CHAPTER 6

AN ITERATIVE IMPROVEMENT PROCEDURE; APPLICATION TO WORLD OR NATIONAL DEVELOPMENT PLANNING; SECOND ORDER SUBDIVISION OF A CLOSED OR OPEN ECONOMY; EXPLICIT TRANSPORTATION COSTS FOR HEAVY PRODUCTS

6.1. The Iterative Improvement Method and its Application to a Closed Economy Model

In this chapter we will outline an iterative improvement method which allows proceeding from the successive solution to the overall optimal solution of a second order spatial distribution model. The need for such a method was pointed out in Sections 5.4 and 5.5, where it was noticed, on the one hand, that the successive solution needs not be the overall optimal one, on the other hand, however, that the successive solution can rather easily be calculated and may be already close to the overall optimal solution. In view of this and of its didactic and organisational advantages it may pay to find first the successive solution, and to obtain after only a few changes the overall optimal solution. This means that such an iterative improvement method allows us to solve the problem of spatial distribution first in what we called in Section 1.2 a first approximation, and to use this solution as a starting basis for solving the problem in a second approximation. It will turn out that this iterative improvement method is relatively easy and can be interpreted meaningfully.

In this section we will illustrate how this iterative improvement method works by applying it to the numerical example of Section 5.3. Moreover, we will provide some intuitive explanation of the several steps of the method. A discussion of some of its mathematical aspects is given in Appendix III. We leave open the possibility that corrections or improvements of this

method can be made, or even that an altogether different method would be more efficient.

In Section 6.2 a simple example of a model for an open economy is formulated and the same method is applied in order to solve it. In Section 6.3 essentially the same method is used for solving a model with transport costs dependent on which flows along the transport routes are dominant.

* The iterative improvement procedure has to start with the information derived in the last steps in reaching the successive solution, that is, the step at which the optimality of the continental distribution solution was checked and the step at which the optimality of the transshipment problem was checked. For the sake of convenience we will repeat these final steps here. We could do so in the form of the tableau's of e.g. Section 3.5 and Section 4.10, when using the uv-method, or in the equivalent form of tableau's like Table 5.4, when using the u-method. We will use this last form (Tables 6.1 and 6.2) as it lends itself well to extensions introduced later on. For the simplex multipliers in the continental distribution problem we will again use the symbols u^h and ${}^r v$, ($h = 3, 4, 5$; $r = 1, 2, 3$; the h's corresponding to the sectors and the r's corresponding to the continents) to correspond with the earlier use of these symbols in Chapters 3 and 4. For the simplex multipliers in the transport or transshipment problem we will now use the symbols u_r, ($r = 1, 2, 3$), the r's corresponding both to the numbering of the equations and to the numbering of the regions.

One fact to be used in the iterative improvement method is that *the total cost of transportation in a sector corresponding to any feasible solution (basic or non basic) of the continental distribution problem can be expressed as a linear function of the production variables of that sector and that the opposites of the values of the simplex multipliers in the corresponding transshipment problem are the coefficients of the production variables in this linear function.*

This can be explained by referring to an important property of the simplex multipliers, which can easily be proved (see page 189 and also BAUMOL *et al.*, 1965). This property says that *for any basic feasible solution of a linear programming problem, with the primary constraints written in the form of equalities, the value of the objective function equals the sum of the products of the constants in the equations and of the values of the simplex multipliers corresponding to these equations.* In our example (cf. Table 6.2) this can be verified for e.g. the (optimal) basic feasible solution of the transshipment

TABLE 6.1

Tableau of the final step (II) in solving the numerical example of the continental distribution problem, without transportation costs

Variables	${}^1y^3$	${}^2y^3$	${}^3y^3$	${}^1y^4$	${}^2y^4$	${}^3y^4$	${}^2y^5$	${}^3y^5$	Constants	Simplex multipliers corresponding to the equations	Values of the simplex multipliers
Coefficients of the constraint equations	1	1	1						0.6	u^3	0
				1	1	1			0.9	u^4	0.3
							1	1	0.6	u^5	−1.0
	1			1					0.5	${}^1v$	3.1
		1			1		1		0.8	${}^2v$	2.7
			1			1		1	0.8	${}^3v$	3.0
Direct cost coefficients ${}^rc^h$	3.1	2.8	3.0	3.5	3.0	3.3	2.3	2.0		Costs of solution	
Basic feasible solution at hand[a)]	0.5	—	0.1	—	0.8	0.1	—	0.6		5.78	
Indirect cost coefficients ${}^r\bar{c}^h$[a)]	(3.1)	2.7	(3.0)	3.4	(3.0)	(3.3)	1.7	(2.0)			
Direct minus indirect cost coefficients ${}^r\tilde{c}^h - {}^r\bar{c}^h$[a)]	x	0.1	x	0.1	x	x	0.6	x			

[a)] The — indicate the non basic variables whose values are set = 0; the ( ) indicate the values of the indirect cost coefficients corresponding to the basic variables and which are equal to, by definition, those of the direct cost coefficients; the x indicate the values of the ${}^r\tilde{c}^h - {}^r\bar{c}^h$ corresponding to the basic variables, and which are by definition = 0.

TABLE 6.2

Tableau of the final step (II) in solving the numerical example of the transshipment problem

Variables	${}^{12}\tilde{t}^{4}$	${}^{13}\tilde{t}^{4}$	${}^{23}\tilde{t}^{4}$	${}^{21}\tilde{t}^{4}$	${}^{31}\tilde{t}^{4}$	${}^{32}\tilde{t}^{4}$	Constants	Simplex multipliers corresponding to the equations	Values of the simplex multipliers
Coefficients in the constraint equations	−1	−1		1	1		$0.4(=0.4-{}^{1}y^{4})$	u_1	0
	1		−1	−1		1	$-0.6(=0.2-{}^{2}y^{4})$	u_2	−0.5
		1	1		−1	−1	$0.2(=0.3-{}^{3}y^{4})$	u_3	0.2
Direct cost coefficients	0.5	0.5	0.7	0.5	0.5	0.7		Costs of solution	
Basic feasible solution at hand[a)]	—	—	0.2	0.4	—	—		0.34	
Indirect cost coefficients ${}^{rr'}\bar{\tilde{t}}^{4}$ a)	−0.5	0.2	(0.7)	(0.5)	−0.2	−0.7			
Direct minus indirect cost coefficients ${}^{rr'}\tilde{t}^{4}-{}^{rr'}\bar{\tilde{t}}^{4}$	1.0	0.3	x	x	0.7	1.4			

a) The − indicate the non basic variables whose values are set = 0; the ( ) indicate the values of the ${}^{rr'}\bar{\tilde{t}}^{4}$ corresponding to basic variables, and which are equal to, by definition, those of the direct cost coefficients; the x indicate the ${}^{rr'}\tilde{t}^{4}-{}^{rr'}\bar{\tilde{t}}^{4}$ corresponding tothe basic variables, and which are by definition = 0.

problem:

$$\min(z_T) = {}^{23}\tilde{\tau}^4\,{}^{23}\tilde{t}^4 + {}^{21}\tilde{\tau}^4\,{}^{21}\tilde{t}^4 = 0.7\times 0.2+0.5\times 0.4 = 0.34$$
$$u_1\times 0.4+u_2\times(-0.6)+u_3\times 0.2 = 0\times 0.4+(-0.5)\times(-0.6)+$$
$$+(0.2)\times(0.2) = 0.34 = \min(z_T).$$

The constants in the equations of the transshipment problem themselves, however, express the deficits (demand minus production) of product 4 per continent. In this simple model they can be expressed in the corresponding production variable with a coefficient -1 (see Table 6.2, column of constants, the terms between brackets). By substituting for the constants these expressions in the production variables, we obtain for the total cost of transportation (z_T), an expression in the production variables ${}^r y^4$, where the simplex multipliers with an opposite sign appear as coefficients:

$$\min(z_T) = u_1(0.4-{}^1y^4)+u_2(0.2-{}^2y^4)+u_3(0.3-{}^3y^4)$$
$$= 0.4u_1+0.2u_2+0.3u_3+(-u_1)\,{}^1y^4+(-u_2)\,{}^2y^4+(-u_3)\,{}^3y^4 \qquad (6.1.1)$$

We remember that the ${}^r y^4$ should satisfy (among other relations):

$${}^1y^4+{}^2y^4+{}^3y^4 = 0.9 \qquad (6.1.2)$$

and

$${}^r y^4 \geq 0, \quad r = 1, 2, 3 \qquad (6.1.3)$$

The expression (6.1.1) is the linear function we were looking for, and the values of the simplex multipliers with an opposite sign indeed turn out to be the coefficients of the variables which appear in it. Because of (6.1.2) this expression always assumes the same value once the ${}^r y^4$ are fixed, although the u_r are only determined apart from a constant, as the reader may verify. In particular this function makes it possible to compute uniquely the effect of small changes in the variables, which are in accordance with (6.1.2); only the differences between the u_r play a role then.

The values of the u_r however correspond only to the particular set of basic variables ${}^{23}\tilde{t}^4$ and ${}^{21}\tilde{t}^4$ in the basic feasible solution at hand. These variables remain in the basis only as long as the constants do not change in sign, i.e. as long as

$$0.4-{}^1y^4 \geq 0; \quad 0.2-{}^2y^4 \leq 0; \quad \text{and } 0.3-{}^3y^4 \geq 0 \qquad (6.1.4)$$

The three inequalities (6.1.4) define a part of a three dimensional space which we will call subspace A_1. At the edges of A_1, that is where one or

more of the inequalities are satisfied as equalities, these u_r values are still valid, but may not be unique (apart from a constant), as different u_r values, corresponding to an alternatively optimal basis, are also valid. In such cases one or both of the basic variables will always have the value zero (a so-called degenerate solution). We will have to use this fact later on.

These three conditions can be written in the form of equations by introducing three nonnegative variables ${}^1s^4_{A_1}$, ${}^2s^4_{A_1}$, ${}^3s^4_{A_1}$. The superscripts of these so-called "slack variables" correspond to those of the production variables in the corresponding inequalities, and the subscripts refer to this particular space of ${}^ry^4$ values defined by the inequalities for which the same u_r values are valid.

In equation form the new conditions read as follows,

$$ {}^1y^4 + {}^1s^4_{A_1} = 0.4 \tag{6.1.5} $$

$$ {}^2y^4 - {}^2s^4_{A_1} = 0.2 \tag{6.1.6} $$

$$ {}^3y^4 + {}^3s^4_{A_1} = 0.3 \tag{6.1.7} $$

$$ {}^rs^4_{A_1} \geq 0 \qquad r = 1, 2, 3 \tag{6.1.8} $$

Thus, the central idea of the iterative improvement method is that as

(*a*) *the costs of transportation can be expressed as a linear function of the variables* ${}^ry^4$ *with the* u_r *as coefficients and as*

(*b*) *the values of the* u_r *variables only remain the same for values of the variables* ${}^ry^4$ satisfying (*6.1.5*)-(*6.1.8*),

we can, after solving the transshipment problem and determining the values of the u_r, *by adding the expression* (*6.1.1*) *for the transport costs to the* (*production*) *cost function of the continental distribution problem, construct a new cost function for the continental distribution problem, which also takes account of the transportation costs, on condition that we add to the orginal constraints of the problem the new constraints* (*6.1.5*)-(*6.1.8*), *which define the subspace* A_1 *in which the new cost function is valid.*

In the case of our example the new cost function of the continental distribution problem reads:

$$ \begin{aligned} z = [z_P] + [\bar{z}_T] = [&3.1\ {}^1y^3 + 2.8\ {}^2y^3 + 3.0\ {}^3y^3 + 3.5\ {}^1y^4 + 3.0\ {}^2y^4 + 3.3\ {}^3y^4 + \\ &+ 2.3\ {}^2y^5 + 2.0\ {}^3y^5] + [0\ {}^1y^4 + 0.5\ {}^2y^4 - 0.2\ {}^3y^4 + 0.4(0) + 0.2(-0.5) + \\ &+ 0.3\ (0.2)] = 3.1\ {}^1y^3 + 2.8\ {}^2y^3 + 3.0\ {}^3y^3 + 3.5\ {}^1y^4 + 3.5\ {}^2y^4 + 3.1\ {}^3y^4 + \\ &+ 2.3\ {}^2y^5 + 2.0\ {}^3y^5 - 0.04 \end{aligned} \tag{6.1.9} $$

where $\bar{z}_T$ is the transport part of the objective function expressed in the variables ${}^r y^4$. The new additional constraints are given by (6.1.5)-(6.1.8). The simplex multipliers corresponding to the new constraints (6.1.5)-(6.1.7) will be indicated by ${}^r w^4_{A_1}$, $r = 1, 2, 3$.

In order to solve this new problem we already have a basic feasible solution available, namely the former solution and in addition the three slack variables ${}^1 s^4_{A_1}$, ${}^2 s^4_{A_1}$ and ${}^3 s^4_{A_1}$, whose values can easily be calculated from (6.1.5)-(6.1.7). We will call solving this new problem round A_1, indicating by A_1 that we solve the problem with a new cost function under the added constraints which define a subspace A_1. We can then refer to the solution of the original continental distribution problem as round A_0.

If the new problem has an optimal solution not on one of the edges of the subspace A_1, then this new optimal solution is the overall optimal solution, the simultaneous solution. A mathematical explanation of this statement can be found in Appendix III. But if the new optimal solution occurs on one or more of the edges of A_1, that is, has one or more of the slack variables $= 0$, then it might not be the overall optimal solution. Indeed, at these edges, at least one of the constants of the transshipment problem becomes $= 0$, and as we made the convention in Section 5.3 that 0 is considered both positive and negative, there will be in general one or more other sets of basic variables, which give an optimal basic feasible solution of the transshipment problem. Such a case, where at least one of the basic variables equals zero, is called a degenerate solution. Such a set of different basic variables would give rise to a different set of u_r values, and there would be a different set of new constraints describing a different subspace A_k of ${}^r y^4$ values, within which these new u_r values remain valid. For each of these other basic feasible solutions we should check whether the optimal solution in round A_1 is also the optimal solution in the other problems, with new basic transport variables, new u_r values, a new cost function and new additional constraints. If this is not the case for one or more of these problems, we should find the optimal solution of this new problem or of one of these new problems, and check again whether this is the optimal solution for any other set of basic variables of the transshipment problem which correspond to this new solution, etc. In this way we will finally reach the overall optimal solution.

Fortunately the check as to whether the optimal solution, found at any round, is the overall optimal solution, can easily be made in the following way. What we want to check is whether any combination of small changes

in the values of the ${}^r y^4$ in the optimal solution, which would bring the ${}^r y^4$ values outside the subspace A_1, could reduce the total costs of production and of transportation. To this end we separate the change in costs of production from the change in costs of transportation. The change in the cost of production can be found in the following way. First we calculate the values of the simplex multipliers for the optimal solution of a slightly different problem, namely one with the same constraints (also the additional one), but with the original cost function. We call these values the ${}^r\bar{w}^4_{A_1}$. Notice next that any small change in the ${}^r y^4$ values from those in the optimal solution is equivalent to an equal change in the corresponding constants of the added equations, while the slack variables keep the same values as in the optimal solution. Therefore *we can calculate the effect of small changes of the* ${}^r y^4$ *values on the production costs by multiplying the changes of the* ${}^r y^4$ *values* (= *the changes of the constants in the added equations*) *by their corresponding* ${}^r\bar{w}^4_{A_1}$:

$$(\Delta z_{\mathrm{P}})_{A_1} = {}^1\bar{w}^4_{A_1}(\Delta\, {}^1y^4)_{A_1} + {}^2\bar{w}^4_{A_1}(\Delta\, {}^2y^4)_{A_1} + {}^3\bar{w}^4_{A_1}(\Delta\, {}^3y^4)_{A_1} \qquad (6.1.10)$$

where $(\Delta\, {}^r y^4)_{A_1}$ indicates that we consider deviations of the values of the production variables in the optimal solution reached at the last iteration of round A_1. As these changes in the ${}^r y^4$ values have to add up to zero – because of (6.1.2) – the change in production costs is exactly determined, even though the ${}^r\bar{w}^4_{A_1}$ are only determined apart from a constant.

The change in the corresponding transport costs can be calculated in a similar way in the transshipment problem. First, however, we must make clear that we are only interested in small changes in the ${}^r y^4$ which cause at least one of the constants in the transshipment problem, which have the value = 0 for the optimal solution of round A_1, to get a sign different from the sign it had during the iterations of round A_1. Only such changes bring us outside of subspace A_1 into another subspace where the transshipment problem has a different set of optimal basic variables, which, however, is also one of the sets of optimal basic variables corresponding to the optimal solution of round A_1. Therefore, only in such a subspace is a set of u_r values valid which is also valid in the optimal solution of round A_1. The changes in each of the constants of the transshipment problem are the same as the changes in the corresponding ${}^r y^4$ values but with an opposite sign. Therefore the change in the cost of transportation due to a small change in the ${}^r y^4$ values equals the sum of the products of these changes in the ${}^r y^4$ values with

an opposite sign and the values of the corresponding u_r, which are valid in view of the signs of the constants in the transshipment problem after the changes in the ${}^r y^4$ values:

$$(\Delta z_1)_{A_1} = -(u_1)_{A_k} \cdot (\Delta\, {}^1 y^4)_{A_1} - (u_2)_{A_k} \cdot (\Delta\, {}^2 y^4)_{A_1} - (u_3)_{A_k} \cdot (\Delta\, {}^3 y^4)_{A_1} \quad (6.1.11)$$

where the subscript A_1 again indicates that we consider deviations from the values in the optimal solution of round A_1, while the subscript A_k indicates that these are the values (of the u_r) which are valid for a subspace A_k, that is, valid in the transshipment problem with the sign of its constants determined after the change in the ${}^r y^4$. In other words the changed values of ${}^r y^4$ are in a subspace A_k. How this method of checking works in practice and how it can be done systematically just by adding and substracting the appropriate ${}^r \bar{w}^4_{A_1}$ and u_r values will be illustrated at the end of this section, when the iterative improvement method is applied to our numerical example.

Summarizing we can represent the iterative improvement method by the following steps:

Round A_0. (a) Calculate the optimal solution of the original continental distribution problem by the u-method, following the steps I, II and III, ending up always with step II. The last iteration we will call iteration A_{0n}.

(b) Calculate the optimal solution of the corresponding transshipment problem, in particular the values of the simplex multipliers u_r.

Round A_1. (a) Construct a new cost function for the continental distribution problem, by adding the $(-u_r)$, $r = 1, 2, 3$, to the appropriate coefficients ${}^r \tilde{c}^4$, and check whether the original solution is still optimal with this new cost function by step II. If it is still optimal, then the original optimal solution is also the overall optimal solution. If the original optimal solution is no longer optimal, go then to step (b).

(b) Introduce the new constraints, under which the u_r used in step (a) remain valid – that is, under which the signs of the constants in the transshipment problem do not change – into the continental distribution problem and solve this problem with the new cost function and with the added constraints by the u-method. The iterations will be called A_{11}, A_{12}, A_{13} ...

If none of the slack variables ${}^r s^4_{A_1}$ introduced in step (b) have become 0 in the optimal solution, then this solution is the overall optimal solution.

(c) If any of these slack variables ${}^r s^4_{A_1}$ have become 0 in the optimal

solution, then investigate, by using (6.1.10) and (6.1.11), whether the total cost of production and transportation can be reduced by any small change in the ${}^r y^4$ values consistent with (6.1.2) which makes the sign change of at least one constant in the transshipment problem corresponding to the variables ${}^r s^4_{A_1}$ with value $= 0$ in the optimal solution. If no such small change in the ${}^r y^4$ values can reduce the total cost, then no other change can do it, and the overall optimal solution has been reached. If, however, any such a small change in the ${}^r y^4$ values turns out to be advantageous, then choose any set of basic variables of the transshipment problem corresponding to such an advantageous change, and proceed to round A_2. If there are more such sets of basic variables to be chosen, then select one of them by one or other criteria, e.g. the criterion of the largest reduction in total cost per average change in the ${}^r y^4$ values in absolute amounts.

Round A_2. (a) During the last step of the preceding round it was shown that the optimal solution reached at step (b) in that round is not the overall optimal one. So proceed to step (b).

(b) Formulate the new cost function by adding the new $(-u_r)$ values corresponding to the new basic variables, to the appropriate cost coefficients ${}^r\tilde{c}^h$ and introduce new constraints under which the new u_r values remain valid – they define a subspace A_2 – leaving out the constraints previously added. Calculate the optimal solution for this new problem calling the iterations $A_{21}, A_{22}, \ldots$

(c) Repeat step (c) of the previous round. Either this step shows that the optimal solution reached in the preceding step (b) is the overall optimal solution, or an indication to consider yet another set of basic variables of the transshipment problem will be given. This would lead to another round A_3, in which the same steps as in round A_2 are repeated. After a finite number of rounds step (c) will show that the overall optimal solution has been reached.

The calculations for the applications of this iterative improvement method to our numerical example can be carried out in extended forms of Tables 6.1 and 6.2. Table 6.1 is extended in the following way:

(a) Above and at the right hand side of the matrix of the coefficients of the original constraint equations room is left for adding at each round new additional constraints and new slack variables.

(b) The right hand part of the table which refers to the simplex multipliers is extended in two ways. First the simplex multipliers ${}^r w_{A_1}^4$ etc. corresponding to the additional constraints are included, and room is left for the values of these multipliers calculated at each iteration of each round. Next, space is left for the simplex multipliers ${}^r \bar{w}_{A_1}^4$ etc. and their values calculated with reference to the original cost function .

(c) Below the original table space is left to indicate the new cost function for each new round and the solutions for each iteration together with the ${}^r\bar{c}^h - {}^r\bar{\bar{c}}^h$ values.

(d) Finally the cells of the matrix of coefficients should be large enough to contain the θ_i's to be used at the beginning of a new iteration to find the new solution. The θ_i are numbered according to the order in which they are used, each θ in the same iteration having the same index.

Table 6.2 is extended in the following way:

(a) At the right hand side of the original tableau new columns are added to indicate for each new round the values of the u_r used in that round, the values of the constants at the end of the round, the ${}^r\bar{w}_{A_1}^4$ to be computed possibly at the end of the round, and for each possible check three columns: one indicating the change in the ${}^r y^4$ values, one indicating the new values of the constants after this change in the ${}^r y^4$ values, and one indicating the values of the u_r (the $(u_r)_{A_k}$) corresponding to the signs of the constants. Below the columns for the checks, room is left to calculate the change in total costs due to the change in the ${}^r y^4$ values and to indicate the sign of this change in total costs.

(b) Below the original tableau new rows have been added to indicate the solutions for each round and each check and the ${}^{rr'}\tilde{\tau}^4 - {}^{rr'}\bar{\tilde{\tau}}^4$ values corresponding to the new values of the constants, possibly with some iterations per check to find the optimal solution.

(c) The cells of the coefficient matrix should be large enough to contain the possible θ_i's to be used when applying the θ-method at the iterations to find the optimal solution for each check.

The extended tables with the values calculated at each round and iteration are shown as Tables 6.3 and 6.4. The changes in the ${}^r y^4$ values for the checks are represented not by $(\Delta\ {}^r y^4)_{A_k}$'s, but in terms of δ_l, $l = 1, 2$. The δ_l are taken to be very small, but nonnegative. As $\sum_{r=1}^{3} (\Delta\ {}^r y^4)_{A_k} = 0$ two δ_l

TABLE

Extended final tableau of the continental distribution

			Original variables							
			${}^{1}y^{3}$	${}^{2}y^{3}$	${}^{3}y^{3}$	${}^{1}y^{4}$	${}^{2}y^{4}$	${}^{3}y^{4}$	${}^{2}y^{5}$	${}^{3}y^{5}$
Coefficients of the added constraint equations	Round A_1					$1+\theta_2$				
							$-\theta_1$ $1-\theta_2$ $-\theta_3$			
								$+\theta_1$ 1 $+\theta_3$		
Coefficients of the original constraint equations			$1-\theta_2$	$+\theta_1$ $1+\theta_2$	$-\theta_1$ 1					
						$1+\theta_2$	$-\theta_1$ $1-\theta_2$ $-\theta_3$	$+\theta_1$ 1 $+\theta_3$		
									1 $+\theta_3$	1 $-\theta_3$
			1			$1+\theta_2$				
				$+\theta_1$ $1+\theta_2$			$-\theta_1$ $1-\theta_2$ $-\theta_3$		1 $+\theta_3$	
					$-\theta_1$ 1			$+\theta_1$ 1 $+\theta_3$		1 $-\theta_3$
Round	Iteration									
A_0	A_{0n}	Original ${}^{r}\tilde{c}^{h}$	3.1	2.8	3.0	3.5	3.0	3.3	2.3	2.0
		Solution ${}^{r}\tilde{c}^{h}-{}^{r}\bar{c}^{h}$	0.5 x	— 0.1	0.1 x	— 0.1	0.8 x	0.1 x	— 0.6	0.6 x
		u_r used New ${}^{r}\tilde{c}^{h}$	3.1	2.8	3	0 3.5	−0.5 3.5	0.2 3.1	2.3	2.0
A_1	A_{11}	Solution ${}^{r}\tilde{c}^{h}-{}^{r}\bar{c}^{h}$	0.5 x	— −0.6	0.1 x	— 0.3	0.8 x	0.1 x	— −0.1	0.6 x
	A_{12}	Solution ($\theta_1 = 0.1$) ${}^{r}\tilde{c}^{h}-{}^{r}\bar{c}^{h}$	0.5 x	0.1 x	— 0.6	— −0.3	0.7 x	0.2 x	— −0.1	0.6 x
	A_{13}	Solution ($\theta_2 = 0.4$) ${}^{r}\tilde{c}^{h}-{}^{r}\bar{c}^{h}$	0.1 x	0.5 x	— 0.6	0.4 x	0.3 x	0.2 x	— −0.1	0.6 x
	A_{14}	Solution ($\theta_3 = 0.1$) ${}^{r}\tilde{c}^{h}-{}^{r}\bar{c}^{h}$	0.1 x	0.5 x	— 0.5	0.4 x	0.2 x	0.3 x	0.1 x	0.5 x

6.3

problem : application of the iterative improvement method.

Slack variables Round A_1			Constants	Simplex multipliers								
				Symbols	Values							
					Round	A_0	A_1					
					Iteration	A_{0n}	A_{11}	A_{12}	A_{13}	A_{14}	with the original ${}^r\tilde{c}^h$ for checking	
${}^1s^4_{A_1}$	${}^2s^4_{A_1}$	${}^3s^4_{A_1}$										
$1-\theta_2$			0.4	${}^1w^4_{A_1}$			0	0	−0.3	−0.3	${}^1\bar{w}^4_{A_1}$	0.2
	$-\theta_1$ $1-\theta_2$ $-\theta_3$		0.2	${}^2w^4_{A_1}$			0	0	0	0	${}^2\bar{w}^4_{A_1}$	0
		$-\theta_1$ 1 $-\theta_3$	0.3	${}^3w^4_{A_1}$			0	0	0	−0.1	${}^3\bar{w}^4_{A_1}$	0.6
			0.6	u^1		0	0	0	0	0		0
			0.9	u^2		0.3	0.1	0.7	0.7	0.7		0.2
			0.6	u^3		−1.0	−1.0	−0.4	−0.4	−0.5		−0.5
			0.5	${}^1v$		3.1	3.1	3.1	3.1	3.1		3.1
			0.8	${}^2v$		2.7	3.4	2.8	2.8	2.8		2.8
			0.8	${}^3v$		3.0	3.0	2.4	2.4	2.5		2.5

0	0	0
0.4 x	0.6 x	0.2 x
0.4 x	0.5 x	0.1 x
— 0.3	0.1 x	0.1 x
— 0.3	0 x	— 0.1

Costs of solution

After iteration A_{11} 6.16

After iteration A_{12} 6.10

After iteration A_{13} 5.98

After iteration A_{14} 5.97

TABLE

Extended final tableau of the transshipment problem

			Variables						Constants in terms of the ${}^{r}y^{4}$	Simplex multi-pliers symbols	Round A_0				
											Values after iteration A_{0n}		Values u_r used in c.d. problem	Values after iteration A_{1n}	
			${}^{12}\tilde{t}^{4}$	${}^{13}\tilde{t}^{4}$	${}^{23}\tilde{t}^{4}$	${}^{21}\tilde{t}^{4}$	${}^{31}\tilde{t}^{4}$	${}^{32}\tilde{t}^{4}$			Constants	u_r		Constants	${}^{r}\bar{\pi}^{4}_{A}$
Coefficients of the constraint equations			-1	-1		1	1		$0.4-{}^{1}y^{4}$	u_1	0.4	0	0	0	0.2
			1		-1	-1		1	$0.2-{}^{2}y^{4}$	u_2	-0.6	-0.5	-0.5	0	0
				1	1		-1	-1	$0.3-{}^{3}y^{4}$	u_3	0.2	0.2	0.2	0	0.6
Round	Check	${}^{rr'}\tilde{\tau}^{4}$	0.5	0.5	0.7	0.5	0.5	0.7							
A_0		Solution	—	—	0.2	0.4	—	—							
		${}^{rr'}\tilde{\tau}^{4}-{}^{rr'}\bar{\tau}^{4}$	1	0.3	x	x	0.7	1.4							
		Solution	—	—	0	0	—	—							
		${}^{rr'}\tilde{\tau}^{4}-{}^{rr'}\bar{\tau}^{4}$	1	0.3	x	x	0.7	1.4							
A_1	1	Solution	—	—	—	—	δ_1	$\delta_2-\delta_1$							
		${}^{rr'}\tilde{\tau}^{4}-{}^{rr'}\bar{\tau}^{4}$	0.3	1	1.4	0.7	x	x							
	2	Solution	δ_1	—	—	—	—	δ_2							
		${}^{rr'}\tilde{\tau}^{4}-{}^{rr'}\bar{\tau}^{4}$	x	0.7	1.4	1	0.3	x							
	3	Solution	—	δ_1	$(\delta_2-\delta_1)$	—	—	—							
		${}^{rr'}\tilde{\tau}^{4}-{}^{rr'}\bar{\tau}^{4}$	0.7	x	x	0.3	1	1.4							
	4	Solution	—	—	—	$(\delta_1-\delta_2)$	δ_2	—							
		${}^{rr'}\tilde{\tau}^{4}-{}^{rr'}\bar{\tau}^{4}$	1	1	0.7	x	x	0.7							
	5	Solution	$(\delta_1-\delta_2)$	δ_2	—	—	—	—							
		${}^{rr'}\tilde{\tau}^{4}-{}^{rr'}\bar{\tau}^{4}$	x	x	0.7	1	1	0							

symbols are enough to represent the three changes as the third change can be represented as a combination of the other two changes. For this combination, however, the sign should be indicated in case one of the δ_l appears with a coefficient -1. This sign indication is equivalent to saying which of the δ_l should be larger than the other one. The following further explanation of the calculations of the iterative improvement method in Tables 6.3 and 6.4 can be given. At the end of round A_1, at iteration A_{14}, a degenerate optimal basic feasible solution is reached, as ${}^{2}s^{4}_{A_1}$ and ${}^{3}s^{4}_{A_1}$ become together $= 0$ at this iteration. Therefore either ${}^{2}s^{4}_{A_1}$ or ${}^{3}s^{4}_{A_1}$ can be considered to be a basic variable in the optimal solution, although its value $= 0$. We consider

4

plication of the iterative improvement method.

Round A_1														
Check 1			Check 2			Check 3			Check 4			Check 5		
a)	b)	c)	a)	b)	c)	a)	b)	c)	a)	b)	c)	a)	b)	c)
$-\delta_1$	$0+\delta_1$	-0.2	$+\delta_1$	$0-\delta_1$	-0.5	$+\delta_1$	$0-\delta_1$	0.2	$-\delta_1$	$0+\delta_1$	0.5	$+\delta_1$	$0-\delta_1$	-0.5
$\delta_1-\delta_2$ (< 0)	$\delta_2-\delta_1$	0	$-(\delta_1+\delta_2)$	$\delta_1+\delta_2$	0	$\delta_2-\delta_1$ (> 0)	$-(\delta_2-\delta_1)$	0	$\delta_1-\delta_2$ (> 0)	$-(\delta_1-\delta_2)$	0	$-(\delta_1-\delta_2)$ (< 0)	$\delta_1-\delta_2$	0
$-\delta_2$	$-\delta_2$	-0.7	$+\delta_2$	$-\delta_2$	0.7	$-\delta_2$	$+\delta_2$	0.7	$+\delta_2$	$-\delta_2$	0	$-\delta_2$	$+\delta_2$	0

	Check 1	Check 2	Check 3	Check 4	Check 5
Change in cost of:					
production	$-0.2\delta_1+0.6\delta_2$	$0.2\delta_1+0.6\delta_2$	$0.2\delta_1-0.6\delta_2$	$-0.2\delta_1+0.6\delta_2$	$0.2\delta_1-0.6\delta_2$
transportation	$-0.2\delta_1+0.7\delta_2$	$0.5\delta_1+0.7\delta_2$	$-0.2\delta_1+0.7\delta_2$	$0.5\delta_1+0$	$0.5\delta_1+0$
total	$-0.4\delta_1+1.3\delta_2$	$0.7\delta_1+1.3\delta_2$	$0 \quad +0.1\delta_2$	$0.3\delta_1+0.6\delta_2$	$0.7\delta_1-0.6\delta_2$
ign of change under the restrictions of the δ_1 and δ_2:	+	+	+	+	+

Change in the ${}^r y^4$
New value constants
Corresponding u_r

${}^2 s^4_{A_1}$ to be the basic variable. As all ${}^r\tilde{c}^h - {}^r\bar{c}^h$ values at iteration A_{14} are non-negative the optimal solution has been found.

As all ${}^r s^4_{A_1}$ have become = 0, and as therefore all the constants in the transshipment problem also have become = 0, in the optimal solution, the check on the overall optimality of the optimal solution is somewhat cumbersome, as all possible changes of the ${}^r y^4$ values and correspondingly of the signs of the constants of the transshipment problem have to be considered. This leads to 5 different checks, as there are altogether 6 different combinations for the constants of the transshipment problem – the changes have to add up to zero – and one of the combinations has been checked

already implicitly in round A_1. The + signs in the last row on the right hand side of Table 6.4 indicate that all checks have shown that any small change in the ${}^r y^4$ values leads to an increase in the total costs and that therefore the optimal solution found at round A_1 is the overall optimal solution. Comparing this solution with the simultaneous solution calculated in Section 5.3, we find that they are indeed the same.

There is too little computational experience to say whether solving the problem by this or a similar iterative improvement method would be more efficient, from the purely computational point of view, than the simultaneous solution method used in Section 5.3. This last method migh tbe computationally quite efficient if the production variables are introduced into the first basic feasible solution in accordance with the separately calculated optimal solution of the original continental distribution problem, and if only after that the transport variables are introduced into this first basic feasible solution. In Section 5.5, however, we have given some other than purely computational arguments in favour of a decomposition method. In particular one may notice that in each of the Tables 6.3 and 6.4 very little of the information used in the other table needs to be known. This implies that only a little information needs to be communicated between two different departments working on the continental distribution problem and the transshipment problem respectively. In particular, after round A_0, for solving the transshipment problem only the surpluses or deficits per area per heavy product need to be known. And conversely, for the second round A_1 of the continental distribution problem only the u_r values and equations (6.1.5)-(6.1.7) need to be derived from the transshipment problem. The checks for overall optimality again carried out in the tableau of the transshipment problem, need additionally only information about the ${}^r\bar{w}^4_{A_1}$.

Another possible advantage of the iterative improvement method is that before applying it one may easily find an upper bound to the gain to be derived from applying it. If this possible gain is considered to be small one may be satisfied with the successive solution. Such an upper bound can be found as follows. No more can be gained than the cost of transportation itself. But from any "gain" by reducing transport costs must be deducted a "loss" of increased production costs. Any change in either cost must be brought about by a change in the production variables. An upper bound to the gain in transport cost per unit change in any production variable is given by the largest difference between the values of the u_r after round A_0 (in our

example this is 0.7 units). A lower bound to the loss in production costs per unit change in any production variable is given by the smallest value of the ${}^r\bar{c}^h - {}^r\bar{\bar{c}}^h$ at the end of round A_0 (in our example this is 0.1 units). The difference between these two figures ($0.7-0.1 = 0.6$ units) is the largest gain one might make per unit change of any variable. The ratio of this difference to the maximal gain in transport cost per unit change in any production variable, multiplied by the total transport costs gives an upper bound for the gain to be made by the iterative improvement method. In our example this upper bound for the gain is therefore $0.6/0.7 \times 0.34 = 0.29$ units. Obviously, if the difference is negative, no gain at all can be made and the successive solution is the overall optimal solution. In practical cases smaller upper bounds might possibly be found.

It is left to the reader to work out cases with more than one heavy sector. One could e.g. also consider sector 3 or 5 to be heavy and solve the problem simultaneously or successively or by applying the iterative improvement method to the successive solution. When solving the problem successively, one obtains a separate transshipment problem for each heavy sector. In applying the iterative improvement method one should be especially careful in stating the additional constraints for the continental distribution problem in each round, which define the area within which the u_r values derived from both transshipment problems remain valid. *

6.2. A Simple Model of an Open Economy with a Heavy Sector and its Solution

In this section we will discuss a simple model for an open economy with one heavy sector. We will again solve it both simultaneously and in successive steps, compare both solutions and indicate the possibility of applying the iterative improvement method to the successive solution. As we apply in principle the same methods as used when solving the numerical example of the closed economy model in the Sections 5.3, 5.4, and 6.1, these methods need little explanation. Occasionally we will point out the necessary small extensions of the methods used hitherto.

For didactic reasons the simple model will be, even numerically, as similar to the former model for a closed economy (see Sections 5.2 and 5.3) as possible. The terminology is different, however, as this is a model for a

national economy, subdivided into regions, instead of a model for a world economy, subdivided into continents. While there are again 5 sectors, the main difference with the former model is that sectors 4 and 5 are international, that is, the products of these sectors can be also imported or exported now. Sector 4 is again the heavy sector whose transport variables appear in the model; its transportation costs exist now, however, not only for interregional transport, but also for international transport.

It is useful also to specify, when introducing variables for exports and imports of a heavy product, whether such products are transported interregionally before being exported or imported, that is before being transported internationally. It is possible, when using a transport cost function in accordance with Assumption 5.1a – as we do in this section – to leave this open and to take account in the cost coefficient for the international transport variable of both the costs of interregional and international transport. However, when using a transport cost function in accordance with Assumption 5.1b – as we will do in Section 6.3 – with return freight at lower costs, then the costs of transporting a product along a certain route depend on the transport of all other goods along that route. Then it is useful to know exactly whether the products which are exported or imported are also transported along interregional routes. Specification of this situation in the model is easy: one simply excludes the possibility of direct exports from or of imports to a region for which the heavy product cannot be transported internationally without being transported interregionally (cf. equation (6.2.10)).

This time we make

Assumption 6.1 (slightly weaker than Assumption 4.1; similar to Assumption 5.2.). Each product can be classified as either regional, or national or international. For non-heavy national or international goods transportation costs between regions or between regions and/or countries respectively can be neglected.

In addition we again make the Assumptions 4.2–4.5 and Assumption 5.1a. Finally we must make

Assumption 6.2 (similar to Assumption 5.3). For each heavy national and international product respectively the increase in demand in each region is proportional to the increase in income of that region.

In the model three regions and five sectors appear as follows:

Sector 1: a regional sector.

Sector 2: a non shiftable national sector, expandable only in region 1.

Sector 3: a shiftable national sector.

Sector 4: a shiftable heavy international sector.

Sector 5: a partly shiftable international sector, expandable only in the regions 2 and 3.

For each region targets for the increases in income are assumed to be given.

The model contains the following equations, all symbols having the same meaning as in Section 5.2 mutatis mutandis.

$$ {}^1y^1 = {}^1\tilde{\eta}^1\ {}^1y \tag{6.2.1} $$

$$ {}^2y^1 = {}^1\tilde{\eta}^2\ {}^2y \tag{6.2.2} $$

$$ {}^3y^1 = {}^1\tilde{\eta}^3\ {}^3y \tag{6.2.3} $$

The increase in income derived from the regional sector in a region is proportional to the increase in income of that region.

$$ {}^1y^2 = \tilde{\eta}^2 y \tag{6.2.4} $$

$$ {}^1y^3 + {}^2y^3 + {}^3y^3 = \tilde{\eta}^3 y \tag{6.2.5} $$

The total increase in income derived from each national sector is proportional to the increase in the national income.

$$ {}^1y^4 + {}^2y^4 + {}^3y^4 + {}^2y^5 + {}^3y^5 = (\tilde{\eta}^4 + \tilde{\eta}^5) y \tag{6.2.6} $$

The total increase in income derived from the international sectors together is proportional to the increase in the national income. This equation will turn out to be redundant, but we nevertheless include it, as we also included in the former model for a closed economy a redundant equation for reasons of symmetry.

$$ {}^1y^1 + {}^1y^2 + {}^1y^3 + {}^1y^4 = {}^1y \tag{6.2.7} $$

$$ {}^2y^1 + {}^2y^3 + {}^2y^4 + {}^2y^5 = {}^2y \tag{6.2.8} $$

$$ {}^3y^1 + {}^3y^3 + {}^3y^4 + {}^3y^5 = {}^3y \tag{6.2.9} $$

The sectoral income increases in a region must add up to the income increase target of that region.

$$^1y^4 - {}^{12}\tilde{t}^4 - {}^{13}\tilde{t}^4 + {}^{21}\tilde{t}^4 + {}^{31}\tilde{t}^4 = {}^1\tilde{\eta}^4\,{}^1y - {}^1\bar{y}^4 + {}^1\tilde{\bar{d}}^4 \tag{6.2.10}$$

$$^2y^4 - {}^{21}\tilde{t}^4 - {}^{23}\tilde{t}^4 + {}^{12}\tilde{t}^4 + {}^{32}\tilde{t}^4 - {}^2\tilde{e}^4 + {}^2\tilde{m}^4 = {}^2\tilde{\eta}^4\,{}^2y - {}^2\bar{y}^4 + {}^1\tilde{\bar{d}}^4 \tag{6.2.11}$$

$$^3y^4 - {}^{31}\tilde{t}^4 - {}^{32}\tilde{t}^4 + {}^{13}\tilde{t}^4 + {}^{23}\tilde{t}^4 - {}^3\tilde{e}^4 + {}^3\tilde{m}^4 = {}^3\tilde{\eta}^4\,{}^3y - {}^3\bar{y}^4 + {}^3\tilde{\bar{d}}^4 \tag{6.2.12}$$

For a heavy international sector these equations, equalizing the supply and demand per region in terms of value added, must be included in the model as the new variables appearing in these equations have non zero coefficients in the objective function; that is, the costs of a solution do vary with the values of these variables. The new symbols ${}^r\tilde{e}^4$ and ${}^r\tilde{m}^4$, $r = 2, 3$, stand for the international transportation for exports and imports respectively of product 4 from, or to, region r. Notice that no such international transport variables exist for region 1. It is assumed that product 4 exported from or imported to this region must be transported interregionally either before or afterwards. The physical situation behind this assumption may be that the regions 2 and 3 are coastal regions, having direct access to international sea transportation, while region 1 lies in the interior of the country.

The secundary restrictions are again:

$$^ry^h \geq 0 \qquad r = 1, 2, 3; \quad h = 1, \ldots, 5; \quad {}^{rr'}\tilde{t}^4 \geq 0 \qquad r, r' = 1, 2, 3;$$
$$^r\tilde{e}^4 \quad \text{and} \quad {}^r\tilde{m}^4 \geq 0 \qquad r = 2, 3 \tag{6.2.13}$$

The objective of the model is again to minimize the total increase in production cost and the total transport costs:

$$z = \underbrace{\sum_{r=1}^{3}\sum_{h=1}^{5} {}^r\tilde{c}^h\,{}^ry^h}_{(z_P)} + \underbrace{\sum_{r=1}^{3}\sum_{r'=1}^{3} {}^{rr'}\tilde{\tau}^4\,{}^{rr'}\tilde{t}^4}_{(z_T)} + \underbrace{{}^2\tilde{\varepsilon}^4\,{}^2\tilde{e}^4 + {}^2\tilde{\mu}^4\,{}^2\tilde{m}^4 + {}^3\tilde{\varepsilon}^4\,{}^3\tilde{e}^4 + {}^3\tilde{\mu}^4\,{}^3\tilde{m}^4}_{(z_{IT})} \tag{6.2.14}$$

For the targets and parameters of the model we assume numerical values equal to those in the model of Section 5.3 (see also Tables 6.5 and 6.6):

$$^1y = {}^2y = {}^3y = 1.0$$
$$^1\tilde{\eta}^1 = {}^2\tilde{\eta}^1 = {}^3\tilde{\eta}^1 = 0.2; \; \tilde{\eta}^2 = 0.1; \; \tilde{\eta}^3 = 0.2; \; {}^1\tilde{\eta}^4 = {}^2\tilde{\eta}^4 = {}^3\tilde{\eta}^4 = 0.3$$
$$^1\bar{y}^4 = 0.9; \quad {}^2\bar{y}^4 = 1.1; \quad {}^3\bar{y}^4 = 1.0; \quad {}^1\tilde{\bar{d}}^4 = {}^2\tilde{\bar{d}}^4 = {}^3\tilde{\bar{d}}^4 = 1.0.$$

* The values of the variables ${}^1y^1$, ${}^2y^1$, ${}^3y^1$ and ${}^1y^2$ are completely determined by the equations (6.2.1)–(6.2.4):

$$^1y^1 = 0.2; \quad {}^2y^1 = 0.2; \quad {}^3y^1 = 0.2; \quad {}^1y^2 = 0.3.$$

TABLE 6.5

Production cost coefficients and income increases by sector and by region[a])

Sectors		Regions 1	2	3	Sectoral increases
Regional	1	3.0	3.0	3.0	0.6
National	2	3.0	x	x	0.3
	3	3.1	2.8	3.0	0.6
International	4	3.5	3.0	3.3	1.5
	5	x	2.3	2.0	
Regional increases		1.0	1.0	1.0	Total 3

[a]) The income increases have been derived from the income targets and the values of the coefficients given above.

TABLE 6.6

Cost coefficients for interregional and international transport[a])

from \ to		Regions 1	2	3	Outside world
Regions	1	x	0.5	0.5	x
	2	0.5	x	0.7	0.6
	3	0.5	0.7	x	0.5
Outside world		x	0.6	0.5	x

[a]) The cost coefficients for international transport contain only that part of the costs which takes up scarce resources of the country concerned.

When we apply the simultaneous solution method there remains a linear programming problem for the other variables, which can be represented in a table similar to Table 5.4. It again gives only the coefficients of the variables of all the equations while all constants are calculated on the basis of the given

Solving simultaneously the numerica

		Variables												
		Production								Tran				
										Interregional				
		${}^{1}y^{3}$	${}^{2}y^{3}$	${}^{3}y^{3}$	${}^{1}y^{4}$	${}^{2}y^{4}$	${}^{3}y^{4}$	${}^{2}y^{5}$	${}^{3}y^{5}$	${}^{12}\tilde{y}^{4}$	${}^{13}\tilde{y}^{4}$	${}^{23}\tilde{y}^{4}$	${}^{21}\tilde{y}^{4}$	${}^{31}\tilde{y}^{4}$
Coefficients of the constraint equations		$-\theta_1$ 1	$+\theta_1$ 1	1										
					$+\theta_1$ 1	1	1	$-\theta_1$ 1	1					
		$-\theta_1$ 1			$+\theta_1$ 1									
			$+\theta_1$ 1			1		$-\theta_1$ 1						
				1			1		1					
					$+\theta_1$ 1					-1	-1		$-\theta_1$ 1	1
						1				1		-1	$-\theta_1$ -1	
							1				1	1		-1
Direct cost coefficients		3.1	2.8	3.0	3.5	3.0	3.3	2.3	2.0	0.5	0.5	0.7	0.5	0.5
Iteration 1	Solution 1 [b])	0.5	0.1	—	—	—	—	0.7	0.8	—	—	—	0.4	—
	Direct minus indirect cost coefficients [b])	x	x	0.5	-0.2	0.1	0.8	—	—	1.0	1.1	0.8	1.0	-0.1
Iteration 2	Solution 2 [b]) ($\theta_1=0.4$)	0.1	0.5	—	0.4	—	—	0.3	0.8	—	—	—	—	—
	Direct minus indirect cost coefficients [b])	x	x	0.5	x	0.1	0.8	x	x	0.8	0.9	0.8	0.2	1.0

[a]) The table shows the coefficients of the equations, the determination of a first basic feasible solution and the the θ-method (for explanation see text).

[b]) For the interpretation of the symbols — and x, see the footnote of Table 6.1.

numerical data. In the same table the calculations to determine feasible solution, and the iterations to reach the optimal soluti represented. This means that the tables for this (open econo which correspond to the Tables 5.4, 5.5 and 5.6 for the for economy) model, are reduced to one table (Table 6.7).

ıing and regional distribution problem with transport costs. [a])

Constants							Corres-ponding equation	Simplex multipliers		
Changed values after setting								Symbols	Values at iteration	
${}^{3}y^{5} = 0.8$ ${}^{3}y^{3} = {}^{3}y^{4} = 0$	${}^{2}y^{5} = 0.7$ ${}^{1}y^{4} = {}^{2}y^{4} = 0$	${}^{2}y^{3} = 0.1$	${}^{1}y^{3} = 0.5$	${}^{21}\tilde{t}^{4} = 0.4$	${}^{2}\tilde{m}^{4} = 0.6$	${}^{3}\tilde{m}^{4} = 0.2$			1	2
		0.5	0				(6.2.5)	u_1	0.5	0.5
0.7	0						(6.2.6)	u_2	0	0
			0				(6.2.7)	u_3	2.6	2.6
	0.1	0					(6.2.8)	u_4	2.3	2.3
0							(6.2.9)	u_5	2	2
				0			(6.2.10)	u_6	1.1	0.9
				0.6	0		(6.2.11)	u_7	0.6	0.6
						0	(6.2.12)	u_8	0.5	0.5

	Costs of solution
Solution 1	5.75
Solution 2	5.67

In solving this model by the simultaneous solution method there is only one difference with the method of Section 5.3, namely in constructing a first basic feasible solution. In order to find the values for the transport variables, the rules given in Section 5.3 should be amended in view of the presence of the international transport variables. Now an interregional

transport variable may take on a positive value, even though the constants in the corresponding equations may not have the same signs as the coefficients of this variable; the signs of the coefficient of the variable in the equation and the constant should be the same only for the corresponding equation in which no international variables appear (i.e. equation (6.2.10)). Therefore, in our example the variable ${}^{21}\tilde{t}^{4}$ can take on a positive value.

In calculating the values of the production variables in the first basic feasible solution we do not simply proceed from left to right, as we did in Section 5.3, but we first give the production variables with the lowest coefficients their maximum value. This procedure is equivalent to the matrix minimum method, just as the procedure in Section 5.3 (proceeding from left to right) was equivalent to the northwest-corner rule.

Next we turn to the successive solution shown in Tables 6.8 and 6.9.

Again, in solving the transshipment problem by the u-method as the second part of the successive solution method, like in solving the problem simultaneously, one is allowed to introduce an interregional transport variable into the basis even though only the sign of its coefficient in an equation without international transport variables matches with the sign of the constant in that equation. A remarkable aspect of this problem, in contradistinction to the transshipment problem in a closed economy, is that the simplex multipliers are now fully determined, and no longer apart from a constant. This is so because the international transport variables prevent any of the equations from being redundant. *

The comparison of the simultaneous and the successive solution can again be represented conveniently in a table like Table 5.7 or in two tables like Table 5.8. We will represent them here only in this last form (Table 6.10).

Comparing the successive and the simultaneous solution shown in Table 6.10, we can make similar remarks to the ones made in the case of the closed economy model. Again in the simultaneous solution considerably less specialization occurs. More specifically, it turns out that in the simultaneous solution the country does not specialize completely in the expansion of sector 5, as in the successive solution; nor does region 1 specialize so strongly in the expansion of sector 3 nor region 2 in that of sector 5 as in the successive solution. The gain in terms of a decrease in total costs when using the simultaneous solution instead of the successive solution are, however, again relatively small: while the costs of all transport reduce by 0.4 units

TABLE 6.8

Solving the national planning and regional distribution problem without transport costs[a)]

	Variables								Constants	Simplex multipliers	
	${}^1y^3$	${}^2y^3$	${}^3y^3$	${}^1y^4$	${}^2y^4$	${}^3y^4$	${}^2y^5$	${}^3y^5$		Symbols	Values
Coefficients	1	1	1						0.6	u^1	0.5
of the				1	1	1	1	1	1.5	u^2	0.0
constraint	1			1					0.5	${}^1v$	2.6
equations		1			1		1		0.8	${}^2v$	2.3
			1					1	0.8	${}^3v$	2.0
Cost coefficients	3.1	2.8	3.0	3.5	3.0	3.3	2.3	2.0	Costs of solution		
Solution	0.5	0.1	—	—	—	—	0.7	0.8	5.09		
${}^r\tilde{c}^h - {}^r\bar{c}^h$	x	x	0.5	0.9	0.7	1.3	x	x			

[a)] The method used is the u-method. In order to find the first basic feasible solution, first the variables with the lowest cost coefficients are introduced into the basis.The changed values of the constants are not represented in the table. For the interpretation of the symbols — and x, see Table 6.1.

– without becoming = 0, however – the costs of production increase by 0.36 units. The production costs of the international sectors together increase even more than their transport costs reduce, but the relatively large reduction in the production costs of the shiftable national sector (sector 3) just tips the balance to the other side. Again we must emphasize that a generalization of these observations referring to this particular example could only be made on the basis of a more theoretical discussion, or of more and more realistic examples. In our opinion, however, the observations on the reduction of specialization and the relatively small gains in total costs might have a more general validity.

On the other hand, a comparison of the successive and simultaneous solutions of the closed economy model with those of the open economy model may be influenced even more by the quantitative and qualitative data of the models, which are exactly the same apart from the difference in character of sectors 4 and 5. One observation, however, might have some more general validity. In the successive solutions the gain from specialization

TABLE 6.9

Solving the transshipment problem[a)]

	Variables										Constants	Simplex multipliers		
												Symbols	Values at iteration 1	Values at iteration 2
	${}^{12}\tilde{t}^4$	${}^{13}\tilde{t}^4$	${}^{23}\tilde{t}^4$	${}^{21}\tilde{t}^4$	${}^{31}\tilde{t}^4$	${}^{32}\tilde{t}^4$	${}^{2}\tilde{e}^4$	${}^{2}\tilde{m}^4$	${}^{3}\tilde{e}^4$	${}^{3}\tilde{m}^4$				
Coefficients of the constraint equations	-1	-1		$1-\theta_1$	$1+\theta_1$						0.4	u_1	1.1	1
	1		-1	-1	$+\theta_1$	1	-1	1		$+\theta_1$	0.2	u_2	0.6	0.6
		1	1		-1	-1			-1	1	0.3	u_3	0.5	0.5
Cost coefficients	0.5	0.5	0.7	0.5	0.5	0.7	0.6	0.6	0.5	0.5	Costs of solution			
Iteration 1 Solution	—	—	—	0.4	—	—	—	0.6	—	0.3	0.71			
Iteration 1 Direct minus indirect cost-coefficients	1.0	1.1	0.8	x	-0.1	0.6	1.2	x	1.0	x				
Iteration 2 Solution ($\theta_1=0.4$)	—	—	—	—	0.4	—	—	0.2	—	0.7	0.67			
Iteration 2 Direct minus indirect cost-coefficients	0.9	1.0	0.8	0.1	x	0	1.2	x	1.0	x				

[a)] The changed values of the constants, which change during the process of finding a first basic feasible solution, are not represented in the table. For the explanation of the symbols — and x, see Table 6.1.

TABLE 6.10

Comparison of the successive and the simultaneous solution of the numerical example of an open economy[a)]

Sectors	The successive solution: Regions 1	2	3	Outside world	Sector totals	Costs of the solution per sector	The simultaneous solution: Regions 1	2	3	Outside world	Sector totals	Costs of the solution per sector
1	3.0 0.2	3.0 0.2	3.0 0.2		0.6	1.8	3.0 0.2	3.0 0.2	3.0 0.2		0.6	1.8
2	3.0 0.3	—	—		0.3	0.9	3.0 0.3	—	—		0.3	0.9
3	3.1 0.5	2.8 0.1	3.0 0		0.6	1.83	3.1 0.1	2.8 0.5	3.0 0		0.6	1.71
4	3.5 0	3.0 0	3.3 0			0	3.5 0.4	3.0 0	3.3 0			1.4
Transport flows (4)		← 0.4 — (0.5)		← 0.7 — (0.5)	1.5	0.67			← 0.3 — (0.5)		1.5	0.27
			← 0.2	— (0.6)				← 0.2		(0.6)		
5	—	2.3 0.7	2.0 0.8			3.21	—	2.3 0.3	2.0 0.8			2.29
Regional targets	1	1	1			Total costs 8.41	1	1	1			Total costs 8.37

[a)] In the upper left hand corner of each cell in the centre part of the table the corresponding cost coefficient is given; in the centre of each cell one finds the value of the corresponding variable in the respective solutions. At the bottom of the row corresponding to the heavy sector the arrows indicate the resulting transport flows. Their volume is placed in the midst of the arrows, while their cost coefficients (between brackets) are placed at the beginning of the arrows. The - indicate non - shiftabilities.

due to the possibility of international trade is larger than in the simultaneous solutions: $8.82-8.41 = 0.41$ units compared with $8.63-8.37 = 0.26$ units (cf. Tables 5.8 and 6.10). The same observation can be formulated differently: the improvement of the simultaneous solution over the successive solution is larger in the closed economy model (0.2 units), than in the open economy model (0.04 units).

* The principles and practical application of the iterative improvement method for proceeding from the successive solution to the simultaneous solution remain the same as in the exposition of this method in Section 6.1 (and Appendix III). The only difference is that for the open economy model the total increase in production in sector 4 is no longer fixed. Therefore, when adding the transport costs for product 4, expressed in the production variables ${}^1y^4$, ${}^2y^4$, ${}^3y^4$ with coefficients $-u_1$, $-u_2$, $-u_3$, to the cost-function of the national-regional planning problem, this time it would matter if the u_r were only determined apart from a constant. However, we saw above in this section that this time these u_r are determined exactly when solving the transport problem.

We will not repeat here the calculations for applying the iterative improvement procedure to the successive solution, since they are much the same as those presented in Section 6.1. One remark, however, can be made with regard to the rough estimate of an upper bound for the reduction in costs which is possible if the successive solution is improved upon by the iterative improvement procedure. This time the maximum reduction in transport costs per unit change in any production variable is not derived from the maximum difference in the values of the simplex multipliers found in the transshipment problem; now it can be derived from the largest value which these simplex multipliers themselves assume. In this case this largest value is 1, as can be seen from the last column of Table 6.9. On the other hand, the smallest increase in production costs per unit change in any production variable is 0.5 as can be seen from the last row of Table 6.8. The largest possible reduction in total costs, when improving the successive solution, can be seen, therefore, to be equal to $(1-0.5)\times 0.67 = 0.335$ units. It is interesting to see that a narrower upper bound can easily be calculated by confining oneself to the ${}^r\tilde{c}^h - {}^r\bar{c}^h$ values of the production variables of the heavy sector which is the obvious way to follow. Then the largest possible reduction in total costs becomes equal to $(1-0.7)\times 0.67 = 0.201$ units.

In the case of a model in which input-output relations are explicitly represented, which would be an extension of the model of Section 4.8. with one or more heavy sectors, the principles of the iterative improvement procedure presented in this chapter and in Appendix III remain applicable. But the expressions of the constants of the transshipment problem in terms of the production variables become more complicated, and hence those of the extra restrictions do so. An investigation into the efficiency of this iterative improvement procedure, as well as an investigation into its possible application to other types of models, is left for later research. *

6.3. The Introduction of Transport Costs Dependent on the Dominant Flow along a Route

In this section we will discuss some aspects and consequences of using Assumption 5.1b instead of Assumption 5.1a about the transport costs in the models of this chapter. In Assumption 5.1b costs of transporting a product along a route in a certain direction are considerably higher than the costs for transporting it in the opposite direction if the total flow of all goods along that route in the former direction is larger than the flow in the latter direction. First we will discuss, however, some general aspects of transportation costs.

The reason why an explicit representation of transportation costs is avoided in the models of this book as much as possible is first of all that the authors are convinced that without such an explicit representation models can already be constructed which give a reasonable first approximation, useful for some planning exercices. A second reason is that, as we saw, such a representation necessitates the introduction of a great number of variables which makes the models extremely complicated.

In Section 1.1 it has already been said that "transportation costs should be considered, in this context, to represent all obstacles to overcome distance, including a number of cultural ones". To re-phrase this: transportation costs here are a name for all those factors which make it advantageous to shorten the distance between the place of production and the place where the product is used, either as final or as interindustrial demand. Therefore such transportation costs should also include such factors as communication costs in the widest sense e.g. between factories or between factories and

consumers. Even apart from the fact that it may not be easy to measure the effect of all such factors, and certainly not to do so directly, it is un-likely that all the effects of such factors will be proportional to the amount of products which must be transported.

An additional difficulty might arise even if one finds that the representation of these effects is possible by means of transport costs proportional to the amounts of a product transported. One may find that the resulting transport cost coefficients are different for transport between factories and for transport between a factory and the consumers, even if the same average distance were covered. This means that different factors are involved in communication between factories in comparison with the communication between a factory and consumers. It would be possible to introduce this distinction into the model, at the cost of an extra index of the transport variables of the sectors concerned. This index would indicate either which sector receives the product as intermediate delivery, or that consumers are the receivers. For all such different transport variables different cost coefficients could be introduced. However, we will not introduce these complications into the models in this book.

Even if we have in mind only the costs of purely physical transportation it is known that a proportional relationship between its costs and the amount of products to be transported is in many cases only a rough approximation of reality. Even over a longer time period there may be some economies of scale, due to the fixed costs of laying the track and making available the vehicles; often the track's capacity must be expanded by more than can be fully utilised within a short period, while the wear and tear may not be proportional to the amounts transported over the track. Another phenomenon of transport costs, which will be taken account of in this section, is that if a dominant flow along a route in one direction clearly exists, the unit transportation costs in the opposite direction become immediately much smaller than the unit costs in the direction of the dominant flow since the empty vessels or vehicles have to return anyway to pick up new loads.

Because of these peculiarities, it is preferable to represent these obstacles to overcome distance only in an implicit and qualitative way by different classes of mobility of the products, and to introduce transport costs only for a few heavy sectors. An additional justification for introducing these classes of mobility is the finding that for Dutch international trade, especially the more complex products, the distance elasticities of trade volumes turn

out to be much more negative than one would expect if only physical obstacles to overcome distance were important and were so in proportion to the distance (KLAASSEN, 1967, p. 43).

In the remainder of this section we will show by means of an example how one could deal with transport costs of heavy sectors only or mainly determined by the amounts of products transported in the direction of the dominant flows along the routes concerned – which include roads and sea routes – of the transport system. This means that we are going to replace Assumption 5.1a about the costs of transportation, by Assumption 5.1b. In fact we use it in its weak but general form, allowing for the fact that transport cost in a non-dominant direction need not be equal to zero. The example we will use is as much as possible the same as that of the previous section. One difference is that now sector 5 will also be assumed to be a heavy one. Obviously, dominant flow transport cost functions are only possible if there is more than one heavy sector.

The main difficulty with a dominant flow transport cost function is that one can no longer calculate directly the costs of transporting a certain product in a certain direction along a route; first one has to know whether the transport concerned is part of a dominant flow along that route. Therefore, for each such transport one must leave open the possibility of two different costs, and correspondingly to each transport variable two cost coefficients are attached, one for the case where the transport concerned is part of a dominant flow, the other one – smaller and possibly even zero – for the case the transport takes place in the direction opposite to the dominant flow. Which cost coefficient is, after all, the relevant one is only determined when the optimal solution is known. This seems to be a vicious circle, from which one could, however, escape by considering systematically all possible alternative cases according to in which direction along each route the dominant flow goes, comparing the optimal solutions of each of these alternative cases and finally choosing the least costly one of them as the really optimal solution. Each such case is characterised not only by a particular transport cost function, but also by a number of restrictions on the transport variables ascertaining that the dominant flows correspond to the particular cost function. Obviously this would lead to a great number of alternative cases being considered. Below, a more efficient method will be presented, by which the number of cases actually to be considered is, to a considerable extent, reduced.

Another difficulty with dominant flow transport cost functions is that one cannot suppose that either the cheapest route or the means of transportation of a product between two regions is known in advance. Take e.g. the following case. Between two regions transportation of the products 4 and 5 (say grain and coal) is possible by train or by ship. Product 4 has to go in the opposite direction to product 5, while the demand for the means of transportation of product 5 is smaller than that for product 4. Assume that, in principle, product 4 can be transported more cheaply by ship and product 5 more cheaply by train. In fact, in this case it might be cheaper to transport both product 4 and 5 by ship, as, in practice product 5 can now be transported in a cheaper way by the returning ships which would otherwise remain empty. This consideration makes it necessary, in principle, to distinguish between each transport variable not only with respect to the type of product and the regions of origin and destination, but also with respect to the means of transportation used. However, in practice the distinction of such alternative means of transport may be relevant only for a limited number of routes and heavy products.

Taking up our numerical example, we already know that the difference with Section 6.2 is that we also suppose sector 5 to be heavy. Therefore, three new constraints for sector 5, similar to those of sector 4, will be introduced. There is, then, no longer a reason to maintain equation (6.2.6). In addition, some more data must be assumed to be known: the demand coefficients for product 5 per region, and the cost coefficients for transportation of product 4 and 5 specified for the two alternative cases, namely, whether the transportation concerned is part of the dominant flow or not. In addition, we suppose that between region 2 and region 3 two alternative means of transport can be used for both product 4 and product 5. The main new elements in comparison with the former models will be the inclusion of alternative formulations of the transport part of the cost function, the introduction of alternative sets of extra restrictions ascertaining the dominant flows to be in the directions corresponding to the alternative transport cost functions, and finally the distinction of different means of transportation.

The cost function can be written symbolically as follows:

$$
\begin{aligned}
z = & \underbrace{\sum_{r=1}^{3} \sum_{h=1}^{5} {}^{r}\tilde{c}^{h}\, {}^{r}y^{h}}_{(z_{\mathrm{P}})} + \\
& + \sum_{h=4,5} \Bigg(\left| \begin{matrix} {}^{12}\tilde{\tau}^{h}_{\mathrm{d}} \\ {}^{12}\tilde{\tau}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{12}\tilde{l}^{h} + \left| \begin{matrix} {}^{13}\tilde{\tau}^{h}_{\mathrm{d}} \\ {}^{13}\tilde{\tau}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{13}\tilde{l}^{h} + \left| \begin{matrix} {}^{21}\tilde{\tau}^{h}_{\mathrm{d}} \\ {}^{21}\tilde{\tau}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{21}\tilde{l}^{h} + \left| \begin{matrix} {}^{23}_{1}\tilde{\tau}^{h}_{\mathrm{d}} \\ {}^{23}_{1}\tilde{\tau}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{23}_{1}\tilde{l}^{h} + \\
& + \left| \begin{matrix} {}^{23}_{2}\tilde{\tau}^{h}_{\mathrm{d}} \\ {}^{23}_{2}\tilde{\tau}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{23}_{2}\tilde{l}^{h} + \left| \begin{matrix} {}^{31}\tilde{\tau}^{h}_{\mathrm{d}} \\ {}^{31}\tilde{\tau}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{31}\tilde{l}^{h} + \left| \begin{matrix} {}^{32}_{1}\tilde{\tau}^{h}_{\mathrm{d}} \\ {}^{32}_{1}\tilde{\tau}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{32}_{1}\tilde{l}^{h} + \left| \begin{matrix} {}^{32}_{2}\tilde{\tau}^{h}_{\mathrm{d}} \\ {}^{32}_{2}\tilde{\tau}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{32}_{2}\tilde{l}^{h} + \\
& \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{(z_{\mathrm{T}^h})} \\
& + \left| \begin{matrix} {}^{2}\tilde{\varepsilon}^{h}_{\mathrm{d}} \\ {}^{2}\tilde{\varepsilon}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{2}\tilde{e}^{h} + \left| \begin{matrix} {}^{2}\tilde{\mu}^{h}_{\mathrm{d}} \\ {}^{2}\tilde{\mu}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{2}\tilde{m}^{h} + \left| \begin{matrix} {}^{3}\tilde{\varepsilon}^{h}_{\mathrm{d}} \\ {}^{3}\tilde{\varepsilon}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{3}\tilde{e}^{h} + \left| \begin{matrix} {}^{3}\tilde{\mu}^{h}_{\mathrm{d}} \\ {}^{3}\tilde{\mu}^{h}_{\mathrm{nd}} \end{matrix} \right| {}^{3}\tilde{m}^{h} \Bigg) \qquad (6.3.1) \\
& \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{(z_{\mathrm{IT}^h})}
\end{aligned}
$$

Here the notation $\left| \begin{matrix} \cdot \\ \cdot \end{matrix} \right|$ means that only one of the coefficients is relevant at a time, depending on whether the corresponding transport flow is a dominant flow or not, as indicated by the lower right hand side subscripts *d* for dominant and *nd* for non-dominant. The lower left hand side subscript of the coefficients and variables related to transport flows between region 2 and region 3 refer to the two different means of transportation.

This function (6.3.1) could be complemented by a table of all possible alternative cases according to which flows are dominant. There are in fact six routes to be considered: between the regions 1 and 2, 1 and 3, 2 and 3 using transport means 1, 2 and 3 using transport means 2, 2 and the outside world, and 3 and the outside world. Therefore, there are altogether already $2^6 = 64$ possible different cases. It serves little purpose to list all these cases in a table. We will just take one case for illustrative purposes in Table 6.11.

In Table 6.12 we indicate by *d* or *nd* which cost coefficients of (6.3.1) must actually be used in the case of Table 6.11.

TABLE 6.11

Example of a combination of dominant and non dominant flows[a])

Route and direction	$(1\rightarrow 2)$	$(1\rightarrow 3)$	${}_1(2\rightarrow 3)$	${}_2(2\rightarrow 3)$	$(2\rightarrow w)$	$(3\rightarrow w)$
Dominance index	*nd*	*nd*	*d*	*d*	*nd*	*d*

[a]) e.g. ${}_2(2\rightarrow 3)$ indicates a flow from region 2 to region 3 by means 2; *d* indicates that the corresponding flow is dominant in the direction indicated, while *nd* indicates that this is not the case.

TABLE 6.12

Coefficients of (6.3.1) corresponding to the case of Table 6.11

Variables	${}^{12}\tilde{t}^h$	${}^{13}\tilde{t}^h$	${}^{21}\tilde{t}^h$	${}^{23}_{1}\tilde{t}^h$	${}^{23}_{2}\tilde{t}^h$	${}^{31}\tilde{t}^h$	${}^{32}_{1}\tilde{t}^h$	${}^{32}_{2}\tilde{t}^h$	${}^{2}\tilde{e}^h$	${}^{2}\tilde{m}^h$	${}^{3}\tilde{e}^h$	${}^{3}\tilde{m}^h$
Coefficient index[a])	*nd*	*nd*	*d*	*d*	*d*	*d*	*nd*	*nd*	*nd*	*d*	*d*	*nd*

[a]) The indices always have the same value for both sectors: $h = 4$ or 5.

The following extra restrictions describe the subspace in which the values of the transport variables must be so that the case of Table 6.11 occurs.

$$({}^{12}\tilde{\tau}^4_{\mathrm{d}}-{}^{12}\tilde{\tau}^4_{\mathrm{nd}})\,{}^{12}\tilde{t}^4+({}^{12}\tilde{\tau}^5_{\mathrm{d}}-{}^{12}\tilde{\tau}^5_{\mathrm{nd}})\,{}^{12}\tilde{t}^5 \leq ({}^{21}\tilde{\tau}^4_{\mathrm{d}}-{}^{21}\tilde{\tau}^4_{\mathrm{nd}})\,{}^{21}\tilde{t}^4+({}^{21}\tilde{\tau}^5_{\mathrm{d}}-{}^{21}\tilde{\tau}^5_{\mathrm{nd}})\,{}^{21}\tilde{t}^5 \tag{6.3.2}$$

$$({}^{13}\tilde{\tau}^4_{\mathrm{d}}-{}^{13}\tilde{\tau}^4_{\mathrm{nd}})\,{}^{13}\tilde{t}^4+({}^{13}\tilde{\tau}^5_{\mathrm{d}}-{}^{13}\tilde{\tau}^5_{\mathrm{nd}})\,{}^{13}\tilde{t}^5 \leq ({}^{31}\tilde{\tau}^4_{\mathrm{d}}-{}^{31}\tilde{\tau}^4_{\mathrm{nd}})\,{}^{31}\tilde{t}^4+({}^{31}\tilde{\tau}^5_{\mathrm{d}}-{}^{31}\tilde{\tau}^5_{\mathrm{nd}})\,{}^{31}\tilde{t}^5 \tag{6.3.3}$$

$$({}^{23}_{1}\tilde{\tau}^4_{\mathrm{d}}-{}^{23}_{1}\tilde{\tau}^4_{\mathrm{nd}})\,{}^{23}_{1}\tilde{t}^4+({}^{23}_{1}\tilde{\tau}^5_{\mathrm{d}}-{}^{23}_{1}\tilde{\tau}^5_{\mathrm{nd}})\,{}^{23}_{1}\tilde{t}^5 \geq ({}^{32}_{1}\tilde{\tau}^4_{\mathrm{d}}-{}^{32}_{1}\tilde{\tau}^4_{\mathrm{nd}})\,{}^{32}_{1}\tilde{t}^4+({}^{32}_{1}\tilde{\tau}^5_{\mathrm{d}}-{}^{32}_{1}\tilde{\tau}^5_{\mathrm{nd}})\,{}^{32}_{1}\tilde{t}^5 \tag{6.3.4}$$

$$({}^{23}_{2}\tilde{\tau}^4_{\mathrm{d}}-{}^{23}_{2}\tilde{\tau}^4_{\mathrm{nd}})\,{}^{23}_{2}\tilde{t}^4+({}^{23}_{2}\tilde{\tau}^5_{\mathrm{d}}-{}^{23}_{2}\tilde{\tau}^5_{\mathrm{nd}})\,{}^{23}_{2}\tilde{t}^5 \geq ({}^{32}_{2}\tilde{\tau}^4_{\mathrm{d}}-{}^{32}_{2}\tilde{\tau}^4_{\mathrm{nd}})\,{}^{32}_{2}\tilde{t}^4+({}^{32}_{2}\tilde{\tau}^5_{\mathrm{d}}-{}^{32}_{2}\tilde{\tau}^5_{\mathrm{nd}})\,{}^{32}_{2}\tilde{t}^5 \tag{6.3.5}$$

$$({}^{2}\tilde{\varepsilon}^4_{\mathrm{d}}-{}^{2}\tilde{\varepsilon}^4_{\mathrm{nd}})\,{}^{2}\tilde{e}^4+({}^{2}\tilde{\varepsilon}^5_{\mathrm{d}}-{}^{2}\tilde{\varepsilon}^5_{\mathrm{nd}})\,{}^{2}\tilde{e}^5 \leq ({}^{2}\tilde{\mu}^4_{\mathrm{d}}-{}^{2}\tilde{\mu}^4_{\mathrm{nd}})\,{}^{2}\tilde{m}^4+({}^{2}\tilde{\mu}^5_{\mathrm{d}}-{}^{2}\tilde{\mu}^5_{\mathrm{nd}})\,{}^{2}\tilde{m}^5 \tag{6.3.6}$$

$$({}^{3}\tilde{\varepsilon}^4_{\mathrm{d}}-{}^{3}\tilde{\varepsilon}^4_{\mathrm{nd}})\,{}^{3}\tilde{e}^4+({}^{3}\tilde{\varepsilon}^5_{\mathrm{d}}-{}^{3}\tilde{\varepsilon}^5_{\mathrm{nd}})\,{}^{3}\tilde{e}^5 \geq ({}^{3}\tilde{\mu}^4_{\mathrm{d}}-{}^{3}\tilde{\mu}^4_{\mathrm{nd}})\,{}^{3}\tilde{m}^4+({}^{3}\tilde{\mu}^5_{\mathrm{d}}-{}^{3}\tilde{\mu}^5_{\mathrm{nd}})\,{}^{3}\tilde{m}^5 \tag{6.3.7}$$

The reason that the coefficients $({}^{rr'}\tilde{\tau}^h_d - {}^{rr'}\tilde{\tau}^h_{nd})$ appear in these inequalities is that the coefficients by which the dominant direction along a route is determined must be only related to that part of the transport costs which is common to the flows in both directions, e.g. the cars and track used, and not to the part of the transport costs which are incurred by the flows separately, e.g. handling costs, for which the coefficients ${}^{rr'}\tilde{\tau}^h_{nd}$ stand. In general these "dominance" coefficients must have the property that, if none of the flows in either direction along a certain route is dominant – that is, if the equality sign holds in the corresponding inequality – then it does not make a difference in transport costs which direction is considered to be dominant. The reader may verify that the coefficients in (6.3.2)-(6.3.7) have this property. The importance of this property will become clear later on, when in Table 6.16 the costs of the solution at line 7 remain the same for both the cost coefficients of line 5 and of line 9.

These restrictions, in inequality form, could again be written as equations by introducing nonnegative slack variables

$$ {}^{12}s,\ {}^{13}s,\ {}^{23}_{1}s,\ {}^{23}_{2}s,\ {}^{2w}s,\ {}^{3w}s, $$

just as we did in the previous section.

For data in our numerical example we will assume numerical values as similar as possible to those in Section 6.2:

$$ {}^1y = {}^2y = {}^3y = 1.0;\ {}^1\tilde{\eta}^1 = {}^2\tilde{\eta}^1 = {}^3\tilde{\eta}^1 = 0.2;\ \tilde{\eta}^2 = 0.1;\ \tilde{\eta}^3 = 0.2; $$
$$ {}^1\tilde{\eta}^4 = {}^2\tilde{\eta}^4 = {}^3\tilde{\eta}^4 = 0.3;\ {}^1\bar{y}^4 = 0.9;\ {}^2\bar{y}^4 = 1.1;\ {}^3\bar{y}^4 = 1.0; $$
$$ {}^1\tilde{d}^4 = {}^2\tilde{d}^4 = {}^3\tilde{d}^4 = 1.0;\ {}^1\tilde{\eta}^5 = 0.1;\ {}^2\tilde{\eta}^5 = 0.2;\ {}^3\tilde{\eta}^5 = 0.3; $$
$$ {}^1\bar{y}^5 = 0;\ {}^2\bar{y}^5 = 0.5;\ {}^3\bar{y}^5 = 0.6;\ {}^1\tilde{d}^5 = 0.2;\ {}^2\tilde{d}^5 = 0.4;\ {}^3\tilde{d}^5 = 0.5. $$

The interpretation of these numerical data is that, while for product 4 the supply and demand in each region has been about in balance, product 5 has never been produced in region 1, region 2 and 3 both being the big suppliers; but the demand in region 1 has also been considerably smaller than in the other regions, and this is reflected in the projected increases in demand in the regions. In order to keep the data as similar as possible to the data in the former models, it is also assumed that until now the products 4 and 5 have been neither exported nor imported. The production cost coefficients are given in table 6.13 and the transport cost coefficient are given in Figure 6.1 or Table 6.14.

TABLE 6.13

Production cost coefficients and income increases by sector and by region

Sectors		Regions 1	2	3	Sectoral increases [a)]
Regional	1	3.0	3.0	3.0	0.6
National	2	3.0	x	x	0.3
	3	3.1	2.8	3.0	0.6
Heavy international	4	3.5	3.0	3.3	1.5
	5	x	2.3	2.0	
Regional increases		1	1	1	Total 3

[a)] The sectoral increases have been derived from the regional targets and the values of the coefficients given above.

The equations of the model are much the same as those of the model of the previous section. In addition to (6.2.1)-(6.2.5) and (6.2.7)-(6.2.9) the following equations for sectors 4 and 5 appear in the model:

$$^{1}y^{4} - {}^{12}\tilde{t}^{4} - {}^{13}\tilde{t}^{4} + {}^{21}\tilde{t}^{4} + {}^{31}\tilde{t}^{4} = {}^{1}\tilde{\eta}^{4}\,{}^{1}y - {}^{1}\bar{y}^{4} + {}^{1}\tilde{d}^{4} \tag{6.3.8}$$

$$^{2}y^{4} - {}^{21}\tilde{t}^{4} - {}^{23}_{1}\tilde{t}^{4} - {}^{23}_{2}\tilde{t}^{4} + {}^{12}\tilde{t}^{4} + {}^{32}_{1}\tilde{t}^{4} + {}^{32}_{2}\tilde{t}^{4} - {}^{2}\tilde{e}^{4} + {}^{2}\tilde{m}^{4} = {}^{2}\tilde{\eta}^{4}\,{}^{2}y - {}^{2}\bar{y}^{4} + {}^{2}\tilde{d}^{4} \tag{6.3.9}$$

$$^{3}y^{4} - {}^{31}\tilde{t}^{4} - {}^{32}_{1}\tilde{t}^{4} - {}^{32}_{2}\tilde{t}^{4} + {}^{13}\tilde{t}^{4} + {}^{23}_{1}\tilde{t}^{4} + {}^{23}_{2}\tilde{t}^{4} - {}^{3}\tilde{e}^{4} + {}^{3}\tilde{m}^{4} = {}^{3}\tilde{\eta}^{4}\,{}^{3}y - {}^{3}\bar{y}^{4} + {}^{3}\tilde{d}^{4} \tag{6.3.10}$$

$$-{}^{12}\tilde{t}^{5} - {}^{13}\tilde{t}^{5} + {}^{21}\tilde{t}^{5} + {}^{31}\tilde{t}^{5} = {}^{1}\tilde{\eta}^{5}\,{}^{1}y - {}^{1}\bar{y}^{5} + {}^{1}\tilde{d}^{5} \tag{6.3.11}$$

$$^{2}y^{5} - {}^{21}\tilde{t}^{5} - {}^{23}_{1}\tilde{t}^{5} - {}^{23}_{2}\tilde{t}^{5} + {}^{12}\tilde{t}^{4} + {}^{32}_{1}\tilde{t}^{5} + {}^{32}_{2}\tilde{t}^{5} - {}^{2}\tilde{e}^{5} + {}^{2}\tilde{m}^{5} = {}^{2}\tilde{\eta}^{5}\,{}^{2}y - {}^{2}\bar{y}^{5} + {}^{2}\tilde{d}^{5} \tag{6.3.12}$$

$$^{3}y^{5} - {}^{31}\tilde{t}^{5} - {}^{32}_{1}\tilde{t}^{5} - {}^{32}_{2}\tilde{t}^{5} + {}^{13}\tilde{t}^{5} + {}^{23}_{1}\tilde{t}^{5} + {}^{23}_{2}\tilde{t}^{5} - {}^{3}\tilde{e}^{5} + {}^{3}\tilde{m}^{5} = {}^{3}\tilde{\eta}^{5}\,{}^{3}y - {}^{3}\bar{y}^{5} + {}^{3}\tilde{d}^{5} \tag{6.3.13}$$

Notice that for transportation between regions 2 and 3 the variables are distinguished according to the means used. Also no variable $^{1}y^{5}$ appears

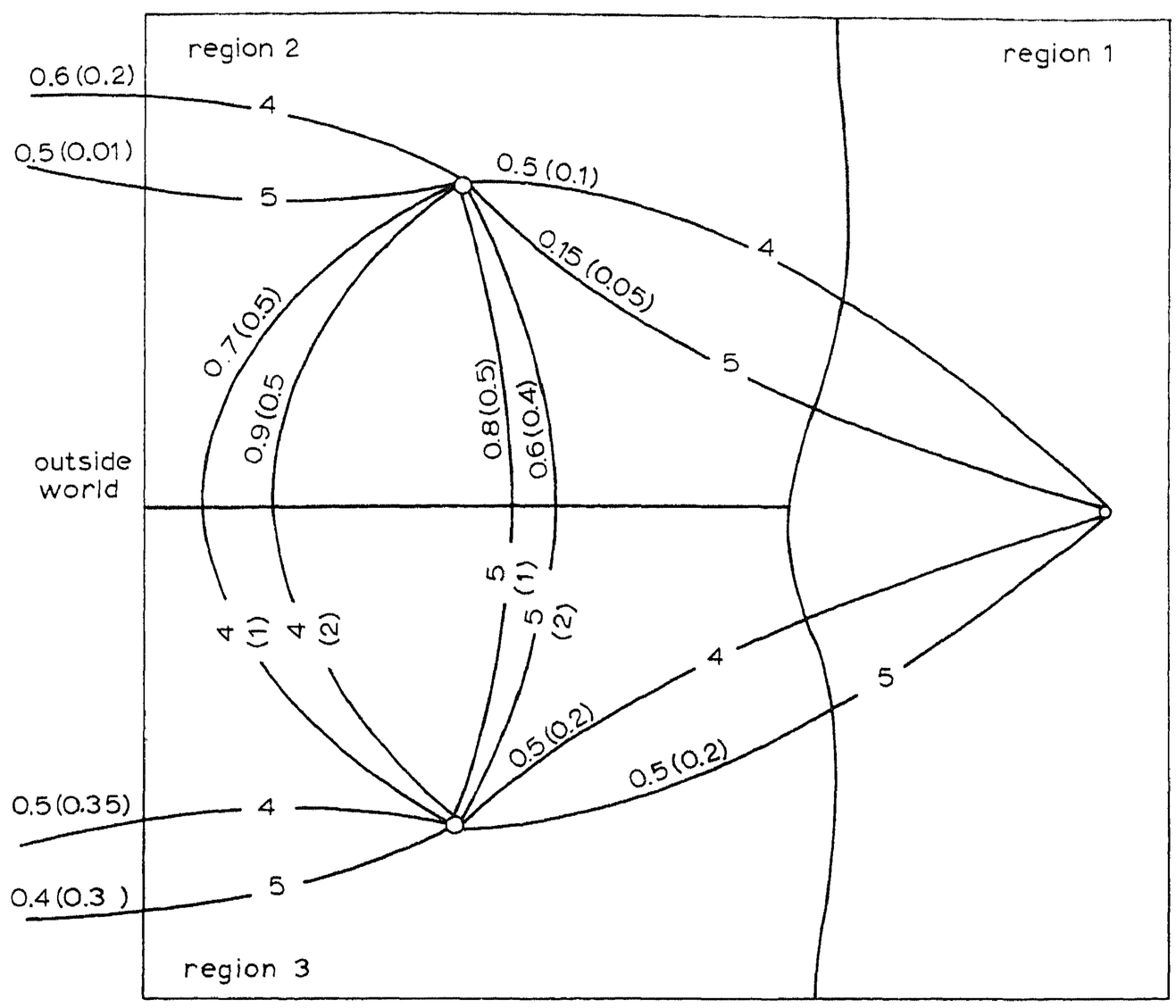

Fig. 6.1. Graphical representation of interregional and international transport routes, and the corresponding dominant and non-dominant cost coefficients. The numbers interrupting the lines indicate the products; the numbers between brackets below these numbers indicate the means of transport, and the numbers on the lines indicate the cost coefficients, of which the non-dominant ones are placed between brackets.

in the equations, as sector 5 is only partly shiftable. Again, as in (6.2.13), all variables should be nonnegative.

Solving this model by evaluating it for each of the 64 alternative cases with the extra restrictions like (6.3.2)-(6.3.7), which are different for each case, is obviously not practicable . Instead we will use a method similar to the iterative improvement method.

* The idea of the method to be used is as follows. We start out solving the model with only the dominant cost coefficients for the transport variables, that is, disregarding the fact that return flows are cheaper. The solution found specifies the transport flows of products 4 and 5 and shows, therefore, which flows are dominant (in this solution). In other words, it shows which

TABLE 6.14

Cost coefficients for interregional and intern

Sector 4

from	to		Regions 1	2		3		Outside world	fror
Regions	1	d	x	0.5		0.5		x	
		nd	x	0.1		0.2		x	
						means 1	means 2		
	2	d	0.5	x		0.7	0.9	0.6	Regior
		nd	0.1	x		0.5	0.5	0.2	
				means 1	means 2				
	3	d	0.5	0.7	0.9	x		0.5	
		nd	0.2	0.5	0.5	x		0.35	
Outside world		d	x	0.6		0.5		x	Outsid
		nd	x	0.2		0.35		x	worl

one of the alternative cases corresponds to this solution. Then we define a new problem, which corresponds to this case. That is, we now take as transport cost coefficients those coefficients (dominant and non-dominant) which correspond to this case, and add the extra restrictions corresponding to this case. We solve this new problem. If in this new optimal solution none of the slack variables in the extra restrictions $= 0$, then this solution is the overall optimal solution. If one or more of these slack variables have become zero, then we have to investigate also all the other alternative cases to which this new solution corresponds. The point is that, if one or more of the inequalities are satisfied as equalities, no dominant flows can be distinguished, so that for the corresponding sectors all alternative cases of dominance have to be examined. If, for all these other alternative cases with their own corresponding transport cost coefficients and their own corresponding extra restrictions, the original solution is still the optimal one, then it is also the overall optimal solution. If this is not the case, then one has to choose one

.14

nternational transport of products 4 and 5

Sector 5

from		to	Regions 1	2		3		Outside world
	1	d	x	0.15		0.5		x
		nd	x	0.05		0.2		x
						means 1	means 2	
egions	2	d	0.15	x		0.8	0.6	0.5
		nd	0.05	x		0.5	0.4	0.01
				means 1	means 2			
	3	d	0.5	0.8	0.6	x		0.4
		nd	0.2	0.5	0.4	x		0.3
utside world		d	x	0.5		0.4		x
		nd	x	0.01		0.3		x

of the cases for which the original solution is not the optimal one, and find the new optimal solution corresponding to this case. This new solution is examined in the same way as the former one, and either its overall optimality is shown, or a new case is found for which a new optimal solution has to be calculated. After a finite number of steps the overall optimal solution will be reached in this way.

This method can be applied not only to the full simultaneous problem, but also to the transshipment problem remaining after a solution for the production variables has been found. If one applies this method only to the transshipment problem the optimal solution found is solely the one of the transshipment problem, and it must be checked, by again applying the iterative improvement procedure, whether a change in the values of the production and transport variables could not give an overall gain in costs. In this section we will use this second decomposition approach; that is, we will apply this method to the transshipment problem.

We will refrain from giving here a full description of all possible steps, as we did in Section 6.1., but confine ourselves to just an example. In doing so we will start from the solution in the production variables which was found in Section 6.2. As a starting solution to the remaining transshipment problem we can use for product 4 the solution of Section 6.2 as well, so that we only need to compute an optimal solution for the transshipment problem of product 5, using the dominant cost coefficients of course. This optimal solution turns out to be:
${}^{21}\tilde{t}^5 = 0.2$; ${}^{31}\tilde{t}^5 = 0.1$; ${}^{3}\tilde{e}^5 = 0.5$; all other transport variables of product 5 = 0. The optimal solution of the transport problem of product 4 was:
${}^{2}\tilde{m}^4 = 0.2$; ${}^{3}\tilde{m}^4 = 0.3$; ${}^{32}\tilde{t}^4 = 0$ (a basic variable); all other transport variables of product 5 = 0 (non-basic variables).

Now we examine each of the six routes to see which flow is dominant, and the results are presented in the following table:

TABLE 6.15

Dominant and non-dominant flows[a])

Route and direction	(1→2)	(1→3)	$_1$(2→3)	$_2$(2→3)	(2→w)	(3→w)
Dominance index	*nd*	*nd*	*nd* and *d*	*nd* and *d*	*nd*	*d*[b])

[a]) Cf. Table 6.11.
[b]) This is so as $0.1 \times 0.5 > 0.15 \times 0.3$.

As a result we find four cases, corresponding to whether we consider the routes $_1$(2→3) and $_2$(2→3) respectively to be dominant or not. We must choose one of these cases and calculate its optimal solution; for this purpose we choose the case represented in Table 6.11. The corresponding extra constraints are already given in (6.3.2)-(6.3.7), which only need to be written as equations. The corresponding transport cost coefficients, given in Table 6.12, need only be replaced by their numerical values.

The new transport problem corresponding to this case can be written in the form of a table with detached coefficients and solved as follows (see Table 6.16).

The following explanation of Table 6.16 and in particular lines 1-10 at the bottom and columns 1-4 at the right hand side can be given. This outline is at the same time the solution procedure.

Line 1-2: the dominant and non-dominant cost coefficients, indicated by *d* and *nd* are placed in the columns of the variables to which they correspond. The slack variables of the added equations (cf. the explanation of line 4) have, by definition, cost coefficients = 0.

Line 3: any basic feasible solution of the combined national-regional planning and transport problem can be taken as a starting solution. The values of the production variables are then used to calculate the constants of the transshipment problem, and the values of the transport variables are used as a first basic feasible solution of the transshipment problem. In this case the overall optimal solution of Section 6.2 completed with the seperately calculated solution of the transshipment problem of Sector 5 with all cost coefficients equal to the dominant ones, is used. This solution determines which of the 64 cases of dominance are relevant, as we set out above, and a further rather arbitrary choice may be made to fully determine which case is relevant. This determines which equations must be added to the problem in order to limit the values of the variables to this case, as well as the corresponding cost coefficients. In our case we have a degenerate basic feasible solution for the transport problem, as only 9 variables are positive, while there are 12 equations. In order to determine the corresponding simplex multipliers we have to indicate which of the variables with values = 0 are also considered to be basic variables. We choose them in such a way that in line 4 (see below) no negative values appear. Some trial and error may be inevitable here. The result is that ${}^{31}\tilde{t}^{4}$, ${}^{23}_{1}s$ and ${}^{23}_{2}s$ are considered to be basic variables.

Column 1: the values of the simplex multipliers corresponding to this first basic feasible solution and the dominant cost coefficients have been calculated in order to show that in line 4 no negative values appear.

Line 4: the direct minus indirect cost coefficients have been calculated on the basis of the dominant cost coefficients and the simplex multiplier values in column 1. The fact that no negative values appear in line 4 shows that this solution is the optimal one for the transport problem *if return flows in a non-dominant direction were not cheaper.*

Line 5 shows the cost coefficients corresponding to the particular case of this solution, taking account of the fact that transport flows in non-domi-

TABLE 6.1

Solving the transshipment problem with cost coe

			Variables for interregional and international transportation																	
			${}^{12}t^4$	${}^{13}t^4$	${}^{23}_{1}t^4$	${}^{23}_{2}t^4$	${}^{21}t^4$	${}^{31}t^4$	${}^{32}_{1}t^4$	${}^{32}_{2}t^4$	${}^{2}e^4$	${}^{2}m^4$	${}^{3}e^4$	${}^{3}m^4$	${}^{12}t^5$	${}^{13}t^5$	${}^{23}_{1}t^5$	${}^{23}_{2}t^5$	${}^{21}t^5$	${}^{31}t^5$
Coefficients of the original equations			−1	−1			1	1												
			1		−1	−1	−1		1	1	−1	1								
				1	1	1		−1	−1	−1			−1	1						
															−1	−1			−θ 1	+θ 1
															1		−1	−1	−θ −1	
																1	1	1		+θ −1
Coefficients of the added equations			0.4				−0.4								0.1				−θ −0.1	
				0.3				−0.3								0.3				+θ −0.3
					0.2				−0.2								0.3			
						0.4				−0.4								0.2		
											0.4	−0.4								
													0.15	−0.15						
Lines[a])																				
1	Cost-	d	0.5	0.5	0.7	0.9	0.5	0.5	0.7	0.9	0.6	0.6	0.5	0.5	0.15	0.5	0.8	0.6	0.15	0.5
2	coefficients	nd	0.1	0.2	0.5	0.5	0.1	0.2	0.5	0.5	0.2	0.2	0.35	0.35	0.05	0.2	0.5	0.4	0.05	0.2
3	First solution		—	—	—	—	—	0	—	—	—	0.2	—	0.3	—	—	—	—	0.2	0.1
4	Direct minus indirect dominant cost coeff.		0.9	1	0.8	1	0.1	x	0.6	0.8	1.2	x	1	x	0.3	1	1.15	0.95	x	x
5	Cost coefficients of first solution		0.1	0.2	0.7	0.9	0.5	0.5	0.5	0.5	0.2	0.6	0.5	0.35	0.05	0.2	0.8	0.6	0.15	0.5
6	Direct minus indirect cost coefficients		0.35	0.7	0.95	1.15	0.25	x	0.25	0.25	0.8	x	0.85	x	0.2	0.7	1.15	0.95	x	x
7	New solution ($\theta = 0.125$)		—	—	—	—	—	0	—	—	—	0.2	—	0.3	—	—	—	—	0.15	0.15
8	Direct minus indirect cost coefficients		0.41	0.7	0.89	1.09	0.19	x	0.31	0.31	0.8	x	0.85	x	0.2	0.7	1.15	0.95	x	x
9	Check : new cost coefficients		0.1	0.2	0.7	0.9	0.5	0.5	0.5	0.5	0.2	0.6	0.1	0.5	0.05	0.2	0.8	0.6	0.15	0.5
10	Check : direct minus indirect cost coefficients		0.41	0.7	0.89	1.09	0.19	x	0.31	0.31	0.8	x	0.6	x	0.2	0.7	1.15	0.95	x	x

[a]) For explanation see text.
[b]) For the explanation of the symbols — and x see Table 6.1.
[c]) Cf. the explanation of line 9 in the text.

ILE 6.16

ost coefficients dependent on the dominant flow. [a,b]

							Slack variables						Constants	Simplex multipliers				
														Symbols	Columns			
t^5	${}^{31}t^5$	${}^{32}_{1}t^5$	${}^{32}_{2}t^5$	${}^{2}e^5$	${}^{2}m^5$	${}^{3}e^5$	${}^{3}m^5$	${}^{12}s$	${}^{13}s$	${}^{23}_{1}s$	${}^{23}_{2}s$	${}^{2w}s$	${}^{3w}s$		1	2	3	4
														$(0.4-{}^{1}y^4=)0$	u_1	1	0.85	0.91	0.91
														$(0.2-{}^{2}y^4=)0.2$	u_2	0.6	0.6	0.6	0.6
														$(0.3-{}^{3}y^4=)0.3$	u_3	0.5	0.35	0.41	0.41
-0	$+\theta$ 1													$(0.3-0=)0.3$	u_4	0.1	0.1	0.14	0.14
-0		1	1	$+\theta$ -1	1									$(0.1-{}^{2}y^5=)-0.2$	u_5	-0.05	-0.05	-0.01	-0.01
	$+\theta$ -1	-1	-1			$-\theta$ -1	1							$(0.2-{}^{3}y^5=)-0.6$	u_6	-0.4	-0.4	-0.36	-0.36
-0 1								-0.1θ 1						0	w_1	0	0	0	0
	$+\theta$ -0.3								$+0.3\theta$ 1					0	w_2	0	0	0	0
		-0.3								-1				0	w_3	0	0	0	0
			-0.2								-1			0	w_4	0	0	0	0
				$+\theta$ 0.49	-0.49							-0.49θ 1		0	w_5	0	0	0	0
						$-\theta$ 0.1	-0.1						-0.1θ $(\mp)^1$ c)	0	w_6	0	0	0.4	-0.6
5	0.5	0.8	0.6	0.5	0.5	0.4	0.4	0	0	0	0	0	0						
5	0.2	0.5	0.4	0.01	0.01	0.3	0.3												
	0.1	—	—	—	—	0.5	—	0.02	0.03	0	0	0.08	0.005						
	x	0.45	0.25	0.45	0.55	x	0.8	x	x	x	x	x	x						
5	0.5	0.5	0.4	0.01	0.5	0.4	0.3	0	0	0	0	0	0						
	x	0.15	0.05	-0.04	0.55	x	0.7	x	x	x	x	x	x						
5	0.15	—	—	0.05	—	0.45	—	0.015	0.045	0	0	0.0555	—						
	x	0.15	0.05	x	0.51	x	0.7	x	x	x	x	x	0.4						
5	0.5	0.5	0.4	0.01	0.5	0.3	0.4	0	0	0	0	0	—						
	x	0.15	0.05	x	0.51	x	0.7	x	x	x	x	x	0.6						

Costs of solutions:

First solution, dominant cost coefficients:	0.550
First solution, correct cost coefficients:	0.505
New solution, correct cost coefficients:	0.503
New solution, alternative cost coefficients:	0.503

nant directions have non-dominant cost coefficients (cf. Tables 6.12 and 6.14).

Column 2: the new values of the simplex multipliers corresponding to the first solution, but to the new cost coefficients, have been calculated.

Line 6: the direct minus indirect cost coefficients have been calculated on the basis of the new cost coefficients and the simplex multiplier values of column 2. The direct minus indirect cost coefficient of ${}^{2}\tilde{e}^{5}$ turns out to be negative, so this variable should be introduced into the basis. The θ-method must be used with a slight modification since the changes in the slack variables amount to θ times a coefficient equal to the corresponding coefficient of the transport variable that is changed. How this works can be seen in the table. It turns out that ${}^{3w}s$ should go out of the basis, and $\theta = 0.05$.

Line 7: the new solution is presented, using $\theta = 0.05$.

Column 3: the new values of the simplex multipliers have been calculated corresponding to the new solution of line 7, and the cost coefficients of line 5.

Line 8: the new direct minus indirect cost coefficients using line 5 and column 3 have been calculated. As all figures are nonnegative the new solution is the optimal one under the added constraints.

Line 9: since in the new solution three slack variables have a value = 0, one should check whether the new solution is still optimal in the seven other cases where at least one of the inequality signs in the new constraints corresponding to these slack variables has been reversed and the transport cost coefficients have been correspondingly changed. Reversing an inequality sign amounts to changing the sign of the coefficient of the corresponding slack variable. First the case where the coefficient of ${}^{3w}s$ changes in sign (indicated by (+) in the corresponding cell) is considered. The new cost coefficients corresponding to this case are inserted.

Column 4: new values of the simplex multipliers have been calculated, using the solution of line 7, and the cost coefficients of line 9.

Line 10: new direct minus indirect cost coefficients have been calculated using column 4 and line 9. As no figure here is negative, the same solution is optimal in the new case, where only the inequality sign in the sixth added equation is reversed.

The other possible cases to be considered, where the inequality signs in the 3rd and 4th added equations change, need not be calculated completely.

As none of the transport variables in these equations is basic, changing their cost coefficients would only change their own direct minus indirect cost coefficients, and not the values of the simplex multipliers. However, in the case of Table 6.16 these direct minus indirect cost coefficients remain positive when the cost coefficients of the variables are changed from the dominant to the non-dominant ones or vice versa. This can be seen by looking consecutively at lines 1, 2, 5 and 8 and at lines 1, 2, 9 and 10. This shows that the solution of line 7 is the optimal solution of the transshipment problem with cost coefficients dependent on the dominant flow. Notice that the total transport costs of this optimal solution are the same for all the 8 cases considered due to the particular property of the dominance coefficients discussed immediately after introducing the inequalities (6.3.2)-(6.3.7). *

In order to compare the optimal solutions of the transshipment problems with and without dominant flows the following graphs may be illustrative.

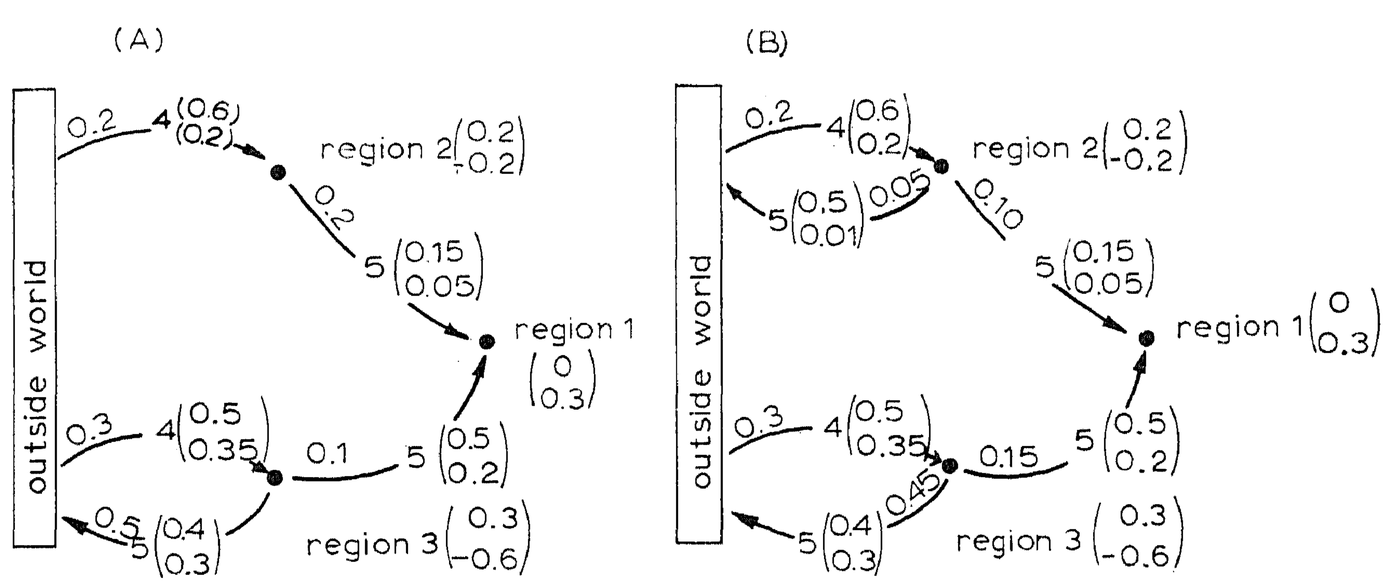

Fig. 6.2. Graphical representation of the optimal solutions of the transport problem with cost coefficients, (*A*) independent of the dominant flows and (*B*) dependent on the dominant flows. Cf. Fig. 6.1. Here the actual flows are represented by arrows in which the product is indicated. Behind this product number, the dominant and non-dominant cost coefficients are indicated between brackets, the higher one being the dominant one. The difference between these two cost coefficients is the dominance coefficient. The size of the flow is indicated above the arrows. This figure times the dominance coefficient determines which direction is dominant. The figures between brackets behind each region's number indicate the net deficit in that region of product 4 (upper) and 5 (lower).

Comparing the two solutions, we can see that the only difference between them is that in the case of cheaper return flows it becomes attractive to

TABLE 6.17

The contracted equations of the trans

	Basic variables in the optimal solution of the transport problem												Cons
	${}^{31}\tilde{t}^{4}$	${}^{2}\tilde{m}^{4}$	${}^{3}\tilde{m}^{4}$	${}^{21}\tilde{t}^{5}$	${}^{31}\tilde{t}^{5}$	${}^{2}\tilde{e}^{5}$	${}^{3}\tilde{e}^{5}$	${}^{12}s$	${}^{13}s$	${}^{23}_{1}s$	${}^{23}_{2}s$	${}^{2w}s$	Cons
Coefficients of the contracted original and added equations	1												0.4
		1											0.2
	−1		1										0.3
				1	1								0.3
				−1		−1							0.1
					−1		−1						0.2
				−0.1				1					0
	−0.3				−0.3				1				0
										−1			0
											−1		0
		−0.4				0.49						1	0
			−0.15				0.1						0
Solution of the contracted equations	1												0.4
		1											0.2
			1										0.7
				1									1.5
					1								−1.2
						1							−1.6
							1						1.0
								1					0.1
									1				−0.2
										1			0
											1		0
												1	0.8

export product 5 from region 2 using the opportunity of the cheap return flows of the vehicles bringing product 4 to region 2. This is so only up to the point where the export flow of product 5 from region 3 would become non-dominant if reduced further, due to an increase in the transport of product 5 from region 3 to region 1. This increase is, in its turn, due to a

6.17

transshipment problem and their solution

Constants in terms of the production variables						
Constant part	Coefficients of the production variables					Equation number
	${}^{1}y^{4}$	${}^{2}y^{4}$	${}^{3}y^{4}$	${}^{2}y^{5}$	${}^{3}y^{5}$	
0.4	−1					(6.3.14)
0.2		−1				(6.3.15)
0.3			−1			(6.3.16)
0.3						(6.3.17)
0.1				−1		(6.3.18)
0.2					−1	(6.3.19)
0						(6.3.20)
0						(6.3.21)
0						(6.3.22)
0						(6.3.23)
0						(6.3.24)
0						(6.3.25)
0.4	−1					(6.3.26)
0.2		−1				(6.3.27)
0.7	−1		−1			(6.3.28)
1.55	−1.5		−1.5		−1	(6.3.29)
−1.25	1.5		1.5		1	(6.3.30)
−1.65	1.5		1.5	1	1	(6.3.31)
1.05	−1.5		−1.5			(6.3.32)
0.155	−0.15		−0.15		−0.1	(6.3.33)
−0.255	0.15		0.45		0.3	(6.3.34)
0						(6.3.35)
0						(6.3.36)
0.8885	−0.735	−0.4	−0.735	−0.49	−0.49	(6.3.37)

reduction in the transport of product 5 from region 2 to region 1. In other words, exports of product 5 from region 2 at a low transport cost rate are "competitive" with exports of product 5 from region 3 at the high transport cost rate.

* Finally we should investigate whether, given this optimal solution of the transshipment problem, a change in the values of the production variables could lead to a different solution of the transshipment problem which would lead to a combined production-transport solution which is cheaper still than the one at hand. In principle this investigation could be made in two ways. One way is to extend the transport problem by introducing the production variables into the simultaneous problem somewhat similarly to that of Section 5.3, and solve this problem along the same lines as the transport problem in this section, starting from the given values of the production variables and the values of the transport variables found in the optimal solution of the transport problem. For this numerical example this is the easiest way, but then in fact the idea of decomposition is given up, together with its possible advantages, mentioned in Section 5.5. Therefore, we leave this way as an exercise to the reader.

The second way of checking the overall optimality of the solution is to go on in the same way as with the decomposition method of Section 6.1. That is, we go back to the regional planning problem, but we change the cost coefficients of the production variables of the heavy sectors by means of the values of the simplex multipliers of the transport problem, and we add restrictions to the problem which ascertain that in the transport problem the same variables continue to constitute the optimal solution. In this case, however, it is somewhat more difficult to find these added restrictions, as we will see below.

We will use this second way as an illustration of checking the overall optimality of the solution, in order to remain in line with the arguments in favour of the decomposition method set out in Section 5.5. The main difficulty is to formulate the additional restrictions on the (production) variables of the regional planning problem which should ascertain that in the transport problem the same variables remain in the optimal solution so that the values of the simplex multipliers in the transport problem remain the same. However, these values of the simplex multipliers may be different for different cases of combinations of dominant directions along the routes. So one should in principle do the exercise for all the 8 different cases for which the same solution remains the optimal one. Each of these cases is characterized by a specific combination of dominant directions along the routes ${}_1(2\rightarrow 3)$, ${}_2(2\rightarrow 3)$ and $(3\rightarrow W)$. However, which direction is considered dominant matters only for routes $(3\rightarrow W)$. We will only work out one case

in detail, namely the case considered in lines 5-8 of Table 6.16. The other cases can then be dealt with more easily.

Once we know which variables are the basic ones in the optimal solution of the transshipment problem, the values of these variables can be determined from the equations of the transshipment problem, all non-basic variables having the value = 0. We will call these equations with the values of the non-basic variables = 0, the equations in contracted form. Solving these new equations means expressing the basic variables in terms of the constants of the transshipment problem, i.e. in terms of the production variables. Using these expressions we can formulate the conditions in the production variables in order that these basic variables remain nonnegative. These conditions are the ones we are looking for. Table 6.17 gives these contracted equations with the detached coefficients and their solution by means of the Gaussian elimination procedure.

From equations (6.3.26)–(6.3.37) the extra conditions for the production variables can be derived, by setting the expressions in the production variables of the solution ≥ 0, and writing them as equations by adding the slack variables ${}^{1}s, \ldots, {}^{12}s$. E.g. from (6.3.26) follows:

$$ {}^{1}y^{4} + {}^{1}s = 0.4 \tag{6.3.38} $$

Only (6.3.35) and (6.3.36) do not lead to extra restrictions in the production variables, as the right hand side expressions (= 0) are independent of the production variables.

Table 6.18 gives the coefficients of the regional planning problem. The extra restrictions derived from (6.3.26)-(6.3.37) have been added, and the cost coefficients of the production variables have been changed by deducting from them the corresponding values of the simplex multipliers of Table 6.16, column 3, in accordance with the same procedure as in Section 6.1. Notice that this time the redundant equation stating that the total income increase from the international sectors is fixed, cf. (6.2.6) is dropped; but it could have been included as well.

The fact that row 5 in Table 6.18 has no negative figures shows that for this case the solution is optimal. The investigation of the other case, corresponding to row 9 in Table 6.16, is easy now. The equations (6.3.14) ... (6.3.25) remain the same, as changing the sign of ${}^{3w}s$ in the last equation of the transshipment problem does not change these equations. The only change is in the cost coefficients of the transshipment problem, and, there-

The extended t

		Production variables							
		${}^{1}y^{3}$	${}^{2}y^{3}$	${}^{3}y^{3}$	${}^{1}y^{4}$	${}^{2}y^{4}$	${}^{3}y^{4}$	${}^{2}y^{5}$	^{3}.
Coefficients of the added restrictions					1				
						1			
					1		1		
					1.5		1.5		
					1.5		1.5		1
					1.5		1.5	1	1
					1.5		1.5		
					0.15		0.15		0
					0.15		0.45		0
					0.735	0.6	0.735	0.49	0
Coefficients of the original equations		1	1	1					
		1			1				
			1			1		1	
				1			1		1
Row 1	Original cost coefficients	3.1	2.8	3.0	3.5	3.0	3.3	2.3	
2	Used u_r first case				0.91	0.6	0.41	−0.01	−
3	New cost coefficients	3.1	2.8	3.0	2.59	2.4	2.89	2.31	
4	Solution checked[a)]	0.1	0.5	—	0.4	—	—	0.3	
5	Direct minus indirect cost coefficients[a)]	x	x	0.15	x	0.09	0.54	x	
6	Used u_r second case				0.91	0.6	0.41	−0.01	−
7	New cost coefficients	3.1	2.8	3.0	2.59	2.4	2.89	2.31	
8	Solution checked[a)]	0.1	0.5	—	0.4	—	—	0.3	
9	Direct minus indirect cost coefficients[a)]	x	x	0.15	x	0.09	0.54	x	

[a)] For explanation of the symbols − and x, see Table 6.1.

ABLE 6.18

of the regional planning problem

Slack variables										Con-stants	Simplex multipliers: Symbols	Corresponding to row 3	Corresponding to row 7
^1s	^2s	^3s	^4s	^5s	^6s	^7s	^8s	^9s	^{10}s				
1										0.4	^1w	−0.02	−0.02
	1									0.2	^2w	0	0
		1								0.7	^3w	0	0
			1							1.55	^4w	0	0
				−1						1.25	^5w	0	0
					−1					1.65	^6w	0	0
						1				1.05	^7w	0	0
							1			0.155	^8w	0	0
								−1		0.255	^9w	0	0
									1	0.8885	^{10}w	0	0
										0.6	v_1	0.49	0.49
										0.5	v_2	2.61	2.61
										0.8	v_3	2.31	2.31
										0.8	v_4	2.36	2.36
0	0	0	0	0	0	0	0	0	0				
0	0	0	0	0	0	0	0	0	0				
−0.2	0.3	0.15	0.15	0.05	0.45	0.015	0.045	0.0615			Cost of solution: Row 4	5.327	
0.0	x	x	x	x	x	x	x	x	x				
0	0	0	0	0	0	0	0	0	0				
−0.2	0.3	0.15	0.15	0.05	0.45	0.015	0.045	0.0615			Row 8	5.327	
0.0	x	x	x	x	x	x	x	x	x				

fore, in the corresponding multipliers in column 4 of Table 6.16. The values of the relevant multipliers in this example remain the same in this new case. This, however, seems to be accidental. We check the solution again for these values of the simplex multipliers of the transport problem (row 6-9 of Table 6.18). As row 9 again shows only nonnegative figures the solution is also optimal for that case. In the other cases neither the contracted equations nor the u_r change, so they need not to be investigated. However, as one of the slack variables (${}^1s$) equals zero, we should apply a check like described in Section 6.1, Table 6.3 and 6.4 to see whether a small increase (by δ) of ${}^1y^4$ could decrease the total costs. For this purpose in Table 6.18 figures like the ${}^r\bar{w}^4_{A_1}$ in Table 6.3 should be calculated in order to determine the change in production costs, and in Table 6.16 a new optimal solution and new u_r should be calculated if ${}^1y^4 = 0.4+\delta$ and ${}^2y^5 = 0.3-\delta$ in order to determine the change in transport costs. (The change in ${}^2y^5$ follows from applying a θ-method change in Table 6.18 after the coefficient of ${}^1s$ has changed from 1 to -1). The computations are left as an exercise to the reader. The result is that with a small increase by δ of ${}^1y^4$ the production costs would increase by 0.9 δ, and the transport costs would decrease by 0.51 δ, which does not give an improvement. *Therefore the solution* (*Table* 6.18, *line* 4 *or* 8) *is the overall optimal one.*

The calculation in advance of an upper bound to the gains to be derived from taking account of explicit transport costs (the second approximation) is less easy for the case of this section, than in the case of Sections 6.1 and 6.2. Probably the best thing to do is to use the calculation for the latter cases also for the case of this section. *

Our conclusion of this discussion of the introduction of cheaper return flows into the problem is that the computational difficulties increase considerably in comparison with the problem without cheaper return flows, although an overall solution can still be reached along the lines indicated in this section. In fact, the difference in the solutions on line 3 and 7 of Table 6.16, showed some advantage could be taken from the explicit introduction of non-dominant cost coefficients, but the gain in total costs was only $0.505-0.503 = 0.002$ units. More computational experience and practical judgment would be necessary in order to decide whether the consideration that return flows are cheaper makes the final solution in actual cases so much better that it is worth the computational complications.