi_{2,1}\phi_{2,3} + \phi_{1,1}\phi_{1,2}\phi_{1,3}\phi_{1,4} \\ &\quad + \phi_{1,1}\phi_{1,2}\phi_{2,4} + \phi_{1,1}\phi_{1,4}\phi_{2,3} - \phi_{2,1}\phi_{2,2}\phi_{2,3}\phi_{2,4} = 1 \end{aligned} \tag{21}$$

is imposed on the parameters, the PAR(2) model contains a single unit root.

When (19) yields two real-valued solutions, one can also analyze the characteristic equation

$$|\Psi_2(z)\Psi_1(z)| = 0. \tag{22}$$

It is easy to see that this equation equals

$$(1 - \beta_1\beta_2\beta_3\beta_4 z)(1 - \alpha_1\alpha_2\alpha_3\alpha_4 z) = 0, \tag{23}$$

and hence that the PAR(2) model has one unit root when either $\beta_1\beta_2\beta_3\beta_4 = 1$ or $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$, and has at most two unit roots when both products equal unity. Obviously, the maximum number of unity solutions to the characteristic equation of a PAR(p) process is equal to p.

The expression (23) shows that one may need to consider a periodic differencing filter to remove the stochastic trend. Consider the simple PAR(1) model

$$y_t = \alpha_s y_{t-1} + \varepsilon_t, \tag{24}$$

which can be written as (5) with

$$\Phi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha_2 & 1 & 0 & 0 \\ 0 & -\alpha_3 & 1 & 0 \\ 0 & 0 & -\alpha_4 & 1 \end{pmatrix} \quad \text{and} \quad \Phi_1 = \begin{pmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

The characteristic equation is

$$|\Phi_0 - \Phi_1 z| = (1 - \alpha_1\alpha_2\alpha_3\alpha_4 z) = 0, \tag{26}$$

and hence the PAR(1) process has a unit root when $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. In case one or more α_s values are unequal to α, that is, when $\alpha_s \neq \alpha$ for all s, and $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$, the y_t process is said to be periodically integrated of order 1 [PI(1)]. Periodic integration of order 2 can similarly be defined in terms of the α_s and β_s parameters in the PAR(2) process using (23). The concept of periodic integration was first defined in Osborn (1988).

As the periodic AR(1) process nests the $y_t = \alpha y_{t-1} + \varepsilon_t$, it is obvious that a unit root in a PAR(1) process implies a unit root in the nonperiodic AR(1) process. The characteristic equation is then $(1-\alpha^4 z)=0$. Hence, when $\alpha=1$, the Y_T process has a single unit root. Also, when $\alpha=-1$, the process Y_T has a unit root. The first case corresponds to the simple random walk process, that is, the case where y_t has a nonseasonal unit root, while the second case corresponds to the case where y_t has a seasonal unit root, see Hylleberg *et al.* (1990). In other words, both the nonseasonal and the seasonal unit root process are nested within the PAR(1) process. This suggests a simple testing strategy, that is, first investigating the presence of a unit root by testing whether $\alpha_1\alpha_2\alpha_3\alpha_4=1$, and second to test whether $\alpha_s=1$ or $\alpha_s=-1$ for all s. Boswijk and Franses (1996) show that, given $\alpha_1\alpha_2\alpha_3\alpha_4=1$, these latter tests are $\chi^2(3)$ distributed. See also Boswijk *et al.* (1997) for testing for so-called seasonal unit roots along a similar line.

Testing for Periodic Integration

To test for periodic integration in the PAR(p) model (1), Boswijk and Franses (1996) consider a likelihood ratio [LR] test. The test statistic equals

$$\mathrm{LR}_{\mathrm{PI}} = n[\ln(SSR_0) - \ln(SSR_a)], \tag{27}$$

where SSR_0 and SSR_a denote the sum of the squared residuals of the estimated PAR(p) model under the restriction of periodic integration and without this restriction, respectively. The latter can be obtained directly from the estimated residuals of the regression model (15). To obtain the residuals under the null, one has to estimate the PAR(p) model under the nonlinear restriction of periodic integration using nonlinear least squares [NLS]. As this restriction may be complex in higher order PAR models, it is more convenient to consider the generalization of (14) to a PAR(p) model, that is,

$$(y_t - \alpha_s y_{t-1}) = \sum_{i=1}^{p-1} \beta_{i,s}(y_{t-i} - \alpha_{s-i} y_{t-1-i}) + \varepsilon_t, \tag{28}$$

with $\alpha_{s-4k}=\alpha_s$ and where the restriction of periodic integration is simply $\alpha_1\alpha_2\alpha_3\alpha_4=1$. Again, this model can be estimated with NLS.

The asymptotic distribution of the $\mathrm{LR}_{\mathrm{PI}}$ test statistic (27) under periodic integration is the same as the asymptotic distribution of the square of the standard unit root t-test of Dickey and Fuller (1979), see Boswijk and Franses (1996). The critical values are given in the first row of Table 15.1 of Johansen (1995). It is also possible to consider a one-sided test by taking the square root of (27). The sign of the resulting statistic is negative if all roots of the characteristic equation (18) are outside the unit circle and positive in all other cases. Under the null hypothesis, this test statistic has the same distribution as Fuller's (1976) τ statistic.

Similar to the standard Dickey-Fuller case, the asymptotic distribution of the test statistic depends on the presence of deterministic terms in the test equation. In the next subsection we discuss the role of intercepts and trends in periodic autoregressions and the appropriate asymptotic distribution of LR_{PI} statistic for periodic integration for different specifications. This discussion is particularly relevant as a trend will dominate out-of-sample forecast patterns.

2.3 Intercepts and Trends

So far, the periodic models did not include any deterministic terms. Seasonal intercepts and seasonal linear trends can be added to (1) in a linear way, that is,

$$y_t = \mu_s + \tau_s T_t + \phi_{1,s} y_{t-1} + .. + \phi_{p,s} y_{t-p} + \varepsilon_t, \tag{29}$$

where $T_t = [(t-1)/4] + 1$ represents an annual linear deterministic trend and μ_s and τ_s $s = 1, 2, 3, 4$ are seasonal dummy and trend parameters. In general, unrestricted periodic processes like (29) can generate data with diverging seasonal trends, which may not be plausible in all practical cases. Common seasonal linear deterministic trends require parameter restrictions on seasonal trend parameters τ_s. Note that the simple restriction $\tau_1 = \tau_2 = \tau_3 = \tau_4$ does not correspond to common seasonal trends, because the τ_s parameters do not represent the slope of the trend in each season.

Periodic Trend-Stationarity

To analyze the role of the linear trend under periodic trend-stationarity we rewrite (29) as

$$(y_t - \mu_s^* - \tau_s^* T_t) = \sum_{i=1}^{p} \phi_{i,s}(y_{t-i} - \mu_{s-i}^* - \tau_{s-i}^* T_{t-i}) + \varepsilon_t, \tag{30}$$

where μ_s^* and τ_s^* are nonlinear functions of the μ_s, τ_s and $\phi_{i,s}$ parameters and where $\mu_{s-4k}^* = \mu_s^*$ and $\tau_{s-4k}^* = \tau_s^*$. This model can easily be estimated using NLS. The restriction for common linear seasonal deterministic trends is given by $\tau_1^* = \tau_2^* = \tau_3^* = \tau_4^*$. This restriction can be tested with a standard likelihood ratio test, which is $\chi^2(3)$ distributed. The restriction for the absence of linear deterministic trends is simply $\tau_1^* = \tau_2^* = \tau_3^* = \tau_4^* = 0$.

Periodic Integration

The presence of a linear deterministic trend in an autoregression for y_t with an imposed unit root corresponds to the presence of quadratic trend in y_t. Likewise, an inclusion of linear deterministic trends in a periodically integrated autoregression [PIAR] assumes the presence of seasonal quadratic trends in y_t. To discuss the role of trends in a PIAR we distinguish three cases: the presence of no quadratic trends [NQT], common (seasonal) linear trends [CLT] and no linear trends [NLT].

To discuss these three cases it convenient to write (29) using (28) as

$$(y_t - \alpha_s y_{t-1} - \mu_s^{**} - \tau_s^{**} T_t) = \sum_{i=1}^{p-1} \beta_{i,s}(y_{t-i} - \alpha_{s-i} y_{t-1-i} - \mu_{s-i}^{**} - \tau_{s-i}^{**} T_{t-i}) + \varepsilon_t \tag{31}$$

with $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ and where μ_s^{**}, τ_s^{**} and $\beta_{i,s}$ are again non-linear functions of μ_s, τ_s and $\phi_{i,s}$ parameters and $\mu_{s-4k}^{**} = \mu_s^{**}$ and $\tau_{s-4k}^{**} = \tau_s^{**}$. Note that it is not possible to write (29) like (30) under the restriction of periodic integration. To analyze the role of the deterministic terms, it is convenient to write (31) in VQ representation. The restrictions on the deterministic elements follow from applying Granger's representation theorem to this VQ representation, see Paap and Franses (1999) for a complete derivation.

For example, it follows that the restriction for NQT in y_t corresponds to

$$\tau_1^{**} + \alpha_1\alpha_3\alpha_4\tau_2^{**} + \alpha_1\alpha_4\tau_3^{**} + \alpha_1\tau_4^{**} = 0, \tag{32}$$

or to the trivial solution $\tau_1^{**} = \tau_2^{**} = \tau_3^{**} = \tau_4^{**} = 0$. To obtain CLT in y_t, one has to impose the four restrictions

$$\tau_s^{**} = (1 - \alpha_s)d \quad \text{for } s = 1, 2, 3, 4, \tag{33}$$

where d is given by

$$d = \mu_4^{**} + \alpha_4\mu_3^{**} + \alpha_3\alpha_4\mu_2^{**} + \alpha_2\alpha_3\alpha_4\mu_1^{**}. \tag{34}$$

Finally, the restriction for the absence of linear deterministic trends [NLT] in y_t is given by

$$\mu_4^{**} + \alpha_4\mu_3^{**} + \alpha_3\alpha_4\mu_2^{**} + \alpha_2\alpha_3\alpha_4\mu_1^{**} = 0 \ \text{ and } \ \tau_1^{**} = \tau_2^{**} = \tau_3^{**} = \tau_4^{**} = 0. \qquad (35)$$

Of course, a special case is the trivial solution $\mu_s^{**} = \tau_s^{**} = 0$ for all s.

All restrictions can be tested with standard likelihood ratio tests. Under the restriction of periodic integration, these tests are asymptotically $\chi^2(\nu)$ distributed, where ν denotes the number of restrictions. Finally, these restrictions are also valid in nonperiodic AR models or PAR models for the first differences of a time series.

Deterministic Components and Testing for Periodic Integration

The inclusion of deterministic components in the test equation for periodic integration changes the asymptotic distribution of the $\mathrm{LR}_{\mathrm{PI}}$ statistic. If one includes only seasonal dummies, the percentiles of the asymptotic distribution of the statistic is tabulated in the first row of Table A.2 of Johansen and Juselius (1990). If one also includes seasonal linear deterministic trends, the asymptotic distribution is given by the square of Fuller's (1976) τ_τ statistic. As this asymptotic distribution has virtually no mass on the positive part of the line, one can simply take the square of the corresponding critical values of the τ_τ statistic. Obviously, the asymptotic distributions of the one-sided LR statistics are the same as the asymptotic distribution of Fuller's (1976) τ_μ and τ_τ statistics.

Finally, it is also possible to perform a joint test on periodic integration and the absence of quadratic (or linear) trends under the null hypothesis. For example, one may test jointly for the presence of periodic integration and the absence of quadratic trends, that is, restriction (32) using a LR test. Hence, one compares specification (29) with (31) under the restriction (32). The asymptotic distribution of this joint test is tabulated in the first row of Table 15.4 of Johansen (1995). Likewise, one may test with a LR test, under the restriction that $\tau_1 = \tau_2 = \tau_3 = \tau_4$, for the presence of periodic integration and for the absence of linear deterministic trends (35). The asymptotic distribution of this joint test is tabulated in the first row of Table 15.2 of Johansen (1995). In the empirical section below, we will apply the various tests.

3 Forecasting

Once the parameters in the PAR models have been estimated, and appropriate parameter restrictions for unit roots and deterministic terms have been imposed, one can use the resultant model for out-of-sample forecasting. In this section, we first consider generating forecasts, and then briefly turn to their evaluation.

Point and Interval Forecasts

Forecasting with PAR models proceeds roughly in the same way as with standard AR models, see Franses (1996a) for an extensive discussion. To illustrate this, we consider the PAR(1) model in (24). The 1-step ahead forecast made at $t = n$ is simply

$$\hat{y}_{n+1} = \mathrm{E}_n[y_{n+1}] = \mathrm{E}[\alpha_s y_n + \varepsilon_{n+1}] = \alpha_s y_n, \tag{36}$$

where we assume that time $n+1$ corresponds to season s. The forecast error $y_{n+1} - \hat{y}_{n+1}$ is ε_{n+1} and hence the variance of the 1-step ahead forecast equals σ^2. Likewise, we can construct the 2-, 3- and 4-steps ahead forecast, which equal

$$\begin{aligned} \hat{y}_{n+2} &= \mathrm{E}_n[y_{n+2}] = \mathrm{E}[\alpha_{s+1}\alpha_s y_n + \varepsilon_{n+2} + \alpha_{s+1}\varepsilon_n] = \alpha_{s+1}\alpha_s y_n \\ \hat{y}_{n+3} &= \mathrm{E}_n[y_{n+3}] = \mathrm{E}_n[\alpha_{s+2} y_{n+2} + \varepsilon_{n+1}] = \alpha_{s+2}\alpha_{s+1}\alpha_s y_n \\ \hat{y}_{n+4} &= \mathrm{E}_n[y_{n+4}] = \mathrm{E}_n[\alpha_{s+3} y_{n+3} + \varepsilon_{n+3}] = \alpha_{s+3}\alpha_{s+2}\alpha_{s+1}\alpha_s y_n. \end{aligned} \tag{37}$$

In case of periodic integration the 4-steps ahead forecast simplifies to $\hat{y}_{n+4} = y_n$. Note that the expressions for the forecasts depend on the season in which you start to forecast.

The forecast errors belonging to these forecasts are

$$\begin{aligned} \hat{y}_{n+2} - y_{n+2} &= \varepsilon_{n+2} + \alpha_{s+1}\varepsilon_{n+1} \\ \hat{y}_{n+3} - y_{n+3} &= \varepsilon_{n+3} + \alpha_{s+2}\varepsilon_{n+2} + \alpha_{s+2}\alpha_{s+1}\varepsilon_{n+1} \\ \hat{y}_{n+4} - y_{n+4} &= \varepsilon_{n+4} + \alpha_{s+3}\varepsilon_{n+3} + \alpha_{s+3}\alpha_{s+2}\varepsilon_{n+2} + \alpha_{s+3}\alpha_{s+2}\alpha_{s+1}\varepsilon_{n+1} \end{aligned} \tag{38}$$

and hence the variances of the forecast errors equal $\sigma^2(1+\alpha_{s+1}^2)$, $\sigma^2(1+\alpha_{s+2}^2+\alpha_{s+2}^2\alpha_{s+1}^2)$ and $\sigma^2(1+\alpha_{s+3}^2+\alpha_{s+3}^2\alpha_{s+2}^2+\alpha_{s+3}^2\alpha_{s+2}^2\alpha_{s+1}^2)$, respectively. These forecast error variances also depend on the season in which one generates forecasts. The variances can be used to construct forecast intervals in the standard way.

In general it is more convenient to use the VQ representation to compute forecasts and forecast error variances. Forecasts can then be generated along the same lines as for

VAR models, see Lütkepohl (1993). Consider again the PAR(1) model in (24). The VQ representation is given by (5) and (25). The forecasts made at $t = n = 4N$ for the next year (the forecasting origin is quarter 4) using the VQ representation are given by

$$\hat{Y}_{N+1} = \mathrm{E}[Y_{N+1}] = \mathrm{E}[\Phi_0^{-1}\Phi_1 Y_N + \Phi_0^{-1}\varepsilon_{N+1}] = \Phi_0^{-1}\Phi_1 Y_N. \tag{39}$$

The forecast errors equal $\hat{Y}_N - Y_N = \Phi_0^{-1}\varepsilon_{N+1}$ and hence the covariance matrix of the forecast errors is simply $\sigma^2(\Phi_0^{-1}(\Phi_0^{-1})')$. It is easy to show that the diagonal elements of this matrix correspond to the forecast error variances derived above.

Likewise, the forecast for 2-years ahead, that is 5- to 8-steps ahead for the quarterly series y_t is given by

$$\hat{Y}_{N+2} = \mathrm{E}[Y_{N+2}] = \mathrm{E}[(\Phi_0^{-1}\Phi_1)^2 Y_N + \Phi_0^{-1}\varepsilon_{N+2} + (\Phi_0^{-1}\Phi_1)\Phi_0^{-1}\varepsilon_{N+1}] = (\Phi_0^{-1}\Phi_1)^2 Y_N. \tag{40}$$

where the corresponding covariance matrix for the forecast errors is given by $\sigma^2(\Phi_0^{-1}(\Phi_0^{-1})' + (\Phi_0^{-1}\Phi_1\Phi_0^{-1})(\Phi_0^{-1}\Phi_1\Phi_0^{-1})')$. The covariances between 1-year ahead and 2-years ahead forecast follow directly from $\mathrm{E}[(\hat{Y}_{N+2} - Y_{N+2})(\hat{Y}_{N+1} - Y_{N+1})'] = \sigma^2(\Phi_0^{-1}(\Phi_0^{-1})'\Phi_1'(\Phi_0^{-1})')$.

Multi-year ahead forecasts can be generated in a similar way. Note that if the series is periodically integrated, it can be shown that the matrix $(\Phi_0^{-1}\Phi_1)$ is idempotent, which may simplify the expressions for the forecasts and forecast error covariances. For instance, it follows from (40) that the 2-years ahead forecasts for Y_T generated by a PIAR(1) model without deterministic elements is equal to the 1-year ahead forecast. This shows that forecasts from a PIAR(1) model are the same as those of the seasonally integrated model $\Delta_4 y_t = y_t - y_{t-4} = u_t$, where u_t is white noise.

Evaluating Forecasts

To compare forecasts generated by PAR models with forecasts from alternative periodic or nonperiodic models, one can consider the familiar Root Mean Squared Prediction Error [RMSPE]. One may also opt for an encompassing test. In brief, one then estimates the following regression equation for the generated forecasts

$$(\hat{y}_{n+h} - y_{n+h}) = \gamma(\hat{y}_{n+h} - \tilde{y}_{n+h}) + \eta_{n+h}, \tag{41}$$

where $\tilde{y}_{n+h}$ is the forecast generated by a competing model, see Clements and Hendry (1993). If $\gamma = 0$, the forecasts $\hat{y}_{n+h}$ encompass forecasts generated by the competing model $\tilde{y}_{n+h}$. This restriction can be tested using a standard F-test. As the variance of the h-step ahead forecast error of PAR models depends on the season in which we start to forecast, one should estimate (41) with different variances for η_{n+h} in each season.

4 Empirical Illustration

Our data concern real nondurable consumption in the United Kingdom on food, alcohol, clothing, energy, other goods, services, and total nondurable consumption (which does not include services). The sample ranges from 1955.I–1994.IV. We use the sample 1955.I–1988.IV for model construction and estimation, and we reserve the period 1989.I–1994.IV for out-of-sample forecasting. All series are log transformed. In Section 4.1, we test for periodicity in the series and construct PAR models. In Section 4.2, we estimate nonperiodic models for the series that turn out to be periodic, as we aim to evaluate these relative to the PAR models in our forecasting exercise. In Section 4.3, we compare forecasts generated by the various models.

4.1 Periodic Models

We construct periodic autoregressions with seasonal dummies and seasonal trends for the seven series under consideration. In the first step we determine the appropriate lag order of the PAR models. This lag order is determined using the BIC criterion in combination with LM tests for (periodic) serial correlation. The estimated lag orders of the PAR models are given in the second column of Table 1. For these lag orders the PAR models pass diagnostic tests for first, and first-to-fourth order serial correlation, and ARCH effects in the residuals. The third column of Table 1 shows the F_{per}-statistics that the autoregressive parameters are the same over the seasons. For three out of seven series, this restriction cannot be rejected at the 5% level of significance. As the main focus of this chapter concerns periodic models, we will not consider these series any further.

For the other four periodic time series, we proceed with testing for the presence of periodic integration. The fourth column of Table 1 shows the outcomes of the $\mathrm{LR}_{\mathrm{PI}}$ test for the presence of periodic integration. None of the $\mathrm{LR}_{\mathrm{PI}}$ statistics is significant at the 5% level of significance, if we compare the results with the squares of the percentiles of the asymptotic distribution of the τ_τ statistic of Fuller (1976) (If we perform a joint test for periodic integration and the absence of quadratic trends (not reported here), we arrive at the same conclusion.). As the remaining roots of the characteristic equation (18) are far outside the unit circle, we do not consider tests for multiple unit roots.

The next step in our model selection strategy concerns testing for restrictions on the deterministic components in the periodic integrated autoregressions. The fifth column of Table 1 shows the outcomes of the LR test for the absence of a quadratic trend (32). If we compare the outcomes with the percentiles of a $\chi^2(1)$ distribution, we conclude that this restriction cannot be rejected for each of the series. The stronger condition $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ is clearly rejected for all series as can be seen from the sixth column of the table, where we mention that this test statistic is asymptotically $\chi^2(4)$ distributed. The seventh column shows the results of the LR test statistic for the restriction of common linear deterministic trends given in (33) in an unrestricted PIAR model. If we compare the results with the percentiles of the $\chi^2(4)$ distribution, we conclude that this restriction is only rejected for the alcohol series.

Finally, we test with a LR test in the resulting PIAR model (31) with the appropriate restrictions on the deterministic terms indicated by the above test results, whether $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ and hence whether the periodic differencing filter $(1 - \alpha_s L)$ can be simplified to the nonperiodic filter $\Delta_1 = (1 - L)$ to obtain stationarity. If this is the case, we end up with a periodic autoregression for the first differences of the time series. Column 8 of Table 1 displays the test results. If we compare the results with the percentiles of the $\chi^2(3)$ distribution, we conclude that the restriction is only valid for the alcohol series. A LR test whether the seasonal differencing filter $(1 + L)$ is appropriate $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -1)$ is not considered here as the estimated α_s parameters are all close to 1 (and hence far from -1).

In the final column of Table 1 we present the final model suggested by the sequence of tests. For the alcohol series we have a PAR model in first differences with no quadratic trend, for clothing and energy we have a PIAR model with a common linear deterministic trend, while for total nondurable consumption we have a PIAR model without quadratic trends.

4.2 Nonperiodic Models

As competing models for our four periodic autoregressions, we consider two nonperiodic models, which roughly correspond to the alternative approaches discussed in the introduction. First, we consider autoregressive models resulting from tests for the presence of (seasonal) unit roots. Second, we consider SARIMA models for the four series, which usually amount to the so-called airline model.

To construct nonperiodic autoregressions for alcohol, energy, clothing and total consumption, we first test for the presence of (seasonal) unit roots using the HEGY test equation of Hylleberg *et al.* (1990), that is

$$\Delta_4 y_t = \sum_{s=1}^{4} \mu_s D_{s,t} + \tau t + \pi_1(1 + L + L^2 + L^3)y_{t-1} + \pi_2(-1 + L - L^2 + L^3)y_{t-1} + (\pi_3 + \pi_4 L)(1 + L^2)y_{t-1} + \sum_{i=1}^{k} \theta_i \Delta_4 y_{t-i} + \varepsilon_t, \quad (42)$$

where $\Delta_4 y_t = (1-L^4)y_t = y_t - y_{t-4}$. The presence of a nonseasonal unit root 1 corresponds to the restriction $\pi_1 = 0$. This can be tested with a t-test. The presence of the three seasonal unit roots, $-1, i, -i$ corresponds to the restriction $\pi_2 = \pi_3 = \pi_4 = 0$, which can be tested with an F-test. Critical values of these tests can be found in Hylleberg *et al.* (1990) and Ghysels *et al.* (1994).

Table 2 shows the results of the tests for unit roots in a nonperiodic autoregression. The second column shows the lags that are included in the test equation (42). These lags are determined using a similar approach as that taken for the periodic models. The third column of this table shows the t-test for $\pi_1 = 0$. This test statistic is not significant at the 5% level of significance for all four variables and hence we cannot reject the presence of a nonseasonal unit root. The test results for the presence of the three seasonal unit roots are given in the fourth column of Table 2. The presence of these seasonal unit roots is rejected for alcohol, energy and clothing series and hence we arrive at an autoregressive model for the first differences of these series with seasonal dummies. For total consumption, we cannot reject the presence of seasonal unit roots and we end up with an autoregressive model for the fourth differences of the series. The last column of the table displays the finally selected models.

The second type of nonperiodic time series models we consider in our forecasting comparison, is the so-called airline model, where one imposes the differencing filter $\Delta_1\Delta_4$ for the series. Using the standard model selection strategy, we find that the following airline model

$$\Delta_1\Delta_4 y_t = (1-\theta L)(1-\lambda L^4)\varepsilon_t \tag{43}$$

is adequate for alcohol, energy and the total consumption. For the clothing series we replace (43) by a moving average model of order 5, where we impose the MA(2) and MA(3) parameters to equal zero.

4.3 Forecast Comparison

In this subsection we report on the performance of the three models in out-of-sample forecasting. We consider 1-, 4-, and 8-step ahead forecasting for y_t in Table 3. We consider similar forecasts fore each of the quarters separately in Table 4. In Table 5 we consider forecasting $\Delta_1 y_t$ and $\Delta_4 y_t$ as this may be relevant in practice even though this transformation does not match with most models.

The results in Table 3 for the RMSPE criterion show that in 4 of the 12 cases the PAR model yields the smallest value, while this occurs for the HEGY-AR and airline model in 6 and 2 cases, respectively. For the energy series the PAR model outperforms the other models on all three horizons. In case the PAR model does not produce the best forecasts, the average difference in RMSPE between the PAR model and the best performing model is 0.46. For the AR-HEGY and the airline model this average difference equals 1.18 and 0.50, respectively. This shows that the PAR model still performs reasonably well if it is not the best forecasting model. This is however not the case for the AR-HEGY model.

The forecasting encompassing test results in the second panel of Table 3 indicate that in 2 cases the forecasts generated by the PAR and the HEGY-AR models encompass each other (4- and 8-step ahead clothing). In most other cases forecasts generated by the PAR model do not encompass forecasts generated by the AR-HEGY model and *vice versa*. The PAR model gets only encompassed by the AR-HEGY model in three cases (1-, 4-, and 8-step ahead total nondurable consumption) and the HEGY-AR model gets only encompassed twice by the PAR model. In contracts, the PAR model encompasses the airline model 5 times, while it gets encompassed by that model only three times. Hence, it seems that PAR models generally outperform airline models, while they do not frequently improve on the HEGY-AR models.

In Table 4 we present the ranks (based on RMSPE) of the three models for each quarter. We observe mixed results, although the PAR model seems most useful for the alcohol and energy series. The last row of the table gives the average rank across the twelve different forecasting runs (4 variables, 3 horizons). Clearly, the PAR model obtains the lowest rank for quarters 1, 2 and 3, while the HEGY-AR models give the most accurate forecast for quarter 4. The airline model appears not to give useful forecasts.

Finally in Table 5 we give the RMSPEs for forecasts of $\Delta_1 y_t$ and $\Delta_4 y_t$, which may sometimes be of interest in practice. In the first panel, concerning $\Delta_1 y_t$, we observe that even though the Δ_1 transformation appears relevant for the alcohol, energy and clothing series, the corresponding forecast are outperformed by PAR models (3 times) and airline models (once). For total consumption we notice that the HEGY-AR model is best for 4- and 8-step ahead forecasts. From the second panel of Table 5, dealing with forecasts for the annual growth rates, we observe that the PAR model beats alternative models in 4 of the 8 cases.

In sum, it seems that a carefully constructed PAR model, when proper account is taken of unit roots and deterministic terms, oftentimes yields better forecasts compared to those generated from HEGY-AR and airline models.

5 Concluding Remarks

In the last few years it could be noticed that periodic time series models became increasingly popular for describing and forecasting univariate seasonal time series. In this chapter we have discussed some important aspects of these models, and we have evaluated their forecasting performance. We showed that when the PAR models are properly specified, that is, when proper care is taken of unit roots and deterministic trends, they tend to outperform often considered alternative models.

A next important step on the research agenda concerns the forecasting properties of multivariate PAR models. These models are considerably more complicated to specify and analyze with respect to unit roots and deterministic terms. It is therefore of significant importance to examine if these efforts results in accurate forecasts.

Tables

Table 1: Specification tests in PAR models for the seven nondurable consumption series.

series	order	F_{per}	LR_{PI}	LR_{NQT}	$\text{LR}_{\tau=\mathbf{0}}$	LR_{CLT}	LR_{Δ_1}	final model
alcohol	2	2.97^{***}	4.70	0.09	33.13^{***}	10.32^{**}	4.94	Δ_1PAR(1), NQT
energy	4	3.11^{***}	0.04	0.06	12.84^{**}	4.39	28.94^{***}	PIAR(4), CLT
food	3	1.67						nonperiodic
clothing	2	2.29^{**}	4.87	2.19	14.30^{***}	4.85	82.93^{***}	PIAR(2), CLT
other	5	1.19						nonperiodic
services	5	1.55						nonperiodic
total	1	7.26^{***}	6.32	0.20	30.36^{***}	70.00^{***}	21.59^{***}	PIAR(2), NQT

Note: The cells contain the values of various F- and LR test statistics. ***, **, * denote significant at 1, 5, 10%, respectively.

Table 2: HEGY tests in nonperiodic AR models.

series	lags	$t(\pi_1)$	$F(\pi_2, \pi_3, \pi_4)$	final model
alcohol	1	-1.05	6.08^{**}	Δ_1, AR(4), SD
energy	0	-1.04	12.76^{***}	Δ_1, AR(3), SD
clothing	1	-1.80	7.11^{**}	Δ_1, AR(4), SD
total	1,4,5	-2.68	1.16	Δ_4, AR(5), const

Note: In the final column we give the selected AR order for the appropriately differenced series and whether this model contains seasonal dummies [SD]. ***, **, * denote significant at 1, 5, 10%, respectively.

Table 3: RMSPEs and encompassing tests for forecasts of y_t generated by the three models. Forecasting sample is 1989.I–1994.IV.

series	horizon	RMSPE ×100 PAR	AR	Airline	encompassing tests[1] $F_{\mathrm{PAR/AR}}$	$F_{\mathrm{AR/PAR}}$	$F_{\mathrm{PAR/Air}}$	$F_{\mathrm{Air/PAR}}$
	1	4.38	3.22	3.23	11.19***	6.46**	15.29***	10.01***
alcohol	4	3.85	4.04	3.66	0.31	12.32***	5.84**	1.06
	8	5.61	8.16	6.57	3.50*	50.97***	0.87	10.65***
	1	3.81	4.70	4.08	0.33	9.53***	4.02*	8.81***
energy	4	3.15	4.04	3.77	4.21**	29.93***	0.43	9.47***
	8	3.84	6.12	5.21	14.76***	51.80**	0.26	9.45***
	1	3.32	2.60	3.15	42.89***	8.93***	35.24***	2.41
clothing	4	3.39	3.30	3.67	0.51	1.22	1.76	7.96**
	8	6.05	6.00	6.53	0.36	0.81	1.62	4.19*
	1	2.14	1.21	1.25	42.25***	1.04	48.78***	1.60
total	4	1.92	1.74	1.63	5.98**	0.89	12.96***	3.47*
	8	3.85	3.60	3.92	3.33*	0.46	5.78**	6.17**

[1] $F_{A,B}$ denotes a F-type statistic for the null hypothesis that forecasts generated by model A encompass forecasts generated by model B, where we allow for seasonal heteroscedasticity in the test equation. ***, **, * denote significant at 1, 5, 10%, respectively.

Table 4: Forecasting rank per quarter[1] for y_t based on the RMSPE. Forecasting sample is 1989.I–1994.IV.

		PAR				AR				airline			
		quarter				*quarter*				*quarter*			
series	horizon	I	II	III	IV	I	II	III	IV	I	II	III	IV
	1	3	3	1	3	1	2	3	2	2	1	2	1
alcohol	4	3	1	1	2	2	3	3	3	1	2	2	1
	8	1	1	1	1	3	3	3	2	2	2	2	3
	1	1	2	2	2	3	1	3	1	2	3	1	3
energy	4	1	1	1	1	3	3	3	2	2	2	2	3
	8	1	1	1	1	3	3	3	2	2	2	2	3
	1	1	3	2	2	2	2	1	1	3	1	3	3
clothing	4	1	3	1	2	2	1	3	1	3	2	2	3
	8	2	1	1	2	1	3	2	1	3	2	3	3
	1	2	3	3	2	1	2	2	1	3	1	1	3
total	4	3	2	3	3	2	1	2	2	1	3	1	1
	8	2	1	3	3	1	2	2	1	3	3	1	2
average rank		1.8	1.8	1.7	2.0	2.0	2.2	2.5	1.6	2.3	2.0	1.8	2.4

[1]Rank 1 corresponds to the smallest RMSPE for the corresponding quarter, while rank 3 corresponds to the largest RMSPE.

Table 5: RMSPE for forecasting quarterly and annual growth rates ($\Delta_1 y_t$ and $\Delta_4 y_t$) for 1989.I–1990.IV.

		quart. growth rates			annual growth rates		
series	horizon	PAR	AR	Air.	PAR	AR	Air.
alcohol	4	5.55	3.69	3.49	3.85	4.04	3.66
	8	6.09	5.67	6.53	3.79	4.71	4.55
energy	4	3.74	4.14	4.03	3.15	4.04	3.77
	8	4.28	4.66	4.29	3.84	4.37	4.09
clothing	4	3.31	2.91	3.34	3.39	3.30	3.67
	8	3.32	3.78	3.67	3.49	3.49	3.71
total	4	1.88	0.98	1.01	1.92	1.74	1.63
	8	1.78	1.18	1.23	2.06	1.97	2.44

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