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Eliciting Discount Functions when Baseline Consumption changes over Time

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Abstract

Many empirical studies on intertemporal choice report preference reversals in the sense that a preference between a small reward to be received soon and a larger reward to be received later reverses as both rewards are equally delayed. Such preference reversals are commonly interpreted as contradicting constant discounting. We show that this interpretation is correct only if baseline consumption to which the outcomes are added, remains constant over time. The difficulty with measuring discounting when baseline consumption changes over time, is that delaying an outcome has two simultaneous effects: (1) due to the change in baseline consumption, it changes the increase in utility from receiving the outcome, and (2) it changes the discount factor applied to this increase in utility. In order to draw conclusions about discounting one needs to disentangle these two effects which seems impossible at first sight (Noor, 2009). Yet, in this paper we propose a way to disentangle the two effects.

Keywords: Hyperbolic discounting; Constant discounting; Preference reversals; Decreasing impatience
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1 Introduction

Many empirical studies show that people's choice behaviour is time inconsistent in the sense that a preference between a small reward to be received soon and a larger reward to be received later, reverses as both rewards are equally delayed. This preference reversal is

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commonly interpreted as contradicting Samuelson's (1937) constant discounting. However, like in Gerber and Rohde (2009), we show that this interpretation is only justified if the baseline consumption to which a decision maker adds the rewards, remains constant over time. Noor (2009) showed that another common approach to falsify constant discounting also only works if baseline consumption is constant over time. This is bad news. Unless we know that baseline consumption remains unchanged over time, we cannot draw any conclusions about the discount function from the usual intertemporal choices that are observed in the literature.

In this paper we do not only provide bad news but we also propose a new approach to verify whether discounting satisfies constant impatience, implying constant discounting, or whether it satisfies decreasing or increasing impatience. Thus, we show under what conditions one can still conclude that choice behaviour is inconsistent with constant discounting and is supported by hyperbolic discounting when we do not know how baseline consumption changes over time.

The difficulty for measuring discounting when baseline consumption changes over time, is that delaying an outcome has two simultaneous effects: (1) due to the change in baseline consumption, it changes the increase in utility from receiving the outcome, and (2) it changes the discount factor applied to this increase in utility. In order to draw conclusions about discounting one needs to disentangle these two effects. At first sight that seems impossible (Noor, 2009). Yet, we found a way to disentangle the two effects.

Key to our approach to disentangle the two effects of delaying an outcome is determining probability equivalents of the delay. Imagine we start with the receipt of an outcome at date 0 and we delay it to date t. Then we determine the probability p that would make 'receiving the outcome at date t for sure' equivalent to 'receiving the outcome at date 0 with probability p'. As we will show, it turns out that eliciting the probability equivalent of a delay for two different outcomes, helps us in measuring the discount function.

2 The Usual Approach

The decision maker (DM) evaluates *temporal prospects* $(t : p : x)$, which give *outcome* $x > 0$ with *probability* p at *date* t and nothing otherwise. Date 0 represents today. We assume that the DMs preferences $\succcurlyeq$ can be represented by[1]

$$V(t : p : x) = p\ \delta(t)\left[u(b_t + x) - u(b_t)\right], \tag{1}$$

where u is a strictly increasing, strictly concave, and twice continuously differentiable *utility function,* where δ is a strictly decreasing *discount function* with $\delta(0) = 1$, and where b_t is *baseline consumption* at date t. Whenever the DM receives an outcome x at date t, he adds it to his baseline consumption and consumes the sum of both, which yields utility $u(b_t+x)$. The increase in utility at date t from receiving the outcome x at date t, therefore, is $[u(b_t + x) - u(b_t)]$, which is equivalent to an increase in utility of $\delta(t)\left[u(b_t + x) - u(b_t)\right]$ at date 0. Thus, baseline consumption determines the increase in utility generated by the receipt of an outcome x. We assume that baseline consumption is riskless, but, like in Noor (2009) it can be viewed as a stand-in for other factors that possibly affect marginal utility. The DM has *decreasing (constant, increasing) absolute risk aversion* if $-u''(x)/u'(x)$ is decreasing (constant, increasing) in x.

A preference reversal typically observed in the literature is

$$(0 : 1 : x) \sim (\tau : 1 : y) \text{ and } (t : 1 : x) \prec (t + \tau : 1 : y) \tag{2}$$

with $y > x > 0$, and $\tau, t > 0$. Based on this preference reversal it is commonly concluded that the discount function cannot be exponential (e.g. $\delta(t) = e^{-rt}$). We will show, though, that this conclusion is only justified if baseline consumption is constant over time. The result follows from a similar line of reasoning as in Gerber and Rohde (2009), but for completeness we repeat the reasoning here.

In line with Prelec (1989, 2004) we say that decreasing impatience holds when the near future is discounted at a higher rate than the far future.

[1] All results in this paper would still hold if the DM would weight probabilities by a probability weighting function w.

Definition 2.1 *Decreasing (constant, increasing) impatience holds if for all $s < t$ and for all $\tau > 0$*

$$\frac{\delta(s)}{\delta(s+\tau)} > \ (=,<) \ \frac{\delta(t)}{\delta(t+\tau)}. \tag{3}$$

Constant impatience is equivalent to the discount function being exponential, i.e. $\delta(t) = e^{-rt}$ for some r. Generalized hyperbolic discounting with $\delta(t) = (1+\alpha t)^{-\beta/\alpha}$ satisfies decreasing impatience (Loewenstein and Prelec, 1992).

If baseline consumption is constant over time, then preferences of type (2) contradict increasing or constant impatience. To show this, a first step is to observe that (2) implies

$$\begin{aligned} \delta(\tau) &= \frac{u(b_0+x)-u(b_0)}{u(b_\tau+y)-u(b_\tau)} \text{ and} \\ \frac{\delta(t+\tau)}{\delta(t)} &> \frac{u(b_t+x)-u(b_t)}{u(b_{t+\tau}+y)-u(b_{t+\tau})}. \end{aligned}$$

If baseline consumption is constant over time then

$$\begin{aligned} \frac{u(b_0+x)-u(b_0)}{u(b_\tau+y)-u(b_\tau)} &= \frac{u(b_0+x)-u(b_0)}{u(b_0+y)-u(b_0)} \text{ and} \\ \frac{u(b_t+x)-u(b_t)}{u(b_{t+\tau}+y)-u(b_{t+\tau})} &= \frac{u(b_0+x)-u(b_0)}{u(b_0+y)-u(b_0)}. \end{aligned}$$

It follows that

$$\delta(\tau) < \frac{\delta(t+\tau)}{\delta(t)}, \tag{4}$$

which contradicts increasing or constant impatience.

Theorem 2.2 *If baseline consumption is constant over time, i.e. if $b_t = b_0$ for all t, then* (2) *contradicts increasing or constant impatience.*

Similarly, if baseline consumption is constant, then

$$(0:1:x) \sim (\tau:1:y) \text{ and } (t:1:x) \succ (t+\tau:1:y) \tag{5}$$

contradicts decreasing or constant impatience.

The above argument no longer holds if baseline consumption changes over time.

Theorem 2.3 *If we do not know how baseline consumption changes over time, then* (2) *does not contradict increasing or constant impatience.*

Similarly, if we do not know how baseline consumption changes over time, then (5) does not contradict decreasing or constant impatience. It follows that if we do not know how baseline consumption evolves over time, then preference reversals like (2) and (5) do not contradict constant (or decreasing or increasing) impatience.

These results are similar in spirit to Noor's (2009) finding that, when baseline consumption is unknown, properties of the money-discount functions do not necessarily translate into properties of the discount function. Noor (2009) dealt with another common approach used in the literature to verify whether discount functions satisfy decreasing, increasing, or constant impatience. If $\psi(x,t)$ is given by $(0:1:\psi(x,t)) \sim (t:1:x)$, then many studies in the literature take the money-discount function $\phi^x(t) = \psi(x,t)/x$ as (an approximation of) the discount function $\delta(t)$, which, as Noor shows, is only justified for small x if baseline consumption is approximately constant.

Thus, in order to determine properties of the discount function, like decreasing, constant, or increasing impatience, preference reversals of type (2) and type (5) are not sufficient if we do not know how baseline consumption changes over time. In the next section we will show how we can test for decreasing, constant, or increasing impatience, without knowing how baseline consumption evolves.

3 New Approach

The previous section and Noor (2009) showed that the common approaches to verify whether discounting satisfies decreasing, constant or increasing impatience, do not work if baseline consumption may change over time. In the following we will introduce an approach to elicit intertemporal preferences, which does allow us to draw conclusions about the mentioned properties of the discount function, even if we do not know a priori how baseline consumption evolves. Key to our approach is the determination of probabilities that are experienced as being equivalent to particular delays, as we will show.

If baseline consumption changes over time, then delaying the receipt of an outcome simultaneously affects the increase in utility due to receiving the outcome, and the factor by which that utility increase is discounted. It seems impossible to disentangle these two effects of a delay. Yet, in order to draw conclusions about the discount function, we need to disentangle these two effects. Surprisingly, in this section we will show that it is possible to disentangle them.

First, assume that the utility function over outcomes at date 0, i.e. $\tilde{u}(x) = u(b_0 + x) - u(b_0)$, is known. It is well-known that this function can be measured from preferences over lotteries that yield outcomes at date 0 only. Consider two outcomes x, y with $y > x > 0$. For both outcomes we determine the probability equivalents of delaying the receipt of the outcomes from date 0 to date t, i.e. we determine p and p' such that

$$(0 : p : x) \sim (t : 1 : x) \quad \text{and} \quad (0 : p' : y) \sim (t : 1 : y).$$

If the DM satisfies decreasing or increasing absolute risk aversion, then these two probability equivalents turn out to be sufficient to determine the change in baseline consumption $b_t - b_0$ and the discount factor $\delta(t)$. This result is stated in the following theorem. Its proof is in the Appendix.

Theorem 3.1 *Let $t > 0$ and $y > x > 0$ and let p and p' be defined by*

$$(0 : p : x) \sim (t : 1 : x) \quad \textit{and} \quad (0 : p' : y) \sim (t : 1 : y).$$

Assume that the function $\tilde{u}(x) = u(b_0 + x) - u(b_0)$ is known. If the decision maker satisfies decreasing or increasing absolute risk aversion, then we have sufficient information to determine $b_t - b_0$ and $\delta(t)$.[2]

If we do not know the utility function $\tilde{u}$ over outcomes at date 0, then the probability equivalents of delays do not allow us to determine the discount function, but they do allow

[2] If the DM would weight probabilities non-linearly, then we would also need to know the probability weighting function for the result to hold.

us to verify whether the DM satisfies decreasing, constant, or increasing impatience, as will be shown in Theorem 3.3.

As a first step, we observe that even if utility $\tilde{u}$ is unknown, probability equivalents of delays allow us to determine whether baseline consumption increases, decreases, or remains unchanged. The following theorem states the results of this first step precisely.

Theorem 3.2 *Let $t > \tau \geq 0$ and $y > x > 0$ and let p and p' be defined by*

$$(\tau : p : x) \sim (t : 1 : x) \quad \textit{and} \quad (\tau : p' : y) \sim (t : 1 : y).$$

Then the following holds true:

(i) If the DM's utility function satisfies decreasing absolute risk aversion, then

$$b_t > (=<)\ b_\tau \iff p' > (=<)\ p.$$

(ii) If the DM's utility function satisfies increasing absolute risk aversion, then

$$b_t > (=<)\ b_\tau \iff p' < (=>)\ p.$$

(iii) If the DM's utility function satisfies constant absolute risk aversion, then $p' = p$ independent of b_τ and b_t.

Theorem 3.2 allows us to prove the following theorem, which shows how probability equivalents of delays together with the typical preference reversals observed in the literature, yield contradictions to constant impatience.

Theorem 3.3 *Let $s \geq 0$ and let $y > x > 0$ and τ be such that*

$$(s : 1 : x) \sim (s + \tau : 1 : y).$$

Let $t > s$ and $Y > X > 0$ and p, p', q, q' be such that

$$(s : p : X) \sim (t : 1 : X) \quad \& \quad (s : p' : Y) \sim (t : 1 : Y), \textit{and}$$
$$(s + \tau : q : X) \sim (t + \tau : 1 : X) \quad \& \quad (s + \tau : q' : Y) \sim (t + \tau : 1 : Y).$$

(i) Let the DM's utility function satisfy decreasing absolute risk aversion. Then

$$(t:1:x) \prec (t+\tau:1:y) \ \ \& \ p' < p \ \& \ q' > q$$

contradicts increasing or constant impatience and

$$(t:1:x) \succ (t+\tau:1:y) \ \ \& \ p' > p \ \& \ q' < q$$

contradicts decreasing or constant impatience.

(ii) Let the DM's utility function satisfy increasing absolute risk aversion. Then

$$(t:1:x) \prec (t+\tau:1:y) \ \ \& \ p' > p \ \& \ q' < q$$

contradicts increasing or constant impatience and

$$(t:1:x) \succ (t+\tau:1:y) \ \ \& \ p' < p \ \& \ q' > q$$

contradicts decreasing or constant impatience.

(iii) If the DM's utility function satisfies constant absolute risk aversion, then $p' = p$ *and* $q' = q$ *and we do not have sufficient information to say whether the discount function satisfies decreasing, increasing, or constant impatience.*

4 Discussion

Like Noor (2009) we showed that a common approach to elicit discount functions only works if baseline consumption remains unchanged over time. As Noor (2009) suggests, it seems impossible to draw conclusions about discount functions when baseline consumption may change over time. Surprisingly, we showed that it *is* possible to draw conclusions about discounting, even when the change in baseline consumption is unknown.

Our main result, Theorem 3.1, shows how the change in baseline consumption and the discount function can simultaneously be determined by eliciting the probability equivalents of a delay for two different outcomes. Thus, we do not only suggest a new way to measure discounting, but also an incentive compatible approach to elicit future baseline consumption. One could simply ask people directly what they expect their future consumption to be, but this would not give them an incentive to report their future consumption truthfully. By inferring the probability equivalents of delays from choices, people do have an incentive to indirectly, but truthfully, reveal their future baseline consumption.

Appendix

Proof of Theorem 3.2: $(\tau : p : x) \sim (t : 1 : x)$ holds if and only if

$$p\delta(\tau)\left[u(b_\tau + x) - u(b_\tau)\right] = \delta(t)\left[u(b_t + x) - u(b_t)\right] \tag{6}$$

Similarly, $(\tau : p' : y) \sim (t : 1 : y)$ holds if and only if

$$p'\delta(\tau)\left[u(b_\tau + y) - u(b_\tau)\right] = \delta(t)\left[u(b_t + y) - u(b_t)\right]. \tag{7}$$

From (6) and (7) it follows that

$$\frac{p}{p'} = \frac{\left[u(b_t + x) - u(b_t)\right]\left[u(b_\tau + y) - u(b_\tau)\right]}{\left[u(b_t + y) - u(b_t)\right]\left[u(b_\tau + x) - u(b_\tau)\right]} \tag{8}$$

Clearly, if $b_t = b_\tau$, then $p' = p$ independent of the DM's degree of risk aversion.

Assume decreasing absolute risk aversion. Then for all $z < x$ and all b we have

$$-\frac{u''(b+x)}{u'(b+x)} < -\frac{u''(b+z)}{u'(b+z)}$$

$$\iff \quad \frac{u''(b+x)}{u'(b+x)} > \frac{u''(b+z)}{u'(b+z)}$$

$$\iff \quad u''(b+x)u'(b+z) > u''(b+z)u'(b+x)$$

$$\Longrightarrow \quad \int_0^x u''(b+x)u'(b+z)dz > \int_0^x u''(b+z)u'(b+x)dz$$

$$\iff \quad \left[u(b+x) - u(b)\right]u''(b+x) > \left[u'(b+x) - u'(b)\right]u'(b+x) \tag{9}$$

$$\iff \quad \frac{u''(b+x)}{u'(b+x)} > \frac{u'(b+x) - u'(b)}{u(b+x) - u(b)}. \tag{10}$$

It follows that

$$\frac{d}{dx}\frac{u'(b+x)-u'(b)}{u(b+x)-u(b)} > 0, \tag{11}$$

which implies

$$\frac{u'(b+x)-u'(b)}{u(b+x)-u(b)} < \frac{u'(b+y)-u'(b)}{u(b+y)-u(b)}. \tag{12}$$

It follows that

$$\frac{d}{db}\left[\frac{u(b+x)-u(b)}{u(b+y)-u(b)}\right] < 0. \tag{13}$$

Thus,

$$b_t > (=<)\ b_\tau \tag{14}$$

if and only if

$$\frac{u(b_\tau+x)-u(b_\tau)}{u(b_\tau+y)-u(b_\tau)} > (=<)\ \frac{u(b_t+x)-u(b_t)}{u(b_t+y)-u(b_t)} \tag{15}$$

if and only if

$$p' > (=<)\ p. \tag{16}$$

With increasing absolute risk aversion the inequalities are always the reverse, and with constant absolute risk aversion the inequalities are equalities, which proves the result.

□

Proof of Theorem 3.3: From $(s:1:x) \sim (s+\tau:1:y)$ we know that

$$\frac{\delta(s)}{\delta(s+\tau)} = \frac{u(b_{s+\tau}+y)-u(b_{s+\tau})}{u(b_s+x)-u(b_s)}.$$

Consider first the case where $(t:1:x) \prec (t+\tau:1:y)$. This is equivalent to

$$\frac{\delta(t)}{\delta(t+\tau)} < \frac{u(b_{t+\tau}+y)-u(b_{t+\tau})}{u(b_t+x)-u(b_t)}.$$

Suppose that $b_t < b_s$ and $b_{t+\tau} > b_{s+\tau}$. Then

$$\frac{\delta(s)}{\delta(s+\tau)} = \frac{u(b_{s+\tau}+y)-u(b_{s+\tau})}{u(b_s+x)-u(b_s)} > \frac{u(b_{t+\tau}+y)-u(b_{t+\tau})}{u(b_t+x)-u(b_t)} > \frac{\delta(t)}{\delta(t+\tau)}. \tag{17}$$

Thus, $b_t < b_s$ and $b_{t+\tau} > b_{s+\tau}$ is only consistent with decreasing impatience. Similarly, $(t:1:x) \succ (t+\tau:1:y)$ together with $b_t > b_s$ and $b_{t+\tau} < b_{s+\tau}$ is only consistent with increasing impatience.

From Theorem 3.2 we know that under decreasing absolute risk aversion $b_t < b_s$ and $b_{t+\tau} > b_{s+\tau}$ hold if and only if $p' < p$ and $q' > q$. Similarly, under decreasing absolute risk aversion $b_t > b_s$ and $b_{t+\tau} < b_{s+\tau}$ hold if and only if $p' > p$ and $q' < q$. Moreover, by Theorem 3.2 under increasing absolute risk aversion $b_t < b_s$ and $b_{t+\tau} > b_{s+\tau}$ hold if and only if $p' > p$ and $q' < q$, and $b_t > b_s$ and $b_{t+\tau} < b_{s+\tau}$ hold if and only if $p' < p$ and $q' > q$.

Theorem 3.2 also implies that, under constant absolute risk aversion, $p' = p$ and $q' = q$ independent of $b_s, b_t, b_{s+\tau}$ and $b_{t+\tau}$. Therefore, from observing p and q we cannot conclude anything about the DM's baseline consumption and hence about the discount function.

□

Proof of Theorem 3.1: Define the function $\tilde{u}(x) = u(b_0 + x) - u(b_0)$. Note that if u satisfies decreasing (increasing) absolute risk aversion, then $\tilde{u}$ satisfies decreasing (increasing) absolute risk aversion.

We have

$$\begin{aligned} u(b_t + x) - u(b_t) &= u(b_0 + b_t - b_0 + x) - u(b_0 + b_t - b_0) \\ &= \tilde{u}(b_t - b_0 + x) - \tilde{u}(b_t - b_0). \end{aligned} \tag{18}$$

Then Eq.(8) with $\tau = 0$ can be rewritten as

$$\frac{p}{p'} = \frac{[\tilde{u}(b_t - b_0 + x) - \tilde{u}(b_t - b_0)]\,[\tilde{u}(y)]}{[\tilde{u}(b_t - b_0 + y) - \tilde{u}(b_t - b_0)]\,[\tilde{u}(x)]} \tag{19}$$

From inequality (13) in the proof of Theorem 3.2 there is exactly one $c_t = b_t - b_0$ which satisfies Eq.(19). Since we know $\tilde{u}$, we have everything we need to determine c_t.

We also know that

$$\delta(t) = p\frac{u(b_0 + x) - u(b_0)}{u(b_t + x) - u(b_t)} = p\frac{\tilde{u}(x)}{\tilde{u}(b_t - b_0 + x) - \tilde{u}(b_t - b_0)}, \tag{20}$$

which is then uniquely defined.

□

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