# Chapter 6

# SEVERAL SECTORS; VARIABLE PRICES

## 6.1. Two-sector Model for Terms of Trade

**6.11** In this chapter, models will be discussed in which prices are among the variables to be planned. In Chap. 1 we discussed the necessity to do so in a number of cases. These cases will now be taken up.

In Sec. 2.14, the Harrod-Domar model was adapted to an open economy. However, no prices were introduced, implying the assumption that exports are salable at the constant price level assumed. We shall now assume that exports can be increased only at a lowering of export prices. This will be the case, for example, for the underdeveloped countries which are the main suppliers on one of the world commodity markets. Imports, in this case mainly consisting of capital and other finished goods, are assumed to be dependent on the national product. Interindustry deliveries are neglected, and two sectors are distinguished: the country concerned and the rest of the world. The price level introduced in the model represents the terms of trade of the country with the rest of the world.[1]

The planning problems that can be studied with the help of this model are, for example, the rate of growth of the national income which leads to balance-of-payments equilibrium and full utilization of the capital supplied and the rate of savings which maximizes the national income at a certain future point of time.

**6.12** The variables used are

$k$ volume of capital
$v$ gross national product
$c$ consumption

[1] See P. J. Verdoorn, Complementarity and Long-range Projections, *Econometrica*, vol. 24, no. 4, p. 429, 1956.

$e$ volume of exports
$i$ volume of imports
$y$ real national income
$p$ terms of trade of country with rest of the world

**6.13** The equations of the model are

$$k = \kappa v \tag{6.13.1}$$
$$\dot{k} = \sigma y \tag{6.13.2}$$
$$v = c + \dot{k} + e - i \tag{6.13.3}$$
$$y = c + \dot{k} + ep - i \tag{6.13.4}$$

As a consequence of changes in the terms of trade, we have to distinguish between the volume of output measured by $v$ and the real national product $y$. The import price level is used as *numéraire.*

$$i = \iota v \tag{6.13.5}$$
$$e = e(p) \tag{6.13.6'}$$

or more specifically

$$e = e_0 p^{\epsilon} \tag{6.13.6}$$

where $e_0$ is a constant and $\epsilon$ the (constant) price elasticity.

$$ep = i \tag{6.13.7}$$

This equation formulates the condition of equilibrium in the balance of payments.

**6.14** The seven equations of the model are just sufficient to determine the seven variables. For example, the solution for the terms of trade $p$ is

$$p_t = \left(p_0 - \frac{\iota}{1+\iota}\right) e^{\frac{\sigma}{\kappa}\frac{1+\iota}{1+\epsilon}t} + \frac{\iota}{1+\iota} \tag{6.14.1}$$

From this equation it follows that the terms of trade improve if $-1 < \epsilon < 0$, and deteriorate if $\epsilon < -1$.

If we want to determine the rate of savings which maximizes $y$ at the point of time $t$, we first derive the solution for $y$, differentiate $y$ with regard to $\sigma$, and put $dy/d\sigma$ equal to zero. We find that

$$\frac{\epsilon}{1+\epsilon}\frac{\iota}{1+\iota} = \left(p_0 - \frac{\iota}{1+\iota}\right) e^{\frac{\sigma}{\kappa}\frac{1+\iota}{1+\epsilon}t} + \frac{\iota}{1+\iota} \tag{6.14.2}$$

This expression enables us to determine the value of $\sigma$ which maximizes $y$ at any point of time $t$. The expression shows that $\sigma$ has to vary inversely proportional to $t$. The relationship between $\sigma$ and the coefficients $\epsilon$ and $\iota$ appears to be fairly complicated.

## 6.2. Open Economy Input-Output Model with World-demand Equations

**6.21** We are now going to take up again the multisector input-output model, which we discussed in Sec. 5.2, dropping the assumption of constant prices in order to express the influence that changes in production volumes may exert on prices and hence on national income. This means that we are giving a more flexible significance to the concept of (comparative) advantages to be derived from international trade.

**6.22** The variables of the model are

$v^h$ production of good $h$
$v^{hh'}$ input of commodity $h$ into current production of $h'$
$w^{hh'}$ input of commodity $h$ into investments in sector $h'$
$c^h$ consumption of good $h$
$e^h$ exports (with negative sign: imports) of good $h$
$p^h$ price level of commodity $h$
$Y$ national income (in money terms)
$S$ savings (in money terms)

**6.23** The relations of the model are

$$w^{hh'} = \frac{\kappa^{hh'}}{\theta} (v^{h'}_{t+\theta} - v^{h'}) \tag{6.23.1}$$

$$S = \sum_h \sum_{h'} w^{hh'} p^h \tag{6.23.2}$$

$$S = \sigma Y \tag{6.23.3}$$

$$Y = \sum_h v^h p^h - \sum_h \sum_{h'} v^{hh'} p^h \tag{6.23.4}$$

$$v^h = c^h + e^h + \sum_{h'} v^{hh'} + \sum_{h'} w^{hh'} \tag{6.23.5}$$

$$c^h p^h = \gamma^h (Y - S) + \Sigma \gamma^{hh'} p^{h'} + \bar{c}^h \bar{p} \tag{6.23.6}$$

As before, we assume that $\Sigma \gamma^h = 1$, $\Sigma \bar{c}^h = 0$, and all $c^h \geq 0$; in addition we now assume that, for all values of $h'$, $\sum_h \gamma^{hh'} = 0$. The consequence of these assumptions again will be that $\Sigma c^h p^h = Y - S$.

$$v^{hh'} = \varphi^{hh'} v^{h'} \tag{6.23.7}$$

$$p^h = \pi^h (v^h) \tag{6.23.8}$$

These equations express the relationship between $p^h$ and $v^h$ as a consequence of world demand. Other variables may occur in the relationship $\pi^h$, but it is assumed that these are exogenous variables, which can be considered given. Some more general assumptions will be discussed in Sec. 6.4.

**6.24** The use of this model will be illustrated again by considering the second stage of a planning problem arising after the rate of savings has been chosen. As in Sec. 5.24, we assume that all $v_0{}^h$ are given as a consequence of previous investments, and that the problem of choosing the production pattern for time period $\theta$ is considered. Again we assume that a maximization of income is aimed at under the side condition of given savings. It is possible, with the aid of relations (6.237) and (6.238), to write $Y$ as a function of $v^h$ only and to apply this formula to year $\theta$.

$$Y_\theta = \sum_h \left(v_\theta{}^h - \sum_{h'} \varphi^{hh'} v_\theta{}^{h'}\right)\pi^h(v_\theta{}^h) \tag{6.24.1}$$

Also savings $S_0$ can be expressed in terms of $v_\theta{}^h$:

$$S_0 = \sum_h \sum_{h'} \frac{\kappa^{hh'}}{\theta} (v_\theta{}^{h'} - v_0{}^{h'}) p_0{}^h \tag{6.24.2}$$

which is a linear function in $v_\theta{}^h$. Our problem will then be to maximize (6.24.1) under the side condition (6.24.2). The solution can be written with the aid of a Lagrangian multiplier $\Psi$.

$$\frac{\partial Y_\theta}{\partial v_\theta} + \Psi \frac{\partial S_0}{\partial v_\theta} = 0 \tag{6.24.3}$$

or

$$\pi^h(v_\theta{}^h) + v_\theta{}^h \frac{d\pi^h}{dv_\theta{}^h} - \frac{d\pi^h}{dv_\theta{}^h} \sum_{h'} \varphi^{hh'} v_\theta{}^{h'} - \sum_{h'} \varphi^{hh'} \pi^{h'}(v_\theta{}^{h'}) + \Psi \sum_{h'} \frac{\kappa^{h'h}}{\theta} p_0{}^{h'} = 0 \tag{6.24.4}$$

In this problem as well as in that of Sec. 5.24 some boundary conditions will have to be respected; again all $v_\theta{}^h - v_0{}^h$ should be positive (or not surpass, when negative, the absolute figure of depreciation allowances). It will depend very much on the nature of the functions $\pi^h$ whether such boundary conditions rather than the maximum conditions (6.24.4) will become active.

## 6.3. Nonlinear Cost Functions

**6.31** The model just described can be further adapted to reality, for the sectors where this is essential, by the introduction of nonlinear cost functions. As already observed, agriculture and mining may operate under increasing marginal costs, and this phenomenon can be taken account of rather easily, as shown in Sec. 5.4. In somewhat loose nonmathematical terms, there are two reasons why increased production in a sector may make its contribution, per unit of capital invested, to future national income less than in the initial situation: increasing costs and decreasing prices. If the sectors contributing most, per unit of invested capital, to national income are of one of these two types, their marginal contribution will be reduced with increasing production, and the optimum will be characterized by marginal contributions which are equal for all sectors in which investment is planned.

**6.32** The situation is less simple for sectors working under decreasing marginal costs, which as a rule are the expression of the existence of indivisibilities of some kind. As already indicated in Chap. 5, the problem of finding the optimum program then can best be solved by making use of partial models, going into details it would be difficult to introduce in a general way for all sectors.

## 6.4. Other Demand Functions

Another adaptation to reality of the model discussed in Sec. 6.2 is to introduce other demand functions.

In the model of Sec. 6.2, the price in each sector was assumed to be a function of gross output in that sector: $p^h = \pi^h(v^h)$. This demand equation, which, however, could also be interpreted as a cost function, is based on the assumption that the internal prices adapt themselves to the corresponding foreign prices and that these prices determine the demand for both home and foreign demand, for final consumption, and other uses.

A more complicated situation arises when uniform domestic and foreign prices can no longer be assumed. Here, for example, separate demand functions for home and foreign demand may be necessary. Foreign demand could then depend on the (relative) price difference at home and abroad.

A further refinement could be to introduce the substitutability or complementarity in the demand equations by making $p^h$ a function not only of $v^h$ but also of the gross output in one or more other sectors $h'$.

Generally, to introduce these refinements in the models would very soon lead to unmanageable results. Here also partial methods could be linked to the over-all planning models.

## 6.5. Alternative Techniques: A Linear Programming Model

**6.51** In this section we shall discuss an example of the use of the linear programming technique in formulating a development model.[1]

Mathematically, a linear programming model is bound to linear equations, which implies in many cases a serious limitation on the number of development problems to which such a model could be applied, or a distortion of reality if the assumption of linearity is made without good reasons. The dynamic nature of nearly all planning models is another difficulty.

[1] For a general exposition of the mathematical and economic aspects of linear programming, see, for example, R. Dorfman, P. A. Samuelson, and R. Solow, "Linear Programming and Economic Analysis," New York, 1958.

The following section describes an interesting attempt by Mr. J. Sandee to develop a planning model for India using linear programming technique.[1] The model takes the situation in 1960 as its starting point and tries to determine an optimum situation for 1970. Our description is formulated in general terms and, therefore, is independent of Mr. Sandee's statistical application to the case of India. Our formulation makes possible comparison with the other models in this book.

**6.52** The variables of the model are

- $v^h$ production of good $h$
- $v^{hh'}$ input of commodity $h$ into current production of $h'$
- $w^{hh'}$ input of commodity $h$ into investments in sector $h'$
- $j$ total volume of investment
- $c^h$ consumption of good $h$
- $c$ total volume of consumption
- $e^h$ exports (imports with negative sign) of good $h$
- $y$ real national product

All variables measure differences between the variables at the end and at the beginning of the planning period.

**6.53** The equations of the model are the following:

$$w^{hh'} = \frac{2}{T}\kappa^{hh'}v^{h'} - 2w_0^{hh'} \tag{6.53.1}$$

The assumption is made that all investments increase (or decrease) linearly with time during the planning period. If the planning period is $T$ years, and if $w_0^{hh'}$ represents the investment flow of commodity $h$ into sector $h'$ in the year before the first year of the planning period, then the flow of investment in the last year of the planning period is equal to $w_0^{hh'} + w^{hh'}$. Total investment during this period is $T(w_0^{hh'} + \frac{1}{2}w^{hh'})$, which can be related to the increase in gross output of sector $h'$.

$$T(w_0^{hh'} + \tfrac{1}{2}w^{hh'}) = \kappa^{hh'}v^{h'}$$

or

$$w^{hh'} = \frac{2}{T}\kappa^{hh'}v^{h'} - 2w_0^{hh'}$$

$$j = \Sigma\Sigma w^{hh'} \tag{6.53.2}$$

$$v^h = c^h + e^h + \Sigma v^{hh'} + \Sigma w^{hh'} \tag{6.53.3}$$

$$c = \Sigma c^h \tag{6.53.4}$$

$$v^{hh'} = \varphi^{hh'}v^{h'} \tag{6.53.5}$$

$$\Sigma e^h = 0 \tag{6.53.6}$$

[1] J. Sandee, "A Demonstration Planning Model for India," Calcutta, 1960.

This equation defines the condition that no changes in the balance-of-payments situation will take place. Imports are considered as negative exports.

$$\bar{e}^h < e^h < \bar{\bar{e}}^h \tag{6.53.7}$$

$$\bar{w}^{hh'} < w^{hh'} < \bar{\bar{w}}^{hh'} \tag{6.53.8}$$

$$\bar{c}^h < c^h < \bar{\bar{c}}^h \tag{6.53.9}$$

The inequalities (.7), (.8), and (.9) are boundary conditions set on the variables $e^h$, $w^{hh'}$, and $c^h$. Both the lower bounds $\bar{e}^h$, $\bar{w}^{hh'}$, and $\bar{c}^h$ as well as the upper bounds $\bar{\bar{e}}^h$, $\bar{\bar{w}}^{hh'}$, and $\bar{\bar{c}}^h$ are determined on *ad hoc* reasonings. The inequalities (.7) and (.9) are substitutes for the export demand and consumption equations.

$$j \leq \frac{\sigma}{1 - \sigma} c \tag{6.53.10}$$

This inequality sets an upper limit on total investment. A marginal propensity to save equal to $\sigma$ implies that per unit increase of income $\sigma$ is invested and $1 - \sigma$ consumed, or per unit increase in consumption investment rises $\sigma/(1 - \sigma)$. The propensity to save is assumed to be constant.

Subject to the above equations and inequalities, total consumption $c$ as a target can be maximized and the allocation of total investment determined.