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Selective Competition

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Abstract

We consider a dynamic (differential) game with three players competing against each other. Each period each player can allocate his resources so as to direct his competition towards particular rivals – we call such competition selective. The setting can be applied to a wide variety of cases: competition between firms, competition between political parties, warfare. We show that if the players are myopic, the weaker players eventually loose the game to their strongest rival. Vice versa, if the players value their future payoffs high enough, each player concentrates more on fighting his strongest opponent. Consequently, the weaker players grow stronger, the strongest player grows weaker and eventually all the players converge and remain in the game.

Key Words: selective competition, dynamic oligopolies, differential games.

JEL Classification: C73, D43.

1 Introduction

Competition lies at the heart of economics and so has been extensively studied. However, there is a class of competition mechanisms that is abound in practice but has not yet been addressed in the literature – those are mechanisms providing a competitor with an ability to target his rivals on individual basis. We group such mechanisms under a common title of selective competition. The examples to follow will illustrate the definition.

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In the classical models of competition (Cournot, Bertrand) a firm cannot target a specific rival. For example, by lowering its price the firm undermines all of its rivals in the market, not a specific one. Recent investigations, however, have brought up the importance of product variety, competition in product characteristics, spatial competition, multiproduct firms (see, for example, surveys by Lancaster, 1990; Gabszewicz and Thisse, 1992; Bailey and Friedlaender, 1982). All these factors provide the possibility to target a particular rival. The examples are: a company developing a product that is closer in a characteristic space or in a location space to that of a particular competitor or a multinational corporation investing relatively more in a market shared with a particular competitor. Unethical practices provide more examples, like launching a fabricated lawsuit against a particular competitor.

Selective competition can be found in other areas of economics as well, the examples include: competition between political parties and their support for specific programs, or international trade and specific trade barriers. Finally, a warfare stays as an ultimate example of selective competition.

This paper shows there are new economics insights to be found in studying the area of selective competition. In case of selective competition there is a strategic consideration that does not arise in case of nonselective competition: a player (a firm, a political party, an army) can influence the balance of powers among his rivals by choosing whom he competes against; in turn, that determines how much this player wins or looses competing with those rivals in the periods to come. In particular, one may intuitively expect the weaker players to direct more resources towards fighting the strongest player rather than fighting each other. Indeed, otherwise the strongest player stands a good chance of forcing the weaker ones out of the game (as time goes by).

As we have said, to the best of our knowledge we are unaware on any research specifically dedicated to the issue of selective competition. For example, any model of selective competition should have three players or more – otherwise the competition cannot be selective, and it should be dynamic – the aforementioned strategic consideration can be studied only in a dynamic setting. The closest matching strand of the literature then is that of dynamic oligopoly models. Though many dynamic competition mechanism are studied: inventories (Kirman and Sobel, 1974), number of adopters and cost learning (Dockner and Jørgensen, 1988), varying levels of exploration of profit opportunities (Ericson and Pakes, 1995), etc., the selective competition mechanisms are not emphasised.

With this paper we set out to raise the question of selective competition and to formally check the intuition that weaker players have incentives to coordinate against their strongest rival.

We develop a model of selective competition that does not focus on case-specific aspects of competition but rather focuses on the general ability to compete selectively. Each player in the model is characterised by his relative power – the amount of resources this player has. The power of a player can be distributed to fight each of the player's rivals. We consider marginal returns of competing against a particular rival to be diminishing, therefore each player chooses to fight both of his opponents. We show that myopic players prefer to fight more with their weakest opponent. Consequently, the strongest player grows in his power and eventually outcompetes the weaker players. Vice versa, we show that if the discount rate is sufficiently small (the future payoffs are valued sufficiently high) and if no player is too strong to start with, then the weaker players concentrate more on fighting their strongest opponent. Consequently, the strongest player becomes weaker over time and all the players converge in their powers and stay in the game.

This latter result may look as a tacit collusion between the weaker players against the strongest one. It is, however, conceptually different. Whereas collusive behaviour in repeated games is sustained by a credible threat that other players are to punish the deviating player, in our game we look for a Markov perfect equilibrium, hence the strategies do not depend upon past actions and so there can be no strategies with punishment. In our case it is the dynamical structure of the game that "punishes" the weaker players: if they are to prefer fighting each other for the sake of immediate gains rather than fighting the strongest player, then the strongest player will grow in his power and will, eventually, outcompete his rivals. If this threat of loosing the game is large enough, then the weaker players will fight more against the strongest player and their behaviour will be alike to that of tacit collusion.

There are two related games that have been studied in the literature: colonel Blotto games (see, e.g., Roberson, 2006) and truel games (Kilgour, 1971).

A colonel Blotto game is a game between two players that share several battlefields. Each player divides his army between the battlefields, a battlefield is won by the larger force, a player who wins more battlefields wins the game. The game of selective competition that we study can be viewed as a game of three players and three battlefields, where each pair of players share a battlefield and where there is no battlefield that is shared by all the players. Then the similarity of our game to colonel Blotto games is the ability of the players to choose how to split their powers against their opponents. The main differences are: 1) there are three players in our game, 2) our game is dynamic – the winner is not realised at once, rather the winner of this round becomes stronger and the game continues.

A truel game is an extension of a duel game. There are three players, each with a gun. Each round each player chooses whom to shoot and kills his opponent with a certain chance that depends upon his skill; if two or more players are still alive the game continues. Like in our game, there is a choice of the opponent, there are dynamics and there is a consideration that killing a certain player influences your chance of survival in the rounds to come. The main differences are: 1) in our game the payoff of the game is a discounted sum of the payoffs in each round, so each round is valuable, whereas in a truel game the payoff is 1 if the player survives and 0 otherwise; 2) in our game if the player is "shot", he does not die at once but rather becomes relatively weaker; 3) in a truel game a player chooses to fight either one opponent or the other, whereas in our game a player chooses *how much* to fight one opponent and *how much* to fight the other (a continuous choice).

So, our game has structural similarities to those of colonel Blotto and truel games, but we think the named differences make our model more appropriate for the aforementioned examples of selective competition.

The rest of the paper is organised as follows. First, keeping in mind the examples of selective competition between firms, we set up the model. Second, we consider a simple case of myopic players and show that only the strongest survive as time goes by. Third, we show that if the discount factor is sufficiently small and if no player is too strong, then there is an equilibrium where all the players converge in power and remain in the game. We conclude briefly.

2 Setup

There are three players, 1, 2, and 3 – firms, political parties, armies, etc. The players are involved in a dynamic competitive game. Each player i at time $t \in [0, \infty)$ is characterised by a state variable $x_i(t)$ being the amount of resources he can use in competition with his rivals at time t. We call this variable the "power" of player i. It can be the market share of a firm, the amount of personnel the firm has, how large and how good its credit resources are or how well the managers are connected; it can be the electoral base or the number of seats in parliament; it can be the number of military units.

For convenience, let $x = (x_1, x_2, x_3)$. The initial state is normalised so that $\sum_i x_i(0) = 1$ (later on we will see that $\sum_i x_i(t) = 1$ for any t) and also no player is too strong to start with. Formally, $x_0 \in X$, where

$$X = \left\{ x \in \mathbb{R}^3 \,\middle|\, \sum_i x_i = 1,\ x_i < \frac{2}{5}\ \forall i \right\}$$

Each player can fight selectively against his rivals. y_{ij} denotes the amount of power player i uses to fight against player j. We consider Markov strategies, i.e. the actions of the players are conditioned upon the state of the game, so y_{ij} are functions of x.

For convenience, let $y_1 = (y_{12}, y_{13})$, $y_2 = (y_{21}, y_{23})$, $y_3 = (y_{31}, y_{32})$ and $y = (y_1, y_2, y_3)$.

Each player uses all his power to fight his opponents[1] and what amount he uses can not be negative, therefore

$$\begin{gathered} y_i \in Y_i(x) \\ Y_i(x) = \left\{ y_i \,\middle|\, y_{ij} \geq 0, \sum\nolimits_j y_{ij} = x_i \right\} \end{gathered} \tag{1}$$

Every "battle" between players i and j has two consequences: 1) the players receive instantaneous payoffs from the battle, 2) their powers change. The instantaneous payoffs can be, for example: profits in case of firms, or the salary and the bonus payments of a top manager; political contributions in case of political parties; access to natural resources in case of warfare for economic reasons.

The instantaneous payoffs for player i when he is fighting player j are given by $\varphi(y_{ij}, y_{ji})$, where 1) $\varphi(0, y_{ji}) = 0$, i.e. if a player doesn't fight, his instantaneous payoffs are always zero; 2) $\varphi(y_{ij}, y_{ji})$ is strictly increasing in y_{ij} and for $y_{ij} > 0$ it is strictly decreasing in y_{ji}; 3) $\varphi(y_{ij}, y_{ji})$ is strictly concave in y_{ij} (decreasing marginal returns).

To have an analytical solution to our model we take a quadratic specification for φ. A general quadratic specification that would also satisfy our assumptions on the relevant domain ($0 \leq y_{ij} \leq 1$, $0 \leq y_{ji} \leq 1$) is

$$\varphi(y_{ij}, y_{ji}) = (a - b_1 y_{ij} - b_2 y_{ji}) y_{ij}$$

where $b_1 > 0$, $b_2 > 0$ and $a \geq 2b_1 + b_2$. To simplify matters we take $b_1 = b_2 = b$, so

$$\varphi(y_{ij}, y_{ji}) = (a - b(y_{ij} + y_{ji})) y_{ij}$$

[1] In our model there are no alternative costs associated with fighting, therefore it is always optimal to use for fighting all the power.

where $b > 0$ and $a \geq 3b$.

Let $\pi_i(y)$ denote the sum of all the instantaneous payoffs that player i receives from fighting his opponents with $\pi_i(y)$. We have

$$\pi_i(y) = \sum_{j \neq i} \varphi(y_{ij}, y_{ji})$$

Per se, the power does not enter the instantaneous payoff function. However, becoming more powerful will yield higher payoffs as more power can be used competing with the rivals thus improving the outcomes of that competition.

If $x(t)$ reaches the boundary of X, the game ends. T denotes the ending time. Formally,

$$T = \inf\{t \geq 0 \,|\, x(t) \notin X\}$$

If the game never ends, then $T = \infty$.

If the game ends, each player i receives a terminal payoffs S_i, the strongest player wins, the weaker players loose:

$$S_i(x) = \begin{cases} M \text{ if } x_i > x_j \ \forall j \neq i \\ 0 \text{ otherwise} \end{cases}$$

where $M > 0$. If the game ends and two of the players are equally strong, they both loose (this assumption is not important for the results).

The rationale for ending the game if the boundary of X is approached is as follows. If one of the players becomes sufficiently strong, it is reasonable to expect him to eventually outcompete his rivals. To simplify the game we stop it at this time and assign a strictly positive payoff of M to the strongest player and a zero payoff to the weaker players.[2] As we will see later on, the results do not depend upon the size of M as long as M is positive, still it is helpful to think of it as of a payoff that is higher than what the strongest player could have got if he was to continue the competition. Loosing, on the other hand, means that a player quits the game (a firm looses its markets, etc) and the stream of the instantaneous payoffs ends – so loosing yields zero payoff.

[2] From $x \in X$ it follows that $x_i > \frac{1}{5}$, so a player i dies if $x_i(t)$ reaches $\frac{1}{5}$. An alternative specification is to say that a player i dies, e.g. a firm goes bankrupt, a political party dissolves, if $x_i(t)$ reaches 0 at some t. Such a specification seems to yield similar results, but requires a numerical solution (see the discussion at the end of section 4.2), so we have chosen against this latter specification.

The payoff for the whole game is the discounted stream of the instantaneous payoffs plus the discounted terminal payoff, so the payoff for player i is

$$U_i = \int_0^T e^{-\delta t} \pi_i(y(x(t)))dt + e^{-\delta T} S_i(x(T)) \tag{2}$$

where δ is a discount factor.

If player i fights player j more than player j fights player i ($y_{ij} > y_{ji}$), then player i becomes more powerful, while player j becomes less powerful. We call such dynamics a power shift. For example, if a company invests more in a market than its rival does, its customer base shall increase relatively to that of the rival; if a political party supports a certain program more than its rival does, its electoral base shall increase relatively to that of the rivalling party, etc. We assume these dynamics to be linear in y:

$$\begin{aligned} \dot{x}_i(t) &= f_i(y(x(t))) \\ f_i(y) &= \sum_{j \neq i} (y_{ij} - y_{ji})\, k \end{aligned} \tag{3}$$

where $k > 0$ stands for the power shift intensity.

We note here that from $\sum_i x_i(0) = 1$ and from (3) it follows that $\sum_i x_i(t) = 1$ for all t.

So, our setup is a differential game with simultaneous play (see Dockner et al., 2000) and we restrict our attention to Markov strategies. The strategies are functions $y(x)$ satisfying (1), the state variables x evolve according to (3) and the objective functions are given by (2).

3 Example: Cournot Competition

In the previous section we did not consider specific cases of selective competition, rather we argued for a setup that can suit cases ranging from spatial competition among firms to warfare. In this section we show, with a particular example, that our setup can also stem from selective Cournot competition with binding capacity constraints.

Suppose there are three universities and three areas (e.g. economics, management and sociology). Suppose that each university is active in two areas only – has two respective departments – and in each area there are only two active universities. Each university i is characterised by the number of professors, x_i, which the university can split between its departments, $\sum_i x_i = 1$. Let y_{ij} denote the number of professors of university i that are in the same area as professors of university j, $\sum_{j \neq i} y_{ij} = x_i$.

The amount of education a university department provides is proportional to the number of professors employed, we take the proportionality coefficient to be one.[3] For example, university 1 employs y_{12} professors in economics and y_{13} professors in sociology, so the supply of education by this university is y_{12} and y_{13} respectively. As for the demand, suppose it is the same in all the areas and is given by $Y = \frac{1}{b}(a - P)$, where P is the admission price and Y is the total amount of education demanded.

Suppose the universities compete a la Cournot and let us neglect the costs for simplicity. Then the profits of university i from an area shared with university j are given by

$$\varphi(y_{ij}, y_{ji}) = P(y_{ij} + y_{ji}) \cdot y_{ij} = (a - b(y_{ij} + y_{ji}))y_{ij}$$

We additionally suppose that the demand for education is high compared to the number of professors to the extend that $a \geq 3b$ (in general terms, the capacity constraints are binding).

Finally suppose that as time goes by, the professors of different universities interact with each other within the same areas and tend to change their appointments toward the larger departments (for reasons of richer environment, better specialisation, etc). If we take these dynamics to be linear, then we get

$$\dot{x}_i = \sum_{j \neq i} (y_{ij} - y_{ji})\, k$$

So, we have presented an example of selective Cournot competition that yields the same game structure, same instantaneous payoffs and same dynamics as in our model. If we further restrict the dynamics to X (a university has to close down if it becomes too small), then this example yields precisely our model.

Real life situations of selective Cournot competition would be more complex, of course, but a simple example of three players is sufficient to study the implications of an ability to compete selectively.

4 Analysis

We consider two cases: a case with myopic players and a general case. In both cases we solve our game for a Markov perfect equilibrium (MPE) and analyse the resulting equilibrium dynamics.

[3]We are free to measure education in any units.

In what follows we denote the best response strategies with $\tilde{y}$ and the equilibrium strategies with $\hat{y}$.

4.1 Myopic Players

The players are myopic if they focus on the current gains only. For a myopic player i the payoff of the game at time t is

$$U_i(t) = \pi_i(y(x(t)))$$

The dynamics of the myopic case are summarised by the following proposition (we limit our attention to a general initial state, when one of the players is strictly stronger than the rest).

Proposition 1. *Suppose, without a loss of generality, that $x_1(0) > x_2(0)$, $x_1(0) > x_3(0)$. Then there exists a unique MPE. Moreover, the equilibrium dynamics are such that the game ends and the strongest player wins, i.e. $T < \infty$ and $x_1(T) > x_2(T)$, $x_1(T) > x_3(T)$*

Proof. Maximising $U_i(t)$ in (y_{ij}, y_{ik}) w.r.t. $y_{ij} + y_{ik} = x_i$ gives a unique best response

$$\tilde{y}_{ij}(x) = \frac{x_i}{2} + \frac{y_{ki}(x) - y_{ji}(x)}{4}$$

(a boundary solution is also possible but it is straightforward to check that it is never attained for $x \in X$).

Given the above best response functions we can solve for a unique equilibrium point. We get

$$\hat{y}_{ij}(x) = \frac{x_i}{2} + \frac{x_k - x_j}{10} \tag{4}$$

As we are considering Markov strategies, (4) constitutes a unique Markov perfect equilibrium.

Plugging (4) into (3) and using $x_1 + x_2 + x_3 = 1$ gives

$$\dot{x}_i(t) = \frac{9k}{5}\left(x_i(t) - \frac{1}{3}\right)$$

As $x \in X$, $x_1(0) > x_2(0)$ and $x_1(0) > x_3(0)$, we have that $x_1(0) > 1/3$ and $x_{2,3} < 1/3$. Consequently, $x_1(t)$ grows over time and

$$\dot{x}_1(t) \geq \frac{9k}{5}\left(x_i(0) - \frac{1}{3}\right) > 0$$

while $x_2(t)$ and $x_3(t)$ decline. Since $\dot{x}_1(t)$ is bounded from below, $x(t)$ eventually reaches the boundary of X, the game ends and $x_1(T) > x_i(T)$ for $i \neq 1$. □

This case illustrates the intuition that if the players are myopic and pursue only their instantaneous payoffs then they may have no incentives to fight more against the stronger player. As a consequence, the weaker players loose.

4.2 Forward-looking Players

If the players are myopic, then the weaker players loose in the equilibrium. The question is, if the players are sufficiently non myopic, i.e. if δ is sufficiently small so that the players value their future profits high enough, will it be the case the dynamics are reversed? We give a positive answer to this question.

Proposition 2. *If $\delta < \frac{4k}{3}$, then there exists an MPE such that for all i $x_i(t) \to \frac{1}{3}$ as $t \to \infty$.*

Proof. We prove the proposition by construction: we state an equilibrium candidate possesing the property that $x_i(t) \to \frac{1}{3}$ and then check that it is an equilibrium indeed. Let

$$\hat{y}_{ij}(x) = \frac{x_i + c(x_k - x_j)}{2} \tag{5}$$

$$c = \frac{1}{18}\left(5\frac{\delta}{k} - 14 - \sqrt{\left(25\frac{\delta}{k} - 76\right)\left(\frac{\delta}{k} - 4\right)}\right) \tag{6}$$

From $\sum_i x_i(t) = 1$, from (3) and from (5) it follows that

$$\dot{x}_i(t) = \frac{3k(c+1)}{2}\left(x_i(t) - \frac{1}{3}\right) \tag{7}$$

If $\delta < \frac{4k}{3}$, then from (6) it follows that $c < -1$. Consequently, from (7) it follows that $x_i(t) \to \frac{1}{3}$ as $t \to \infty$.

Let us now prove that (5) constitute an MPE. To do so we need to show that $\hat{y}_i$ is a best response to $\hat{y}_j$ and $\hat{y}_k$. All the possible strategies of player i can be divided into two classes: those strategies that eventually end the game ($T < \infty$) – let it be class $\mathcal{B}$, and those that do not ($T = \infty$) – class $\mathcal{A}$. We proceed as follows. First, we restrict the strategies of player i to class $\mathcal{A}$

and show that in this class the strategy $\hat{y}_i$, as given by (5), is indeed a best response strategy. Second, we extend this result to $\mathcal{A} \cup \mathcal{B}$.

So, let the strategies of player i be restricted to class $\mathcal{A}$. Let us compute the value function V of player i if every player follows strategy $\hat{y}$ and if the game starts at $x(0) = x$. Solving (7) gives

$$x_i(t) = \left(x_i - \frac{1}{3}\right) e^{3k(c+1)/2 \cdot t} + \frac{1}{3}$$

Therefore (also using $x_1 + x_2 + x_3 = 1$) we have[4]

$$V_i(x) = \int_0^\infty e^{-\delta t} \pi_i(\hat{y}(x(t)))dt = \\ c_1 \left(x_i - \frac{1}{3}\right)^2 + c_2 \left(x_i - \frac{1}{3}\right) + c_3 + c_4(x_k - x_j)^2 \quad (8)$$

where

$$\begin{cases} c_1 = \dfrac{b(3c-1)}{4(\delta - 3k(c+1))} \\ c_2 = \dfrac{12a + b(3c-5)}{6(2\delta - 3k(c+1))} \\ c_3 = \dfrac{3a-b}{9\delta} \\ c_4 = -\dfrac{bc(3c-1)}{4(\delta - 3k(c+1))} \end{cases} \quad (9)$$

Consider now the Hamilton-Jacobi-Bellman equations:

$$\hat{y}_i(x) \in \arg\max_{y_i \in Y_i(x)} \left(\pi_i(y_i, \hat{y}_{-i}(x)) + \sum_j \frac{\partial V_i(x)}{\partial x_j} f_j(y_i, \hat{y}_{-i}(x)) \right) \quad (10)$$

$$\delta V_i(x) = \pi_i(\hat{y}(x)) + \sum_j \frac{\partial V_i(x)}{\partial x_j} f_j(\hat{y}(x)) \quad (11)$$

If these equations are satisfied for all $x \in X$, then $\hat{y}_i$ is a best response to $\hat{y}_{-i}$ (when the strategies of player i are limited to class $\mathcal{A}$, so that $x(t)$ never leaves X) – see Dockner et al. (2000, chapters 3 and 4).

Equation (11) is automatically satisfied by the way V is constructed. We now check equation (10). Let

$$g(y_i, x) = \pi_i(y_i, \hat{y}_{-i}(x)) + \sum_j \frac{\partial V_i(x)}{\partial x_j} f_j(y_i, \hat{y}_{-i}(x))$$

[4] See the appendix for the details of the derivation.

Using (5), (8) and the definitions for π_i, f_i to expand $g(y_i, x)$ and maximising the result w.r.t. $y_{ij} + y_{ik} = x_i$ gives

$$\tilde{y}_{ij}(x) = \frac{x_i + d(x_k - x_j)}{2} \tag{12}$$

$$d = \frac{1-c}{4} - \frac{ck(3c-1)}{2(\delta - 3k(c+1))} \tag{13}$$

Strategy $\hat{y}_i$ is a best response strategy if (5) coincides with (12), i.e. if $c = d$. We check it now. Using (13) to expand $c = d$ and simplifying gives

$$18c^2 + \left(28 - 10\frac{\delta}{k}\right)c + \left(2\frac{\delta}{k} - 6\right) = 0$$

It is straightforward to check that c as defined in (6) is a solution to the above equation. Hence $c = d$ and $\hat{y}_i$ is a best response.

In principle, it is possible that a corner solution is obtained when maximising $g(y_i, x)$, however it is never a case for $x \in X$.

Consider now an arbitrary strategy $\grave{y}_i(x) \in \mathcal{B}$. With a class $\mathcal{B}$ strategy the game ends at some T (that is determined by $y_i(x)$). Let

$$y_i^n(x,t) = \begin{cases} \grave{y}_i(x) & \text{if} \quad t \le T - \epsilon_n \\ \hat{y}_i(x) & \text{if} \quad t > T - \epsilon_n \end{cases}$$

where ϵ_n is a sequence, $\epsilon_n > 0$ and $\lim_{n\to\infty} \epsilon_n = 0$. This strategy $y_i^n(x,t)$ belongs to $\mathcal{A}$, therefore it gives the same or a lower payoff than the best response strategy $\hat{y}_i(x)$, i.e.

$$\int_0^\infty e^{-\delta t}\pi_i(\hat{y}(x(t)))dt \ge \int_0^\infty e^{-\delta t}\pi_i(y^n(x(t)))dt = \int_0^{T-\epsilon_n} e^{-\delta t}\pi_i(\grave{y}(x(t)))dt + \int_{T-\epsilon_n}^\infty e^{-\delta t}\pi_i(\hat{y}(x(t)))dt$$

Taking the limit as $n \to \infty$ gives

$$\int_0^\infty e^{-\delta t}\pi_i(\hat{y}(x(t)))dt \ge \int_0^T e^{-\delta t}\pi_i(\grave{y}(x(t)))dt + V_i(x(T))$$

On the other hand, the payoff from employing strategy $\grave{y}_i(x)$ is

$$\int_0^T e^{-\delta t}\pi_i(\grave{y}(x(t)))dt + S_i(x(T))$$

Therefore, if $S_i(x(T)) \le V_i(x(T))$, then $\hat{y}_i$ is the optimal strategy in class $\mathcal{A} \cup \mathcal{B}$ as well.

As $x(0) \in X$, then from the definition of X it follows that $x_i(0) < \frac{2}{5}$. Whatever the strategy $\grave{y}(x)$ is, from (3), from (5) and from $x_1 + x_2 + x_3 = 1$ it follows that

$$\dot{x}_i(t) \le \frac{3k(c+1)}{2}\left(x_i(t) - \frac{1}{3}\right)$$

Consequently, $x(T) < \frac{2}{5}$. At the same time, $x(T)$ belongs to the boundary of X. So, if it was true that $x_i(T) > x_j(T)$ for all $j \ne i$, then it should have been that $x_i(T) = \frac{2}{5}$. As it is not, we have that $x_i(T) \le x_j(T)$ for at least some $j \ne i$. Therefore $S_i(x(T)) = 0$. But from $\varphi(\hat{y}_{ij}(x), \hat{y}_{ji}(x)) > 0$ it follows that $V_i(x(T)) > 0$.

So, $S_i(x(T)) \le V_i(x(T))$ and $\hat{y}_i(x)$ is a best response strategy when all possible strategies are considered (class $\mathcal{A} \cup \mathcal{B}$).

In words, a weaker player can choose a strategy to reach the boundary of X, but doing so is not optimal. As for the strongest player, he may prefer to reach the boundary if he is still the strongest player when he does so, but he cannot achieve such dynamics if his rivals are playing the equilibrium strategies. $\square$

So, for a sufficiently small δ there is an equilibrium such that the strongest player declines in his power while the weaker players improve in their powers. Consequently, all the players converge. A notable property of this equilibrium is that each player fights his strongest opponent more.

5 Concluding Remarks

If there are ways to compete selectively, then for a sufficiently small δ everyone competes more against the stronger rival, consequently the players converge in their power, and oligopolistic competition is sustainable – it does not boil out to a monopoly.

We have analysed but a basic setup of selective competition and two possible extensions are worth mentioning – stochastic dynamics and multiple players. Arguably, both extensions would bring the model closer to judging real life situations as outcomes of competition are scarcely deterministic and many examples we talked about (e.g., multiproduct firms) often involve more than three players. The main question here will stay the same: is it more difficult or more easy for the weaker rivals to tacitly coordinate against the strongest one given stochastic dynamics or given multiple (more than three) players in the game? Answers to this question can help explain and predict the degree of convergence and the number of players in relevant situations.

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Appendix

Here we give a detailed derivation of (8), (9).

Let $z_i = x_i - \frac{1}{3}$. As $x_1 + x_2 + x_3 = 1$, so $z_1 + z_2 + z_3 = 0$. Next we derive $\pi_i(\hat{y}(z))$.

First,

$$\hat{y}_{ij}(x) = \frac{x_i + c(x_k - x_j)}{2} = \frac{z_i + c(z_k - z_j)}{2} + \frac{1}{6}$$

Then (using $\sum_i z_i = 0$ where appropriate)

$$\begin{aligned}
\pi_i(\hat{y}(z)) = (a - b(\hat{y}_{ij} + \hat{y}_{ji}))\hat{y}_{ij} + (a - b(\hat{y}_{ik} + \hat{y}_{ki}))\hat{y}_{ik} = \\
\left(a - b\left(\frac{z_i + c(z_k - z_j)}{2} + \frac{z_j + c(z_k - z_i)}{2} + \frac{1}{3}\right)\right) \cdot \\
\left(\frac{z_i + c(z_k - z_j)}{2} + \frac{1}{6}\right) + \\
\left(a - b\left(\frac{z_i + c(z_j - z_k)}{2} + \frac{z_k + c(z_j - z_i)}{2} + \frac{1}{3}\right)\right) \cdot \\
\left(\frac{z_i + c(z_j - z_k)}{2} + \frac{1}{6}\right) = \\
\left(a - \frac{b}{3}\right)\left(z_i + \frac{1}{3}\right) - \frac{b(3c-1)}{2}\left(z_k\left(\frac{z_i + c(z_k - z_j)}{2} + \frac{1}{6}\right) + \right. \\
\left. z_j\left(\frac{z_i + c(z_j - z_k)}{2} + \frac{1}{6}\right)\right) = \\
\frac{b(3c-1)}{4} z_i^2 + \frac{12a + b(3c-5)}{12} z_i + \frac{3a - b}{9} - \frac{bc(3c-1)}{4}(z_k - z_j)^2
\end{aligned}$$

Let $m = 3k(c+1)/2$, then $z_i(t) = z_i e^{mt}$. So,

$$\begin{aligned}
V_i(z) = \int_0^\infty e^{-\delta t} \pi_i(\hat{y}(z(t)))dt = \\
\int_0^\infty e^{-\delta t}\left(\frac{b(3c-1)}{4}\left(z_i e^{mt}\right)^2 + \frac{12a + b(3c-5)}{12} z_i e^{mt} + \right. \\
\left. \frac{3a - b}{9} - \frac{bc(3c-1)}{4}\left(z_k e^{mt} - z_j e^{mt}\right)^2\right) dt = \\
\frac{b(3c-1)}{4}\frac{1}{\delta - 2m} z_i^2 + \frac{12a + b(3c-5)}{12}\frac{1}{\delta - m} z_i + \\
\frac{3a - b}{9}\frac{1}{\delta} - \frac{bc(3c-1)}{4}\frac{1}{\delta - 2m}(z_k - z_j)^2
\end{aligned}$$

Plugging in $z_i = x_i - \frac{1}{3}$ and $m = 3k(c+1)/2$ gives precisely (8) and (9).