INVENTORY CONTROL AND REGENERATIVE PROCESSES: THEORY

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ABSTRACT. In this paper we will discuss a general framework for single item inventory models based on the theory of regenerative processes. After presenting without proof the main theorems for regenerative processes we analyze in detail how the different single item models can be embedded within this general theory. This facilitates to write down the expressions for the average cost associated with an arbitrary costrate function f, and some of the service measures, which appear most frequently in the literature.

1. INTRODUCTION

Ford Harris' famous paper on the EOQ model in 1913 (cf. [8]) was the first of many publications on inventory theory. At present, thousands of papers have appeared in the management science and operations research literature. One may wonder why so much research is done on inventory models. The explanation is simply that in practice one encounters many different situations and each one requires a tailor-made analysis. For example, there may be differences with respect to the following aspects: number of locations and echelons, number of products, demand process, cost structure, service requirements and measurement, possible moments of placing a replenishment order, the way a stockout is handled, and the lead time of replenishment orders. Since so many different situations can be analyzed, we feel that there is a need to develop a general framework. Such a framework will help to improve the understanding of the models that appeared in the literature. In this paper the average cost and the most widely known service measures will be derived for a number of basic inventory models using a general framework which is presented in the next section. This framework is based on the theory of regenerative processes. It can be shown that most inventory models satisfy the so-called regenerative property which allows for a nice derivation of the average cost and service measures. We will restrict ourselves to inventory systems with a single product, a single location, backordering of stockouts and deterministic lead times. In most of the inventory literature such a system is considered and for an overview the reader is referred to Chikán (cf. [5]). A more recent discussion of the classical single item inventory

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models is given by de Kok (cf. [11]). Observe, the results of these models serve as a basis for multi-location, multi-echelon and multi-item systems (e.g. [9], [4]). It is hopefully clear now that a proper understanding of the classical single item inventory models is important and to achieve this we structured the paper in the following way. The theory of regenerative processes is briefly discussed in Section 2. Its application to inventory models with complete back-ordering is presented in Section 3. In this section we also impose a general condition satisfied by all classical single item inventory models and using this condition it is possible to give a unified and easy proof of all the relevant expressions. In Section 4 this condition is checked for all the classical inventory models and at the same time the expressions for the average cost and service measures are derived for each model. In Section 5 some conclusions are presented. Finally we observe that no computational results are reported. This is the topic of a subsequent paper (cf. [2]) in which the Laplace transforms of the objective functions are calculated. These Laplace transforms are used in a newly developed robust Laplace transform inversion algorithm (cf, [10]) and as a result we obtain computations with machine precision.

2. Regenerative Processes

The theory of regenerative processes plays a major role within stochastic models. For these stochastic processes many properties and results are presented in the book of Asmussen (cf. [1]). Since for our purpose it is sufficient to consider only pure regenerative processes, we will not discuss delayed regenerative processes. The goal of this section is to give an overview of the theory of pure regenerative processes and we first start with a simplified version of a pure regenerative process. Observe that random variables are denoted by boldface characters while the set T either denotes the set $[0,\infty)$ or $\mathbb{N}\bigcup\{0\}$.

Definition 2.1. A stochastic process $\mathbf{X}=\{\mathbf{X}(t):t\in T\}$ with metric state space E is called a pure regenerative process if there exists a positive constant $\sigma\in T$ such that for every $n\in\mathbb{N}\bigcup\{0\}$ the distribution of the shifted stochastic process $\{\mathbf{X}(t+n\sigma):t\in T\}$ is independent of n.

A more general definition of a pure regenerative process is given by the next one.

Definition 2.2. A stochastic process $\mathbf{X}=\{\mathbf{X}(t):t\in T\}$ with metric state space E is called a pure regenerative process if there exists an increasing sequence $\sigma_n,\ n\in\mathbb{N}\bigcup\{0\}$ with $\sigma_0:=0$ of random points belonging to T, satisfying

1. The random variables $\sigma_{n+1}-\sigma_n\ ,n\in\mathbb{N}\bigcup\{0\}$, are independent and identically distributed with right continuous cumulative distribution function F_σ satisfying $F_\sigma(0)=0$ and $F_\sigma(\infty)=1$.
2. For each $n\in\mathbb{N}\bigcup\{0\}$ the post-σ_n process
$$\{\mathbf{X}(t+\sigma_n)\ :\ t\in T\}$$
is independent of $\sigma_0,\dots,\sigma_n$.
3. The distribution of $\{\mathbf{X}(t+\sigma_n)\ :\ t\in T\}$ is independent of n.

In case the difference $\sigma_{n+1}-\sigma_n,\ n\in\mathbb{N}\bigcup\{0\}$, is degenerate and given by $\sigma=\sigma_{n+1}-\sigma_n>0$ with probability one then it is obvious that the definition of a pure regenerative process as mentioned in Definition 2.2 reduces to Definition 2.1. Moreover, in most applications with $T=[0,\infty)$ a pure regenerative process

$\mathbf{X}$ is càdlàg. This means that the sample paths of the stochastic process $\mathbf{X}$ are right continuous with left limits $I\!P$-almost surely with $I\!P$ denoting the probability measure of the underlying probability space. For pure regenerative processes the next result is easy to verify.

Theorem 2.3. *If the stochastic process $\mathbf{X} = \{\mathbf{X}(t) : t \in T\}$ is a pure regenerative process with metric state space E and the increasing sequence σ_n, $n \in I\!N \bigcup \{0\}$ and $\Phi : E \to I\!R$ is a Borel measurable function, then the process $\Phi \circ \mathbf{X} := \{\Phi(\mathbf{X}(t)) : t \in T\}$ is a pure regenerative process with the same increasing sequence. Moreover, if $T = [0, \infty)$ and Φ is a continuous function then the stochastic process $\Phi \circ \mathbf{X}$ is càdlàg if $\mathbf{X}$ is càdlàg.*

To introduce a cost structure on a pure regenerative process we consider a nonnegative Borel measurable function $f : E \to I\!R$, called the costrate function. Using this function we denote by $\mathbf{C} = \{\mathbf{C}(t) : t \in T\}$ with $T = [0, \infty)$, the stochastic cumulative cost process given by

$$\mathbf{C}(t) := \int_0^t f(\mathbf{X}(s))ds,$$

while for $T = I\!N \bigcup \{0\}$ it is given by

$$\mathbf{C}(t) := \sum_{n=0}^{t} f(\mathbf{X}(n)).$$

We always assume that this stochastic cumulative cost process $\mathbf{C}$ is well-defined and $I\!E\mathbf{C}(t) < \infty$ for every $t \in T$. In order to keep the size of this paper limited, we only consider pure regenerative processes with $T = [0, \infty)$. Observe that the next result can be easily rewritten for the discrete time version.

Theorem 2.4. *If $\mathbf{X} = \{\mathbf{X}(t) : 0 \leq t < \infty\}$ is a pure regenerative process with the increasing sequence $\sigma_n, n \in I\!N \bigcup \{0\}$ of random points satisfying $I\!E\sigma_1 < \infty$ and equipped with a nonnegative Borel measurable function f satisfying*

$$I\!E\left(\int_0^{\sigma_1} f(\mathbf{X}(s))ds\right) < \infty$$

then it follows that

$$\lim_{t \uparrow \infty} \frac{1}{t} I\!E\mathbf{C}(t) = \frac{1}{I\!E\sigma_1} I\!E\left(\int_0^{\sigma_1} f(\mathbf{X}(s))ds\right).$$

By obvious reasons the above limit is called the *average cost* with respect to the costrate function f. The proof of the above result uses a standard renewal argument and the weak renewal theorem (cf. [1], [13]). In the next section we show how the theory of pure regenerative processes can be applied in the analysis of single item inventory models (cf. [15]).

3. A General Framework for Single Item Inventory Models

In single item inventory control the decision maker is dealing with two objectives. First of all, he likes to control the cost of keeping inventory, and secondly, he likes to maintain a certain service level (cf. [16]). Therefore the decision maker faces two main questions. These questions are when to order and how much to order. Clearly the question of how much to order depends on the demand process the decision maker is expecting in the future. Moreover, if at some time in the future a stockout occurs, the decision maker needs to know whether the excess demand is lost or can be backlogged. To model this situation, we need to distinguish between the so-called lost sales case or the backlog case. Lost sales means that any demand arising when the system is out of stock is lost. Backlogging implies that the demand occurring during the stockout period will be filled as soon as a new replenishment arrives. In this paper we will only consider the backlog case. Clearly, the amount of backlog depends on the time it takes before a new replenishment order arrives. Therefore we introduce the following assumption with respect to the arrival of orders.

Assumption 3.1. *If an order is placed at some time t this order arrives at the facility at time $t + L$ with $L > 0$ a given fixed constant.*

The constant L in Assumption 3.1 is called the *lead time*. When L equals zero this corresponds to instantaneous replenishments. By the above observations the inventory manager faces two different forms of risk. The first type of risk is the probability of not being able to satisfy demand directly from stock. The second type of risk is the generation of costs due to unnecessarily high average inventory levels. To model the possibility of not being able to satisfy demand directly, the following service measures are used (cf. [15]):

- *No-stockout service measure or P_1-measure*: the fraction of cycles in which a stockout does not occur. Observe that the time interval between the arrival of two consecutive replenishment orders is called a cycle.
- *Fill rate service measure or P_2-measure*: the fraction of demand satisfied directly from stock.
- *Ready rate service measure or P_3-measure*: the fraction of time that the inventory level is positive.

Since usually managers aim to minimize total costs we need to assign penalty costs to the occurance of events described by the above service measures (cf. [15]). The most often used penalty costs are given by the following:

- *Fixed cost (b_1) per stockout occasion*: for each stockout a fixed cost b_1 is determined and the total cost depends only on the number of stockouts and it is independent of the magnitude or duration of the stockout.
- *Fractional charge (b_2) per unit short*: for each unit short a fixed cost b_2 is determined and the total cost depends only on the total number of items short and not on the total time items are short.
- *Fractional charge (b_3) per unit short per unit time*: for each unit short a fixed cost b_3 is determined per unit of time that this item is short and so the total cost depends on the number of short items and the time that these items are short.

Before presenting how these costs can be incorporated in our analysis, let us introduce the basic processes which describe the inventory system. To describe the behaviour of the inventory level we need to introduce the *demand process* for a single item. For this purpose we consider a stochastic process $\mathbf{D} = \{\mathbf{D}(t) : t \geq 0\}$, with state space $[0, \infty)$ or $\mathbb{N} \bigcup \{0\}$. The random variable $\mathbf{D}(t)$ represents the total demand up to time t. By its definition the process $\mathbf{D}$ is increasing and it is assumed that this process is càdlàg. In case the state space is $[0, \infty)$ the item is called *indivisible* (for example, gasoline) and in case the state space is $\mathbb{N} \bigcup \{0\}$ the item is called *divisible*. It is assumed that the customers arrive according to some renewal process $\mathbf{N} = \{\mathbf{N}(t) : t \geq 0\}$ with independent and identically distributed interarrival times $\mathbf{T}_i, i \in I\!N$. Moreover, the nth arriving customer has a random demand $\mathbf{Y}_n$ and the random variables $\mathbf{Y}_n, n \in I\!N$ are independent and identically distributed with common cumulative distribution function F_Y satisfying $F_Y(0) = 0$ and $F_Y(\infty) = 1$. Observe, in case the item is indivisible, it is always assumed that the cumulative distribution function F_Y is continuous. We also assume that these random variables, representing the individual demands, are independent of the arrival process $\mathbf{N}$. Moreover, to define the cost assigned to an inventory model governed by some inventory control rule, to be specified later, we need to introduce the following different inventory processes. These processes are defined on the same state space as the demand process. First of all, consider the stochastic process $\mathbf{I} = \{\mathbf{I}(t) : t \geq 0\}$ with

$$\mathbf{I}(t) := \text{actual stock on the shelves at time } t.$$

This process is called the on-hand stock process and sometimes it is also referred to as the inventory level process. Since we assume that the demand process is càdlàg this process is also càdlàg. Another inventory process is given by the càdlàg process $\mathbf{B} = \{\mathbf{B}(t) : t \geq 0\}$ with $\mathbf{B}(0) = 0$ and

$$\mathbf{B}(t) := \text{amount of items backlogged at time } t.$$

Using the definition of the on-hand stock process and the backlog process we obtain the so-called netstock or net inventory process $\mathbf{IN} = \{\mathbf{IN}(t) : t \geq 0\}$ given by

$$\mathbf{IN}(t) := \mathbf{I}(t) - \mathbf{B}(t)$$

for every $t \geq 0$. Again $\mathbf{IN}$ is càdlàg and by the definition of the backlog process, it follows that $\mathbf{IN}(t) > 0$ implies $\mathbf{B}(t) = 0$ and $\mathbf{IN}(t) < 0$ implies $\mathbf{I}(t) = 0$. The definition of the netstock process implies that the cost of an inventory system, governed by some inventory control rule, depends only on the stochastic process $\mathbf{IN}$. Hence, if one can show that this process is a pure regenerative process, we can apply the results from Section 2 to derive the average cost in a fast and efficient way. To verify the regenerative structure of this process we need to consider the càdlàg stochastic process $\mathbf{O} = \{\mathbf{O}(t) : t \geq 0\}$ with $\mathbf{O}(0) := 0$ and

$$\mathbf{O}(t) := \text{amount of ordered and not yet delivered items at time } t.$$

The related, so-called inventory position process $\mathbf{IP} = \{\mathbf{IP}(t) : t \geq 0\}$ given by

$$\mathbf{IP}(t) := \mathbf{IN}(t) + \mathbf{O}(t),$$

for every $t \geq 0$, now plays an important role. First of all it should be clear that the decision to order depends on the inventory position process. Secondly, it is relatively easy to show that under certain conditions on the demand process the inventory position process $\mathbf{IP}$ is a pure regenerative process for the different inventory control models. Since it is shown by Sahin (cf. [14]) that there exists an easy relation, in case of backlogging, between the inventory position process and the netstock process this enables us to verify that a shifted version of the netstock process is a pure regenerative process. To start with this verification we first list the relation between the netstock and inventory position processes. The proof of this relation is easy and can be found in Sahin (cf. [14]).

Theorem 3.2. *If backordering occurs with a fixed lead time L and the stochastic process $\mathbf{D}$ is càdlàg then it follows for every $t \geq 0$ that*

$$\mathbf{IN}(t + L) = \mathbf{IP}(t) - (\mathbf{D}(t + L) - \mathbf{D}(t)), \ I\!P - \textit{almost surely.}$$

In the remainder of this section we assume that the following assumption holds. This assumption will be verified for the different classical single item inventory models.

Assumption 3.3. *The joint stochastic process* $\{(\mathbf{IP}(t), \mathbf{D}(t+L) - \mathbf{D}(t)) : t \geq 0\}$ *is a pure regenerative process with an increasing sequence* σ_n, $n \in \mathbb{N} \bigcup \{0\}$, *satisfying* $\mathbb{E}\sigma_1 < \infty$ *and this increasing sequence contains as a subset the times that a replenishment order is issued.*

It is very important to realize that Assumption 3.3 plays a crucial role in our analysis. The next result shows the connection between the theory of pure regenerative processes and inventory control models and it is an easy consequence of Theorem 2.3 and Theorem 3.2.

Theorem 3.4. *If the stochastic process* $\{(\mathbf{IP}(t), \mathbf{D}(t+L) - \mathbf{D}(t)) : t \geq 0\}$ *associated with an inventory model and a given control rule satisfies Assumption* 3.3, *then it follows that the stochastic process* $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t+L) : t \geq 0\}$ *is a pure regenerative process with the same increasing sequence.*

Based on the previous results, the next step will be to give a general expression for the different service measures and shortage costs, in case Assumption 3.3 holds. These expressions can be specialized to the different single item inventory models. Observe it is possible to give a general expression solely based on Assumption 3.3. To start we first aim to give an expression for the no-stockout service measure, and the average number β_1 of stockout occasions.

Theorem 3.5. *If the stochastic process* $\{(\mathbf{IP}(t), \mathbf{D}(t+L) - \mathbf{D}(t)) : t \geq 0\}$ *associated with an inventory model and a given control rule satisfies Assumption* 3.3, *the stockout probability* $1 - P_1$ *is given by*

$$1 - P_1 = \mathbb{P}\{\mathbf{D}((\sigma_1 + L)-) > \mathbf{IP}(0)\} - \mathbb{P}\{\mathbf{D}(L) > \mathbf{IP}(0)\}.$$

Moreover, the average number of stockout occasions is given by

$$\beta_1 = \frac{1 - P_1}{\mathbb{E}\sigma_1}.$$

Proof. By Assumption 3.3 we know that the sequence $\sigma_1, \sigma_2, \ldots$ of regeneration points contains the random points in time when an order is placed, and this order arrives at the facility L time units later. Therefore, in the random interval $[\sigma_n + L, \sigma_{n+1} + L)$, $n \in \mathbb{N} \bigcup \{0\}$ *at most one* order arrives, and if so, it will arrive at $\sigma_n + L$. Hence, the càdlàg netstock process is monotone decreasing on this interval due to the arrival of demands. Moreover, for the same reason, it is also monotone decreasing on the interval $[0, L)$. Since by definition a stockout occurs if the netstock process drops from a nonnegative value to a negative value, it follows by the previous observations that there is *at most one* stockout occasion in the random interval $[\sigma_n + L, \sigma_{n+1} + L)$ for every $n \in \mathbb{N} \bigcup \{0\}$. Let $A_n, n \in \mathbb{N} \bigcup \{0\}$ denote the event that a stockout occurs in $[\sigma_n + L, \sigma_{n+1} + L)$, and A_{-1} in $[0, L)$. By this definition we obtain that

$$A_k = \{\mathbf{IN}(\sigma_k + L) \geq 0, \mathbf{IN}((\sigma_{k+1} + L)-) < 0\}, \tag{3.1}$$

for every $k \in \mathbb{N} \bigcup \{0\}$, and

$$\begin{aligned} 1 - P_1 &= \text{fraction of cycles that there is a stockout} \\ &= \lim_{n \uparrow \infty} \frac{\mathbb{E}\left(\sum_{k=-1}^{n} \mathbf{1}_{A_k}\right)}{n} = \lim_{n \uparrow \infty} \frac{\sum_{k=0}^{n} \mathbb{P}\{A_k\}}{n}. \end{aligned} \tag{3.2}$$

Since Theorem 3.4 implies that the stochastic process $\tilde{\mathbf{IN}}$ is regenerative, it follows for every $k \in I\!N \bigcup \{0\}$ that

$$\mathbb{P}\{A_k\} = \mathbb{P}\{\mathbf{IN}(L) \geq 0, \mathbf{IN}((\sigma_1 + L)-) < 0\}. \tag{3.3}$$

In order to evaluate this probability we observe that

$$\mathbb{P}\{\mathbf{IN}((\sigma_1 + L)-) < 0\} = \mathbb{P}\{\mathbf{IN}(L) < 0, \mathbf{IN}((\sigma_1 + L)-) < 0\} + \mathbb{P}\{A_k\}, \tag{3.4}$$

for every $k \in I\!N \bigcup \{0\}$. Moreover, since the sample paths of $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t + L) : t \geq 0\}$ are decreasing on $[0, \sigma_1)$, we obtain that

$$\mathbb{P}\{\mathbf{IN}(L) < 0, \mathbf{IN}((\sigma_1 + L)-) < 0\} = \mathbb{P}\{\mathbf{IN}(L) < 0\}. \tag{3.5}$$

These observations, together with relations (3.3), (3.4), (3.5) and Theorem 3.2 imply that

$$\begin{aligned} \mathbb{P}\{A_k\} &= \mathbb{P}\{\mathbf{IN}((\sigma_1 + L)-) < 0\} - \mathbb{P}\{\mathbf{IN}(L) < 0\} \\ &= \mathbb{P}\{\mathbf{D}((\sigma_1 + L)-) > \mathbf{IP}(0)\} - \mathbb{P}\{\mathbf{D}(L) > \mathbf{IP}(0)\}. \end{aligned} \tag{3.6}$$

Substituting this into relation (3.2), the first part of the theorem is verified.

If we denote

$$\mathbf{N}_\sigma(t) := \sup\{n \in I\!N \bigcup \{0\} : \sigma_n \leq t\},$$

representing the total number of completed cycles up to time $t+L$, then the average number of stockout occasions β_1 is given by

$$\beta_1 = \lim_{t \uparrow \infty} \frac{\mathbb{E}\left(\sum_{k=0}^{\mathbf{N}_\sigma(t)} \mathbf{1}_{A_k}\right)}{t}. \tag{3.7}$$

As already observed the stochastic process $\tilde{\mathbf{IN}}$ is regenerative and this implies for arbitrary $k \in I\!N \bigcup \{0\}$ and relation (3.1) that the event

$$\{\mathbf{N}_\sigma(t) \geq k\} = \{\sigma_k \leq t\}$$

is independent of the event $\mathbf{1}_{A_k}$. This observation together with the weak renewal theorem (cf. [13]) and relations (3.7),(3.6) and (3.2) implies that

$$\beta_1 = \lim_{t \uparrow \infty} \frac{\mathbb{E}(1 + \mathbf{N}_\sigma(t))\mathbb{P}\{A_0\}}{t} = \lim_{t \uparrow \infty} \frac{(1 + \mathbb{E}\mathbf{N}_\sigma(t))(1 - P_1)}{t} = \frac{1 - P_1}{\mathbb{E}\sigma_1}.$$

This shows the second part of the theorem. □

Next we determine the fraction of demand satisfied directly from stock and we also give an expression for the average number of items short.

Theorem 3.6. *If the stochastic process* $\{(\mathbf{IP}(t), \mathbf{D}(t+L) - \mathbf{D}(t)) : t \geq 0\}$ *associated with an inventory model and a given control rule satisfies Assumption* 3.3, *then it follows that the average number* β_2 *of items short is given by*

$$\beta_2 = \frac{\mathbb{E}(\max\{\mathbf{D}((\sigma_1 + L)-) - \mathbf{IP}(0), 0\}) - \mathbb{E}(\max\{\mathbf{D}(L) - \mathbf{IP}(0), 0\})}{\mathbb{E}(\sigma_1)}.$$

Moreover, if $a = \lim_{t \uparrow \infty} \frac{\mathbb{E}(\mathbf{D}(t))}{t} > 0$ *exists, the fraction* P_2 *of demand satisfied directly from stock is given by*

$$P_2 = 1 - \frac{\beta_2}{a}.$$

Proof. Introduce the stochastic process $\mathbf{NIS} := \{\mathbf{NIS}(t) : t \geq 0\}$ with

$$\mathbf{NIS}(t) := \text{the total number of items short up to time } t.$$

By definition we obtain:

$$\beta_2 := \lim_{t\uparrow\infty} \frac{\mathbb{E}(\mathbf{NIS}(t+L))}{t+L} \quad \text{and} \quad P_2 := \lim_{t\uparrow\infty} \frac{\mathbb{E}(\mathbf{D}(t+L)) - \mathbb{E}(\mathbf{NIS}(t+L))}{\mathbb{E}(\mathbf{D}(t+L))}.$$

This implies for the P_2 measure that:

$$P_2 = 1 - \lim_{t\uparrow\infty} \frac{\mathbb{E}(\mathbf{NIS}(t+L))}{\mathbb{E}(\mathbf{D}(t+L))} = 1 - \frac{\beta_2}{a},$$

and so we only have to derive an expression for β_2.

By Assumption 3.3 *at most* one replenishment order arrives in the random interval $[\sigma_n + L, \sigma_{n+1} + L)$, and if so, it will arrive at $\sigma_n + L$. Hence, the càdlàg backlog process $\mathbf{B} = \{\mathbf{B}(t) : t \geq 0\}$ has a possible jump at $\sigma_n + L$ and it is monotone increasing on the interval $[\sigma_n + L, \sigma_{n+1} + L)$. Denoting by $\mathbf{NIS}_n$, $n \in \mathbb{N} \bigcup \{0\}$ the total number of items short on the interval $[\sigma_n + L, \sigma_{n+1} + L)$, it follows by the previous observations that

$$\begin{aligned} \mathbf{NIS}_n &= \mathbf{B}((\sigma_{n+1}+L)-) - \mathbf{B}(\sigma_n + L) \\ &= \max\{-\mathbf{IN}((\sigma_{n+1}+L)-), 0\} - \max\{-\mathbf{IN}(\sigma_n + L), 0\}, \end{aligned} \tag{3.8}$$

for every $n \in \mathbb{N} \bigcup \{0\}$. Introducing now the random variable $\mathbf{NIS}_{-1}$, which represents the total number of items short in the interval $[0, L)$, we similarly obtain

$$\mathbf{NIS}_{-1} = \mathbf{B}(L-) = \max\{-\mathbf{IN}(L-), 0\}.$$

By Theorem 3.4 and relation (3.8) the random variables $\mathbf{NIS}_n$, $n \in \mathbb{N} \bigcup \{0\}$ are identically distributed with

$$\begin{aligned} \mathbf{NIS}_n &\stackrel{d}{=} \max\{-\mathbf{IN}((\sigma_1+L)-), 0\} - \max\{-\mathbf{IN}(L), 0\} \\ &= \max\{\mathbf{D}((\sigma_1+L)-) - \mathbf{IP}(0), 0\} - \max\{\mathbf{D}(L) - \mathbf{IP}(0), 0\}, \end{aligned} \tag{3.9}$$

and '$\stackrel{d}{=}$' denoting 'distributed as'. Moreover, again by Theorem 3.4 and relation (3.8) it follows that the random variable $\mathbf{NIS}_n$ is independent of the random variables $\sigma_1, \ldots, \sigma_n$. Now it is easy to verify that

$$\sum_{n=-1}^{\mathbf{N}_\sigma(t)} \mathbf{NIS}_n \leq \mathbf{NIS}(t) \leq \sum_{n=-1}^{\mathbf{N}_\sigma(t)+1} \mathbf{NIS}_n \tag{3.10}$$

for every $t \geq 0$. By a similar argument as used in Theorem 3.16 of Ross [13] and the argument used at the end of the proof of Theorem 3.5, we obtain that

$$\lim_{t\uparrow\infty} \frac{\mathbb{E}\left(\mathbf{NIS}_{N(t)+1}\right)}{t} = 0.$$

and this implies together with (3.10), that

$$\beta_2 = \lim_{t\uparrow\infty} \frac{\mathbb{E}\left(\mathbf{NIS}(t+L)\right)}{t+L} = \frac{\mathbb{E}\left(\mathbf{NIS}_0\right)}{\mathbb{E}\sigma_1}.$$

Applying now relation (3.9) yields the desired result. □

Finally we derive the expression for the ready rate measure.

Theorem 3.7. *If the stochastic process* $\{(\mathbf{IP}(t), \mathbf{D}(t+L) - \mathbf{D}(t)) : t \geq 0\}$ *associated with an inventory model and a given control rule satisfies Assumption* 3.3*, then it follows that the ready rate has the form*

$$P_3 = \frac{\mathbb{E}\left(\int_L^{\sigma_1+L} \mathbf{1}_{\{D(y)>IN(0)\}} dy\right)}{\mathbb{E}(\sigma_1)}.$$

Proof. To determine the ready rate P_3 we need to evaluate the expression

$$\lim_{n\uparrow\infty} \frac{\mathbb{E}\left(\int_0^t \mathbf{1}_{\{I(y)>0\}} dy\right)}{t}$$

Since $\mathbf{I}(t) = \max\{\mathbf{IN}(t), 0\}$ it follows by Theorem 3.4 and Theorem 2.4 that

$$\lim_{t\uparrow\infty} \frac{\mathbb{E}\left(\int_0^t \mathbf{1}_{\{I(y)>0\}} dy\right)}{t} = \lim_{t\uparrow\infty} \frac{\mathbb{E}\left(\int_0^t \mathbf{1}_{\{I(y+L)>0\}} dy\right)}{t} = \frac{\mathbb{E}\left(\int_L^{\sigma_1+L} \mathbf{1}_{\{D(y)<IN(0)\}} dy\right)}{\mathbb{E}(\sigma_1)}.$$

□

This concludes our general discussion on single item inventory models. In the next section we will apply the results of this section to a number of basic inventory models. In all the cases we assume that the costs of inventory, related to the netstock process, are determined by a cost rate function f. Moreover, every time an order is placed a fixed setup cost K has to be paid.

4. Analysis of the inventory models

In the inventory literature some of the basic control rules have been studied extensively (cf. [15]). The (s, S) policy is the most frequently analyzed, though very often in the discrete time case (i.e. periodic review (s, S) policy with unit period lengths). To analyze this model, Sahin (cf. [14]) also used the theory of regenerative processes. However, his approach is less probabilistic and more based on the analysis of the underlying renewal equation. This makes his proofs more complicated and long. Also, inventory costs are based on period ending inventory and approximations of the cost formula are used for optimization purposes. Following the ideas of Chen and Zheng (cf. [3]) we also analyze the (s, nQ) model. The above inventory models are examples of the so called continuous review systems. However, in many practical situations, it is only possible to place replenishment at certain points in time. If this holds, the above continuous review policies are replaced by the periodic review (R, S), (R, s, S) and (R, s, nQ) policies. An important advantage of these models is the easy coordination of replenishment orders for different items.

The remainder of this section is organized as follows: first, we will discuss the periodic review inventory models described above. For each model, we will show that the regenerative property is satisfied and an expression for the average cost and the service measures is derived. Thereafter, a similar analysis is presented for the continuous review models.

4.1. **Periodic review inventory models.** In this section we subsequently analyze the (R, S), (R, s, S) and (R, s, nQ) models.

4.1.1. *The (R, S) inventory model.* Under this rule every R time units the inventory position process is inspected and an order is placed if the level of the inventory position process at the inspection time is below S. Moreover, the size of the order is such that all excess demand up to that time will be backlogged and the inventory position process is raised to order-up-to level S. Hence, the variables $R > 0$ and $S > 0$ are decision variables and need to be chosen optimally depending upon the chosen cost structure. It is assumed that the demand process $\mathbf{D}$ associated with the (R, S) inventory control model is a compound Poisson process with arrival rate $\lambda > 0$, and the used nonnegative costrate function f satisfies

$$\mathbb{E}\left(\int_0^t f(\mathbf{IN}(s))ds\right) < \infty$$

for every $t \geq 0$. Without loss of generality, the initial inventory level is set at S, or, equivalently, $\mathbf{IP}(0) = \mathbf{IN}(0) = S$. If this does not hold, we need to apply the theory of delayed regenerative processes (cf. [1]). It is now possible to show the following key result for the (R, S) model. Observe that the proof includes both divisible and indivisible items.

Theorem 4.1. *For any (R, S) model with a compound Poisson demand process the stochastic process $\{(\mathbf{IP}(t), \mathbf{D}(t + L) - \mathbf{D}(t)) : t \geq 0\}$ is a pure regenerative process with the increasing sequence of points given by $nR, n \in \mathbb{N} \bigcup \{0\}$. Moreover, the netstock process $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t + L) : t \geq 0\}$ is a pure regenerative process with the same sequence of increasing points.*

Proof. To prove the first part of this result we only need to verify Definition 2.1. To show that this holds, we first observe (cf. [12]) that the shifted stochastic process

$$\mathbf{D}_{nR} := \{\mathbf{D}(t + nR) - \mathbf{D}(nR) : t \geq 0\}$$

has the same distribution as the process $\mathbf{D}$ for every $n \in \mathbb{N}$, and so, the stochastic process $\mathbf{D}_{nR}$ is again a compound Poisson process. By the definition of the (R, S) rule we obtain that $\mathbf{IP}(nR) = S$, and the process $\{\mathbf{IP}(t+nR) : t \geq 0\}$ is completely determined by $\mathbf{IP}(nR)$ and the stochastic process $\mathbf{D}_{nR}$. This implies that the joint stochastic process $\{(\mathbf{IP}(t + nR), \mathbf{D}(t + nR + L) - \mathbf{D}(t + nR)) : t \geq 0\}$ is a function, independent of n, of the stochastic process $\mathbf{D}_{nR}$, and so, by the previous observations its distribution is independent of n. Clearly, the second part is an immediate consequence of the first part and Theorem 3.4. □

It is also easy to see by the definition of the (R, S) policy, that the history of the inventory position process $\mathbf{IP}$ up to time s is completely determined by the history of the demand process up to time s, given by $\{\mathbf{D}(t) : t \leq s\}$. This implies, due to the stationary and independent increments of a compound Poisson process (cf. [12]) that the random variables $\mathbf{IP}(t+nR)$ and $\mathbf{D}(t+nR+L) - \mathbf{D}(t+nR)$ are independent for any $t \geq 0$ and $n \in \mathbb{N} \bigcup \{0\}$. Moreover, for every $0 \leq t < R$, the relation $\mathbf{IN}(t + L) = S - \mathbf{D}(t + L)$ holds. It is now possible to identify the average cost of an (R, S) model.

Theorem 4.2. *For any (R, S) model with a compound Poisson demand process it follows that the average cost $\Phi(R, S)$ of an (R, S) policy with nonnegative cost rate function f and ordering cost $K > 0$ is given by*

$$\Phi(R, S) = \frac{K(1 - \exp(-\lambda R)) + \int_0^R \mathbb{E}f(S - \mathbf{D}(t + L))dt}{R}.$$

Proof. Introducing the function $C : [0, \infty) \to \mathbb{R}$ given by

$$C(t) := \mathbb{E}\left(\int_0^t f(\mathbf{IN}(s))ds\right),\ t \geq 0$$

we obtain for every $t \geq 0$ that

$$C(t+L) - C(L) = \mathbb{E}\left(\int_L^{t+L} f(\mathbf{IN}(s))ds\right) = \mathbb{E}\left(\int_0^t f(\mathbf{IN}(s+L))ds\right).$$

Since by Theorem 4.1 the stochastic process $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t+L) : t \geq 0\}$ is a pure regenerative process with regeneration points $nR, n \in \mathbb{N} \bigcup \{0\}$ it follows by Theorem 2.4 and the finiteness of $C(L)$ that

$$\lim_{t\uparrow\infty} \frac{1}{t} C(t) = \lim_{t\uparrow\infty} \frac{1}{t}\left(C(t+L) - C(L)\right) = \frac{1}{R}\mathbb{E}\left(\int_0^R f(\mathbf{IN}(t+L))dt\right).$$

Using now $\mathbf{IN}(t+L) = S - \mathbf{D}(t+L)$ for every $0 \leq t < R$ this yields that

$$\lim_{t\uparrow\infty} \frac{1}{t} C(t) = \frac{1}{R}\mathbb{E}\left(\int_0^R f(S - \mathbf{D}(t+L))dt\right).$$

Applying now the renewal reward theorem (cf. Ross [13]) it is easy to verify by the memoryless property of the compound Poisson process that the average ordering costs are given by $\frac{1}{R}K(1 - \exp(-\lambda R))$. As a consequence, the expression of the average cost follows by adding the two components. □

By the regenerative structure of the shifted netstock process $\tilde{\mathbf{IN}}$ it should be clear that it is sufficient and necessary to assume that

$$\mathbb{E}\left(\int_0^R f(\mathbf{IN}(s+L))ds\right) < \infty$$

in order to obtain finite average costs. If we want to derive the expressions for the different service measures then we can use Theorem 4.1 and the general formulas for these measures presented in Section 3. Observe, if we consider an (R, S) inventory model with a compound Poisson demand process then the average demand rate a is given by $\lambda \mathbb{E}\mathbf{Y}_1$, by Theorem 4.1 the random variable σ_1 equals R, and $\mathbf{IP}(0) = S$. Applying now Theorems 3.5 up to 3.7 we immediately obtain the following result.

Theorem 4.3. *For any (R, S) model, with a compound Poisson demand process it follows that the stockout probability $1 - P_1$ is given by*

$$1 - P_1 = \mathbb{P}\{\mathbf{D}(R+L) > S\} - \mathbb{P}\{\mathbf{D}(L) > S\}.$$

Moreover, the average number β_2 of items short has the form

$$\beta_2 = \frac{\mathbb{E}(\max\{\mathbf{D}(R+L) - S, 0\}) - \mathbb{E}(\max\{\mathbf{D}(L) - S, 0\})}{R},$$

while the fraction P_2 of demand satisfied directly from stock equals

$$P_2 = 1 - \frac{\beta_2}{\lambda \mathbb{E}\mathbf{Y}_1}.$$

Finally, the ready rate measure P_3 boils down to

$$P_3 = \frac{\int_L^{R+L} \mathbb{P}\{D(y) > S\}dy}{R}.$$

The expressions in Theorems 4.2 and 4.3 may look complicated, but as will be shown in [2] it is easy to give analytical formulas for associated Laplace transforms. Hence we may apply (cf. [2]) a newly developed Laplace transform inversion algorithm (cf. [10]) to generate accurate computations.

In the next section we consider a generalization of the (R, S) inventory model.

4.1.2. *The* (R, s, S) *inventory model.* Under this rule every R time units the inventory position process is inspected and an order is placed if the level of the inventory position process is below level $s \leq S$. If this is not the case no order is triggered and one waits until the next inspection time. The order size is now similarly determined as for the (R, S) rule, and the decision variables are S, s and R. As before, the demand process is a compound Poisson process with arrival rate $\lambda > 0$, and the used nonnegative costrate function f satisfies

$$\mathbb{E}\left(\int_0^t f(\mathbf{IN}(s))ds\right) < \infty$$

for every $t \geq 0$. Assume without loss of generality that the initial inventory level is set at S, or, equivalently, $\mathbf{IP}(0) = \mathbf{IN}(0) = S$. To analyze the (R, s, S) model we define iteratively the stopping times $\sigma_n, n \in \mathbb{N} \bigcup \{0\}$ $(\sigma_0 := 0)$, with respect to the demand process $\mathbf{D}$ given by

$$\sigma_1 := \min\{nR : \mathbf{D}(nR) > S - s, n \in \mathbb{N}\}. \tag{4.1}$$

This random variable represents the first inspection time, where the total demand exceeds the quantity $S - s$. This results in the inventory level dropping below level s and so a replenishment order is triggered. We define now inductively

$$\sigma_{n+1} := \sigma_n + \min\{mR : \mathbf{D}(\sigma_n + mR) - \mathbf{D}(\sigma_n) > S - s, m \in \mathbb{N}\}, \quad n \in \mathbb{N}. \tag{4.2}$$

Since $\sigma_n, n \in \mathbb{N}$ is an increasing sequence of stopping times with respect to the compound Poisson process $\mathbf{D}$, we obtain by Theorem 32 of Protter (cf. [12]) that for every $n \in \mathbb{N}$ the stochastic process

$$\mathbf{D}_{\sigma_n} := \{\mathbf{D}(t + \sigma_n) - \mathbf{D}(\sigma_n) : t \geq 0\}$$

is again a compound Poisson process, with the same distribution as $\mathbf{D}$, and this stochastic process $\mathbf{D}_{\sigma_n}$ is independent of the process $\{\mathbf{D}(t) : t \leq \sigma_n\}$. Moreover, it follows that the event $\sigma_1 < \sigma_2 < \ldots < \sigma_n \leq t$ is completely determined by the history of the process $\mathbf{D}$ up to time t (this is the definition of a stopping time), and so by the previous result we also obtain that the stochastic process $\mathbf{D}_{\sigma_n}$ is independent of the random variables $\sigma_0, \sigma_1, \ldots, \sigma_n$. By these observations it follows that the random variables $\sigma_{n+1} - \sigma_n, n \in \mathbb{N}$ are independent and identically distributed. To determine their distribution we observe for every $k \in \mathbb{N} \bigcup \{0\}$ that

$$\mathbb{P}\{\sigma_{n+1} - \sigma_n > kR\} = \mathbb{P}\{\mathbf{D}(kR) \leq S - s\} = F_{D(R)}^{k^*}(S - s).$$

In the above formula $F_{D(R)}^{k^*}$ denotes the k-fold convolution of the cumulative distribution function $F_{D(R)}$, given by

$$F_{D(R)}(x) := \mathbb{P}\{\mathbf{D}(R) \leq x\}.$$

Using the above observations it is easy to show the following result.

Theorem 4.4. *For any* (R, s, S) *model with a compound Poisson demand process the stochastic process* $\{(\mathbf{IP}(t), \mathbf{D}(t + L) - \mathbf{D}(t)) : t \geq 0\}$ *is a pure regenerative process with the increasing sequence of random points given by relation* (4.1) *and*

(4.2). *Moreover, the netstock process* $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t+L) : t \geq 0\}$ *is a pure regenerative process with the same sequence of increasing points.*

Proof. To show the regenerative property of the considered stochastic processes we need to check Definition 2.2. We already showed before Theorem 4.4 that the differences $\sigma_{n+1} - \sigma_n, n \in \mathbb{N} \bigcup \{0\}$ with σ_n iteratively defined by (4.1) and (4.2) are independent and identically distributed, and hence the first condition of Definition 2.2 is verified. To show that the second condition of Definition 2.2 also holds we observe by the definition of the (R, s, S) rule that $\mathbf{IP}(\sigma_n) = S$, and therefore the stochastic process $\{\mathbf{IP}(t + \sigma_n) : t \geq 0\}$ is completely determined by the demand process $\mathbf{D}_{\sigma_n}$. By this observation it follows that the joint stochastic process

$$\{(\mathbf{IP}(t + \sigma_n), \mathbf{D}(t + \sigma_n + L) - \mathbf{D}(t + \sigma_n)) : t \geq 0\}$$

is a function, independent of n, of the stochastic process $\mathbf{D}_{\sigma_n}$. This implies by the observation before Theorem 4.4 that the process

$$\{(\mathbf{IP}(t + \sigma_n), \mathbf{D}(t + \sigma_n + L) - \mathbf{D}(t + \sigma_n)) : t \geq 0\}$$

is independent of $\sigma_0, \ldots, \sigma_n$ and this verifies the second condition of Definition 2.2. Since the process $\{\mathbf{IP}(t+\sigma_n), \mathbf{D}(t+\sigma_n+L) - \mathbf{D}(t+\sigma_n) : t \geq 0\}$ is a function of the stochastic process $\mathbf{D}_{\sigma_n}$ and the distribution of $\mathbf{D}_{\sigma_n}$ is independent of n, condition 3 of Definition 2.2 also holds. Hence we have verified the first part of the theorem and the second part is an immediate consequence of this part and Theorem 3.4. □

Since the demand process is a compound Poisson process it follows again by the observation before Theorem 4.4 that the random variables $\mathbf{IP}(t + \sigma_n)$ and $\mathbf{D}(t + \sigma_n + L) - \mathbf{D}(t + \sigma_n)$ are independent for every given $t \geq 0$ and $n \in \mathbb{N} \bigcup \{0\}$. Moreover, if $0 \leq t < \sigma_1$ we obtain the relation $\mathbf{IN}(t + L) = S - \mathbf{D}(t + L)$. This observation enables us to identify the average cost of an (R, s, S) model.

Theorem 4.5. *For any* (R, s, S) *model, with a compound Poisson demand process and the stopping time* σ_1 *defined by relation* (4.1)*, the average cost* $\Phi(R, s, S)$ *of an* (R, s, S) *policy with nonnegative costrate function* f *and ordering cost* $K > 0$*,is given by*

$$\Phi(R, s, S) = \frac{K + \mathbb{E}\left(\int_0^{\sigma_1} f(S - \mathbf{D}(t + L))dt\right)}{\mathbb{E}\sigma_1}$$

with $\mathbb{E}\sigma_1 = RU_0(S-s)$ *and* $U_0(t) := \sum_{n=0}^{\infty} F_{D(R)}^{n^*}(t)$ *the renewal function associated with the distribution* $F_{D(R)}$.

Proof. Introduce the random variable $\tau := \min\{n \in \mathbb{N} \bigcup \{0\} : \mathbf{D}(nR) > S - s\}$. By the definition of the stopping time σ_1 it follows that $\sigma_1 = R\tau$ and so $\mathbb{E}\sigma_1 = R\mathbb{E}\tau$. Observe now for every $n \in \mathbb{N} \bigcup \{0\}$ that

$$\mathbb{P}\{\tau \geq n + 1\} = \mathbb{P}\{\mathbf{D}(nR) \leq S - s\} = F_{D(R)}^{n^*}(S - s)$$

which implies

$$\mathbb{E}(\tau) = \sum_{n=0}^{\infty} \mathbb{P}\{\tau \geq n + 1\} = 1 + \sum_{n=1}^{\infty} F_{D(R)}^{n^*}(S - s) = U_0(S - s).$$

Since $F_{D(R)}(0) = \exp(-\lambda R) < 1$, and hence $U_0(S - s) < \infty$ (cf. [13]), we obtain that $\mathbb{E}\sigma_1 = RU_0(S - s) < \infty$. As before, we introduce the function $C : [0, \infty) \to \mathbb{R}$

given by

$$C(t) := I\!E\left(\int_0^t f(\mathbf{IN}(s))ds\right)$$

and this yields for every $t \geq 0$ that

$$C(t+L) - C(L) = I\!E\left(\int_L^{t+L} f(\mathbf{IN}(s))ds\right) = I\!E\left(\int_0^t f(\mathbf{IN}(s+L))ds\right).$$

Since $C(L)$ is finite and by Theorem 4.4 the stochastic process $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t+L) : t \geq 0\}$ is a pure regenerative process with regeneration points $\sigma_n, n \in I\!N \bigcup \{0\}$ given by (4.1) and (4.2) it follows by Theorem 4.4 and Theorem 2.4 that

$$\lim_{t\uparrow\infty} \frac{1}{t} C(t) = \frac{I\!E\left(\int_0^{\sigma_1} f(\mathbf{IN}(t+L))dt\right)}{I\!E\sigma_1} = \frac{I\!E\left(\int_0^{\sigma_1} f(S - \mathbf{D}(t+L))dt\right)}{I\!E\sigma_1}.$$

Again by the renewal reward theorem we obtain that the average ordering costs are given by $K/I\!E\sigma_1$ and combining the two components yields the desired result. □

If we want to derive the expressions for the different service measures then we can use Theorem 4.4 and the general formulas for these measures presented in Section 3. Observe, if we consider an (R,s,S) inventory model with a compound demand process, then the average demand rate a is given by $\lambda I\!E\mathbf{Y}_1$, σ_1 is given by relation (4.1), and $\mathbf{IP}(0) = S$. Applying now Theorems 3.5 up to 3.7 we immediately obtain the following result.

Theorem 4.6. *For any (R,s,S) model, with a compound Poisson demand process it follows that the stockout probability $1 - P_1$ is given by*

$$1 - P_1 = I\!P\{\mathbf{D}(\sigma_1 + L) > S\} - I\!P\{\mathbf{D}(L) > S\}.$$

Moreover, the average number β_2 of items short has the form

$$\beta_2 = \frac{I\!E(\max\{\mathbf{D}(\sigma_1+L) - S, 0\}) - I\!E(\max\{\mathbf{D}(L) - S, 0\})}{RU_0(S-s)},$$

while the fraction P_2 of demand satisfied directly from stock equals

$$P_2 = 1 - \frac{\beta_2}{\lambda I\!E\mathbf{Y}_1}.$$

Finally, the ready rate measure P_3 boils down to

$$P_3 = \frac{I\!E\left(\int_L^{\sigma_1+L} \mathbf{1}_{\{D(t)>S\}} dt\right)}{RU_0(S-s)}.$$

The expressions in Theorems 4.5 and 4.6 may look complicated, but as will be shown in [2] it is easy to give analytical formulas for associated Laplace transforms. Hence we may apply (cf. [2]) a newly developed Laplace inversion algorithm to generate accurate computations.

Clearly, for $s = S$ we recover the simplified formulas derived in the previous subsection for the (R,S) model. Moreover, if each arriving customer has unit demand, the above formulas simplify considerably. We leave the details to the reader. In the next subsection we discuss the last periodic review model.

4.1.3. *The (R, s, nQ) inventory model.* According to this inventory rule at every $R > 0$ time units the inventory position process is inspected and an order is placed if the level of the inventory position process is below or at level $s \leq S$. If this is not the case no order is triggered and one waits until the next inspection time. The order size is chosen to be an integer multiple of Q, such that after ordering all excess demand will be backlogged and the inventory position process will be between s and $s + Q$. Again, the demand process is a compound Poisson process with arrival rate $\lambda > 0$ and the nonnegative costrate function f satisfies

$$\mathbb{E}\left(\int_0^t f(\mathbf{IN}(s))ds\right) < \infty \text{ for every } t \geq 0.$$

To show that the inventory position process, equipped with some suitably chosen initial distribution, is a pure regenerative process, we first consider the embedded stochastic process $\{\mathbf{IP}_n : n \in \mathbb{N} \bigcup \{0\}\}$ given by $\mathbf{IP}_n := \mathbf{IP}(nR)$. This means that we only consider the review moments of the inventory control model and by the definition of this control rule it follows for every $n \in \mathbb{N} \bigcup \{0\}$ that

$$\mathbf{IP}_{n+1} = \mathbf{IP}_n - (\mathbf{D}((n+1)R) - \mathbf{D}(nR)) + \mathbf{c}_n Q$$

with $\mathbf{c}_n := \inf\{k \in \mathbb{N} \bigcup \{0\} : kQ > s - \mathbf{IP}_n + (\mathbf{D}((n+1)R) - \mathbf{D}(nR))\}$. Since the demand process is compound Poisson, an equivalent representation is given by

$$\mathbf{IP}_{n+1} = \mathbf{IP}_n - \mathbf{B}_n + \mathbf{c}_n Q,$$

with $\mathbf{c}_n := \inf\{k \in \mathbb{N} \bigcup \{0\} : kQ > s - \mathbf{IP}_n + \mathbf{B}_n\}$, and $\mathbf{B}_n$, $n \in \mathbb{N}$, a sequence of independent and compound Poisson distributed random variables. By the above representation it is clear that for divisible items the process $\{\mathbf{IP}_n : n \in \mathbb{N} \bigcup \{0\}\}$ is a stationary Markov chain with state space $\{s+1, \ldots, s+Q\}$, while for indivisible items it is a Markov chain with state space $(s, s+Q]$. We will now determine the invariant distribution of this Markov chain. Observe, that this invariant distribution has already been computed on page 246 of Hadley and Whitin (cf. [7]) for divisible items and to be complete we list their argument. Introducing the transition probabilities

$$a_{ij} = \mathbb{P}\{\mathbf{IP}_{n+1} = s + j \,|\, \mathbf{IP}_n = s + i\},\ i, j \in \{1, \ldots, Q\},$$

and

$$p_k = \mathbb{P}\{\mathbf{B}_n = k\},\ k \in \mathbb{N} \bigcup \{0\},$$

it follows that

$$a_{ij} = \begin{cases} \sum_{n=0}^{\infty} p_{i-j+nQ} & \text{if } 1 \leq j \leq i \leq Q \\ \sum_{n=1}^{\infty} p_{i-j+nQ} & \text{if } 1 \leq i < j \leq Q \end{cases} \tag{4.3}$$

By this observation it is easy to verify that $\sum_{j=1}^{Q} a_{ij} = 1$ for $i = 1, \ldots, Q$ and

$$\sum_{i=1}^{Q} a_{ij} = \sum_{i=1}^{j-1} \sum_{n=1}^{\infty} p_{i-j+nQ} + \sum_{i=j}^{Q} \sum_{n=0}^{\infty} p_{i-j+nQ} = \sum_{k=0}^{\infty} p_k = 1,$$

for every $1 \leq j \leq Q$, and so the transition matrix (a_{ij}) is double stochastic. The invariant distribution of the Markov chain $\{\mathbf{IP}_n : n \in \mathbb{N} \bigcup \{0\}\}$ is now given by the uniform distribution on $\{s+1, \ldots, s+Q\}$ (easy to check by substitution). This implies, with $\stackrel{d}{=}$ denoting "distributed as", that $\mathbf{IP}_n \stackrel{d}{=} s + Q\mathbf{U}$, for every $n \in \mathbb{N}$ if $\mathbf{IP}_0 \stackrel{d}{=} s + Q\mathbf{U}$ and $\mathbf{U}$ is a discrete uniformly distributed random variable on $\{\frac{1}{Q}, \ldots, 1\}$. Consulting Section VI.11 of Feller (cf. [6]), it is easy to verify that a similar result holds for the Markov chain $\mathbf{IP}$ associated with an indivisible item and in this case it follows that $\mathbf{IP}_n \stackrel{d}{=} s + Q\mathbf{U}$, for every $n \in \mathbb{N}$ if $\mathbf{IP}_0 \stackrel{d}{=} s + Q\mathbf{U}$ and $\mathbf{U}$ is a uniformly distributed random variable on $(0, 1]$. If we consider now the stochastic process $\{(\mathbf{IP}(t), \mathbf{D}(t + L) - \mathbf{D}(t)) : t \geq 0\}$ satisfying $\mathbf{IP}(0) \stackrel{d}{=} s + Q\mathbf{U}$ and the random variable $\mathbf{U}$ is independent of the demand process $\mathbf{D}$, then by a similar argument as used for the (R, S) model, and having $\mathbf{IP}(nR) \stackrel{d}{=} \mathbf{IP}(0)$ for every $n \in \mathbb{N}$, we obtain that the distribution of the stochastic process

$$\{(\mathbf{IP}(t + nR), \mathbf{D}(t + nR + L) - \mathbf{D}(t + nR)) : t \geq 0\}$$

is independent of n. By this observation the next result follows immediately.

Theorem 4.7. *For any (R, s, nQ) model with a compound Poisson demand process the stochastic process $\{(\mathbf{IP}(t), \mathbf{D}(t + L) - \mathbf{D}(t)) : t \geq 0\}$, satisfying $\mathbf{IP}_0 \stackrel{d}{=} s + Q\mathbf{U}$ and $\mathbf{U}$ independent of the demand process $\mathbf{D}$, is a pure regenerative process with the increasing sequence of points given by nR, $n \in \mathbb{N} \bigcup \{0\}$. Moreover, the shifted netstock process $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t + L) : t \geq 0\}$ is a pure regenerative process with the same sequence of points.*

Since in particular

$$\mathbf{IN}(t + L) = s + Q\mathbf{U} - \mathbf{D}(t + L),\ 0 \leq t < R,$$

the average cost of the above model is easy to derive. Observe, the random variable $\mathbf{U}$ is independent of the demand process $\mathbf{D}$ and uniformly distributed on $(0, 1]$ for an indivisible item, and discrete uniformly distributed on $\{\frac{1}{Q}, \ldots, 1\}$ for a divisible item.

Theorem 4.8. *For any (R, s, nQ) model with $\mathbf{IP}_0 \stackrel{d}{=} s + Q\mathbf{U}$ and $\mathbf{U}$ independent of the compound Poisson demand process $\mathbf{D}$, it follows that the average cost $\Phi(R, s, Q)$ with nonnegative costrate function f and ordering cost $K > 0$ is given by*

$$\Phi(R, s, Q) = \frac{K(1 - \mathbb{E}F_{D(R)}(Q\mathbf{U})) + \int_0^R \mathbb{E}(f(s + Q\mathbf{U} - \boldsymbol{D}(t + L)))dt}{R}.$$

Proof. Apply Theorem 4.7 and use a similar argument as in Theorem 4.2. □

In the article of Y.S. Zheng and F.Chen (cf. [3]) a detailed analysis of the average cost of a related model is presented. Moreover, they discuss a simple algorithm to compute the optimal inventory control rule. To compare their model with the above model we observe, that the periodic review model discussed in [3] is a discrete time inventory model with unit review periods, or equivalently $R = 1$. Moreover, in [3] costs are charged on period ending inventory levels. This means in our framework that instead of $T = [0, \infty)$ one should take $T = \mathbb{N} \bigcup \{0\}$. Since as already observed, the results in this paper can be easily adapted for $T = \mathbb{N} \bigcup \{0\}$, it follows that the average cost $\Phi(R,s,Q)$ of an (R,s,nQ) model in this case and R necessarily an integer is given by

$$\Phi(R,s,Q) = \frac{K(1-\mathbb{E}(F_{D(R)}(Q\mathbf{U}))) + \sum_{k=1}^{R} \mathbb{E}(f(s+Q\mathbf{U}-\boldsymbol{D}(k+L)))}{R},$$

and so, we obtain their average cost by substituting $R = 1$. If we want to derive the expressions for the different service measures then we can use Theorem 4.7 and the general formulas for these measures presented in Section 3. Observe, if we consider an (R,s,nQ) inventory model with a compound Poisson demand process then the average demand rate a is given by $\lambda\mathbb{E}\mathbf{Y}_1$, by Theorem 4.7 the random variable σ_1 equals R, and $\mathbf{IP}(0) = s + Q\mathbf{U}$. Applying now Theorems 3.5 up to 3.7 we immediately obtain the following result.

Theorem 4.9. *For any (R,s,nQ) model with $\mathbf{IP}_0 \stackrel{d}{=} s + Q\mathbf{U}$, $\mathbf{U}$ independent of the compound Poisson demand process $\mathbf{D}$, it follows that the stockout probability $1-P_1$ is given by*

$$1 - P_1 = \mathbb{P}\{\mathbf{D}(R+L) > s + Q\mathbf{U}\} - \mathbb{P}\{\mathbf{D}(L) > s + Q\mathbf{U}\}.$$

Moreover, the average number β_2 of items short has the form

$$\beta_2 = \frac{\mathbb{E}(\max\{\mathbf{D}(R+L) - s - Q\mathbf{U}, 0\}) - \mathbb{E}(\max\{\mathbf{D}(L) - s - Q\mathbf{U}, 0\})}{R},$$

while the fraction P_2 of demand satisfied directly from stock equals

$$P_2 = 1 - \frac{\beta_2}{\lambda\mathbb{E}\mathbf{Y}_1}.$$

Finally, the ready rate measure P_3 boils down to

$$P_3 = \frac{\int_L^{R+L} \mathbb{P}\{\mathbf{D}(t) > s + Q\mathbf{U}\}dt}{R}.$$

The expressions in Theorems 4.8 and 4.9 may look complicated, but as will be shown in [2] it is easy to give analytical formulas for associated Laplace transforms. Hence we may apply (cf. [2]) a newly developed Laplace transform inversion algorithm (cf. [10]) to generate accurate computations.

4.2. **Continuous review inventory models.** In this section we subsequently analyze the (s,S) and (s,nQ) inventory model.

4.2.1. *The (s,S) inventory model.* Under this rule an order is triggerd in the moment the level of the inventory position drops below $s < S$ and the size of the order is such that all excess demand is backlogged and the level of the inventory position process is raised to order-up-to level S. It is assumed that the demand process associated with the (s,S) policy is a compound renewal process and the used nonnegative costrate function f satisfies

$$\mathbb{E}\left(\int_0^t f(\mathbf{IN}(s))ds\right) < \infty$$

for every $t \geq 0$. Just as for the (R,S) and (R,s,S) model, we assume without loss of generality that the initial inventory level is set at S or equivalently $\mathbf{IP}(0) = \mathbf{IN}(0) = S$. Observe, by the definition of an (s,S) policy, that we can identify iteratively an increasing sequence σ_n, $n \in \mathbb{N} \bigcup \{0\}$, $\sigma_0 := 0$ of stopping times with respect to the compound renewal demand process $\mathbf{D}$, given by

$$\sigma_1 = \min\{t > 0 \, : \, \mathbf{D}(t) > S - s\} \tag{4.4}$$

and

$$\sigma_{n+1} = \sigma_n + \min\{t > 0 \, : \, \mathbf{D}(t + \sigma_n) - \mathbf{D}(\sigma_n) > S - s\}, \; n \in \mathbb{N}. \tag{4.5}$$

The random times σ_n represent the times that an order is placed and are a subset of the random arrival times of customers. By this observation it follows that the stochastic process

$$\mathbf{D}_{\sigma_n} := \{\mathbf{D}(t + \sigma_n) - \mathbf{D}(\sigma_n) : t \geq 0\}$$

is again an compound renewal process and it is independent of $\{\mathbf{D}(t) : t \leq \sigma_n\}$. Since the random variables σ_n are stopping times with respect to the demand process this also implies that the process $\mathbf{D}_{\sigma_n}$ is independent of the random variables $\sigma_0, \ldots, \sigma_n$, $n \in \mathbb{N}$. Finally, by these remarks and the relations (4.4) and (4.5) it is easy to check that the random variables $\sigma_{n+1} - \sigma_n$, $n \in \mathbb{N} \bigcup \{0\}$ are independent and identically distributed. Hence, by a similar proof as for the (R, s, S) model, one can show the next result for an (s, S) model.

Theorem 4.10. *For any (s, S) model with a compound renewal demand process, the stochastic process $\{(\mathbf{IP}(t), \mathbf{D}(t + L) - \mathbf{D}(t)) : t > 0\}$ is a pure regenerative process with the increasing sequence σ_n, $n \in \mathbb{N} \bigcup \{0\}$ of random points given by* (4.4) *and* (4.5). *Moreover, the netstock process $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t + L) : t \geq 0\}$ is a pure regenerative process with the same sequence of increasing points.*

Proof. We already showed that the differences $\sigma_{n+1} - \sigma_n, n \in \mathbb{N} \bigcup \{0\}$ with σ_n iteratively defined by (4.4) and (4.5) are independent and identically distributed and hence the first condition of Definition 2.2 is verified. We observe by the definition of the (s, S) rule that $\mathbf{IP}(\sigma_n) = S$ and therefore the stochastic process $\{\mathbf{IP}(t + \sigma_n) : t \geq 0\}$ is completely determined by the demand process $\mathbf{D}_{\sigma_n}$. By this observation it follows that the joint stochastic process

$$\{(\mathbf{IP}(t + \sigma_n), \mathbf{D}(t + \sigma_n + L) - \mathbf{D}(t + \sigma_n)) : t \geq 0\}$$

is a function of the stochastic process $\mathbf{D}_{\sigma_n}$ and this implies that the process

$$\{(\mathbf{IP}(t + \sigma_n), \mathbf{D}(t + \sigma_n + L) - \mathbf{D}(t + \sigma_n)) : t \geq 0\}$$

is independent of $\sigma_0, \ldots, \sigma_n$. This verifies the second condition of Definition 2.2. By the observation that the process $\{(\mathbf{IP}(t+\sigma_n), \mathbf{D}(t+\sigma_n+L)-\mathbf{D}(t+\sigma_n)) : t \geq 0\}$ is a function of the stochastic process $\mathbf{D}_{\sigma_n}$ and the distribution of $\mathbf{D}_{\sigma_n}$ is independent of n, the third condition of Definition 2.2 also holds. Hence $\{(\mathbf{IP}(t), \mathbf{D}(t+L)-\mathbf{D}(t)) : t > 0\}$ is a pure regenerative process and the regenerative property of the process $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t + L) : t \geq 0\}$ is an immediate consequence of Theorem 3.4. ☐

To identify the average cost of an (s, S) model with costrate function f we observe that

$$\mathbf{IN}(t + L) = S - \mathbf{D}(t + L) \;\text{ for } 0 \leq t < \sigma_1.$$

Moreover, we introduce the stopping time $\nu(S-s)$ with respect to the individual demand process $\{\mathbf{Y}_i : i \in I\!N\}$ and this random variable is given by

$$\nu(S-s) := \min\{n \in I\!N : \sum_{i=1}^{n} \mathbf{Y}_i > S-s\}.$$

It is easy to verify that $\sigma_1 = \sum_{i=1}^{\nu(S-s)} \mathbf{T}_i$. Since the individual demands $\mathbf{Y}_i, i \in I\!N$ are independent of the renewal arrival process the stopping time $\nu(S-s)$ is also independent of the random variables $\mathbf{T}_i, i \in I\!N$. By this observation we obtain (cf. [13]) that

$$I\!E(\sigma_1) = I\!E\left(\sum_{i=1}^{\nu(S-s)} \mathbf{T}_i\right) = I\!E(\mathbf{T}_1) I\!E\nu(S-s).$$

To compute the expectation $I\!E\nu(S-s)$ we observe for every $n \in I\!N$ and $\mathbf{Y}_0 := 0$ that

$$I\!P\{\nu(S-s) \geq n+1\} = I\!P\{\sum_{i=0}^{n} \mathbf{Y}_i \leq S-s\} = F_Y^{n^*}(S-s).$$

and so,

$$I\!E\nu(S-s) = \sum_{n=0}^{\infty} I\!P\{\nu(S-s) \geq n+1\} = 1 + \sum_{n=1}^{\infty} F_Y^{n^*}(S-s) = U_0(S-s),$$

with U_0 denoting the renewal function associated with the cumulative distribution function F_Y. Due to $F_Y(0) = 0$ this implies (cf. [13]) $I\!E(\sigma_1) = I\!E\mathbf{T}_1 U_0(S-s) < \infty$. It is now possible to prove the following result, which includes both the case of a divisible and indivisible item.

Theorem 4.11. *For any (s,S) model with a compound renewal process it follows that the average cost $\Phi(s,S)$ of an (s,S) policy with nonnegative costrate function f and ordering cost $K > 0$ is given by*

$$\Phi(s,S) = \frac{K + I\!E\left(\int_0^{\sigma_1} f(S - \mathbf{D}(t+L))dt\right)}{I\!E(\mathbf{T}_1)U_0(S-s)}$$

with σ_1 given by (4.4) and U_0 the renewal function associated with the cumulative distribution function F_Y.

Proof. By a similar argument as in the proof for the (R,s,S) model it follows by Theorem 4.10 and Theorem 2.4 that

$$\lim_{t\uparrow\infty} \frac{1}{t} C(t) = \frac{I\!E\left(\int_0^{\sigma_1} f(\mathbf{IN}(t+L))dt\right)}{I\!E\sigma_1} = \frac{I\!E\left(\int_0^{\sigma_1} f(S - \mathbf{D}(t+L))dt\right)}{I\!E\sigma_1}.$$

Again by the renewal reward theorem we obtain that the average ordering costs are given by $K/I\!E\sigma_1$, and combining the two components yields the desired result. □

If we want to derive the expressions for the different service measures then we can use Theorem 4.10 and the general formulas for these measures presented in Section 3. Observe, if we consider an (s,S) inventory model with a compound renewal demand process with arrival rate $\lambda > 0$, then the average demand rate a is given by $\lambda I\!E\mathbf{Y}_1$, σ_1 is given by relation (4.4), and $\mathbf{IP}(0) = S$. Applying now Theorems 3.5 up to 3.7 we immediately obtain the following result.

Theorem 4.12. *For any (s, S) model, with a compound renewal demand process it follows that the stockout probability $1 - P_1$ is given by*

$$1 - P_1 = \mathbb{P}\{\mathbf{D}(\sigma_1 + L) > S\} - \mathbb{P}\{\mathbf{D}(L) > S\}.$$

Moreover, the average number β_2 of items short has the form

$$\beta_2 = \frac{\mathbb{E}(\max\{\mathbf{D}(\sigma_1 + L) - S, 0\}) - \mathbb{E}(\max\{\mathbf{D}(L) - S, 0\})}{\mathbb{E}\mathbf{T}_1 U_0(S - s)},$$

while the fraction P_2 of demand satisfied directly from stock equals

$$P_2 = 1 - \frac{\beta_2}{\lambda \mathbb{E}\mathbf{Y}_1}.$$

Finally, the ready rate measure P_3 boils down to

$$P_3 = \frac{\mathbb{E}\left(\int_L^{\sigma_1 + L} \mathbf{1}_{\{D(t) > S\}} dt\right)}{\mathbb{E}\mathbf{T}_1 U_0(S - s)}.$$

The expressions in Theorems 4.11 and 4.12 may look complicated, but as will be shown in [2] it is easy to give analytical formulas for associated Laplace transforms. Hence we may apply (cf. [2]) a newly developed Laplace inversion algorithm to generate accurate computations. Finally we will present in the next subsection the (s, nQ) inventory model.

4.2.2. *The (s, nQ) inventory model.* According to this inventory rule an order is triggered at the moment the inventory position drops below or equals the reorder level s. The order size is chosen to be an integer multiple of Q, such that after ordering all excess demand is backordered and the inventory position process will be between s and $s + Q$. This model is also discussed by F.Chen and Y.S. Zheng (cf. [3]) for a divisible item. However, in their paper the correct formula for the average cost is given without presenting a detailed proof. As before, we will show that this model also fits within the general framework of regenerative processes. Again, we assume that the demand process is a compound renewal process and the costrate function f satisfies

$$\mathbb{E}\left(\int_0^t f(\mathbf{IN}(s))ds\right) < \infty$$

for every $t \geq 0$. To start our analysis of this model we introduce the random variables $\mathbf{S}_n$, $n \in \mathbb{N} \bigcup \{0\}$, with $\mathbf{S}_0 := 0$ and $\mathbf{S}_n$ the arrival moment of the nth customer, $n \in \mathbb{N}$. By the definition of the renewal arrival process $\mathbf{N} = \{\mathbf{N}(t) : t \geq 0\}$ it follows that $\mathbf{S}_n = \sum_{k=1}^n \mathbf{T}_k$, $n \in \mathbb{N}$ and we consider now the càdlàg inventory position process $\mathbf{IP}$ evaluated at the arrival moments $\mathbf{S}_n$. Introducing the random variables $\{\mathbf{IP}_n : n \in \mathbb{N} \bigcup \{0\}\}$, given by $\mathbf{IP}_n = \mathbf{IP}(\mathbf{S}_n)$, it follows by a similar proof as for the (R, s, nQ) model (cf. [7]) that the stochastic process $\{\mathbf{IP}_n : n \in \mathbb{N} \bigcup \{0\}\}$ is a Markov chain with state space $(s, s + Q]$ for an indivisible item and state space $\{s + 1, \ldots, s + Q\}$ for a divisible item. Moreover, the invariant distribution is the same as for the embedded Markov chain in the (R, s, nQ) model. As before, the random variable $\mathbf{U}$ is uniformly distributed on $(0, 1]$ for an indivisible item and discrete uniformly distributed on $\{\frac{1}{Q}, \ldots, 1\}$ for a divisible item. It is now possible to show the following result.

Theorem 4.13. *For any (s, nQ) model with a compound renewal demand process, the stochastic process $\{(\mathbf{IP}(t), \mathbf{D}(t+L) - \mathbf{D}(t)) : t \geq 0\}$ satisfying $\mathbf{IP}(0) \stackrel{d}{=} s + Q\mathbf{U}$ and $\mathbf{U}$ is independent of the demand process $\mathbf{D}$, is a pure regenerative process with the increasing sequence σ_n, $n \in I\!\!N \bigcup \{0\}$ of random points given by $\mathbf{S}_n$. Moreover, the netstock process $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t+L) : t \geq 0\}$ is a pure regenerative process with the same sequence of increasing points.*

Proof. By the definition of a compound renewal process it follows that the differences $\sigma_{n+1} - \sigma_n$, $n \in I\!\!N \bigcup \{0\}$ are independent and identically distributed, and hence the first condition of Definition 2.2 trivially holds. To check the second condition of Definition 2.2, we observe by the definition of the (s, nQ) rule that the stochastic process $\{\mathbf{IP}(t + \sigma_n) : t \geq 0\}$ is a function of the random variable $\mathbf{IP}(\sigma_n)$ and the stochastic process

$$\mathbf{D}_{\sigma_n} := \{\mathbf{D}(t + \sigma_n) - \mathbf{D}(\sigma_n) : t \geq 0\}.$$

Since $\mathbf{IP}(\sigma_n)$ is completely determined by the individual demands $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ and the initial inventory level $\mathbf{IP}(0)$, this implies by the definition of the compound renewal demand process that the random variable $\mathbf{IP}(\sigma_n)$ is independent of $\mathbf{S}_0, \ldots, \mathbf{S}_n$. Moreover, the stochastic process $\mathbf{D}_{\sigma_n}$, by the definition of σ_n, is also independent of $\mathbf{S}_0, \ldots, \mathbf{S}_n$. By these observations, the joint stochastic process

$$\{(\mathbf{IP}(t + \sigma_n), \mathbf{D}(t + \sigma_n + L) - \mathbf{D}(t + \sigma_n)) : t \geq 0\},$$

being a function of $\mathbf{IP}(\sigma_n)$ and $\mathbf{D}_{\sigma_n}$ is independent of $\mathbf{S}_0, \ldots, \mathbf{S}_n$ and this verifies the second condition of Definition 2.2. Since by the definition of σ_n the process $\mathbf{D}_{\sigma_n}$ has the same distribution as $\mathbf{D}$, and by the observation before Theorem 4.13 the distribution of $\mathbf{IP}(\sigma_n)$ is independent of n, we finally obtain that the third condition of Definition 2.2 holds. Hence, $\{(\mathbf{IP}(t), \mathbf{D}(t+L) - \mathbf{D}(t)) : t \geq 0\}$ is a pure regenerative process, and the regenerative property of the process $\tilde{\mathbf{IN}} = \{\mathbf{IN}(t+L) : t \geq 0\}$ is an immediate consequence of Theorem 3.4. □

To identify the average cost of an (s, nQ) model we observe, since $\mathbf{IP}(\sigma_n)$ is a function of $\mathbf{IP}(0)$ and $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$, that this random variable is independent of the compound renewal process $\mathbf{D}_{\sigma_n}$. Moreover, if the conditions of Theorem 4.13 hold, it follows for every $0 \leq t < \sigma_1$ that $\mathbf{IN}(t+L) = s + Q\mathbf{U} - \mathbf{D}(t+L)$ with the random variable $\mathbf{U}$ independent of the demand process $\mathbf{D}$. Using Theorem 4.13 and the above relation, one can apply the same arguments as before, and so we list without proof the next result.

Theorem 4.14. *For an (s, nQ) model with $\mathbf{IP}(0) \stackrel{d}{=} s + Q\mathbf{U}$ and $\mathbf{U}$ independent of the compound renewal demand process $\mathbf{D}$, it follows that the average cost $\Phi(s, Q)$ of an (s, nQ) inventory model, with nonnegative costrate function f and ordering cost $K > 0$, is given by*

$$\Phi(s, Q) = \frac{K(1 - I\!\!E(F_Y(Q\mathbf{U}))) + I\!\!E\left(\int_0^{\mathbf{T}_1} f(s + Q\mathbf{U} - \mathbf{D}(y + L))dy\right)}{I\!\!E\mathbf{T}_1}.$$

If we want to derive the expressions for the different service measures then we can use Theorem 4.13 and the general formulas for these measures presented in Section 3. Observe, if we consider an (s, nQ) inventory model with a compound renewal demand process with arrival rate $\lambda > 0$, then the average demand rate a is given by $\lambda I\!\!E\mathbf{Y}_1$, by Theorem 4.13 the random variable σ_1 is given by $\mathbf{S}_1 = \mathbf{T}_1$, and

$\mathbf{IP}(0) = s + Q\mathbf{U}$. Applying now Theorems 3.5 up to 3.7 we immediately obtain the following result.

Theorem 4.15. *For any (s, nQ) model with $\mathbf{IP}_0 \stackrel{d}{=} s + Q\mathbf{U}$, $\mathbf{U}$ independent of the compound renewal demand process $\mathbf{D}$, it follows that the stockout probability $1 - P_1$ is given by*

$$1 - P_1 = \mathbb{P}\{\mathbf{D}(\mathbf{T}_1 + L) > s + Q\mathbf{U}\} - \mathbb{P}\{\mathbf{D}(L) > s + Q\mathbf{U}\}.$$

Moreover, the average number β_2 of items short has the form

$$\beta_2 = \lambda \left(\mathbb{E}(\max\{\mathbf{D}(\mathbf{T}_1 + L) - s - Q\mathbf{U}, 0\}) - \mathbb{E}(\max\{\mathbf{D}(L) - s - Q\mathbf{U}, 0\}) \right),$$

while the fraction P_2 of demand satisfied directly from stock equals

$$P_2 = 1 - \frac{\beta_2}{\lambda \mathbb{E}\mathbf{Y}_1}.$$

Finally, the ready rate measure P_3 boils down to

$$P_3 = \lambda \left(\int_L^{\mathbf{T}_1 + L} \mathbb{P}\{\mathbf{D}(t) > s + Q\mathbf{U}\} dt \right).$$

The expressions in Theorems 4.14 and 4.15 may look complicated, but as will be shown in [2] it is easy to give analytical formulas for associated Laplace transforms. Hence we may apply (cf. [2]) a newly developed Laplace transform inversion algorithm (cf. [10]) to generate accurate computations.

5. Conclusions

The general framework of regenerative processes enables us to derive the average cost and the most well known service measures for any of the classical single item inventory control models in an easy and efficient way. After we constructed this framework it only has to be checked for every model if it fits into this framework. In a subsequent paper (cf. [2]) computations are presented.

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