Logarithmic residues of analytic Banach algebra valued functions possessing a simply meromorphic inverse

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Econometric Institute Report EI2001-06

Dedicated to T. Ando on the occasion of his seventieth birthday.

Abstract

A logarithmic residue is a contour integral of a logarithmic derivative (left or right) of an analytic Banach algebra valued function. For functions possessing a meromorphic inverse with simple poles only, the logarithmic residues are identified as the sums of idempotents. With the help of this observation, the issue of left versus right logarithmic residues is investigated, both for connected and nonconnected underlying Cauchy domains. Examples are given to elucidate the subject matter.

1 Introduction

Let B be a complex Banach algebra with unit element e. A logarithmic residue in B is a contour integral of a logarithmic derivative of an analytic B–valued function f. There is a left version and there is a right version of this notion. The left version corresponds to the left logarithmic derivative $f'(\lambda)f(\lambda)^{-1}$, the right version to right logarithmic derivative $f(\lambda)^{-1}f'(\lambda)$.

The first to consider integrals of this type – in a vector valued context – was L. Mittenthal [M]. His goal was to generalize the spectral theory of a single Banach algebra element (case $f(\lambda) = \lambda e - b,\ b \in B$). He gave sufficient conditions for a logarithmic residue to be an idempotent. The conditions are very restrictive.

Logarithmic residues also appear in the paper [GS1] by I.C. Gohberg and E.I. Sigal. The setting there is that $B = \mathcal{B}(X)$, the Banach algebra of all bounded operators on a complex Banach space, and f is Fredholm operator valued. For such functions Gohberg and Sigal introduced the concept of algebraic multiplicity. It turns out that the algebraic multiplicity of f with respect to a given contour is equal to the trace of the corresponding (left/right) logarithmic residues (see also [GKL] and [GGK]).

Further progress was made in [BES2-5]. In these papers, logarithmic residues are studied from different angles and perspectives. The problems dealt with are of the following type.

1. *If a logarithmic residue vanishes, does it follows that f takes invertible values inside the (integration) contour*? This question was first posed in [B]. The answer turns out to depend very much on the underlying Banach algebra. For certain important classes it is positive, for other (equally relevant) classes it is negative.

2. *What kind of elements are logarithmic residues*? Here a strong connection with (sums of) idempotents appears (cf. also [BES1]). As for the problem posed under 1, the answer depends on the Banach algebra under consideration too.

3. *How about left versus right logarithmic residues*? In all situations where a definite answer could be obtained, the set of left logarithmic residues coincides with the set of right logarithmic residues. In some situations it was possible to identify the pairs of left and right logarithmic residues associated with one single function f (and the same integration contour).

4. *What can be said about the topological properties of the set of logarithmic residues*? In some cases it was possible, for instance, to identify the connected components of this set.

The present paper is concerned with logarithmic residues of Banach algebra valued functions $f(\lambda)$ possessing a simply meromorphic inverse $f(\lambda)^{-1}$. The latter means that $f(\lambda)^{-1}$ is meromorphic with poles of order one. Attention is paid to problems 2, 3 and (to a lesser extent) 1. An outline of the paper reads as follows.

Section 2 is partly of a preliminary nature in the sense that it contains definitions and notations. In another part it deals with problem 2. For the functions under consideration, the logarithmic residues turn out to coincide with the sums of idempotents. In particular, the set of (left/right) logarithmic residues of B-valued analytic function possessing a simply meromorphic inverse is equal to the set of sums of idempotents in B. In such generality nothing sensible can be said about problem 4 (cf. [BES3-5] and [PT]).

Section 3 is the core of the paper and deals with the issue of left versus right logarithmic residues (problem 3). A distinction is made between the case where the underlying Cauchy domain is connected and where it is not.

Section 4 contains additional remarks and (counter)examples. One of the counterexamples – based on the main result of Section 3 – exhibits a function whose left logarithmic residue vanishes while its right logarithmic residue does not. This example has relevance in connection with problem 1. Another counterexample features several interesting properties. Among other things it shows that logarithmic residues in matrix algebras can fail to belong to the closure of the algebra generated by the idempotents (cf. [BES3-4]).

2 Preliminaries and first results

Throughout this paper , B will be a complex Banach algebra with unit element e. If f is a B-valued function with domain Δ, then f^{-1} stands for the function given by $f^{-1}(\lambda) = f(\lambda)^{-1}$ with domain the set of all $\lambda \in \Delta$ such that $f(\lambda)$ is invertible. If Δ is an open subset of $\mathbb{C}$ and $f : \Delta \to B$ is analytic, then so is f^{-1} on its domain. The derivative of f will be denoted by f'. The *left*, respectively *right, logarithmic derivative* of f is the function given by $f'(\lambda)f^{-1}(\lambda)$, respectively $f^{-1}(\lambda)f'(\lambda)$, with the same domain as f^{-1}.

Logarithmic residues are contour integrals of logarithmic derivatives. To make this notion more precise, we shall employ bounded Cauchy domains (in $\mathbb{C}$) and their (positively oriented) boundaries. For a discussion of these notions, see, for instance [TL].

Let D be a bounded Cauchy domain. The (positively oriented) boundary of D will be denoted by ∂D. We write $\mathcal{A}_\partial(D;B)$ for the set of all B-valued functions f with the following properties: f is defined and analytic on an open neighborhood of the closure $\overline{D}(= D \cup \partial D)$ of D and f takes invertible values on all of ∂D (hence f^{-1} is analytic on a neighborhood of ∂D). For $f \in \mathcal{A}_\partial(D;B)$, one can define

$$LR_{left}(f;D) = \frac{1}{2\pi i}\int_{\partial D} f'(\lambda)f^{-1}(\lambda)d\lambda, \tag{2.1}$$

$$LR_{right}(f;D) = \frac{1}{2\pi i}\int_{\partial D} f^{-1}(\lambda)f'(\lambda)d\lambda. \tag{2.2}$$

The elements of the form (2.1) or (2.2) are called *logarithmic residues* in B. More specifically, we call $LR_{left}(f;D)$ the *left* and $LR_{right}(f;D)$ the *right logarithmic residue of f with respect to D.*

It is convenient to also introduce a local version of these concepts. Given a complex number λ_0, we let $\mathcal{A}(\lambda_0;B)$ be the set of all B-valued functions f with the following properties: f is defined and analytic on an open neighborhood of λ_0 and f takes invertible values on a deleted neighborhood of λ_0. For $f \in \mathcal{A}_\partial(\lambda_0;B)$, one can introduce

$$LR_{left}(f;\lambda_0) = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\rho} f'(\lambda)f^{-1}(\lambda)d\lambda, \tag{2.3}$$

$$LR_{right}(f;\lambda_0) = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\rho} f^{-1}(\lambda)f'(\lambda)d\lambda, \tag{2.4}$$

where ρ is a positive number such that both f and f^{-1} are analytic on an open neighborhood of the punctured closed disc with center λ_0 and radius ρ. The orientation of the integration contour $|\lambda-\lambda_0| = \rho$ is, of course, taken positively, that is counterclockwise. Note that the right hand sides of (2.3) and (2.4) do not depend on the choice of ρ. In fact (2.3), respectively (2.4), is equal to the coefficient of $(\lambda-\lambda_0)^{-1}$ in the Laurent expansion at λ_0 of the left, respectively the right, logarithmic derivative of f at λ_0. Obviously, $LR_{left}(f;\lambda_0)$, respectively $LR_{right}(f;\lambda_0)$, is a left, respectively right, logarithmic residue in the sense of the definitions given in the preceding paragraphs (take for D the disc with radius ρ centered at λ_0). We call $LR_{left}(f;\lambda_0)$ and $LR_{right}(f;\lambda_0)$ the *left* and *right logarithmic residue of f at λ_0*, respectively.

In certain cases, the study of logarithmic residues with respect to bounded Cauchy domains can be reduced to the study of logarithmic residues with respect to points. The typical situation is as follows. Let D be a bounded Cauchy domain, let $f \in \mathcal{A}_\partial(D;B)$ and suppose f takes invertible values on all of D, except in a finite number of distinct points $\lambda_1,\ldots,\lambda_n \in D$. Then

$$LR_{left}(f;D) = \sum_{j=1}^{n} LR_{left}(f;\lambda_j),$$

$$LR_{right}(f;D) = \sum_{j=1}^{n} LR_{right}(f;\lambda_j).$$

This occurs, in particular, when f^{-1} is meromorphic on D, a state of affairs that we will encounter below.

Let $\lambda_0 \in \mathbb{C}$ and let h be a B-valued function defined and analytic on a neigborhood of λ_0. We say that h has a *simple* pole at λ_0 if λ_0 is a pole of h of order one.

Proposition 2.1. *Let $\lambda_0 \in \mathbb{C}$, let $f \in \mathcal{A}(\lambda_0;B)$, and suppose f^{-1} has a simple pole at λ_0. Write p and q for the left and right logarithmic residue of f at λ_0, respectively, i.e.,*

$$p = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\rho} f'(\lambda)f^{-1}(\lambda)d\lambda,$$

$$q = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\rho} f^{-1}(\lambda)f'(\lambda)d\lambda,$$

where ρ is positive and sufficiently small. Then p and q are nonzero idempotents. Also p and q are similar, i.e., $p = s^{-1}qs$ for some invertible $s \in B$.

Proof. Write

$$f(\lambda) = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j a_j, \qquad f'(\lambda) = \sum_{j=1}^{\infty} j(\lambda - \lambda_0)^{j-1} a_j,$$

$$f^{-1}(\lambda) = \frac{1}{\lambda - \lambda_0} b_{-1} + \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j b_j.$$

Then several identities hold. We list the following:

$$a_0 b_{-1} = b_{-1} a_0 = 0,$$

$$a_0 b_0 + a_1 b_{-1} = b_0 a_0 + b_{-1} a_1 = e,$$

$$a_0 b_1 + a_1 b_0 + a_2 b_{-1} = b_1 a_0 + b_0 a_1 + b_{-1} a_2 = 0,$$

$$p = a_1 b_{-1}, \qquad e - p = a_0 b_0,$$

$$q = b_{-1} a_1, \qquad e - q = b_0 a_0.$$

Clearly $p^2 = a_1 b_{-1} a_1 b_{-1} = a_1(e - b_0 a_0) b_{-1} = p - a_1 b_0 a_0 b_{-1} = p$ and, analogously, $q^2 = q$. Put

$$s = b_0 a_0 b_0 + b_{-1} a_1 b_{-1}, \qquad t = a_0 b_0 a_0 + a_1 b_{-1} a_1.$$

Then $st = (e - q)^3 + q^3 + b_0 a_0 b_0 a_1 b_{-1} a_1 = e - b_0 a_0 (b_{-1} a_2 + b_1 a_0) b_{-1} a_1 = e$. Similarly, $ts = e$. So s is invertible with inverse t. Note that $s = b_0(e - p) + b_{-1} p = (e - q) b_0 + q b_{-1}$. Hence $sp = b_{-1} p = b_{-1} a_1 b_{-1} = q b_{-1} = qs$. We conclude that p and q are similar. Finally, if $p = 0$ or $q = 0$, then $p = q = 0$, and it follows that $a_0 b_0 = b_0 a_0 = e$ and $b_{-1} = 0$. This contradicts the assumption that f^{-1} has a pole of order one at λ_0. □

The requirement in Proposition 2.1 that f^{-1} has a simple pole at λ_0 is essential. If f^{-1} has nonsimple poles, then the logarithmic residue need not even belong to the closure of the subalgebra of B generated by the idempotents. An example is given in Section 4.

A B-valued function h is called *simply meromorphic* on an open set $\Delta \subset \mathbb{C}$ if h is meromorphic on Δ and all poles of h are simple.

Theorem 2.2. *Let $x \in B$, where B is a complex Banach algebra, and let D be a bounded Cauchy domain in $\mathbb{C}$. The following statements are equivalent:*

(i) x is a sum of idempotents in B;

(ii) x is the left logarithmic residue with respect to D of a function $f \in \mathcal{A}_\partial(D; B)$ such that f^{-1} is simply meromorphic on D;

(iii) x is the right logarithmic residue with respect to D of a function $f \in \mathcal{A}_\partial(D; B)$ such that f^{-1} is simply meromorphic on D.

Proof. Suppose (ii) holds. Then the number of poles of f^{-1} in D is finite. Thus x is a sum of left logarithmic residues of f at a point. Applying Proposition 2.1, we see that (ii) implies (i). Similarly, (iii) implies (i). It remains to prove that (i) implies (ii) and (iii). Here the complexity of the arguments depends very much on the "shape" of D.

Assume $x = p_1 + \cdots + p_n$, where $p_1, \ldots, p_n$ are idempotents in B. Let $D_1, \ldots, D_k$ be the connected components of D. When $k \geq n$, the situation is rather simple and the argument is just a slight modification of the proof of [BES3], Proposition 2.1. Indeed, choose $\lambda_1, \ldots, \lambda_n$ in $D_1, \ldots, D_n$ respectively, and let $f \in \mathcal{A}_\partial(D; B)$ be such that

$$f(\lambda) = \begin{cases} e - p_j + (\lambda - \lambda_j)p_j, & \lambda \in \overline{D}_j;\ j = 1, \ldots, n, \\ \\ e, & \lambda \in \overline{D}_j;\ j = n+1, \ldots, k. \end{cases}$$

Then one verifies without difficulty that

$$LR_{left}(f; D) = \sum_{j=1}^{n} LR_{left}(f; \lambda_j) = \sum_{j=1}^{n} p_j,$$

$$LR_{right}(f; D) = \sum_{j=1}^{n} LR_{right}(f; \lambda_j) = \sum_{j=1}^{n} p_j.$$

Things are considerably more complicated when $k < n$. Of course it suffices to consider the case $k = 1$ where D itself is connected. This situation is covered by the following theorem which is a slight reformulation of the result obtained by one of the authors (T. Ehrhardt) in [E]. $\square$

Theorem 2.3. *Let $p_1, \ldots, p_n$ be nonzero idempotents in the complex Banach algebra B and let $\lambda_1, \ldots, \lambda_n$ be distinct (but otherwise arbitrary) complex numbers. Then there exists an entire analytic B-valued function f such that f takes invertible values on all of $\mathbb{C}$, except for $\lambda_1, \ldots, \lambda_n$, where f^{-1} has simple poles, while in addition,*

$$LR_{left}(f; \lambda_j) = LR_{right}(f; \lambda_j) = p_j, \qquad j = 1, \ldots, n.$$

For completeness, we mention that the function f constructed in [E] is a product of $3n$ factors, each of them is a function of the form $e - p + \varphi(\lambda)p$ where p is one of the given idempotents and φ is an entire scalar function.

3 Left versus right logarithmic residues

Next we take on the issue of left versus right logarithmic residues. We begin with a result which holds for arbitrary bounded (so possibly nonconnected) Cauchy domains.

Theorem 3.1. *Let x and y be elements in the complex Banach algebra B and let n be a nonnegative integer. The following statements are equivalent:*

(i) There exists a bounded Cauchy domain D and a function $f \in \mathcal{A}_\partial(D; B)$ such that f^{-1} is simply meromorphic on D , f^{-1} has exactly n simple poles in D and

$$x = LR_{left}(f; D) = \frac{1}{2\pi i} \int_{\partial D} f'(\lambda) f^{-1}(\lambda) d\lambda,$$

$$y = LR_{right}(f; D) = \frac{1}{2\pi i} \int_{\partial D} f^{-1}(\lambda) f'(\lambda) d\lambda;$$

(ii) There exist nonzero idempotents $p_1, \ldots, p_n \in B$ and invertible elements $s_1, \ldots, s_n \in B$ such that

$$x = \sum_{j=1}^{n} p_j, \qquad y = \sum_{j=1}^{n} s_j^{-1} p_j s_j.$$

Proof. The implication (i) $\Rightarrow$ (ii) is immediate from Proposition 2.1 and the fact that f^{-1} has only a finite number of (simple) poles in D. So let us turn to the implication (ii) $\Rightarrow$ (i). The argument will be a slight modification of the proof of [BES3], Proposition 2.1 (cf. also the proof of Theorem 2.2 above).

Let D be a bounded Cauchy domain with n connected components $D_1, \ldots, D_n$, choose $\lambda_1, \ldots, \lambda_n$ in $D_1, \ldots, D_n$ respectively, and let $f \in \mathcal{A}_\partial(D; B)$ be such that

$$f(\lambda) = \Big(e - p_j + (\lambda - \lambda_j)p_j\Big)s_j, \qquad \lambda \in \overline{D}_j;\ j = 1, \ldots, n.$$

Then, for $\lambda \in \overline{D}_j \backslash \{\lambda_j\},\ j = 1, \ldots, n,$

$$f'(\lambda)f^{-1}(\lambda) = \frac{1}{\lambda - \lambda_j} p_j, \qquad f^{-1}(\lambda)f'(\lambda) = \frac{1}{\lambda - \lambda_j} s_j^{-1} p_j s_j,$$

and hence

$$LR_{left}(f; D) = \sum_{j=1}^{n} LR_{left}(f; \lambda_j) = \sum_{j=1}^{n} p_j = x,$$

$$LR_{right}(f; D) = \sum_{j=1}^{n} LR_{right}(f; \lambda_j) = \sum_{j=1}^{n} s_j^{-1} p_j s_j = y.$$

Note that f^{-1} is simply meromorphic on D. □

We remark that the statement (i) in the previous theorem is an assertion about the existence of a function f and a suitable Cauchy domain D. Later on we will analyze the more complicated situation, where the Cauchy domain D is prescribed and possibly connected.

We continue our discussion of left versus right logarithmic residues, but now with underlying Cauchy domains that are required to be connected. It is convenient to establish two lemmas. The first one – Lemma 3.2 – is modelled after certain factorisation results for (semi-)Fredholm operator valued analytic functions (see, for instance, [GS2] and [T]; cf. also [BES5], Proposition 3.1 and the discussion presented there exhibiting a connection with [GKL]); the second one – Lemma 3.3 – is an interpolation result.

Lemma 3.2. *Let Δ be a non-empty open subset of $\mathbb{C}$ and let $f : \Delta \to B$ analytic. Suppose f takes invertible values on Δ, except in a finite number of distinct points $\lambda_1, \ldots, \lambda_n$ where f^{-1} has simple poles. Then there exist nonzero idempotents $p_1, \ldots, p_n$ in B and an analytic function $g : \Delta \to B$ such that g takes invertible values on all of Δ and*

$$f(\lambda) = \left(\prod_{j=1}^{n} \Big(e - p_j + (\lambda - \lambda_j)p_j\Big) \right) g(\lambda), \qquad \lambda \in \Delta.$$

In products written in the Π-notation and involving possibly noncommuting factors, the order of the factors corresponds to the order of the indices. So in the above product, the first factor is $e - p_1 + (\lambda - \lambda_1)p_1$, the all but last factor is $e - p_n + (\lambda - \lambda_n)p_n$ and the last factor is $g(\lambda)$. There is an analogue to Lemma 3.2 where $g(\lambda)$ is the first instead of the last factor in the factorization of f. We shall comment on this point in Section 4. As we shall also see there, the condition that f^{-1} is simply meromorphic in Lemma 3.2 is essential.

Proof. If $n = 0$, then f itself takes invertible values on all of Δ and we can put $g = f$. So assume n is positive. The proof goes by induction.
Write

$$f(\lambda) = \sum_{j=0}^{\infty} (\lambda - \lambda_1)^j a_j,$$

$$f^{-1}(\lambda) = \frac{1}{\lambda - \lambda_1} b_{-1} + \sum_{j=0}^{\infty} (\lambda - \lambda_1)^j b_j,$$

and set $p_1 = a_1 b_{-1},\ t_1 = a_0 b_0 a_0 + a_1 b_{-1} a_1$. From (the proof of) Proposition 2.1 we know that p_1 is a nonzero idempotent and that t_1 is invertible (with inverse $b_0 a_0 b_0 + b_{-1} a_1 b_{-1}$). Introduce

$$f_1(\lambda) = \begin{cases} \left(e - p_1 + \frac{1}{\lambda - \lambda_1} p_1\right) f(\lambda), & \lambda \in \Delta;\ \lambda \neq \lambda_1, \\ t_1, & \lambda = \lambda_1. \end{cases}$$

Then f_1 is analytic on $\Delta \backslash \{\lambda_1\}$. From $p_1 a_0 = 0$ and $(e - p_1)a_0 + p_1 a_1 = t_1$ we can conclude that $f_1(\lambda) \to t_1 = f_1(\lambda_1)$ when $\lambda \to \lambda_1$. Hence f_1 is analytic on all of Δ. Clearly f_1 takes invertible values on Δ, except in the points $\lambda_2, \ldots, \lambda_n$ where f_1^{-1} has simple poles. Here we used that $f_1(\lambda_1) = t_1$ is invertible. The (induction) argument can now be completed by observing that the identity

$$f(\lambda) = \left(e - p_1 + (\lambda - \lambda_1)p_1\right) f_1(\lambda)$$

holds on all of Δ. □

We shall write $\mathcal{G}(B)$ for the group of invertible elements in B. The connected component of $\mathcal{G}(B)$ containing the unit element e will be denoted by $\mathcal{G}_1(B)$.

Lemma 3.3. *Let $s_1, \ldots, s_n \in \mathcal{G}_1(B)$ and let $\lambda_1, \ldots, \lambda_n$ be distinct complex numbers. Then there exists an entire function $h : \mathbb{C} \to B$ such that h takes invertible values on all of $\mathbb{C}$ and*

$$h(\lambda_k) = s_k, \qquad h'(\lambda_k) = 0, \qquad k = 1, \ldots, n.$$

The condition that $s_1, \ldots, s_n \in \mathcal{G}_1(B)$ may be replaced by the requirement that $s_1, \ldots, s_n$ belong to precisely one and the same connected component of $\mathcal{G}(B)$. This is clear from the fact that the connected component of $\mathcal{G}(B)$ containing $s \in \mathcal{G}(B)$ is equal to $\{\, st \mid t \in \mathcal{G}_1(B) \,\}$ or, alternatively, $\{\, ts \mid t \in \mathcal{G}_1(B) \,\}$. Conversely, if h is as in Lemma 4.3, then necessarily $s_1, \ldots, s_n$ belong to precisely one and the same connected component of $\mathcal{G}(B)$.

Proof. From [R, Theorem 10.44] we know that s_k can be written as

$$s_k = \exp\Big(s_k(1)\Big) \cdots \exp\Big(s_k(m_k)\Big)$$

with $s_k(1), \dots, s_k(m_k)$ in B. Choose scalar polynomials $r_1, \dots, r_n$ with

$$r_j(\lambda_k) = \delta_{jk}, \qquad r_j'(\lambda_k) = 0, \qquad j, k = 1, \dots, n$$

(δ_{jk} is the Kronecker delta), and put

$$h_j(\lambda) = \exp\Big(r_j(\lambda)s_j(1)\Big) \cdots \exp\Big(r_j(\lambda)s_j(m_j)\Big).$$

Then $h_j : \mathbb{C} \to B$ is analytic and takes invertible values on all of $\mathbb{C}$. Also

$$\begin{array}{ll} h_j(\lambda_k) = e, & j, k = 1, \dots, n;\ j \neq k, \\ h_k(\lambda_k) = s_k, & k = 1, \dots, n, \\ h_j'(\lambda_k) = 0, & j, k = 1, \dots, n. \end{array}$$

The function $h(\lambda) = h_1(\lambda) \cdots h_n(\lambda)$ now has the desired properties. □

Theorem 3.4. *Let D be a connected bounded Cauchy domain in $\mathbb{C}$, let n be a nonnegative integer and let x and y be elements in the complex Banach algebra B. The following statements are equivalent:*

(i) There exists a function $f \in \mathcal{A}_\partial(D; B)$ such that f^{-1} is simply meromorphic on D, f^{-1} has exactly n simple poles in D and

$$x = LR_{left}(f; D) = \frac{1}{2\pi i}\int_{\partial D} f'(\lambda)f^{-1}(\lambda)d\lambda;$$

$$y = LR_{right}(f; D) = \frac{1}{2\pi i}\int_{\partial D} f^{-1}(\lambda)f'(\lambda)d\lambda,$$

(ii) There exist nonzero idempotents $p_1, \dots, p_n \in B$, invertible elements $s_1, \dots, s_n \in \mathcal{G}_1(B)$ and $s \in \mathcal{G}(B)$ such that

$$x = \sum_{j=1}^{n} p_j, \qquad y = s^{-1}\left(\sum_{j=1}^{n} s_j^{-1}p_js_j\right)s.$$

Note that (ii) can be rephrased as follows:

(iii) *There exist nonzero idempotents $p_1, \dots, p_n \in B$ and invertible elements $t_1, \dots, t_n \in \mathcal{G}(B)$, all belonging to precisely one and the same connected component of $\mathcal{G}(B)$, such that*

$$x = \sum_{j=1}^{n} p_j, \qquad y = \sum_{j=1}^{n} t_j^{-1}p_jt_j.$$

As a preliminary to the proof of Theorem 3.4 we make two observations. If $v \in B$ and $v^2 = 0$, then $e + \mu v \in \mathcal{G}_1(B)$ for all $\mu \in \mathbb{C}$. Also, if $p \in B$ and $p^2 = p$, then $e - p + \mu p \in \mathcal{G}_1(B)$ for all nonzero $\mu \in \mathbb{C}$. The proof of implication (ii) ⇒ (i) will provide additional information about the freedom one has in choosing the function f.

Proof. Suppose (i) holds. The function f takes invertible values on D, except in a finite number of distinct points $\lambda_1, \ldots, \lambda_n$ where f^{-1} has simple poles. Clearly

$$x = \sum_{j=1}^{n} LR_{left}(f;\lambda_j) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{|\lambda-\lambda_0|=\rho} f'(\lambda) f^{-1}(\lambda) d\lambda,$$

$$y = \sum_{j=1}^{n} LR_{right}(f;\lambda_j) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{|\lambda-\lambda_0|=\rho} f^{-1}(\lambda) f'(\lambda) d\lambda,$$

where ρ is positive and sufficiently small. We shall now investigate the connection between $LR_{left}(f;\lambda_k)$ and $LR_{right}(f;\lambda_k)$, $k = 1, \ldots, n$.

According to Lemma 3.2, we can factorize f as

$$f(\lambda) = \left(\prod_{j=1}^{n} \Big(e - q_j + (\lambda - \lambda_j) q_j \Big) \right) g(\lambda), \quad \lambda \in D.$$

Here $q_1, \ldots, q_n$ are nonzero idempotents in B, $g : D \to B$ is analytic and g takes invertible values on all of D. For $k = 1, \ldots, n$, put

$$a_k(\lambda) = \prod_{j=1}^{k-1} \Big(e - q_j + (\lambda - \lambda_j) q_j \Big),$$

$$f_k(\lambda) = e - q_k + (\lambda - \lambda_k) q_k,$$

$$b_k(\lambda) = \prod_{j=k+1}^{n} \Big(e - q_j + (\lambda - \lambda_j) q_j \Big).$$

Then $f(\lambda) = a_k(\lambda) f_k(\lambda) b_k(\lambda) g(\lambda)$. Note that $a_k(\lambda_k)$ and $b_k(\lambda_k)$ are invertible. In fact

$$a_k(\lambda_k) = \prod_{j=1}^{k-1} \Big(e - q_j + (\lambda_k - \lambda_j) q_j \Big)$$

and

$$b_k(\lambda_k) = \prod_{j=k+1}^{n} \Big(e - q_j + (\lambda_k - \lambda_j) q_j \Big)$$

belong to $\mathcal{G}_1(B)$.

First we consider the left logarithmic residue of f at λ_k. Suppressing the variable λ, we have

$$f^{-1} = g^{-1} b_k^{-1} f_k^{-1} a_k^{-1},$$

$$f' = a_k' f_k b_k g + a_k f_k' b_k g + a_k f_k b_k' g + a_k f_k b_k g',$$

and hence

$$f' f^{-1} = a_k' a_k^{-1} + a_k f_k' f_k^{-1} a_k^{-1} + a_k f_k b_k' b_k^{-1} f_k^{-1} a_k^{-1} + a_k f_k b_k g' g^{-1} b_k^{-1} f_k^{-1} a_k^{-1}.$$

Now $f_k(\lambda) = e - q_k + (\lambda - \lambda_k)q_k$, $f_k^{-1}(\lambda) = e - q_k + (\lambda - \lambda_k)^{-1}q_k$ and $f_k'(\lambda) = q_k$ with $q_k^2 = q_k$. It follows that

$$\begin{aligned} LR_{left}(f;\lambda_k) &= a_k(\lambda_k)q_k a_k(\lambda_k)^{-1} + a_k(\lambda_k)(e-q_k)b_k'(\lambda_k)b_k(\lambda_k)^{-1}q_k a_k(\lambda_k)^{-1} \\ &\quad + a_k(\lambda_k)(e-q_k)b_k(\lambda_k)g'(\lambda_k)g(\lambda_k)^{-1}b_k(\lambda_k)^{-1}q_k a_k(\lambda_k)^{-1} \\ &= a_k(\lambda_k)(q_k + \tilde{v}_k)a_k(\lambda_k)^{-1} \end{aligned}$$

where $\tilde{v}_k \in B$ is given by $\tilde{v}_k = (e - q_k)\Big(b_k'(\lambda_k)b_k(\lambda_k)^{-1} + b_k(\lambda_k)g'(\lambda_k)g(\lambda_k)b_k(\lambda_k)^{-1}\Big)q_k$. Clearly $\tilde{v}_k q_k = \tilde{v}_k$ and $q_k\tilde{v}_k = 0$. Hence $q_k + \tilde{v}_k = (e + \tilde{v}_k)\, q_k\, (e - \tilde{v}_k)$. But then

$$LR_{left}(f;\lambda_k) = a_k(\lambda_k)(e+\tilde{v}_k)q_k(e-\tilde{v}_k)a_k(\lambda_k)^{-1}.$$

Since $\tilde{v}_k^2 = 0$, we have $e + \tilde{v}_k \in \mathcal{G}_1(B)$ and $(e+\tilde{v}_k)^{-1} = e - \tilde{v}_k$. Put $\tilde{s}_k = a_k(\lambda_k)(e+\tilde{v}_k)$. Then $\tilde{s}_k \in \mathcal{G}_1(B)$ and $LR_{left}(f;\lambda_k) = \tilde{s}_k q_k \tilde{s}_k^{-1}$.

Next we look at the right logarithmic residue of f at λ_k. Again suppressing the variable λ, we have

$$f^{-1}f' = g^{-1}g + g^{-1}b_k^{-1}b_k'g + g^{-1}b_k^{-1}f_k^{-1}f_k'b_k g + g^{-1}b_k^{-1}f_k^{-1}a_k^{-1}a_k'f_k b_k g.$$

It follows that

$$\begin{aligned} LR_{right}(f;\lambda_k) &= g(\lambda_k)^{-1}b_k(\lambda_k)^{-1}q_k b_k(\lambda_k)g(\lambda_k) \\ &\quad + g(\lambda_k)^{-1}b_k(\lambda_k)^{-1}q_k a_k(\lambda_k)^{-1}a_k'(\lambda_k)(e-q_k)b_k(\lambda_k)g(\lambda_k) \\ &= g_k(\lambda_k)^{-1}b_k(\lambda_k)^{-1}(e-\hat{v}_k)q_k(e+\hat{v}_k)b_k(\lambda_k)g(\lambda_k), \end{aligned}$$

where $\hat{v}_k \in B$ is given by $\hat{v}_k = q_k a_k(\lambda_k)^{-1}a_k'(\lambda_k)(e-q_k)$. Since $\hat{v}_k^2 = 0$, we have $e + \hat{v}_k \in \mathcal{G}_1(B)$ and $(e+\hat{v}_k)^{-1} = e - \hat{v}_k$. Put $\hat{s}_k = (e+\hat{v}_k)b_k(\lambda_k)$. Then $\hat{s}_k \in \mathcal{G}_1(B)$ and $LR_{right}(f;\lambda_k) = g(\lambda_k)^{-1}\hat{s}_k^{-1}q_k\hat{s}_k g(\lambda_k)$.

Combining the results obtained so far, we get

$$\begin{aligned} x &= LR_{left}(f;D) = \sum_{k=1}^{n} \tilde{s}_k q_k \tilde{s}_k^{-1}, \\ y &= LR_{right}(f;D) = \sum_{k=1}^{n} g(\lambda_k)^{-1}\hat{s}_k^{-1}q_k\hat{s}_k g(\lambda_k). \end{aligned}$$

Put $p_k = \tilde{s}_k q_k \tilde{s}_k^{-1}$, $s = g(\lambda_0)$ and $s_k = \tilde{s}_k\hat{s}_k g(\lambda_k)s^{-1}$, where $\lambda_0 \in D$ is arbitrary. Then $p_1, \ldots, p_n$ are nonzero idempotents in B and

$$x = \sum_{k=1}^{n} p_k, \qquad y = s^{-1}\left(\sum_{k=1}^{n} s_k^{-1}p_k s_k\right)s.$$

It remains to prove that $s_1, \ldots, s_n \in \mathcal{G}_1(B)$.

We know already that $\tilde{s}_k$ and $\hat{s}_k$ are in $\mathcal{G}_1(B)$. So what we need to show is that $g(\lambda_k)s^{-1} = g(\lambda_k)g(\lambda_0)^{-1}$ belongs to $\mathcal{G}_1(B)$. Consider the function $g_0(\lambda) = g(\lambda)g(\lambda_0)^{-1}$. Clearly g_0 is continuous (even analytic) on D and $g_0(\lambda_0) = e$. Also g_0 takes invertible values on all of D. Since D is connected, it follows that the range of g_0 is contained in $\mathcal{G}_1(B)$. In particular the elements of the form $g(\lambda_k)g(\lambda_0)^{-1}$ are in $\mathcal{G}_1(B)$. This completes the proof of the implication (i) $\Rightarrow$ (ii).

Next we turn to the implication (ii) $\Rightarrow$ (i). Suppose x and y have the representation as in (ii). We shall prove the following version of (i): *Given a connected bounded Cauchy domain and distinct points $\lambda_1, \ldots, \lambda_n$ in D, there exists a function $f : \mathbb{C} \to B$ such that*

(a) *f is entire analytic on all of $\mathbb{C}$,*

(b) *f takes invertible values on all of $\mathbb{C}$, except for $\lambda_1, \ldots, \lambda_n$; in particular $f \in \mathcal{A}_\partial(D; B)$;*

(c) *f^{-1} has simple poles at $\lambda_1, \ldots, \lambda_n$;*

(d) *$LR_{left}(f; \lambda_j) = p_j$ and $LR_{right}(f; \lambda_j) = s^{-1}s_j^{-1}p_j s_j s$, $j = 1, \ldots, n$; hence x is the left and y is the right logarithmic residue of f with respect to D.*

The argument is as follows. Let $h : \mathbb{C} \to B$ be as in the interpolation result Lemma 3.3 and let $f_0 : \mathbb{C} \to B$ be an analytic function such that f_0 takes invertible values on all of $\mathbb{C}$, except for $\lambda_1, \ldots, \lambda_n$, where f_0^{-1} has simple poles, and

$$LR_{left}(f_0; \lambda_j) = LR_{right}(f_0; \lambda_j) = p_j, \quad j = 1, \ldots, n.$$

For the existence of f_0, see Ehrhardt's Theorem (Theorem 2.3 above). Introduce $f(\lambda) = f_0(\lambda)h(\lambda)s$. Then $f : \mathbb{C} \to B$ is a function which obviously satisfies (a)-(c). It remains to establish (d).

Take ρ positive and sufficiently small. Then

$$LR_{left}(f; \lambda_j) = \frac{1}{2\pi i}\int_{|\lambda-\lambda_j|=\rho} f_0'(\lambda)f_0^{-1}(\lambda)\, d\lambda + \frac{1}{2\pi i}\int_{|\lambda-\lambda_j|=\rho} f_0(\lambda)h'(\lambda)h^{-1}(\lambda)f_0^{-1}(\lambda)\, d\lambda.$$

The first term in the right hand side is equal to p_j. The second term vanishes because f_0^{-1} has a simple pole at λ_j and $h'(\lambda_j) = 0$. Hence $LR_{left}(f; \lambda_j) = p_j$. Also, with a similar reasoning,

$$\begin{aligned} LR_{right}(f; \lambda_j) &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_j|=\rho} s^{-1}h^{-1}(\lambda)f_0^{-1}(\lambda)f_0'(\lambda)h(\lambda)s\, d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{|\lambda-\lambda_j=\rho} s^{-1}h^{-1}(\lambda)h'(\lambda)s\, d\lambda \\ &= s^{-1}h(\lambda_j)^{-1}p_j h(\lambda_j)s, \end{aligned}$$

and the desired result, namely $LR_{right}(f; \lambda_j) = s^{-1}s_j^{-1}p_j s_j s$, follows from $h(\lambda_j) = s_j$. □

A comparison of Theorems 3.1 and 3.4 suggests that there is a difference (as far as the issue of left versus right logarithmic residues is concerned) between working with connected or working with possibly nonconnected Cauchy domains. As yet we do not have a concrete example substantiating this suggestion. An obstacle is that it is generally impossible to describe the (sums of) idempotents in Banach algebras.

4 Remarks and examples

We begin this section by returning to factorization result obtained in Lemma 3.2. On the basis of the assumptions of Lemma 3.2, the following alternative conclusion can be reached, too. *There exist nonzero idempotents $q_1, \ldots, q_n \in B$ and an analytic function $h : \Delta \to B$ such that h takes invertible values on all of Δ and*

$$f(\lambda) = h(\lambda)\left(\prod_{j=1}^{n}\Big(e - q_j + (\lambda - \lambda_j)q_j\Big)\right), \quad \lambda \in \Delta.$$

Comparing this factorization with the one in Lemma 3.2, we note that the idempotents $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ are necessarily similar, i.e., there exist invertible elements $s_1, \ldots, s_n \in \mathcal{G}(B)$ such that $q_k = s_k^{-1} p_k s_k$, $k = 1, \ldots, n$.

To see this, we argue as follows. From the first part of the proof of Theorem 3.4 we see that p_k, $LR_{left}(f; \lambda_k)$ and $LR_{right}(f; \lambda_k)$ are mutually similar. Analogously we have that q_k, $LR_{left}(f; \lambda_k)$ and $LR_{right}(f; \lambda_k)$ are mutually similar. But then the same conclusion holds for $p_k, q_k, LR_{left}(f; \lambda_k)$ and $LR_{right}(f; \lambda_k)$; cf. also Proposition 2.1.

In this context the following general observation is of interest (cf. the proof of [BES5] Proposition 3.1).

Remark 4.1. Let Δ be a non-empty open subset of $\mathbb{C}$, let $g : \Delta \to B$ be analytic, let $p \in B$ be an idempotent and let $\alpha \in \Delta$. Suppose g takes invertible values on all of Δ. Then there exist an idempotent $q \in B$ and an analytic function $h : \Delta \to B$ such that h takes invertible values on all of Δ, q is similar to p and

$$\Big(e - p + (\lambda - \alpha)p\Big) g(\lambda) = h(\lambda) \Big(e - q + (\lambda - \alpha) q\Big), \quad \lambda \in \Delta. \tag{4.1}$$

This is the reasoning. Put $q = g^{-1}(\alpha) p g(\alpha)$. Then q is an idempotent similar to p. Introduce

$$h(\lambda) = \begin{cases} \Big(e - p + (\lambda - \alpha)p\Big) g(\lambda) \left(e - q + \frac{1}{\lambda - \alpha} q\right), & \lambda \in \Delta; \lambda \neq \alpha, \\ g(\alpha) + (e - p) g'(\alpha) g^{-1}(\alpha) p g(\alpha), & \lambda = \alpha. \end{cases}$$

Then h is analytic on $\Delta \backslash \{\alpha\}$ and takes invertible values there. Also $h(\lambda) \to h(\alpha)$ when $\lambda \to \alpha$, so h is analytic on all of Δ. A direct computation shows that $h(\alpha)$ is invertible with inverse

$$h(\alpha)^{-1} = g^{-1}(\alpha) - g^{-1}(\alpha)(e - p) g'(\alpha) g^{-1}(\alpha) p.$$

Finally, the desired identity (4.1) holds. □

Remark 4.2. It is possible to combine Theorems 3.1 and 3.4 into one single result, thereby actually providing some extra information. The details are as follows.

Let x and y be elements in the complex Banach algebra B, let D be a bounded Cauchy domain in $\mathbb{C}$ with connected components $D_1, \ldots, D_m$, let $n_1, \ldots, n_m$ be nonnegative integers, and let $\lambda_{k1}, \ldots, \lambda_{kn_k}$ be distinct points in D_k $(k = 1, \ldots, m)$. The following statements are equivalent:

(i) There exists a function $f \in \mathcal{A}_\partial(D; B)$ such that f takes invertible values on D except in the points λ_{kj} where f^{-1} has simple poles and

$$x = LR_{left}(f; D) = \frac{1}{2\pi i} \int_{\partial D} f'(\lambda) f^{-1}(\lambda)\, d\lambda,$$

$$y = LR_{right}(f; D) = \frac{1}{2\pi i} \int_{\partial D} f^{-1}(\lambda) f'(\lambda)\, d\lambda;$$

(ii) The elements x and y admit a representation

$$x = \sum_{k=1}^{m} \sum_{j=1}^{n_k} p_{kj}, \quad y = \sum_{k=1}^{m} s_k^{-1} \left(\sum_{j=1}^{n_k} s_{kj}^{-1} p_{kj} s_{kj} \right) s_k$$

where $s_k \in \mathcal{G}(B)$, $s_{kj} \in \mathcal{G}_1(B)$ and p_{kj} are nonzero idempotents in B $(j = 1, \ldots, n_k;$ $k = 1, \ldots, m)$.

The verification is left to the reader.

Let D be a bounded Cauchy domain and let $f \in \mathcal{A}_\partial(D; B)$. If f takes invertible values on all of D, then obviously $LR_{left}(f; D) = LR_{right}(f; D) = 0$. Inspired by the scalar case ($B = \mathbb{C}$), one may ask whether the converse is also true (cf. problem 1 in the Introduction). In [BES2] it is shown that in general the answer is negative. However, it is also demonstrated there that for large and interesting classes of Banach algebras (for instance the polynomial-identity Banach algebras), the fact that $LR_{left}(f; D)$ *or* $LR_{right}(f; D)$ vanishes does imply that f takes invertible values on all of D. For such algebras, one has of course that $LR_{left}(f; D) = 0$ if and only if $LR_{right}(f; D) = 0$. The following (nonexotic) example, involving a connected Cauchy domain and an entire function f, shows that in general it can happen that precisely one of $LR_{left}(f; D)$ and $LR_{right}(f; D)$ vanishes.

Example 4.3. Let H be an infinite dimensional Hilbert space and let $\mathcal{B}(H)$ be the Banach algebra of all bounded linear operator on H. According to [PT] , each bounded linear operator on H can be written as a sum of five projections on H (i.e., idempotents in $\mathcal{B}(H)$). Let P_1 be a projection on H such that both P_1 and $I - P_1$ are nonzero. Choose projections $P_2, \ldots, P_6$ on H such that $-P_1 = P_2 + \cdots + P_6$, that is $P_1 + \cdots + P_6 = 0$. Write

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : \text{Im } P_1 \oplus \text{Ker } P_1 \to \text{Im } P_1 \oplus \text{Ker } P_1,$$

and introduce

$$N_1 = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} : \text{Im } P_1 \oplus \text{Ker } P_1 \to \text{Im } P_1 \oplus \text{Ker } P_1,$$

where $N : \text{Ker } P_1 \to \text{Im } P_1$ is a nonzero bounded linear operator. Here Ker and Im signal null spaces and ranges, while $\oplus$ stands for a direct (possibly nonorthogonal) sum. The fact that N can be chosen to be a nonzero operator is due to the nontriviality of Ker P_1 and Im P_1. Clearly $P_1 N_1 \neq N_1 P_1$ and $N_1^2 = 0$. The latter implies that $S_1 = I - N_1$ belongs to $\mathcal{G}_1(\mathcal{B}(H))$, the first that $P_1 S_1 \neq S_1 P_1$.

Let D be any connected bounded Cauchy domain. By Theorem 3.4, there exists a function $F \in \mathcal{A}_\partial(D; B)$ such that F^{-1} is simply meromorphic on D, F^{-1} has (at most) six simple poles in D and

$$LR_{left}(F; D) = P_1 + P_2 + P_3 + P_4 + P_5 + P_6,$$

$$LR_{right}(F; D) = S_1^{-1} P_1 S_1 + P_2 + P_3 + P_4 + P_5 + P_6.$$

It now follows that $LR_{left}(F; D) = 0$ and $LR_{right}(F; D) = S_1^{-1} P_1 S_1 - P_1 \neq 0$. Note that F can even be chosen to be entire. An example involving five instead of six projection operators can be constructed with the help of [BES1], Example 3.1. □

Our next example shows that the factorization result Lemma 3.2 need not hold in the absence of the condition that the poles of f^{-1} are simple.

Example 4.4. Let $m \geq 2$ and let B_m be the Banach algebra of all $m \times m$ matrices $(a_{ij})_{i,j=1}^m$ such that

$$a_{ij} = 0, \qquad i, j = 1, \ldots, m;\ i > j,$$

$$a_{ii} = a_{11}, \qquad i = 1, \ldots, m.$$

In other words, B_m is the Banach subalgebra of $\mathbb{C}^{m\times m}$ consisting of all upper triangular $m\times m$ matrices with constant diagonal. Observe that B_m is inverse closed in $\mathbb{C}^{m\times m}$, i.e., if $A\in B_m$ and A is invertible in $\mathbb{C}^{m\times m}$, then so is A in B_m (and, of course, the inverses of A in B_m and $\mathbb{C}^{m\times m}$ coincide). It is evident that the only idempotents in B_m are the $m\times m$ zero matrix and the $m\times m$ identity matrix. For completeness (cf. Example 4.3) we observe that if D is a bounded Cauchy domain, $f\in\mathcal{A}_\partial(D;B)$ and either $LR_{left}(f;D)$ or $LR_{right}(f;D)$ vanishes, then f takes invertible values on all of D (and so $LR_{left}(f;D)$ and $LR_{right}(f;D)$ both vanish). Note also that B_m, being a subalgebra of $\mathbb{C}^{m\times m}$, is a polynomial identity algebra (see [AL]).

Now let N be an upper triangular $m\times m$ matrix with zeros on the diagonal. Then N is nilpotent. We assume that the order of nilpotency n of N is larger than one (so $N^n=0$ and $N^{n-1}\neq 0$, where $2\leq n\leq m$). Put $F(\lambda)=\lambda I-N$. Then $F:\mathbb{C}\to B_m$ is entire, F takes invertible values on all of $\mathbb{C}$, except in the origin where $F^{-1}(\lambda)=\lambda^{-1}I+\lambda^{-2}N+\cdots+\lambda^{-n}N^{n-1}$ has a pole of order n.

Let Δ be an open subset of $\mathbb{C}$ containing the origin. By analogy with Lemma 3.2, one might conjecture F to admit a factorization

$$F(\lambda)=\left(\prod_{j=1}^{n}\Big(I-P_j+\lambda P_j\Big)\right)G(\lambda),\quad \lambda\in\Delta$$

where $P_1,\ldots,P_n$ are idempotents in B_m, $G:\Delta\to B_m$ is analytic and G takes invertible values on all of Δ. This, however, is not true. Indeed, since $0\neq -N=F(0)$, none of the idempotents $I-P_1,\ldots,I-P_n$ can vanish; but then $I-P_j=I,\ j=1,\ldots,n$. Hence $-N=F(0)=G(0)$, contradicting the invertibility of $G(0)$. □

The Banach algebra B_m in Example 4.4 can be used to extract some additional information. For this, we begin by observing that B_m is generated by $m-1$ upper triangular nilpotent $m\times m$ matrices. In particular, B_2 is generated by a single matrix. Hence each logarithmic residue in B_2 is a sum of idempotents in B_2. This follows from [BES4], Theorem 3.2, but of course in this special case it is easy to see directly that the logarithmic residues in B_2 are just the (nonnegative) integer multiples of the 2×2 identity matrix.

For $m\geq 3$, the situation is completely different. Focussing on $m=3$, we obtain the following example which is an improvement in two respects of [BES3], Example 2.4, where it was shown that there exist logarithmic residues which are not the sum of idempotents. First, Example 4.5 involves a matrix algebra (cf. [BES4] for more information about the matrix case). Second, the logarithmic residues in question not only fail to be a sum of idempotents, in fact they do not even belong to (the closure of) the algebra generated by the idempotents. The logarithmic residue constructed in [BES3], Example 2.4 – although not a sum of idempotents – does belong to this algebra. Finally, Example 4.5 corroborates the fact – already clear form the scalaer case – that the requirement in Proposition 2.1 that the pole of f^{-1} has order one is essential.

Example 4.5. Introduce $F:\mathbb{C}\to B_3$ by

$$F(\lambda)=\begin{pmatrix}\lambda & \lambda^2 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}.$$

Then F is entire and F takes invertible values on all of $\mathbb{C}$, except in the origin. A straightforward computation shows that

$$F'(\lambda)F^{-1}(\lambda)=\begin{pmatrix}\lambda^{-1} & 1 & -\lambda^{-1}\\ 0 & \lambda^{-1} & -\lambda^{-2}\\ 0 & 0 & \lambda^{-1}\end{pmatrix},\quad \lambda\neq 0,$$

$$F^{-1}(\lambda)F'(\lambda) = \begin{pmatrix} \lambda^{-1} & 1 & \lambda^{-1} \\ 0 & \lambda^{-1} & -\lambda^{-2} \\ 0 & 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \neq 0.$$

Hence the left and right logarithmic residue of F at the origin are given by

$$LR_{left}(F;0) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$LR_{right}(F;0) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Although both these matrices are sums of idempotents in $\mathbb{C}^{3\times 3}$, neither of them is a sum of idempotents in B_3. Indeed, the sums of idempotents is B_3 are just the (non-negative integer) multiples of the 3×3 identity matrix. So actually, $LR_{left}(F;0)$ and $LR_{right}(F;0)$ do not even belong to the (closure of) the algebra generated by the idempotents in B_3.
For completeness we mention that the logarithmic residues in B_3 coincide with the matrices of the form

$$k \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where k is a non-negative integer and $\alpha \in \mathbb{C}$ is arbitrary. □

It is worthwile to compare Example 4.4 and 4.5 with the results of [BES5], Section 3: one sees that most of the conclusions that can be drawn from [BES5] when f is viewed as a $\mathbb{C}^{m\times m}$-valued function fail to have an analogue in the B_m context.

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