Chapter 2

ONE SCARCE FACTOR; ONE SECTOR

2.1. Macromodel; No Gestation Lag; No Depreciation

2.11 The models to be discussed in this chapter are the simplest models conceivable, thought to reflect the one phenomenon most characteristic of development, that is, the accumulation of capital. The one scarce factor considered is capital, and no other scarce factor is assumed to exist. Notwithstanding their extreme simplicity, these models can sometimes be used to make a first rough exploration of a country's growth process and to demonstrate some very fundamental relationships. These models or related ones have been introduced and discussed by R. F. Harrod and E. D. Domar.[1]

2.12 The variables used are

k capital stock[2]
y national income
j investment

The equations assumed are

$$\dot{k} = j \tag{2.12.1}$$

This equation states that, in the absence of a gestation lag and of depreciation, the rate of increase $\dot{k}$ ($= dk/dt$) of capital stock equals investment.

$$k = \kappa y \tag{2.12.2}$$

[1] R. F. Harrod, "Towards a Dynamic Economics," London, 1948; Evsey D. Domar, "Essays in the Theory of Economic Growth," New York, 1957.

[2] Since the role to be played by this variable depends on the production laws assumed, its definition will have to be adapted to these laws.

This equation expresses the assumption of a fixed *capital coefficient* (or *capital-output ratio*), representing a very simple production function.

$$j = \sigma y \tag{2.12.3}$$

This equation states that investment (assumed equal to savings) shows a fixed ratio σ to income; σ may be called the *savings ratio.*

2.13 The model admits a very simple solution of its system of equations, informing us about the speed of development.

$$\sigma y = j = \dot{k} = \kappa \dot{y} \tag{2.13.1}$$

or

$$\frac{\dot{y}}{y} = \frac{\sigma}{\kappa} \tag{2.13.2}$$

meaning that the rate of growth of income (and hence of both other variables) equals σ/κ.

As an example, let us take $\sigma = 0.12$ and $\kappa = 3$ years. Then evidently $\dot{y}/y = 0.04$ per annum; income, capital, and investment all grow 4 per cent per annum. Development over time of income can be represented by

$$y_t = y_0 e^{\sigma t/\kappa} \tag{2.13.3}$$

where y_0 is income at time $t = 0$. This income, or, alternatively, the initial value of capital (k_0) or investments (j_0), has to be given in order that the development path may be determined.

The formulas may be interpreted as the solution of an analytical problem in which σ and κ are given and the rate of development follows. Inversely, a political problem may be solved with them by considering the *desired rate of growth* of income ω given and calculating the *required rate of savings* σ'.

$$\sigma' = \omega\kappa \tag{2.13.4}$$

2.14 The model may be supplemented with more variables and equations which do not change the relationships already discussed. This will always be the case if the new variables are dependent on the variables already discussed without changing the equations already discussed. The simplest example is the addition of the variable c, standing for consumption and satisfying the relation $c = y - j$.

Other variables may be added for an *open* country, namely, imports i and exports e, as well as gross product v.* The relations to be added may be

$$i = \iota v \tag{2.14.1}$$

$$v = y + i = c + j + e \tag{2.14.2}$$

$$e = i \tag{2.14.3}$$

* The use of the word "gross" here means that product is taken at the *final* stage, that is, when reaching the consumer, the investor, or the country to which exported. For a nation as a whole v is also called *total resources.*

The latter equation is a consequence of our assumption $c = y - j$ and expresses the well-known equivalence between internal financial equilibrium and balance-of-payments equilibrium. It should be noted, however, that there is an implicit assumption in these equations, namely, that exports to the volume of e are salable at the (constant) price level assumed.

2.2. Macromodel; No Gestation Lag; with Depreciation and Replacement

2.21 The models to be treated in this section are characterized by the assumption of a finite lifetime Θ of all capital goods. This assumption makes it desirable to distinguish between the stock b of equipment or capital goods and the stock k of capital. The difference between the two concepts is to be found in the fact that an individual machine remains a constant volume of equipment until it is scrapped, whereas its contribution to the capital stock falls because of its depreciation. Our assumption implies that no obsolescence occurs; otherwise the contribution to b may not be constant. In order not to complicate the model unnecessarily, linear depreciation will be assumed. The model brings out some interesting features of development under these assumptions.

2.22 The variables used are

b volume of equipment
k volume of capital
v gross product
d depreciation allowances
r replacement
c consumption
s savings
j^G gross investment
j net investment
y net product

The equations of the model are, with their motivations,

$$\dot{b} = j^G - r \qquad (2.22.1)$$

The net addition to the stock of equipment can be found by deducting replacement from gross investment.

$$\dot{k} = s \qquad (2.22.2)$$

The net additions to capital are equal to savings.

$$b = \kappa' v \qquad (2.22.3)$$

Gross product is taken to be proportional to the volume (or *capacity*) of equipment, κ' representing a *gross capital coefficient.*

$$j = j^G - d \tag{2.22.4}$$

Net investment equals gross investment minus depreciation.

$$r_t = j^G_{t-\Theta} \tag{2.22.5}$$

Replacement equals gross investment one lifetime before.

$$d = \frac{b}{\Theta} \tag{2.22.6}$$

Depreciation allowances are equal to the new value of total equipment divided by lifetime. The essence of what is here called new value is that it is value without deduction of depreciation.

$$y = v - d \tag{2.22.7}$$

Income equals gross product minus depreciation. Since no imports are assumed to exist in this version of the model, no deduction of imports is necessary.

$$y = c + s \tag{2.22.8}$$

From the spending side, income equals consumption plus savings. No lags are assumed to occur in this relationship.

$$s = j \tag{2.22.9}$$

Savings are equal to net investment.

$$s = \sigma y \tag{2.22.10}$$

Savings are a portion σ of income, where σ is the savings rate.

2.23 Also, this system of relations admits a relatively simple solution, although less simple than the previous model. Since the system is linear, the usual method will do, which consists in assuming a solution of the shape[1]

$$j^G = j_0{}^G e^{\omega t} \tag{2.23.1}$$

where $j_0{}^G$ is an arbitrary constant representing the initial value of j^G and ω a constant which will have to satisfy some condition to be found from the system of equations.

It can be easily found that

$$r_t = j^G_{t-\Theta} = j_0{}^G e^{\omega(t-\Theta)} \tag{2.23.2}$$

$$\dot{b} = j^G - r = j_0{}^G e^{\omega t}(1 - e^{-\omega\Theta}) \tag{2.23.3}$$

[1] See, e.g., Lyman M. Kells, "Elementary Differential Equations," p. 87, New York and London, 1935.

from which it follows that[1]

$$b = \frac{1}{\omega} j_0{}^G e^{\omega t}(1 - e^{-\omega\Theta}) \tag{2.23.4}$$

From b we can derive v and d:

$$v = \frac{b}{\kappa'} \qquad \text{and} \qquad d = \frac{b}{\Theta}$$

leading to

$$j^G - d = j = s = \sigma y = \sigma(v - d) = \sigma\left(\frac{1}{\kappa'} - \frac{1}{\Theta}\right) b \tag{2.23.5}$$

Upon filling out the expressions for j^G, d, and b, we find the condition

$$\omega = \left(\frac{\sigma}{\kappa'} + \frac{1 - \sigma}{\Theta}\right)(1 - e^{-\omega\Theta}) \tag{2.23.6}$$

A complete solution for j^G and the other variables consists of as many terms of the shape shown in (2.23.1) as there are roots of Eq. (2.23.6).

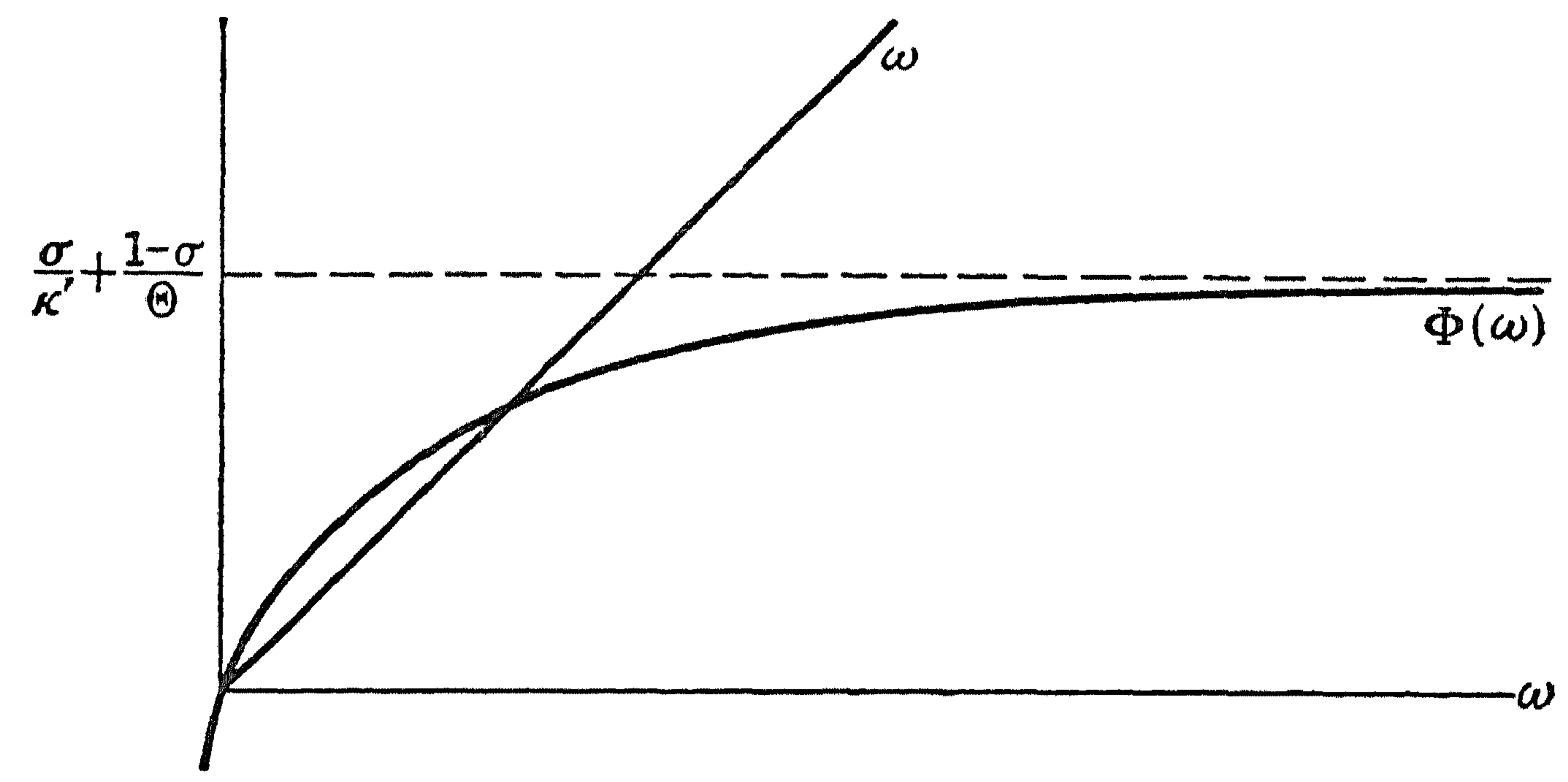

Fig. 1

Roots may be real or complex; complex roots correspond to fluctuating movements of the variables. The nature of real roots—to which, as long as $\omega > 0$, monotonically rising movements of the variables correspond—can be illustrated by Fig. 1. Let the right-hand side of (2.23.6) be represented by

$$\Phi(\omega) = \left(\frac{\sigma}{\kappa'} + \frac{1 - \sigma}{\Theta}\right)(1 - e^{-\omega\Theta})$$

Then the root ω_0 will be represented by the point of intersection between the straight line with slope 1 (of which the ordinates are ω) and the curve with ordinates $\Phi(\omega)$. It can be shown that for $\sigma > 0$ and $\Theta > \kappa'$, both

[1] No additive constant can be added to (2.23.4), since then the relation (2.22.5) would be violated.

realistic assumptions, there is always such an intersection point. An explicit solution is not possible, but for small values of $\omega\Theta$ the expression $e^{-\omega\Theta}$ can be approximated by $1 - \omega\Theta + \frac{1}{2}\omega^2\Theta^2 \cdots$ leading to

$$\omega = \left(\frac{\sigma}{\kappa'} + \frac{1 - \sigma}{\Theta}\right)\left(\omega\Theta - \frac{1}{2}\,\omega^2\Theta^2\right)$$

or

$$\omega = 2\sigma\left(\frac{1}{\kappa'} - \frac{1}{\Theta}\right)$$

For large values of Θ, on the other hand, the Domar-Harrod result $\omega = \sigma/\kappa'$ will be obtained, as can be seen directly from (2.23.6). Evidently, there are no other real roots.

2.24 The solutions can be used for the *analytical problem* to explain development with given values of the coefficients, including the savings ratio σ, and with given initial values of the variables, such as j_0^G, etc. In such a problem also the complex roots of (2.23.6) will have to play their part. Only when a sufficient number of terms of the general form (2.23.1) [where, however, the constant factors are not now identical to j_0^G, since that only applies if only one term is taken, as in (2.23.1)] is introduced is it possible to solve the analytical problem with given initial values of the variables. These initial values may, however, be such as to let the system move along cycles.

For the solution of the *political problem*, the situation is different. There is scope here to consider as the aim a pattern of *development without cyclical setbacks*. In our mathematical language, this means that only one root, namely the real one, of Eq. (2.23.6) is relevant to us and accordingly Eqs. (2.23.1) to (2.23.5) are valid, with j_0^G equal to the initial value of j^G, whereas all other initial values can be calculated from it. This means that, in order to warrant a noncyclical movement, certain relationships between the initial variables must be satisfied. If, in addition, a certain rate of development ω_0 is desired, the rate of savings σ_0 can be derived from (2.23.6); it should be

$$\sigma_0 = \frac{\omega_0/(1 - e^{-\omega_0\Theta}) - 1/\Theta}{1/\kappa' - 1/\Theta} \qquad (2.24.1)$$

It will again easily be seen that for $\Theta = \infty$ this expression coincides with (2.13.4).

2.25 Once a development pattern of this kind is followed, there will exist fixed ratios between the variables. This is a consequence, of course, of a number of simplifying assumptions implied in our model. Later we shall discuss models where this somewhat unrealistic feature has been removed. It seems interesting, however, to calculate some of the ratios that may make sense, if only approximately, under more general conditions. We shall calculate k/y, k/b, and r/d.

Starting from (2.23.5) we have

$$\dot{k} = s = \sigma\left(\frac{1}{\kappa'} - \frac{1}{\Theta}\right)b \qquad (2.25.1)$$

Using (2.23.4) we find

$$\dot{k} = \frac{\sigma}{\omega}\left(\frac{1}{\kappa'} - \frac{1}{\Theta}\right)j_0{}^{G}e^{\omega t}(1 - e^{-\omega\Theta}) \qquad (2.25.2)$$

By integration over time and by requiring that k should approach zero for $t = -\infty$, where the other variables so far considered also vanish, we have

$$k = \frac{\sigma}{\omega^2}\left(\frac{1}{\kappa'} - \frac{1}{\Theta}\right)j_0{}^{G}e^{\omega t}(1 - e^{-\omega\Theta}) \qquad (2.25.3)$$

We can now calculate the three ratios mentioned:

$$\frac{k}{y} = \frac{\sigma}{\omega} \qquad \frac{k}{b} = \frac{\sigma}{\omega}\left(\frac{1}{\kappa'} - \frac{1}{\Theta}\right) \qquad \frac{r}{d} = \frac{\omega\Theta}{e^{\omega\Theta} - 1} \qquad (2.25.4)$$

For the political problem it is desirable to express them in terms of ω, as can be done explicitly with the aid of (2.23.6); the results are

$$\frac{k}{y} = \frac{1/(1 - e^{-\omega\Theta}) - 1/\omega\Theta}{1/\kappa' - 1/\Theta} \qquad \frac{k}{b} = \frac{1}{1 - e^{-\omega\Theta}} - \frac{1}{\omega\Theta} \qquad \frac{r}{d} = \frac{\omega\Theta}{e^{\omega\Theta} - 1} \qquad (2.25.5)$$

For very small and very large values of $\omega\Theta$ these expressions can be further simplified. The results are shown below.

	k/y	k/b	r/d
$\omega\Theta$ small	$\frac{1/2}{1/\kappa' - 1/\Theta}$	½	1
$\omega\Theta$ large	κ'	1	0

2.26 The model remains simpler if instead of Eq. (2.22.3), representing a purely technological production function, the more customary but less clear relationship

$$k = \kappa y \qquad (2.26.1)$$

is maintained. In this case, as in Sec. 2.14, k, s, or j and y remain an inner circle of variables, the movements of which are independent from the equations outside (2.22.2), (2.22.10), and (2.26.1). We have

$$\dot{k} = \frac{\sigma}{\kappa}k \qquad (2.26.2)$$

and hence $k = k_0 e^{\omega t}$ satisfies, with

$$\omega = \frac{\sigma}{\kappa} \tag{2.26.3}$$

As a consequence $y = (k_0/\kappa)e^{\sigma t/\kappa}$.

The remaining variables may be determined by first considering b, which has to satisfy

$$\dot{b} = j^G - j^G_{t-\Theta} = \sigma y + \frac{b}{\Theta} - \sigma y_{t-\Theta} - \frac{b_{t-\Theta}}{\Theta} \tag{2.26.4}$$

This is a linear nonhomogeneous difference-differential equation. The general solution consists of two parts: (1) the general solution of the homogeneous equation

$$\Theta \dot{b}_t = b_t - b_{t-\Theta} \tag{2.26.5}$$

and (2) a particular solution of the nonhomogeneous equation.[1] It is easily seen that the general solution of (2.26.5) runs

$$b_t = b^{00} + b^{01}t \tag{2.26.6}$$

If, however, we require that for $t = -\infty$, b_t should be zero, it will appear that this solution is discarded again.

A particular solution of the nonhomogeneous equation may be attempted by assuming $b_t = b_0 e^{\omega t}$ where $\omega = \sigma/\kappa$, as in (2.26.3). This solution is admissible only when b_0 satisfies

$$\omega = (1 - e^{-\omega\Theta})\left(\frac{\sigma}{\kappa}\frac{k_0}{b_0} + \frac{1}{\Theta}\right) \tag{2.26.7}$$

Since σ has to be adapted, in the political problem, to the desired rate of growth ω, this expression can then be transformed into

$$\frac{k_0}{b_0} = \frac{1}{1 - e^{-\omega\Theta}} - \frac{1}{\omega\Theta} \tag{2.26.8}$$

It is interesting to note that this formula is identical with the one shown for k/b in (2.25.5).

The remaining variables can be derived from the equations connecting them with the variables already determined.

2.3. Macromodel with Gestation Lag; No Depreciation

2.31 A somewhat unrealistic feature of the models so far discussed is the absence of a gestation lag. Each unit of investment, however small, is supposed immediately to add to the capital stock. It will now be assumed that a time lag θ occurs between the start of any investment process (say building) and the addition to the capital stock of a finished

[1] See, e.g., Kells, *op. cit.*, p. 93.

capital good. The introduction of this phenomenon necessitates the addition of assumptions about the investment process during this time period. We shall first make the simplest conceivable hypothesis, namely, that the process requires a uniform input of effort during the period θ.

It is useful to introduce as a new variable "investment finished," to be indicated by j'_t. By definition we shall then have

$$\dot{k}_t = j'_t \tag{2.31.1}$$

Total investment activity j_t at any time period t is now the total of activities started and not yet finished, that is, activities of which the time of finishing is between t and $t + \theta$. Since all are running at an even pace, activity j is only an unweighted average:

$$j_t = \frac{1}{\theta} \int_t^{t+\theta} j' \, dt' \tag{2.31.2}$$

This expression can be transformed with the aid of (2.31.1):

$$j_t = \frac{1}{\theta} k_{t'} \Big|_t^{t+\theta} = \frac{1}{\theta} (k_{t+\theta} - k_t) \tag{2.31.2'}$$

Adding Eqs. (2.12.3) and (2.12.2), we have a system of four equations for our four variables.

2.32 The solution of the system can be found by expressing j_t in terms of k_t with the aid of the last two equations, yielding

$$j_t = \frac{\sigma}{\kappa} k_t = \frac{1}{\theta} (k_{t+\theta} - k_t) \tag{2.32.1}$$

which may be rewritten

$$k_{t+\theta} = \left(1 + \frac{\theta\sigma}{\kappa}\right) k_t \tag{2.32.2}$$

The same equation will also hold for the other variables. During a period θ, capital will have grown in the ratio $1 + \theta\sigma/\kappa$. Disregarding fluctuations with a period smaller than θ, we may write the solution

$$k_t = k_0 \left(1 + \frac{\theta\sigma}{\kappa}\right)^{t/\theta} \tag{2.32.3}$$

from which we can derive

$$\frac{\dot{k}_t}{k_t} = \frac{1}{\theta} \ln \left(1 + \frac{\theta\sigma}{\kappa}\right) \tag{2.32.4}$$

It is easily seen that for small values of $\theta\sigma/\kappa$ this becomes identical to σ/κ, that is, the rate of growth found in Eq. (2.13.2) holding for the model without gestation lag. For larger values of $\theta\sigma/\kappa$, the deviations from the rate previously found may be considerable; growth will be slower.

2.33 The investment process may be of a different kind. Another simple example is that of a point input at the beginning of the lag, followed by the same point output we assumed in the previous case. Here we simply have

$$j_t = j'_{t+\theta} \tag{2.33.1}$$

Consequently we find

$$\dot{k}_t = j'_t = j_{t-\theta} = \frac{\sigma}{\kappa} k_{t-\theta} \tag{2.33.2}$$

Again writing $k_t = k_0 e^{\omega t}$, we find that ω will have to satisfy

$$\omega = \frac{\sigma}{\kappa} e^{-\omega\theta} \tag{2.33.3}$$

Again, for small values of $\omega\theta$ we find $\omega = \sigma/\kappa$.

2.4. The Optimum Rate of Development[1]

2.41 A practical problem of considerable importance consists in deciding upon the rate of growth of production to be chosen. The well-known fact that in communist countries these rates and, as a consequence, the rates of saving applied are almost double those of noncommunist countries[2] illustrates the wide differences in decisions taken. The question may therefore be asked whether economic science can give a clue to a numerical choice.

Attempts made by the present authors seem to justify a negative answer. Nevertheless, it seems worth while to describe the attempts and their results. An attempt was made so to interpret the problem that the optimum rate of development was the one maximizing utility over time and to utilize the scarce efforts made at measuring the relevant properties of the utility function. Maximization of utility as a device is in any case superior to maximization of consumption, which again is better than maximization of income. The specific interpretation given to utility was the assumption that it depends only on consumption in the same time unit; this may be too restrictive, as will be discussed later, but no measurements at all are available for the dependence of utility of consumption at other time periods.

2.42 Various types of *utility functions* were assumed; it appeared desirable to introduce a minimum level $\bar{c}$ of consumption, below which

[1] This section may be skipped by the reader interested only in practical planning.

[2] In the United States the average savings ratio over a business cycle has long been 11 to 12 per cent; for the United Kingdom a slightly higher figure has been found. At present, savings ratios in Western Europe are higher. Savings ratios in communist countries cannot be compared easily, for several reasons, but figures of 25 per cent in terms of Western concepts have been mentioned.

marginal utility becomes infinitely large. Further exploration taught the authors that it is also desirable to introduce a maximum or saturation level c^m above $\bar{c}$, that is, $c = c^m + \bar{c}$, at which the marginal utility is zero. The utility function to be used in this section will be written

$$u = \left(\frac{c^m}{c - \bar{c}} - 1\right)^v \tag{2.42.1}$$

where u is marginal utility and v a constant. Its value was derived from Frisch's well-known estimates[1] of the flexibility of marginal utility, originally based on the assumption that sugar is an "independent" commodity, an assumption later removed, however, to some extent. In order to arrive at one single value for v, it was necessary to complete Frisch's estimates by a few more assumptions. Frisch estimated the flexibility of marginal utility for two groups of workers, an American group, for which he found -1, and a French group, for which he found -3.5. Our assumptions are that (1) the same utility function holds for both and that (2) the level of consumption of French workers, at the time of measurement, was one-half the level of American workers.

Flexibility being defined as $(\partial u/\partial c)\, c/u$, it appears to be

$$-\frac{vc^m}{(c - \bar{c})^2}\left(\frac{c^m}{c - \bar{c}} - 1\right)^{v-1}\frac{c}{[c^m/(c - \bar{c}) - 1]^v} = -\frac{vc^m c}{c^m + \bar{c} - c}\,\frac{1}{c - \bar{c}} \tag{2.42.2}$$

Indicating French consumption by c^F and hence American consumption by $2c^F$, we have, according to Frisch,

$$\frac{2c^m vc^F}{(c^m + \bar{c} - 2c^F)(2c^F - \bar{c})} = 1 \qquad \text{and} \qquad \frac{c^m vc^F}{(c^m + \bar{c} - c^F)(c^F - \bar{c})} = 3.5 \tag{2.42.3}$$

Adding the further assumption that c^m is large in comparison with $2c^F$, we have, approximately,

$$\frac{2vc^F}{2c^F - \bar{c}} = 1 \qquad \frac{vc^F}{c^F - \bar{c}} = 3.5 \tag{2.42.4}$$

From these equations we find that $v = 0.6$ and $\bar{c}/c^F = 5/6$. This second result does not seem to be unrealistic, and may justify some confidence in the result for v too.

No *discount* for future consumption was applied in the belief that for a country's planning, future generations should count as much as present generations. According to this philosophy, a discount may be realistic for the individual's plans but not necessarily for a nation's. It is not difficult to introduce discounts for future consumption when so desired,

[1] Ragnar Frisch, "New Methods of Measuring Marginal Utility," Tübingen, 1932.

but the question then arises at what level the discount should be put. Instead of a discount, a finite horizon T may be introduced; a similar question then comes up about its length.

Population was assumed to remain constant. It is not difficult, however, to assume a certain rate of growth π and change the formulas accordingly. In a general way this will raise the optimum rate of savings in the well-known way, that is, by $\pi\kappa$, when a fixed capital-output ratio is assumed.

This, in fact, was the *production function* utilized. As long as capital is the scarcest factor, this assumption may be a proper approximation. It appeared to be very difficult, if not impossible, to find an explicit solution if a more complicated production function were assumed to exist, for example, the Cobb-Douglas function.

2.43 The *problem of the optimum rate of development* was then given the following formal shape. Given an initial income y_0 and a capital-output ratio κ (implying, if one likes, an initial capital stock $k_0 = \kappa y_0$) and given the utility function (2.42.1), what program $c(t)$ of consumption (implying a program of saving and hence of capital expansion) yields maximum satisfaction over time $\int_0^\infty U(t')\,dt'$, where U is total utility of consumption at time t'?

Evidently the maximum is one with a *side condition*, namely, that at any time t (using symbols as before), $c + s = y$ or, with $s = j = \dot{k} = \kappa\dot{y}$

$$c + \kappa\dot{y} = y \tag{2.43.1}$$

Apart from this side condition, we shall also consider two *boundary conditions*, namely,

$$c \geq \bar{c} \qquad s \geq 0 \tag{2.43.2}$$

As long as these boundary conditions are not active, that is, income is actually distributed between some positive savings and a volume of consumption surpassing the subsistence minimum $\bar{c}$, for all time units considered, that is, for $0 \leq t \leq \infty$, the maximum requires that the marginal utility of consumption at moment t equals the total marginal utility of the additional consumption in the future to be obtained from giving up one unit of consumption at time t. Since the increase in future production made possible by giving up one unit of consumption is $1/\kappa$ for all the future, the condition runs

$$u_t = \frac{1}{\kappa}\int_t^\infty u_t\,dt' \tag{2.43.3}$$

It does not matter that this future production may not actually be consumed but partly saved; this decision can be separated from the one at time t. If this future decision again obeys (2.43.3), the marginal utility of the corresponding addition to production can be measured either on the consumption or on the savings side; these two are equal.

Equation (2.43.3) may be replaced by one which is simpler to handle, by replacing both sides by their derivatives with regard to time. We must later test, however, whether (2.43.3) is then also satisfied, depending on the integration constant applied. The new equation runs

$$\frac{du}{dt} = -\frac{u}{\kappa} \tag{2.43.4}$$

Since u depends on t via c, it is better to rewrite it

$$\kappa \frac{du}{dc} \dot{c} = -u$$

Using (2.42.1), we obtain

$$\dot{c}' = \frac{c^m - c'}{\kappa v c^m} c' \tag{2.43.5}$$

where $c' = c - \bar{c}$. Since this equation only contains the variable c' and not the other variable y of our system, its integration can be undertaken separately. Equation (2.43.5) is the well-known differential equation of the logistic curve; the integral may be written

$$c' = \frac{c^m}{1 + Be^{-t/\kappa v}} \tag{2.43.6}$$

where $B = e^{t_0/\kappa v}$ is an arbitrary constant which may be replaced by t_0, the time at which $c' = \frac{1}{2}c^m$, that is, half the level of the asymptote.

Our result means that consumption will gradually have to approach, but never have to reach, the saturation level $c^m + \bar{c}$.

2.44 The next step consists in integrating Eq. (2.43.1) for y, which can now be written

$$y' - \kappa \dot{y}' = \frac{c^m}{1 + Be^{-t/\kappa v}} \tag{2.44.1}$$

when

$$y' = y - \bar{c} \tag{2.44.2}$$

and evidently represents a nonhomogeneous first-order linear differential equation. A standard method to deal with the left-hand side of the equation is to calculate the derivative of $y'e^{-t/\kappa}$ with respect to time:

$$\frac{d}{dt} y'e^{-t/\kappa} = e^{-t/\kappa}\left(\dot{y}' - \frac{y'}{\kappa}\right)$$

According to (2.44.1), this expression must be equal to

$$-\frac{c^m e^{-t/\kappa}}{\kappa(1 + Be^{-t/\kappa v})}$$

Hence we have

$$y'e^{-t/\kappa} = -\frac{c^m}{\kappa}\int \frac{e^{-t'/\kappa}\,dt'}{1 + Be^{-t'/\kappa v}} \tag{2.44.3}$$

It appears possible to carry out explicitly this integration for integer values of $1/v$. Since our estimate $v = 0.6$ is a rough approximation, it seems worth while carrying out the integration for $v = 0.5$ or $1/v = 2$. This can be done with the aid of the substitution

$$Be^{-2t'/\kappa} = t''^2$$

where t'' is a new integration variable. It follows that $e^{-t'/\kappa}\sqrt{B} = t''$

and

$$-\frac{\sqrt{B}}{\kappa}e^{-t'/\kappa}\,dt' = dt''$$

The integral (2.44.3) now becomes

$$\frac{\kappa}{c^m}y'e^{-t/\kappa} = \frac{\kappa}{\sqrt{B}}\int \frac{dt''}{1 + t''^2} \tag{2.44.4}$$

or

$$y' = \frac{c^m}{\sqrt{B}}e^{t/\kappa}\arctan t'' + \bar{y} = \frac{c^m}{\sqrt{B}}e^{t/\kappa}\arctan e^{-t/\kappa}\sqrt{B} + \bar{y}$$

where $\bar{y}$ is an arbitrary constant the value of which has to be determined with the aid of boundary conditions.

A natural boundary condition, to be added to the initial condition that y_0 be given, is that for $t = \infty$, y approaches c; in fact, there is no reason to save at the saturation level, because there is no reason to surpass that level. Economic development finds its natural end when saturation is approached. Since for $t = \infty$

$$\frac{\arctan e^{-t/\kappa}\sqrt{B}}{e^{-t/\kappa}\sqrt{B}} = 1$$

$y' = c^m + \bar{y}$ and hence $\bar{y} = 0$.

The solution for y therefore is

$$y_t = \frac{c^m}{e^{-t/\kappa}\sqrt{B}}\arctan e^{-t/\kappa}\sqrt{B} + \bar{c} \tag{2.44.5}$$

This implies that

$$y_0 = c^m\frac{\arctan\sqrt{B}}{\sqrt{B}} + \bar{c} \tag{2.44.6}$$

Production and hence capital must develop, according to (2.44.5), along a curve very similar to a logistic; both curves are characterized by a moderate slope in the early phases, increasing in acceleration until a certain point and then slowing down again and approximating a horizontal asymptote.

2.45 The nature of the movements found may be illustrated by substituting for the last section an alternative setup which seems more illuminating from the mathematical point of view and not even a bad approximation from a practical economic point of view. Writing instead of $\dot{y}$ in Eq. (2.43.1) $(y_t - y_{t-\kappa})/\kappa$, that is, the average rate of

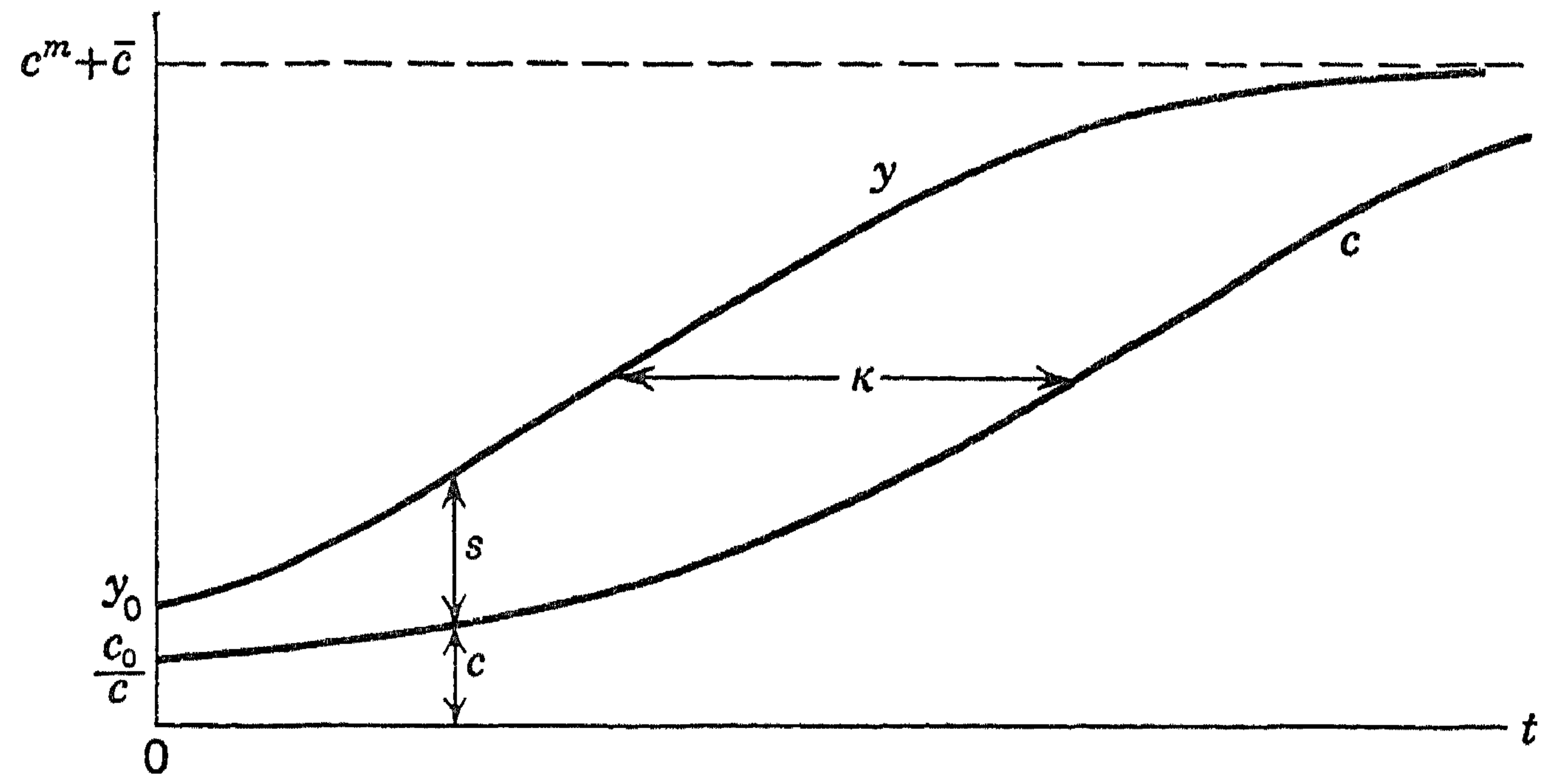

Fig. 2

increase over the last κ time units (practically some three to four years), we find that this equation takes a particularly simple shape

$$c_t + y_t - y_{t-\kappa} = y_t \tag{2.45.1}$$

leading immediately to the solution

$$y_{t-\kappa} = c_t$$

or

$$y_t = c_{t+\kappa} \tag{2.45.2}$$

The practical justification for this alternative setup may be that investments are based on the rate of increase in income experienced in the last three to four years, rather than the last small time unit, indeed appealing to the economist. The nature of the result then becomes very simple: both consumption and production have to move along a logistic, the only difference between the two being a time lag of κ time units (cf. Fig. 2). The logistic for y has to be located so as to leave an intercept on the vertical axis of y_0.

Savings are measured by the vertical distance between the two curves. In the beginning they are small; they may become quite substantial, absolutely as well as relatively speaking, but in the end, when half the saturation level of consumption has been passed, they diminish and finally peter out.

2.46 We may finish this exercise by making an attempt at measuring the rates of saving following from our formulas under various circumstances. As a consequence of our first boundary condition (2.43.2), no savings are envisaged whenever $y_0 = \bar{c}$. As a consequence, no development will take place either, and consumption and production will remain on the subsistence level. Initial savings will have to be positive, however, whenever $y_0 > \bar{c}$; in the beginning they may be small, but they will increase, and during the full swing of the development process, they will have to be considerable. This may be shown by calculating the maximum rate of savings following from the formulas. Writing again

$$t'' = e^{-t/\kappa}\sqrt{B} \tag{2.46.1}$$

we have

$$c = \frac{c^m}{1 + t''^2} + \bar{c} \tag{2.46.2}$$

and

$$y = \frac{c^m \arctan t''}{t''} + \bar{c} \tag{2.46.3}$$

from which we derive, for the savings rate σ,

$$\sigma = 1 - \frac{c}{y} = 1 - \frac{c^m/(1 + t''^2) + \bar{c}}{(c^m/t'') \arctan t'' + \bar{c}} \tag{2.46.4}$$

It is more elegant now to introduce $t''' = \arctan t''$, or $t'' = \tan t'''$; writing also A for $c^m/\bar{c}$, we find

$$\sigma = \frac{1 - (\sin 2t''')/2t'''}{1 + \dfrac{1}{A}\dfrac{\tan t'''}{t'''}} \tag{2.46.5}$$

Since

$$y = \bar{c}\left(A\frac{t'''}{\tan t'''} + 1\right)$$

and $t'''/\tan t'''$ is a decreasing function of t''', t''' evidently has its highest value for the initial value y_0 of y and then decreases.

As can be seen from (2.46.5), the maximum value of σ depends only on A, at least as long as y_0 is below the value of y corresponding with that maximum σ. For y_0 low, in comparison with $c^m + \bar{c}$, this always applies. Computation shows that σ_{max} is quite high, as is illustrated by the following figures:

A	σ_{max}
10	0.63
100	0.86
500	0.94

2.47 From the values shown it appears that the model used leads to unrealistically high values for the optimum rate of savings. In fact, the formulas have the tendency to recommend austerity in order to reach the saturation level "as soon as possible." This may be illustrated by calculating the rate of increase in consumption for the level $\frac{1}{2}c^m + \bar{c}$, that is, halfway between the subsistence level and the saturation level and asking ourselves how much time it would take, at this rate of increase, to reach the saturation level. From formula (2.43.5) it follows that $\dot{c} = \dot{c}' = c^m/2$ for $c' = \frac{1}{2}c^m$, meaning that, at that speed, the distance from $\bar{c}$ to $c^m + \bar{c}$ takes 2κ years or 6 to 8 years only. Even though the actual time will be longer, since the speed at halfway is the maximum, it illustrates the order of magnitude involved.

There seem to be two main reasons why actual savings rates are so much lower. On the one hand, individuals do discount future consumption, although we disregarded this phenomenon. On the other hand, savings programs near the ones implied in our formulas always mean that the present generation is suffering for the coming generations to an extent that is not generally considered proper. If, therefore, the element of "more justice in the relations between generations" were brought into our utility concept, we would obtain lower savings as the optimum program. Since no attempts to measure the preference schedules implied are known to us, we have not tried to generalize our findings along these lines. Our conclusion with regard to the question whether economic science can indicate an optimum rate of development tends to be negative therefore.