Chapter 3

SEVERAL SCARCE FACTORS; ONE SECTOR

3.1. Population-Capital-Technical–Change Model; Continuous Substitution between Factors

3.11 We shall now consider models with more than one scarce factor of production. In order to show the consequences of this generalization in the simplest possible way, we shall stick to the assumption of only one sector, that is, one product. Our models will have only two scarce factors, to be called capital and labor. They can also be land and labor, or any other pair. It is possible to generalize the results in order to apply them to the cases of a larger number of factors.

To begin with, it will be assumed that the two factors are perfectly substitutable. More restricted possibilities of substitution will be considered later (see Sec. 3.4). Since the simplest example of perfect substitution between two factors is represented by the well-known Cobb-Douglas production function, and since observation does not contradict the main implications of this function, we shall use this function throughout this section and the next few. The function will be given the so-called generalized shape

$$v = (1 + \epsilon)^t a^\lambda k^\mu \tag{3.11.1}$$

where v is production volume

- a is quantity of labor employed, or employment
- k is quantity of capital used
- ϵ is the annual rise in efficiency[1]
- λ and μ are the elasticities of production with regard to labor and capital, respectively
- t is time

[1] As will be clear from the formula, ϵ indicates the annual rise in production with the aid of constant quantities of labor and capital, and hence implies the consequences of changes in working hours.

We shall develop most of our formulas, however, on the specific assumption that $\lambda + \mu = 1$, meaning that there are, for the country considered, no economies of scale. This assumption could be justified by pointing to the fact that for the majority of industries the optimum size of the enterprise is small in comparison with the total market. As a consequence, expansion of production will, as a rule, have to be obtained by increasing the number of enterprises of optimum size, implying constant returns or no economies of scale.

3.12 The variables of the model are, including those already mentioned,

v volume of production
a quantity of labor employed or employment
k quantity of capital used
k^0 quantity of capital in existence
P size of population
l real wage rate
m real income per unit of capital
l^0 real wage rate considered normal or appropriate

This last variable enters into the labor supply function (see Sec. 3.13) as a psychological reference variable indicative of the gradual development in the ideas of the population—probably based on past experience—as to what is a normal, an appropriate, or a reasonable wage level.

3.13 The following relations constitute the model. The *production function* already mentioned represents the collection of technological possibilities:

$$v = (1 + \epsilon)^t a^\lambda k^{1-\lambda} \tag{3.13.1}$$

The meaning of the constants ϵ and λ has already been discussed in Sec. 3.11.

Since perfect competition between employers will be assumed, *demand for labor* and *for capital* will be such as to equate their price to their marginal productivities. Hence

$$l = \frac{\partial v}{\partial a} = \lambda(1 + \epsilon)^t \left(\frac{k}{a}\right)^{1-\lambda} \tag{3.13.2}$$

$$m = \frac{\partial v}{\partial k} = (1 - \lambda)(1 + \epsilon)^t \left(\frac{a}{k}\right)^{\lambda} \tag{3.13.3}$$

The *supply of labor* connects the proportion a/P of the population willing to work with the ratio l/l_0 of the wage rate to the "normal" wage rate.

$$\frac{l}{l^0} = \left(\frac{a}{P}\right)^{\alpha} \tag{3.13.4}$$

where α is the flexibility of supply ($1/\alpha$ is the elasticity of supply).

The *background variables* l^0 and P appearing in this equation are supposed to move in an autonomous way.

$$l^0 = l_0^0(1 + \Omega)^t \tag{3.13.5}$$
$$P = P_0(1 + \pi)^t \tag{3.13.6}$$

where l_0^0 and P_0 are the values at $t = 0$ of l^0 and P, respectively, and Ω and π are the annual rates of increase of the normal wage rate and population, successively.

Equations (3.13.4) to (3.13.6) may be combined to

$$l = \left[\frac{a}{P_0(1 + \pi)^t}\right]^\alpha l_0^0(1 + \Omega)^t \tag{3.13.4'}$$

The *supply of capital* k depends on the total quantity in existence, k^0, and the rate of interest m both having a positive influence on k. We shall write the supply function in the following form:

$$m = \left(\frac{k}{k^0}\right)^\beta \tag{3.13.7}$$

Here β is the supply flexibility. It would have been possible to introduce a "normal interest rate" m^0, as in the case of the supply of labor, but we shall not go into this in much detail here, since, as a rule, the supply of capital is assumed to be inelastic, meaning that $k = k^0$, irrespective of the value of m^0 and m.

The *formation of capital*, finally, is assumed to be linked with national product or income by the simple equation

$$\dot{k}^0 = \sigma v \tag{3.13.8}$$

where σ represents the propensity to save, supposed constant. Because of the special shape of the production function, the shares of income going to capital and labor are constant. Even when savings are assumed to be made by capital owners only, there is no objection to the assumption of a constant value of σ.

Equations (3.13.1) to (3.13.3), (3.13.4′), (3.13.7), and (3.13.8) enable us to calculate the development over time of the six variables a, k^0, k, v, l, and m.

3.14 As an example, we shall explicitly determine the time shape of development for the simplest case where the *supply elasticity both of labor and of capital are zero*, or α and β infinite. In fact these assumptions simplify Eqs. (3.13.4′) and (3.13.7); they become

$$a = P_0(1 + \pi)^t \tag{3.14.1}$$
$$k = k^0 \tag{3.14.2}$$

Combining them with (3.13.1) and (3.13.8), we have

$$\dot{k} = \sigma v = \sigma(1 + \epsilon)^t P_0^{\lambda}(1 + \pi)^{\lambda t} k^{1-\lambda}$$

or

$$\dot{k}k^{\lambda-1} = \sigma P_0^{\lambda}[(1 + \epsilon)(1 + \pi)^{\lambda}]^t \tag{3.14.3}$$

Since $(d/dt)k^{\lambda} = \lambda k^{\lambda-1}\dot{k}$, we have

$$\frac{d}{dt} k^{\lambda} = \lambda\sigma P_0^{\lambda}[(1 + \epsilon)(1 + \pi)^{\lambda}]^t = \lambda\sigma P_0^{\lambda} e^{t \ln [\ \ ]}$$

where [ ] stands for the expression between [ ] in (3.14.3). Integrating, we get

$$k^{\lambda} = \frac{\lambda\sigma P_0^{\lambda}}{\ln [\ \ ]} e^{t \ln [\ \ ]} + C \tag{3.14.4}$$

where C is an integration constant depending on the initial value of k.

It will be assumed that our units are so chosen as to make, for $t = 0$, $a_0 = k_0 = v_0 = P_0 = 1$.

Substituting $t = 0$ in (3.14.4), we obtain for C

$$C = 1 - \frac{\lambda\sigma}{\ln [\ \ ]}$$

and hence, for small values of ϵ and π,

$$k^{\lambda} = 1 + \frac{\lambda\sigma}{\epsilon + \lambda\pi} \{[(1 + \epsilon)(1 + \pi)^{\lambda}]^t - 1\} \tag{3.14.5}$$

It is clear that the time shape of k is, even in this case, less simple than most statistical methods of trend determination assume.

In order to study the influence exerted by the various data on the rate of development of production, we may rewrite Eq. (3.13.1) in logarithmic form:

$$\ln v = t \ln (1 + \epsilon) + \lambda \ln P_0 + \lambda t \ln (1 + \pi) + (1 - \lambda) \ln k$$

differentiate it with respect to time,

$$\frac{\dot{v}}{v} = \ln (1 + \epsilon) + \lambda \ln (1 + \pi) + (1 - \lambda) \frac{\dot{k}}{k} \tag{3.14.6}$$

apply it to time $t = 0$, use (3.13.8), and assume ϵ and π to be small,

$$\left(\frac{\dot{v}}{v}\right)_0 = \epsilon + \lambda\pi + (1 - \lambda)\sigma$$

This equation is illuminating in that it shows the influence of the four constants ϵ, λ, π, and σ on the initial rate of growth. While the rate of increase in efficiency ϵ has a coefficient 1, the rate of population growth π has a coefficient λ and the propensity to save a coefficient $1 - \lambda$.

3.15 A general solution of the type (3.14.5) is less easily obtained if the supply of labor is not completely inelastic. It is possible, however, to obtain results of the type (3.14.6) for cases where λ and α have been given numerical values. Examples have been worked out for $\lambda = 3/4$, a value often quoted in statistical publications by Professor Douglas and his associates. Since very little is known about the supply elasticity of labor, various values have been assumed for it. The results are shown in Table 3.1.

TABLE 3.1. VALUES FOR $\dot{k}$, $\dot{a}$, AND $\dot{v}$ AT TIME $t = 0$, FOR VARIOUS VALUES OF α
($a = P_0 = k = v = 1$; $\lambda = 3/4$)

α	0	$1/2$	∞	-1
$\dot{k}$	σ	σ	σ	σ
$\dot{a}$	$\sigma + 4\epsilon - 4\Omega$	$\frac{\sigma}{3} + \frac{2}{3}\pi + \frac{4}{3}\epsilon - \frac{4}{3}\Omega$	π	$-\frac{\sigma}{3} + \frac{4}{3}\pi - \frac{4}{3}\epsilon + \frac{4}{3}\Omega$
$\dot{v}$	$\sigma + 4\epsilon - 3\Omega$	$\frac{\sigma}{2} + \frac{1}{2}\pi + 2\epsilon - \Omega$	$1/4\sigma + 3/4\pi + \epsilon$	$\pi + \Omega$

As presented here, the results are contributions to the solution of the analytical problem to explain the rate of growth in employment and production with the aid of such data as the rate of savings σ, the rate of growth ϵ of efficiency, the rate of growth π of the population, and the rate of increase Ω in desired wages. Interestingly enough, the results depend to a considerable extent on the supply elasticity of labor α. Even so, some general conclusions are possible, for example, that an acceleration in the growth of population (that is, an increase in π) will as a rule be accompanied by a lower acceleration in production, and by an equal acceleration in production only when $\alpha = -1$. Likewise, an acceleration in capital formation, that is, an increase in σ, will at most lead to an equal expansion in production and employment (for $\alpha = 0$), but as a rule to a smaller relative expansion.

The results may also be used for the solution of policy problems, for example, in order to determine the values of σ and Ω in order to obtain given rates of increase in production and employment. While using the table, the reader should be aware of the units in which the variables have been expressed (indicated in the table heading).

3.16 A few words may be added about the case in which

$$\lambda + \mu > 1 \tag{3.16.1}$$

The main difficulty which such a model presents is that incomes accruing to labor and to capital, respectively, add up to more than total national product. This points to the necessity to raise taxes from one or both of these groups, in order to cover the losses on the operation of such an economy, losses comparable to those incurred when production requires fixed costs. Another alternative is that employers act as monopolists vis-à-vis the factors of production, or as oligopolists, which is more likely. The case therefore requires a careful specification of the structural and institutional setup before it can be presented as a realistic picture of an economy as a whole.

Such a difficulty does not exist if $\lambda + \mu < 1$. Here, a profit remains in the hands of the organizers of production, which is a more realistic feature than a loss.

3.2. Models with Various Types of Technological Change

3.21 Before presenting some examples of models in this field, we must make a few general remarks on the concept of technological change and the appropriate instruments of analysis. Some instruments that are appropriate at a primitive stage of analysis lose their meaning in a more sophisticated approach. Whether such a sophisticated approach is applicable, however, depends on factual details.

By the primitive approach we mean the one which describes each individual process of production by the numerical inputs and outputs characterizing it. Taking capital and labor as the only inputs and considering the process the production of one unit of a certain product, we can represent each input combination by a point in a two-dimensional diagram, plotting the quantity of labor along the horizontal and the quantity of capital along the vertical axis (see Fig. 3). Process 1 will then be more labor-consuming and less capital-intensive than process 2, and the transition from process 1 to process 2 is a technological change characterized by saving labor, a laborsaving change, since $a_2 < a_1$. At the same time it requires more capital per unit of product, and we may also speak therefore of substitution of labor by capital. A pure case of laborsaving, in this context, would be a change from 1 to 3, where only $a_3 < a_1$, but $k_3 = k_1$. Clearly one could also conceive of pure capital-saving changes and of changes where capital is substituted by labor (for example, a change from 2 to 3 and one from 1 to 4).

Consideration of all individual processes is a rather cumbersome affair, with very restricted possibilities to generalize any results found. Of course the facts may be that way, and that cannot be helped. More powerful conclusions can be drawn if the facts obey a certain logical structure. An important feature of technological reality is that at any single moment not one single process to produce a given commodity is known, but often a set of them, which can be represented by a set of points in our diagram, for example, the set 2, 1, 4. In other cases, a continuous collection of processes may be known. In fact, even when only discrete processes are known, such as 2, 1, and 4, there is a continuous set of possibilities, since 2 and 1 can be carried out simultaneously in any proportion, meaning that all points of the line 12 are also technological possibilities. For various purposes, such a continuous set of points may be sufficiently approximated by a curve, often called the *curve of technological possibilities.*

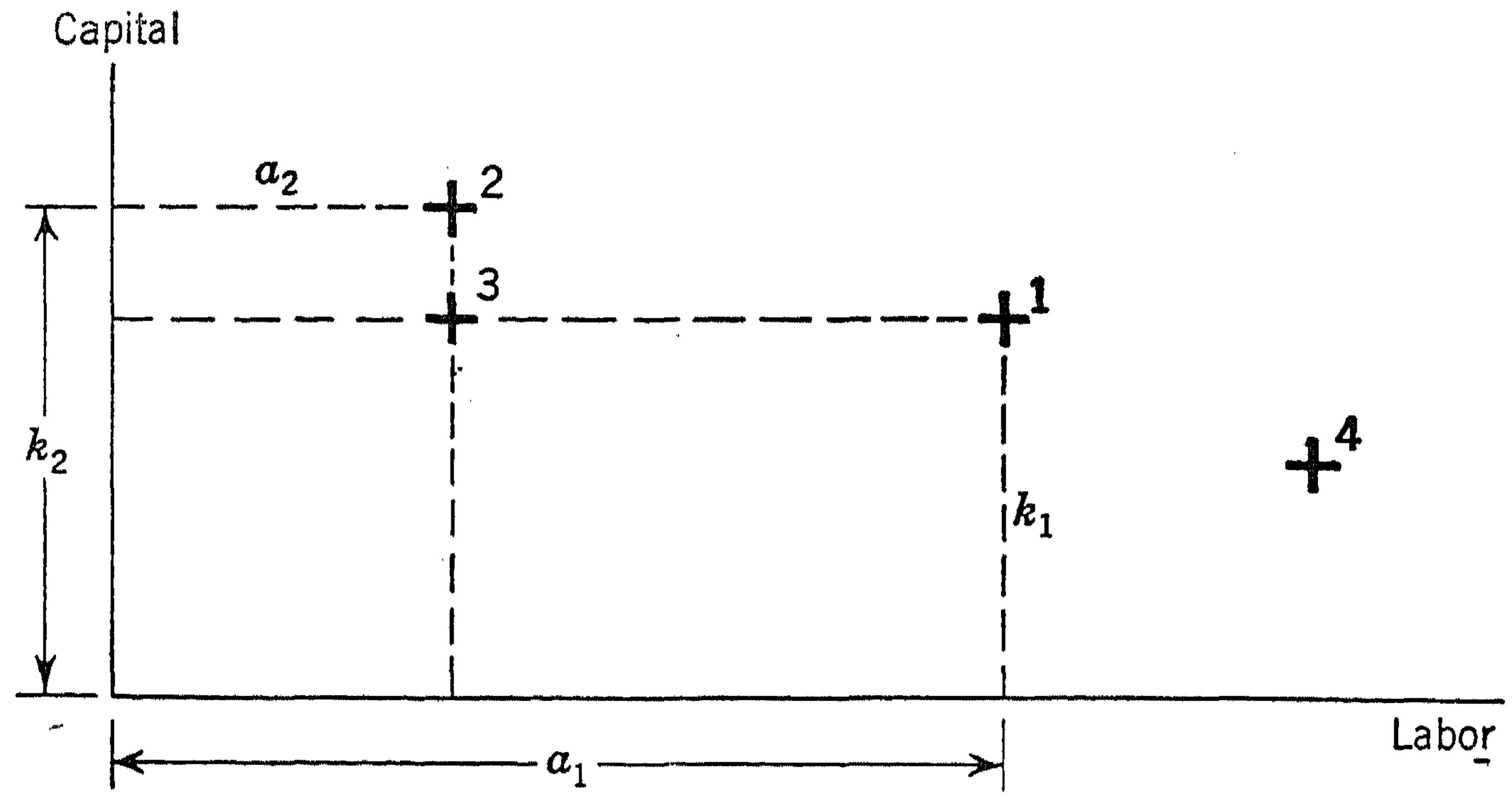

Fig. 3

The Cobb-Douglas curve is an example; for the unit of product it is represented, at any given time, by the formula

$$1 = \Gamma a^{\lambda} k^{1-\lambda} \tag{3.21.1}$$

or

$$\frac{1}{a} = \Gamma \left(\frac{k}{a}\right)^{1-\lambda} \tag{3.21.2}$$

In the last version, the left-hand side indicates labor productivity, and this appears to be a very simple function of capital per worker, or capital intensity.

If now, by a change in technological knowledge, the curve of technological possibilities shifts, we no longer have the possibility of speaking of laborsaving changes in the simple way we did before, since we cannot say that any particular point on the second curve corresponds to a given point on the first curve. The natural way to characterize the change is to indicate how the parameters of the curve, occurring in its formula, have changed. It depends on the number and nature of the parameters, how many types of change there are, and what their nature is. Appropriate names can then be given to each kind of change.

Since the Cobb-Douglas function is one of the very few production functions presented and tested so far, it makes sense first to study technological changes applying to that function. Since it contains two parameters in its simplest version (3.21.2), and three when the exponent of capital in its more general version (3.11.1) is left free, there are two or three types of changes to be distinguished.

We shall call a rise in Γ a *rise in general efficiency*, since it implies that the quantity of product obtained with the help of any combination of production factors increases in the same proportion. If such a rise occurs at a uniform rate, Γ can be given the form chosen in (3.11.1), namely $(1 + \epsilon)^t$.

The second type of change that may occur is a rise or fall in λ. The nature of a change in λ may be understood by a study of its consequences. Perhaps the most striking consequence of a fall in λ is that the portion of national product accruing to labor falls. In this sense the change means a saving in labor costs and may be called a laborsaving change. The phrase is not a happy one, however. It is more exact to speak of a fall in the *elasticity of production* with respect to labor, since λ simply is the elasticity and nothing else. It should be realized, at the same time, that changes in λ or μ do not always represent technological progress. Only changes in λ or μ or both, leaving an addition to the product, constitute progress. Such changes $d\lambda$ and $d\mu$ therefore must obey the condition

$$a^{\lambda+d\lambda}k^{\mu+d\mu} > a^{\lambda}k^{\mu}$$

or

$$d\lambda \ln a + d\mu \ln k > 0 \tag{3.21.3}$$

3.22 We shall now indicate some consequences for economic development of a systematic change in λ and μ, characterized by the condition, previously adhered to, that $\lambda + \mu = 1$ and by the relation

$$\lambda = \lambda_0 + \lambda' t \tag{3.22.1}$$

It is possible to apply the same method as in Sec. 3.15. The results in Table 3.2 are obtained.[1]

TABLE 3.2. VALUES FOR $\dot{a}$ AND $\dot{v}$ AT TIME $t = 0$ FOR VARIOUS VALUES OF α ($a = P_0 = k = v = 1$; $\lambda_0 = 3/4$)

α	0	½	∞	-1
$\dot{a}$	$\sigma - 16/3\lambda' + 4\epsilon - 4\Omega$	$\frac{\sigma}{3} - \frac{16}{9}\lambda' + \frac{2}{3}\pi + \frac{4}{3}\epsilon - \frac{4}{3}\Omega$	π	$-\frac{\sigma}{3} + \frac{16}{9}\lambda' + \frac{4}{3}\pi - \frac{4}{3}\epsilon + \frac{4}{3}\Omega$
$\dot{v}$	$\sigma - 4\lambda' + 4\epsilon - 3\Omega$	$\frac{\sigma}{2} - \frac{4}{3}\lambda' + \frac{1}{2}\pi + 2\epsilon - \Omega$	$1/4\sigma + 3/4\pi + \epsilon$	$4/3\lambda' + \pi + \Omega$

3.23 For practical purposes, it is interesting to answer the question what values of ϵ, a general rise in efficiency, and of λ', a shift in the elasticity of production with respect to labor, are realistic.

[1] Taken from H. Banerji, Technical Progress and the Process of Economic Development, *NUFFIC*, The Hague, 1956.

Over very long periods not very much of a change in λ has been observed: the distribution of income between labor and capital has long been one of a constant ratio. To be sure, this phenomenon has been accompanied by two other phenomena which should not be lost sight of: (1) a persistent reduction in the number of working hours and (2) a change in the relative numbers of workers and independent producers. The latter phenomenon implies that per capita incomes of workers and of the economy as a whole need not have moved in proportion.

A shift in Γ equivalent to a fairly regular increase in general efficiency has occurred, however, and roughly of the order of 1 per cent per annum. A check on this figure can be made with the aid of the empirical fact that the capital-output ratio has been roughly constant in some developed countries over about a century.

TABLE 3.3. SOME REALISTIC VALUES (IN PER CENT) OF λ, ω, AND π AND THE RESULTING VALUES OF ϵ REQUIRED IN ORDER THAT THE CAPITAL COEFFICIENT BE CONSTANT

ω	π	$\omega - \pi$	ϵ	
			$\lambda = \frac{3}{4}$	$\lambda = \frac{2}{3}$
4	1.5	2.5	1.9	1.7
4	1.0	3.0	2.3	2.0
4	0.5	3.5	2.6	2.3
3	1.5	1.5	1.1	1.0
3	1.0	2.0	1.5	1.3
3	0.5	2.5	1.9	1.7
2.5	1.5	1.0	0.8	0.7
2.5	1.0	1.5	1.1	1.0
2.5	0.5	2.0	1.5	1.3

Writing the Cobb-Douglas production function in its most general version (3.11.1), and assuming inelasticity in the labor supply, we can derive an expression for the capital-output ratio.

$$\kappa = \frac{k}{v} = P_0^{-\lambda} \frac{k^{1-\mu}}{(1+\epsilon)^t(1+\pi)^{\lambda t}} \tag{3.23.1}$$

In order that this ratio be constant over time, the fraction at the right-hand side must be constant; since capital and output have to move proportionately also, the rate of increase per annum of k must be ω (that is, the rate of increase in production). This requires that

$$\omega(1 - \mu) = \epsilon + \lambda\pi \tag{3.23.2}$$

or

$$\epsilon = \omega(1 - \mu) - \lambda\pi \tag{3.23.3}$$

If we assume that $\lambda + \mu = 1$, this reduces to

$$\epsilon = \lambda(\omega - \pi) \tag{3.23.4}$$

Some realistic values of λ, ω, and π have been summarized in Table 3.3, together with the resulting values of ϵ.

3.24 It is also possible to prove that, with the model discussed, the capital-output ratio has a tendency to become constant in the course of time. Starting from Eq. (3.13.1) and (3.14.4), we obtain for the capital-output ratio

$$\kappa = \frac{k}{v} = \frac{k^\lambda}{(1+\epsilon)^t P_0{}^\lambda (1+\pi)^{\lambda t}}$$
$$= \frac{\lambda\sigma}{\ln\,[(1+\epsilon)(1+\pi)^\lambda]} + \frac{C}{[(1+\epsilon)(1+\pi)^\lambda]^t P_0{}^\lambda \ln\,[\quad]}$$

which for growing values of t approaches the limit

$$\kappa_\infty = \frac{\lambda\sigma}{\ln\,[(1+\epsilon)(1+\pi)^\lambda]} \sim \frac{\lambda\sigma}{\epsilon + \lambda\pi} \tag{3.24.1}$$

This result[1] may be interpreted to mean that the constancy of the capital coefficient need not be considered a technical datum, but rather the result of a growth process. According to this formula, the capital coefficient depends on the two technical coefficients λ and ϵ, as well as on the rate of savings σ and the rate of population growth π. A high capital coefficient should be expected, according to this formula, in countries showing a high savings ratio and a low rate of population growth.

3.3. Models with Accounting Prices

3.31 Models of the type discussed in the preceding sections of this chapter may be used to estimate the influence of a *policy of accounting prices* for production factors. Such a policy is characterized by (1) the imposition on the public sector of a calculation system which uses accounting prices for labor and capital instead of market prices, and (2) the introduction, in the private sector, of a system of taxes and subsidies on the use of factors to induce private entrepreneurs to base their calculations on accounting prices also. If the accounting price of labor is lower than its market price, a subsidy will have to be offered, while a tax will be imposed wherever the accounting price is higher than the market price. It is proposed to estimate the effects of such a policy on employment, production, and development, as well as on public finance.

[1] Due to Professor P. J. Verdoorn; see The Role of Capital in Long-term Projection Models, *Cahiers écon. (Bruxelles)*, vol. 1, p. 49, 1959.

Following Dr. A. Qayum,[1] we have assumed that accounting prices are applied only to new investments.

3.32 The variables used in the model are

- a total employment
- $\bar{a}$ employment in sector without accounting prices, to be called *nonaccounting sector*
- $\tilde{a}$ employment in sector under accounting prices, to be called *accounting sector*
- k total capital in use
- $\bar{k}$ capital used in nonaccounting sector
- $\tilde{k}$ capital used in accounting sector
- v national product
- l market wage rate
- l' accounting wage rate
- m market interest rate
- m' accounting interest rate
- G taxes (subsidies are considered negative taxes)
- Φ_1 labor-capital ratio in accounting sector
- Φ_0 labor-capital ratio without accounting prices

Development over time will be calculated for all these variables; it is assumed that the policy of accounting prices is introduced at time $t = 0$. The formulas will be so devised that they show the development at any level of accounting prices introduced; this level as well as the level of market prices will be assumed constant over time, however, implying that, without accounting prices, the endowment with capital and labor would have grown proportionately.

3.33 The relations of the model are: capital in the nonaccounting sector is assumed to diminish at a rate $\delta' = 1 - \delta$, where δ is the rate of depreciation of capital goods—treated as a constant percentage applied to the existing stock.

$$\bar{k}_t = \bar{k}_0\delta'^t \qquad (3.33.1)$$

Since the labor-capital ratio in the nonaccounting sector has been chosen on the basis of market prices, it remains at the same level; hence employment in this sector diminishes proportionately to capital.

$$\bar{a}_t = \bar{a}_0\delta'^t \qquad (3.33.2)$$

Capital in the accounting sector grows for two reasons: depreciation allowances are re-invested and new savings are invested in it.

$$\dot{\tilde{k}}_t = \delta\bar{k}_t + \sigma v_t \qquad (3.33.3)$$

Here σ is the propensity to save, as before.

[1] A. Qayum, "Theory and Policy of Accounting Prices," Amsterdam, 1960.

Employment in the accounting sector remains in a fixed proportion to capital, because accounting prices remain fixed. This implies that we assume no technological change, that is, $\epsilon = 0$.

$$\tilde{a}_t = \Phi_1 \tilde{k}_t \tag{3.33.4}$$

Total production, and hence income, is composed of production in the two sectors; using the preceding equations, we get

$$v_t = \bar{a}_t^{\lambda} \bar{k}_t^{1-\lambda} + \tilde{a}_t^{\lambda} \tilde{k}_t^{1-\lambda} = \bar{a}_0^{\lambda} \bar{k}_0^{1-\lambda} \delta'^t + \Phi_1^{\lambda} \tilde{k}_t = v_0 \delta'^t + \Phi_1^{\lambda} \tilde{k}_t \tag{3.33.5}$$

Market wage rates are equal to marginal productivity of labor in the nonaccounting sector:

$$l = \lambda \left(\frac{\bar{k}_t}{\bar{a}_t}\right)^{1-\lambda} = \lambda \left(\frac{\bar{k}_0}{\bar{a}_0}\right)^{1-\lambda} = \lambda \Phi_0^{\lambda-1} \tag{3.33.6}$$

Market interest rates are equal to marginal productivity of capital in the nonaccounting sector:

$$m = (1 - \lambda) \left(\frac{\bar{a}_t}{\bar{k}_t}\right)^{\lambda} = (1 - \lambda)\Phi_0^{\lambda} \tag{3.33.7}$$

Accounting wage rates are equal to marginal productivity of labor in the accounting sector:

$$l' = \lambda \left(\frac{\tilde{k}_t}{\tilde{a}_t}\right)^{1-\lambda} = \lambda \Phi_1^{\lambda-1} \tag{3.33.8}$$

Accounting interest rates are equal to marginal productivity of capital in the accounting sector:

$$m' = (1 - \lambda) \left(\frac{\tilde{a}_t}{\tilde{k}_t}\right)^{\lambda} = (1 - \lambda)\Phi_1^{\lambda} \tag{3.33.9}$$

Taxes are equal to taxes on capital in the accounting sector minus subsidies on labor in that sector; both are proportional to the differences between accounting and market prices, whichever is higher, and to the quantities employed in the accounting sector:

$$G = (m' - m)\tilde{k}_t - (l - l')\tilde{a}_t \tag{3.33.10}$$

These ten equations determine the ten variables $\bar{a}$, $\tilde{a}$, $\bar{k}$, $\tilde{k}$, v, l, l', m, m', and G. If we so wish, a and k can be found from two additional balance equations:

$$a = \bar{a} + \tilde{a} \tag{3.33.11}$$
$$k = \bar{k} + \tilde{k} \tag{3.33.12}$$

3.34 The two problems we proposed to solve can easily be dealt with in the following way.

Combining Eqs. (3.33.3), (3.33.1), and (3.33.5), we obtain

$$\begin{aligned}\dot{\tilde{k}}_t &= \delta \delta'^t k_0 + \sigma(v_0 \delta'^t + \Phi_1^{\lambda} \tilde{k}_t) \\ &= (\delta k_0 + \sigma v_0)\delta'^t + \sigma \Phi_1^{\lambda} \tilde{k}_t\end{aligned} \tag{3.34.1}$$

This is a nonhomogeneous differential equation of the first order in $\tilde{k}_t$; its general solution consists of two parts, I and II, where I is the general solution of the homogeneous equation

$$\dot{\tilde{k}}_t = \sigma\Phi_1{}^\lambda \tilde{k}_t \tag{3.34.2}$$

and II a particular solution of the nonhomogeneous equation (3.34.1). Solution I is:

$$\tilde{k}_t{}^{\mathrm{I}} = \tilde{K}_0{}^{\mathrm{I}} e^{\sigma\Phi_1{}^\lambda t} \qquad \tilde{K}_0{}^{\mathrm{I}} \text{ arbitrary} \tag{3.34.3}$$

For solution II assume that $\tilde{k}_t{}^{\mathrm{II}} = \tilde{K}_0{}^{\mathrm{II}}\delta'^t$. Then $\dot{\tilde{k}}_t{}^{\mathrm{II}} = \tilde{K}_0{}^{\mathrm{II}}\delta'^t \ln \delta'$; substituting into (3.34.1), we get

$$\tilde{K}_0{}^{\mathrm{II}} \ln \delta' = \delta k_0 + \sigma v_0 + \sigma\Phi_1{}^\lambda \tilde{K}_0{}^{\mathrm{II}}$$

from which it follows that $\tilde{K}_0{}^{\mathrm{II}}$ has to satisfy the condition

$$\tilde{K}_0{}^{\mathrm{II}} = \frac{\delta k_0 + \sigma v_0}{\ln \delta' - \sigma\Phi_1{}^\lambda} \tag{3.34.4}$$

Hence the general solution is

$$\tilde{k}_t = \tilde{K}_0{}^{\mathrm{I}} e^{\sigma\Phi_1{}^\lambda t} + \frac{\delta k_0 + \sigma v_0}{\ln \delta' - \sigma\Phi_1{}^\lambda}\,\delta'^t \tag{3.34.5}$$

The arbitrary constant $\tilde{K}_0{}^{\mathrm{I}}$ has to be found from the initial condition that for $t = 0$, $\tilde{k}_0 = 0$; this requires

$$\tilde{K}_0{}^{\mathrm{I}} = -\frac{\delta k_0 + \sigma v_0}{\ln \delta' - \sigma\Phi_1{}^\lambda} \tag{3.34.6}$$

Thus the solution becomes

$$\tilde{k}_t = \frac{\delta k_0 + \sigma v_0}{\sigma\Phi_1{}^\lambda - \ln \delta'}\left(e^{\sigma\Phi_1{}^\lambda t} - \delta'^t\right) \tag{3.34.7}$$

The denominator may be written approximately as $\sigma\Phi_1{}^\lambda + \delta$, since for small δ, $\ln \delta' = \ln (1 - \delta) = -\delta$.

The expression for $\tilde{k}_t$ is positively dependent on Φ_1, meaning that capital develops more quickly if Φ_1 is put higher, that is, if the accounting price for labor is put relatively low and the accounting price for capital relatively high. Since both employment and production are again positively dependent on $\tilde{k}_t$ and on Φ_1, all these variables will grow faster, the higher Φ_1 is. The limit set to this growth is evidently full employment of the labor force.

In order to determine the financial burden to the government, we combine Eqs. (3.33.10) and (3.33.6) to (3.33.9), yielding

$$G = (\Phi_1{}^\lambda - \Phi_0{}^\lambda + \lambda\Phi_0{}^\lambda - \lambda\Phi_0{}^{\lambda-1}\Phi_1)\tilde{k}_t \tag{3.34.8}$$

Writing $\Phi_1 = \varphi\Phi_0$, we find

$$G = (\varphi^\lambda - 1 + \lambda - \lambda\varphi)\Phi_0{}^\lambda \bar{k}_t \tag{3.34.9}$$

For values of φ slightly above 1 this is of second order only in $\varphi - 1$, meaning that as a first approximation taxes and subsidies are of equal size and that there is a very small net burden on the government. For values of φ well above 1 this is no longer so; thus, for $\varphi = 1.5$ and $\lambda = 2/3$ we find

Tax on capital:	$\frac{1}{3}(1.5^{2/3} - 1)\bar{k}_t = 0.10\bar{k}_t$
Subsidy on labor:	$\frac{2}{3}(1.5 - 1.5^{2/3})\bar{k}_t = 0.12\bar{k}_t$
Net burden:	$-G = 0.02\bar{k}_t$

3.4. Models with Restricted Substitution between Factors

3.41 If the number of processes is small, there may be a point in specifying them exactly and in developing the various cases or configurations that may present themselves. This method recommends itself if the problems of a single industry with clearly distinct processes are at stake, or if the economy is of an extremely simple type. A good example of one industry in which these conditions prevail is Indian cotton weaving. Sometimes similar differences between large-scale and small-scale business may play a role. For a more complicated economy as a whole it is doubtful whether the method is of much help. Even if in a single industry there is a choice between two methods only, as a rule the choice between industries represents another possibility of choosing between capital-intensive and labor-intensive processes, and the number of possibilities may turn out to be large.

3.42 Let us consider two processes characterized by upper indexes 1 and 2, where 1 is the more labor-intensive process. The following variables will be considered.

a^1, a^2	employment in processes 1 and 2
k^1, k^2	capital used in processes 1 and 2
v^1, v^2	product obtained in the two processes
m^1, m^2	profit rates earned in the two processes
l	wage rate, supposed uniform

As data we shall consider

P	total population, as far as employable
k^0	total capital stock available
g^1, g^2	labor productivity in the two processes
κ^1, κ^2	capital coefficients of the two processes
$\bar{l}$	minimum wage level, by custom or law

3.43 We shall not study the models in this class in much detail but only briefly indicate the configurations conceivable, as mentioned. The relation with development planning is that these configurations may materialize in succession, if capital per head of the population moves along an upward trend. Over very long periods this may indeed be so. Most of our other planning models will refer to the period of capital scarcity and labor abundance, which is the common situation in underdeveloped countries, although not necessarily the only possible situation.

It appears that five configurations or phases may thus be distinguished.

I. All capital used in process 1, leaving still a surplus of labor
II. All capital used in process 1, supplying employment to all labor
III. Capital and labor distributed over both processes
IV. Capital and labor fully employed only in process 2
V. All labor employed in process 2, leaving some capital unused

Which of these configurations materializes evidently depends on the ratio k_0/P, the dependency being

Case I: $$\frac{k^0}{P} < \frac{k^1}{a^1} = \kappa^1 g^1$$

Case II: $$\frac{k^0}{P} = \kappa^1 g^1$$

Case III: $$\kappa^2 g^2 > \frac{k^0}{P} > \kappa^1 g^1$$

Case IV: $$\frac{k^0}{P} = \kappa^2 g^2$$

Case V: $$\frac{k^0}{P} > \kappa^2 g^2$$

The relationship can be shown most clearly in a graphic way:

$$\begin{array}{ccccccccc} & & & \mathrm{II} & & \mathrm{IV} & & & \\ & & \mathrm{I} & \downarrow & \mathrm{III} & \downarrow & \mathrm{V} & & \\ \hline 0 & & & \kappa^1 g^1 & & \kappa^2 g^2 & & \rightarrow & \dfrac{k^0}{P} \end{array}$$

Under free competition of factors, the price of labor in case I and that of capital in case V would be zero. In case III, where there is coexistence of processes 1 and 2, wages will have to satisfy the condition $m^1 = m^2$ since otherwise capital would be withdrawn from one of the processes. This condition can be elaborated by using the definition of profit rates and of capital-output ratios.

$$m^1 = \frac{v^1 - la^1}{\kappa^1 v^1} = \frac{v^2 - la^2}{\kappa^2 v^2} = m^2 \tag{3.43.1}$$

Since, in addition, $v^1 = g^1a^1$ and $v^2 = g^2a^2$, the condition becomes, expressed in terms of data,

$$m^1 = m^2 = m^{\text{III}} = \frac{g^1 - l}{\kappa^1 g^1} = \frac{g^2 - l}{\kappa^2 g^2}$$

or

$$l^{\text{III}} = g^1 g^2 \frac{\kappa^2 - \kappa^1}{\kappa^2 g^2 - \kappa^1 g^1} \tag{3.43.2}$$

In cases II and IV, wage and profit rates are, within certain limits, undetermined. With our assumption about a minimum wage $\bar{l}$, this will be a lower limit to l in case II. Evidently this implies that $\bar{l} < l^{\text{III}}$.

In case I, therefore, we have $l^{\text{I}} = \bar{l}$ and consequently

$$m^{\text{I}} = \frac{g^1 - \bar{l}}{\kappa^1 g^1}$$

In case V, we will have $m^2 = 0$ and hence $l^{\text{V}} = g^2$ unless there is some minimum set to m also.

3.44 Sometimes there may be scope to combine the ideas underlying Secs. 3.1 and 3.4 by assuming that within certain limits of k/a, continuous substitution is possible, whereas outside these limits no further substitution is possible.

3.5. The Role of Aims in the Design of Policy

3.51 The main subject of this book is the presentation of economic models which may be helpful in planning development. Models themselves do not say anything about the use that may be made of them, as was already set out in Chap. 1. In fact, very different policy devices may be obtained with the aid of the same model. This difference evidently depends on the aims set for a development policy. It may also depend on the means used or excluded. In this section we shall demonstrate by some examples how widely different policy devices may sometimes be obtained with the same model. We do so at this stage because the models we have been discussing are sufficient to make our point. It could be repeated later when more complicated models have been discussed, but then it would be a cumbersome affair, not adding very much to the clarity of exposition. We shall first treat some examples and finish with some conclusions (see Sec. 3.55).

3.52 In this section we use a one-sector one-factor model as discussed in Sec. 2.1. Our variables will be capital k_t, income y_t, and consumption c_t. Our initial position k_0 (and hence y_0) is given. The policy instrument considered is the rate of saving $\sigma = (y - c)/y$, supposed to be constant over time and to be chosen once forever. In order to show the influence of aims set, we choose consumption at different times as the aim of development: in Example 1, consumption now; in Example 2, consumption at some later time T; in example 3, total consumption over a period $0 \leq t \leq T$.

With our attention to be focused on consumption, it is useful to express the development of this variable over time in terms of the data of our problem. This can be done in the following way, if we remember that with a constant savings ratio σ all variables develop proportionately to $e^{(\sigma/\kappa)t}$ (see Sec. 2.13):

$$c_t = y_t - \dot{k}_t = \frac{k_t}{\kappa} - \dot{k}_t = \left(\frac{k_0}{\kappa} - \frac{\sigma}{\kappa} k_0\right) e^{(\sigma/\kappa)t} = \frac{k_0}{\kappa}(1 - \sigma)e^{(\sigma/\kappa)t} \quad (3.52.1)$$

Example 1. Consumption now

$$c_0 = \frac{k_0}{\kappa}(1 - \sigma) \quad (3.52.2)$$

This evidently becomes a maximum if we choose $\sigma = 0$; no savings, no development.

Example 2. Consumption at time T

$$c_T = \frac{k_0}{\kappa}(1 - \sigma)e^{(\sigma/\kappa)T} \quad (3.52.3)$$

This will be a maximum for $dc_T/d\sigma = 0$, or

$$\frac{T}{\kappa}(1 - \sigma)e^{(\sigma/\kappa)T} - e^{(\sigma/\kappa)T} = 0$$

$$\frac{T}{\kappa} - \frac{T}{\kappa}\sigma - 1 = 0$$

$$\sigma = \frac{T - \kappa}{T}$$

Since κ is around 3 years, we find, for some values of T,

$T =$	5	10	15	20
$\sigma =$	0.4	0.7	0.8	0.85

We also see that up to $T = 3$, the policy device obtained suggests no savings (taking into account the boundary condition that savings should not be negative) or even negative savings—supposing that we can dissave up to the amount of k_0.

Example 3. Consumption over period $0 \leq t \leq T$

$$\bar{c} = \frac{k_0}{\kappa}(1 - \sigma)\int_0^T e^{(\sigma/\kappa)t'}\,dt'$$

We can easily find

$$\bar{c} = \frac{k_0}{\sigma}(1 - \sigma)(e^{(\sigma/\kappa)T} - 1) \quad (3.52.4)$$

Upon differentiation and putting the derivative equal to zero, we obtain a transcendental equation in σ, which is not explicitly solvable. The simplest procedure to estimate σ is to try, for given numerical values of T and κ. It is easily seen that T and κ appear only in the combination T/κ; therefore it is only this that matters.

$\frac{T}{\kappa} =$	0	0.55	1.4	2.0	2.1	2.3	2.4	2.6	2.7	3.5	5.3	∞
$\sigma =$	$-\infty$	-5.4	-0.72	0	0.10	0.18	0.25	0.31	0.37	0.57	0.75	1

Figures up to $T/\kappa = 2$ do not make sense evidently and will have to be replaced by boundary conditions. The remaining figures show that, depending on the length of the period T, any savings figure can be found. Assuming $\kappa = 3$ years, we find that for $T = 7.2$ years, such a realistic figure as 18 per cent is the optimum rate of savings, but that for $T = 8.1$ years, we find already 37 per cent and for $Y = 16$ years, 75 per cent.

3.53 In this section we use a one-sector two-factors model with continuous substitution between factors, as discussed in Sec. 3.14, without the rise in general efficiency (the simplest Cobb-Douglas model). Initial capital k_0 is supposed given; so is the population P, not supposed to be fully employed during the period considered. Let the aim be income v_T at time T and the instrument the choice of technology, which may simply be represented by the number a employed with the capital available. For the sake of simplicity, we assume a to be constant, but other assumptions are possible as well.

Income $\qquad v = a^{\lambda}k^{1-\lambda} \qquad$ and $\qquad \dot{k} = \sigma v = \sigma a^{\lambda}k^{1-\lambda} \qquad (3.53.1)$

This leads to $\qquad k^{\lambda} = \lambda\sigma a^{\lambda}t + k_0^{\lambda} \qquad (3.53.2)$

and to $\qquad v = a^{\lambda}k^{1-\lambda} = a^{\lambda}(\lambda\sigma a^{\lambda}t + k_0^{\lambda})^{(1-\lambda)/\lambda} \qquad (3.53.3)$

It is clear that, at any time T, v_T will be larger the larger a is and that, for the model assumed, the most labor-intensive technology will be the best device. Here, then, there is no great influence of the aim chosen on the policy device.

3.54 In this section we use a one-sector two-factors model with a choice between two processes of production yielding different savings rates. The variables considered are product v^1 or v^2, employment a^1 or a^2, capital k^1 or k^2, and profit rate m. The data are

Capital coefficients:	$\kappa^1 > \kappa^2$
Labor productivities:	$g^1 > g^2$
Savings ratios:	$\sigma^1 > \sigma^2$

These savings ratios are ratios of savings to profits and not to total income; the profits are k^1m and k^2m and hence savings σ^1k^1m or σ^2k^2m. Finally, initial capital stock k_0 is given.

We assume that the aim is to maximize employment at time T, the policy instrument being the choice of technology.

Suppose that process 1 is chosen; then we have

Volume of production: $$v_t^1 = \frac{k_t^1}{\kappa^1} \tag{3.54.1}$$

Volume of employment: $$a_t^1 = \frac{v_t^1}{g^1} = \frac{k_t^1}{\kappa^1 g^1} \tag{3.54.2}$$

Profits are $k_t^1 m$, savings $\sigma^1 k_t^1 m$, and hence

$$\dot{k}_t^1 = \sigma^1 k_t^1 m$$

yielding $$k_t^1 = k_0 e^{\sigma^1 m t} \tag{3.54.3}$$

It follows that employment at time T is

$$a_T^1 = \frac{k_0 e^{\sigma^1 m T}}{\kappa^1 g^1} \tag{3.54.4}$$

Since $\kappa^1 g^1 > \kappa^2 g^2$, employment at time $t = 0$ will be larger if process 2 is chosen. Since, however, $\sigma^1 > \sigma^2$, capital and hence employment will grow faster if process 1 is chosen. From a certain value of T on, therefore, process 1 will yield maximum employment. Again it depends on which time T is chosen whether one process or the other has to be selected.

3.55 Some provisional conclusions may be drawn from the examples presented and from general experience about related subjects. Clearly the most important conclusion is that a precise formulation of the aims of development is necessary in order to narrow down the range of uncertainty in the choice of policy instruments. The practical importance of this conclusion can be illustrated by mentioning some examples of the wide divergencies among policies adopted in different countries. We have already mentioned the large differences in the rate of savings. A second example is that of the large differences in trade policy and the choice of industries connected with it. Whereas some countries try to develop export industries in order to have the full advantage of the international division of labor, other countries develop import-replacing industries and tend toward autarchy. The remarkable trend from the latter toward the former point of view in the communist countries may be noted. A third example finally is the famous discussion about the choice of technology, in which some authors favor the establishment of capital-intensive and others labor-intensive industries. Whereas our last section (3.54) shows that again the aim chosen influences the answer, our next preceding section (3.53), in comparison with 3.54, also shows that the details of the model sometimes matter. Thus in this particular problem much depends on whether investments are financed out of savings made from profits or out of other sources, for example, taxes. In case they depend on taxes, they are connected with income as a whole and not only with profits; and the method of production yielding the maximum initial income will be the best one.

Important as a more precise formulation of aims may be, it still does not follow that such a formulation is sufficient to remove divergencies. As we have discussed previously, economic science is not yet able, to cite an important example, to solve the problem of the optimum rate of development (see Sec. 2.4). How much less will practical policy arguments be able to settle some of the biggest differences!