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STOCHASTIC DOMINANCE: CONVEXITY AND SOME EFFICIENCY TESTS[†]

Andrey M. Lizyayev[‡]

ABSTRACT

This paper points out the importance of Stochastic Dominance (SD) efficient sets being convex. We review classic convexity and efficient set characterization results on SD efficiency of a given portfolio relative to a diversified set of assets and generalize them in the following aspects. First, we broaden the class of individual utilities in Rubinstein (1974) that lead to two-fund separation. Secondly, we propose a linear programming SSD test that is more efficient than that of Post (2003) and expand the SSD efficiency criteria developed by Dybvig and Ross (1982) onto the Third Order Stochastic Dominance and further to Decreasing Absolute and Increasing Relative Risk Aversion Stochastic Dominance. The efficient sets for those are finite unions of convex sets.

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1. INTRODUCTION

Stochastic Dominance (SD) is a probabilistic concept of superiority among different random variables. Unlike parametric criteria such as Mean-Variance analysis, SD accounts for the whole range of distribution function, rather than its particular characteristics such as central moments. Although SD finds applications in a huge variety of areas ranging from medicine to agriculture (see, e.g., Bawa 1982; and Levy 1992, 2006 for a survey and references, Eeckhoudt *et al* 2009 for recent applications), this paper focuses on its use in the area of finance. In financial decision making one has to select efficient portfolios from an available portfolio possibilities set on the basis of a tradeoff among their expected returns, associated risk of having extreme losses and the potential of earning excessive gains.

We consider the expected utility framework where individuals select portfolios maximizing the expected value of their utility function which can capture different individual risk attitudes such as risk aversion, risk neutrality, risk seeking, or a combination of those for different levels of wealth. The non-parametric nature of SD criteria allows us to identify efficient portfolios without having to specify the utility functions explicitly. Instead, it employs some general restrictions such as non-satiation and risk aversion. The set of all portfolios supported by some utility function in a given class is called efficient set. It turns out that convex efficient sets have a special economic content and hence necessary and sufficient conditions leading to the convexity of efficient sets have been puzzling researchers for more than three decades. Rubinstein (1974) has shown that when preferences of all investors are similar enough, two-fund separation results. On the other hand, Dybving and Ross (1982) proved that if no assumptions on investors' preferences are made other than concavity and monotonicity, the efficient set is generally non-convex. In line with this concave and monotone utility class Ross (1978) derived some assumptions on the distribution function of returns that lead to k-fund separation. Among recent researchers, Versijp (2007) reviewed Rubinstain's result in relation to stochastic dominance and asset pricing models.

In this paper we point out the importance of Stochastic Dominance efficient sets being convex, review classic convexity and efficient set characterization results and generalize them in the following aspects. First, we broaden the class of individual utilities in Rubinstein (1974) that lead to two-fund separation. Second, we expand the SSD efficiency criteria developed by Dybvig and Ross (1982) onto the Third Order Stochastic Dominance and further to Decreasing Absolute and Increasing Relative Risk Aversion Stochastic Dominance. The efficient sets for those are finite unions of intersections of convex sets. We also give a linear programming SSD efficiency test which is more efficient than that of Post (2003) in case of unrestricted short sales.

This paper is focused on portfolio efficiency with respect to a diversified portfolio possibilities set, normally a polytope whose vertices are the assets available to investors. Further, we consider discrete distribution of returns due to its interpretability via empirically observed data, as well as tractability of the computational methods involved.

This paper is organized as follows. Section 2 provides general assumptions and problem formulation, Section 3 points out the importance of efficient sets being convex and reviews the associated necessary and sufficient conditions, Section 4 suggests some efficiency tests for a given portfolio relative to various economically meaningful classes of utility functions, and finally Section 5 summarizes the major results and concludes the paper.

2. GENERAL FRAMEWORK

Consider a single period investment decision making problem under uncertainty in a classic expected utility framework, in which:

1. Investors select investment portfolios to maximize the expected utility of the return on their investment portfolio. Let $U = \{u: \mathbb{R} \to \mathbb{R}\}$ denote the class of von Neuman-Morgenstern utility functions and X be the m-by-n matrix of returns of n available assets in m states of the world. The probability of occurrence of state i is denoted π_i. Naturally, $0 < \pi_i < 1$, i=1…m, and $\sum_{i=1..m} \pi_i = 1$. Investors are uncertain about which of the states of nature will occur, but they know the underlying probabilities of the states with certainty.
2. Investors may diversify between available assets. Denote $\lambda \in \mathbb{R}^n$ for a vector of portfolio weights. Unless otherwise specified, we assume that short sales are allowed and unrestricted. The portfolio possibilities set then becomes
$\Lambda \equiv \{\lambda \in \mathbb{R}^n: \lambda^T e = 1\}$, and the set of all available allocations is
$M_X = \{x \in \mathbb{R}^m \mid x = X\lambda, \lambda \in \Lambda\}$.
3. If a riskless asset is available in the market, it will be used either as part of X (a column with equal components), or separately (in which case X will be the set of risky assets only), whichever is more convenient.

A given portfolio $\tau \in \Lambda$ is **optimal** for an investor with utility $u \in U$ if and only if

$$\mathrm{E}u(X\tau) = \sup_{\lambda\in\Lambda} \mathrm{E}u(X\lambda) \qquad (1)$$

where Eu denotes the expected value of u.

If $\pi_1, \ldots, \pi_m$ are probabilities of occurrence of the states of the world, then (1) becomes

$$\sum_{i=1}^{m} \pi_i u\left(x^i\tau\right) = \sup_{\lambda\in\Lambda} \sum_{i=1}^{m} \pi_i u\left(x^i\lambda\right) \qquad (2)$$

where x^i is the i^{th} row of X.

In practical applications full information about utility functions is not available, and (2) cannot be verified directly. This provides the rationale for relying on a set of general assumptions, rather than a full specification of the utility function. The Second Order Stochastic Dominance criterion (**SSD**) restricts attention to the class of strictly increasing and concave utility functions, modeling thereby non-satiable and risk-averse preferences. The Third Order SD (**TSD**) assumes that in addition to SSD utilities are positively skewed. A portfolio τ is said to be **optimal** in given utility class U if and only if there exists $u\in U$ such that τ is optimal for u in the sense of (1).

A portfolio τ is **efficient** if it is not dominated by any other portfolio, that is $\forall\lambda \in \Lambda\backslash\{\tau\}\ \ \exists\, u \in U$: E$u(X\lambda)$ < E$u(X\tau)$. If both Λ and U are convex (as it will be the case in this paper), efficiency is equivalent to optimality due to the Minimax theorem (see e.g. Post (2003)) and thus the two concepts will be henceforth used interchangeably.

An individual investor with utility $u \in U$ is facing the following portfolio allocation problem:

$$\max_{\lambda} \sum_{i=1}^{m} \pi_i u\left(x^i\lambda\right) - v\left(\lambda^T e - 1\right) \qquad (3)$$

where v is the Lagrange multiplier. It is known that linearity constraints do not alter convexity. So if we assume $u(x)$ to be strictly concave and twice continuously differentiable in x, it will remain concave in λ. In fact, the Hessian of u with respect to λ is

H$u = -\,Y^{\mathrm{T}}Y$, where Y is an m-by-n matrix defined by $Y_{ij} = -x_{ij}\sqrt{-u''\left(x^i\lambda\right)}$.

Therefore, Hu is always negative semidefinite, and $u(\cdot)$ is concave in λ.
In light of above mentioned, the necessary and sufficient conditions for τ to be the solution of (3) is that there exists $v \in \mathbb{R}$ such that

for all $j = 1..n$ $$\sum_{i=1}^{m} \pi_i x_{ij} u'\left(x^i \tau\right) = v \qquad (4)$$

Note that if a risk-free asset is available, due to (4) there should hold: $\sum_{i=1}^{m} \pi_i u'\left(x^i \tau\right) = \frac{v}{r_F}$,

where r_F is the risk-free return. Thus, the optimality condition (4) takes on the form:

for all risky assets j $$\sum_{i=1}^{m} \pi_i \left(x_{ij} - r_F\right) u'\left(x^i \tau\right) = 0 \qquad (5)$$

We could as well relax the twice continuous differentiability assumption for u; (4) would still hold in optimality if u' is substituted by ∂u – any vector from the supergradient correspondence.

3. CONVEXITY

Program (3) represents portfolio formation of an individual investor having a particular well-behaved utility function $u()$. In macroeconomic settings one would like to study the aggregate investment decision of a large group of individuals, assuming all of them to be well-behaved, for instance non-satiable risk averters. A reasonable theoretical model should allow us to judge about all investors by a small number of large composite portfolios. The largest of those, the total value-weighted aggregate portfolio is generally referred to as the market portfolio, efficiency of which has been a starting point of many asset pricing theories, including the capital asset pricing model (CAPM).

The simplest case when the market portfolio is efficient is two-fund separation, where any optimal portfolio is a linear combination of two assets, normally a risk-free asset and the market portfolio. In such economy any individual investor will hold a share of the same (risky) market portfolio and will invest the rest of his/her wealth in the risk-free asset available, i.e. either borrowing or lending at the risk-free rate. Various assumptions lead to two fund separation, such as: mean-variance setting (when investment decision is a tradeoff only between mean and variance of underlying portfolio, see Markowitz (1952, 1987)), homogeneity of preferences (see Rubinstein (1974)), joint normal distribution of asset returns which is a common assumption of the traditional Capital Asset Pricing Model (CAPM, see Cochrane (2005) for an overview) and quadratic utility functions.

Despite its theoretical appeal, two fund separation is extremely restrictive and is very unlikely to hold in practice. A straightforward generalization of the concept, preserving the market portfolio efficiency, is k-fund separation, where each efficient portfolio is a linear combination of k fixed mutual funds. Naturally, k-fund separation is of practical and theoretical interest only when $k << n$. Generally k-fund separation holds in complete markets (see e.g. Dybving and Ross (1982)). Ross (1978) derives a necessary and sufficient condition for k-fund separation which however involves returns only; the result is hard to generalize on possible variation of individual preferences and is somewhat uninformative.[1]

A natural further generalization of k-fund separability is convexity of efficient set. Indeed, the market portfolio is nothing else than a convex combination of all individual portfolios (with unknown positive weights), and therefore convexity of efficient set suffices for efficiency of the market portfolio. Indeed, we may assume without loss of generality that individual assets are optimal for at least one investor with a well-behaved utility function and therefore those assets are efficient (if however we do have an asset whose returns are strictly dominated by another marketed asset or fund, we may as well discard it, as no rational investor will invest in it). Clearly, the market portfolio is now an interior point of a polyhedron whose vertices are all efficient, and if the whole efficient set is known to be convex, the efficiency of the market portfolio automatically follows. In addition to being an implication of various asset pricing theories, efficiency of the market portfolio has an intuitive economic interpretation. Observing the popularity of large composite index funds (which proxy the market portfolio) among many individual and institutional investors in practice, many researchers argue that even heterogeneous investors models inconsistent with the two fund separation should imply efficiency of the market portfolio. Moreover, 2- and k-fund separation are just particular cases of efficient sets being convex.

Conditions leading to convexity of efficient sets have been challenging researchers for more than three decades already, as it could deliver interesting aggregation results for the models of heterogeneous investors. If an economy is close to satisfy k-fund separation, there is no need for active investment, as every investor is better off investing into k available mutual funds (with specific allocation among those funds determined individually for each investor) and saving on transaction costs associated with actively trading strategies. The case when k-fund separation does not hold but the efficient set is convex is still of theoretical interest, as one could study utility

[1] In fact, Theorem 3 in Ross (1978) can be seen as a refinement of the classic definition of k-fund separability, as both are given solely in terms of returns and both assume k generating factors.

preferences that support large composite portfolios, or test implication of heterogeneous investors models, refining utility class on basis of observed individual allocations and composite market indices.

The convexity puzzle can be tackled from two different perspectives – returns on underlying assets and preferences of individual investors. The former would lead the reader towards arbitrage pricing theories and various factor models, while the latter remains undiscovered. Dybvig and Ross (1982) do show on a simple example that SSD efficient set is generally non-convex. However, SSD efficient set comprises portfolios optimal for all non-satiable risk-averse preferences many of which are known to be unrealistic. For that reason, after reviewing and providing a more constructive proof of the results in Rubinstein (1974) related to 2-fund separation, we shall summarize the result of Dybvig and Ross (1982) and give efficiency tests for some refined utility classes containing far less unrealistic preferences than all risk averters.

3.1 HOMOGENEOUS PREFERENCES

Rubinstein (1974) considers three heterogeneous investors models in which individual preferences are modeled according to the following utility functions:

(a) $u(x) \sim -\exp(-x/\mathrm{A})$, $\quad$ A>0
(b) $u(x) \sim -\ln(\mathrm{A} + x)$, $\quad$ A>0
(c) $u(x) \sim (\mathrm{A} + \mathrm{B}x)^{1-b}/(1-b)$, $\quad$ A>0, B>0, b>0, $b \neq 1$

Rubinstein shows that the two-fund separation results if all agents have the same taste parameters B and beliefs π, but may have different parameters A's in (a) and (b) and A's and B's in (c). He assumes availability of a risk-free asset and requires in addition that B=1/b in (c). Below we sketch a more constructive proof of the two fund separation than the original one of Rubinstein and show that varying B's across individuals will not alter the two fund separation, as long as the agents have the same power parameter b, even if $b \neq 1/\mathrm{B}$, thereby generalizing the result of Rubinstein.

Since all the functions above are strictly concave and twice continuously differentiable, sufficient and necessary condition for portfolio optimality is (4). Let r_F be the risk free rate, X – all risky assets. It is convenient to split the portfolio into its risk-free investment α and the remaining risky part $(1-\alpha)$. The portfolio allocation program now becomes

$$\max_{\lambda} \sum_{i=1}^{m} \pi_i u\left(1 + \alpha r_F + (1-\alpha) x^i \lambda\right) - v\left(\lambda^T e - 1\right) \tag{6}$$

The optimality conditions (4) are now:

$$\begin{cases} \sum_{i=1}^{m} \pi_i x_{ij} \left(1-\alpha\right) u'\left(1+\alpha r_F + \left(1-\alpha\right) x^i \tau\right) = v, & \forall\, j = 1..n \\ \sum_{i=1}^{m} \pi_i \left(r_F - x^i \tau\right) u'\left(1+\alpha r_F + \left(1-\alpha\right) x^i \tau\right) = 0 \end{cases} \tag{7}$$

Let us start with the exponential utility class.

(a) $u(x, \mathrm{A}) = -\exp(-x/\mathrm{A})$, $\quad$ A>0.

Suppose a portfolio (α_1, $(1-\alpha_1)\tau$) is optimal for $u(x, \mathrm{A}_1)$. By (7) this happens if and only if

$$\begin{cases} \sum_{i=1}^{m} \pi_i x_{ij} \left(1-\alpha_1\right) \exp\left(-\left(1+\alpha_1 r_F + \left(1-\alpha_1\right) x^i \tau\right) / A_1\right) = v_1, & \forall\, j = 1..n \\ \sum_{i=1}^{m} \pi_i \left(r_F - x^i \tau\right) \exp\left(-\left(1+\alpha_1 r_F + \left(1-\alpha_1\right) x^i \tau\right) / A_1\right) = 0 \end{cases} \tag{8}$$

One can check by straightforward substitution to (8) that for any $\mathrm{A}_2>0$ the optimal portfolio for investor with utility $u(x, \mathrm{A}_2)$ will be (α_2, $(1-\alpha_2)\tau$), for some α_2 for $\alpha_2 = 1 - \mathrm{A}_2/\mathrm{A}_1(1-\alpha_1)$. This proves that the efficient portfolio corresponding to $u(x, \mathrm{A}_2)$ has the same composition of risky assets. Due to uniqueness of the solution to (7), and continuity of $\mathrm{f}(\mathrm{A}_2) = 1 - \mathrm{A}_2/\mathrm{A}_1(1-\alpha_1)$ as a function of A_2, the two-fund separation follows.

Note that we can make derivations above only if $\alpha_1 \neq 1$, that is, not all the budget is invested in the riskless asset. The portfolio (α =1, 0) will not be optimal for any agent $u(x, \mathrm{A})$ with A>0, except only for the case when X happens to satisfy the second equation in (8) for $\alpha = 1$. However, the risk-free asset will always be asymptotically efficient as risk aversion increases.

(b) $u(x) = -\ln(\mathrm{A} + x)$, $\quad$ A>0

Similarly to (a) one can show that portfolio (α_2, $1-\alpha_2$) where $\alpha_2 = \dfrac{A_1+\alpha_1\left(1+r_F\right)-A_2\left(1-\alpha_1\right)}{A_1+1+r_F}$ is optimal for $u(x, \mathrm{A}_2)$ whenever (α_1, $1-\alpha_1$) is optimal for $u(x, \mathrm{A}_1)$.

(c) $u(x) = (\mathrm{A} + \mathrm{B}x)^{1-b}/(1-b)$, A>0, B>0, b>0, $b\neq 1$

Again, a portfolio (α_1, $(1-\alpha_1)\,\tau$) is optimal for $u(x, \mathrm{A}_1, \mathrm{B}_1)$ if and only if

$$\begin{cases} \sum_{i=1}^{m} \pi_i x_{ij}\left(1-\alpha_1\right)\left(A_1+B_1\left(1+\alpha_1 r_F+\left(1-\alpha_1\right)x^i\tau\right)\right)^{-b}=v_1, & \forall\, j=1..n \\ \sum_{i=1}^{m} \pi_i\left(r_F-x^i\tau\right)\left(A_1+B_1\left(1+\alpha_1 r_F+\left(1-\alpha_1\right)x^i\tau\right)\right)^{-b}=0 & \end{cases}$$

Similar equations for $u(x, \mathrm{A}_2, \mathrm{B}_2)$

$$\sum_{i=1}^{m} \pi_i x_{ij}\left(1-\alpha_2\right)\left(A_2+B_2\left(1+\alpha_2 r_F+\left(1-\alpha_2\right)x^i\tau\right)\right)^{-b}=$$

$$\sum_{i=1}^{m} \pi_i x_{ij}\left(1-\alpha_1\right)\frac{\left(1-\alpha_2\right)}{\left(1-\alpha_1\right)}*\begin{bmatrix}\left(A_1+B_1\left(1+\alpha_1 r_F+\left(1-\alpha_1\right)x^i\tau\right)\right)\dfrac{\left(1-\alpha_2\right)}{\left(1-\alpha_1\right)}\dfrac{B_2}{B_1}-\dfrac{A_1B_2}{B_1}\dfrac{\left(1-\alpha_2\right)}{\left(1-\alpha_1\right)}- \\ -B_1\left(1+\alpha_1 r_F\right)\dfrac{\left(1-\alpha_2\right)}{\left(1-\alpha_1\right)}\dfrac{B_2}{B_1}+A_2+B_2\left(1+\alpha_2 r_F+\left(1-\alpha_2\right)x^i\tau\right)\end{bmatrix}^{-b}=$$

$$v_1\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{1-b}\left(\frac{B_2}{B_1}\right)^{-b}\equiv v_2, \qquad \forall j=1..n$$

are satisfied for

$$\alpha_2 \equiv \frac{A_2B_1\left(\alpha_1-1\right)+B_2B_1\left(1+r_F\right)+A_1B_2}{B_2\left(r_FB_1+A_1+B_1\right)},$$

which proves the two-fund separation even for the case of different taste parameters.

Note that this proof generalizes Rubinstein's result, as agents are allowed to have different taste parameters (B's) now, provided they agree on the power parameter b, while Rubinstein explicitly assumes B = 1/b.

Note also that one explicit assumption behind the derivations above is that the number of underlying assets (including the riskless one) is less than or equal to the number of states: $n + 1 \leq m$.

4. GENERALIZING PREFERENCES: NON-CONVEXITY AND SOME HIGHER ORDER EFFICIENCY TESTS

So far we have analyzed the set of utilities leading to two-fund separation. Although this is a particular case of convex SD efficient sets, it only allows for homogeneous utilities among all investors in the sense that the preferences of all investors are assumed to be parameterized by one or two single parameters which implies that investors have very similar tastes and as a result take similar investment decisions. Therefore we would like to broaden the class of individual utility functions to allow for heterogeneity among investors. The question is whether the efficient sets for those extended utility classes remain convex. Consider first the set of all risk-averse and non-satiable investors.

Dybvig and Ross (1982) give a simple example of a non-convex second order SD (i.e. when $U = U_2 = \{u\colon \mathbb{R}\to\mathbb{R}\colon u'(x)>0,\ u''(x)<0,\ \forall x\in\mathbb{R}\}$) efficient set with 3 assets and 4 states. They state the following necessary and sufficient conditions for SSD efficiency of portfolio x^0.

An allocation $x^0\in M_X$ is efficient in U_2 if and only if there exists $z^0 \in \mathbb{R}^m$ such that:

(i) $x^{\mathrm{T}}z^0$ is constant for $x\in M_X$ (9)

(ii) $x_i^0 < x_j^0 \Rightarrow \dfrac{z_i^0}{\pi_i} \geq \dfrac{z_j^0}{\pi_j},\ \forall\, i, j$

(iii) $z^0 > 0$

Vector z^0 can be interpreted as a vector of marginal utility rationalizing portfolio x. Condition (i) holds only if short sales are allowed and unrestricted. Otherwise (i) holds only for strictly interior points. In general (i) reads: $(x^0)^T z^0 \geq x^{\mathrm{T}}z^0,\ \ \forall x\in M_X$. Conditions

(ii) and (iii) reflect the existence of a strictly concave supporting utility function and the equality sign may be changed, depending on the properties of the utility class considered. For instance, for strictly concave functions there should hold: $\frac{z_i^0}{\pi_i} \geq \frac{z_j^0}{\pi_j} \Rightarrow x_i^0 \leq x_j^0$; if the functions are in addition differentiable then $\frac{z_i^0}{\pi_i} > \frac{z_j^0}{\pi_j} \Rightarrow x_i^0 < x_j^0$.

The non-convexity example of Dybvig and Ross is both disappointing and challenging. It shows on one hand that even relative to the set of rather well-behaved preferences the market portfolio can be inefficient. On the other hand, the result challenges us to examine more refined utility classes – after all, if the utility set is restricted to nearly-homogeneous investors like in Rubinstein (1974), not only convexity follows, but even two-fund separation. Taking this into account, below we derive efficiency tests for some higher order SD criteria. In that section we analyse the case when short sales are allowed and unrestricted, for two reasons. As a matter of fact, most of the efficiency tests published by far assume away short sales. However, efficiency of a given portfolio in the unrestricted case implies its efficiency in the restricted case, too, whereas a portfolio efficient relative to a restricted portfolio possibilities set may very well be inefficient with respect to the same set with the short sales restriction relaxed. Therefore, the unrestricted case can be seen as a generalization of the restricted short sales and has a practical advantage of not having to specify the exact boundaries for short sales. Moreover, as we shall show further in this section, some algorithms proposed below have superior properties in terms of computational complexity relative to traditional methods in the case when no short sales are assumed. Finally, some of the efficiency tests are only applicable when short sales are restricted, for instance Post (2003) test assumes the portfolio possibilities set to be a polyhedron, so the formulation of the test includes the vertices of this set explicitly.

4.1 SSD EFFICIENCY

Although many SSD efficiency tests have been proposed already, see Post (2003), Dentcheva and Ruszczynski (2003), Kuosmanen (2004), Post and Versijp (2007) among others, we shall focus on linear programming formulations only, since such methods have the lowest computational complexity which is often a burden for real-life data sets, particularly when it comes to repeating the test many times for statistical inference and bootstrapping or high dimensionality of the data (see for instance Dentcheva and Ruszczynski, 2006). The least computationally demanding SSD efficiency test known by far is Post (2003). In this section we derive another LP test that is even more efficient than that of Post (2003) in the case when short sales are unrestricted. In the following section we derive a TSD efficiency test, also exploiting the special structure of portfolio possibilities set and thereby improving its computational complexity.

Consider a given portfolio x. As the ordering of states of the world is not relevant, we may assume without loss of generality that x is sorted in ascending order:

$x_1 \leq x_2 \leq \ldots \leq x_m$.

In order to determine if x is SSD efficient, we need to find a supporting gradient vector z. First note, that the condition

$$\forall \alpha \in \Lambda: \ (X\alpha)^{\mathrm{T}} z = C \tag{10}$$

is equivalent to

$X^{\mathrm{T}} z = Ce$.

We are interested in the case when the market is incomplete and $m > n$. Without loss of generality we may assume that the first n rows of X are linearly independent. Partitioning X into X_1 (first n rows) and X_2 (the rest $(m\text{–}n)$ rows), we may write:

$$X^T z = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^T z = X_1^T z_{1:n} + X_2^T z_{(n+1):m} = Ce$$

Therefore, the general solution of (10) has m–n free parameters and can be expressed as

$$z = \begin{bmatrix} \left(X_1^T\right)^{-1}\left(Ce - X_2^T \beta\right) \\ \beta \end{bmatrix}, \tag{11}$$

where β is $(m\text{–}n)$-parameter vector. Since only the ordering of elements of z matters, z can be normalized, so that $C = 1$.

Given the criteria above, the portfolio x is efficient if and only if there exists a decreasing positive vector z satisfying (11). If exists, it is also a strictly interior point to the following set:

$$\{\beta \in \mathbb{R}^{m-n} \text{ such that } D\begin{bmatrix} -\left(X_1^T\right)^{-1} X_2^T \\ I_{m-n} \end{bmatrix}\beta \leq -D\begin{bmatrix} \left(X_1^T\right)^{-1} e \\ 0_{m-n} \end{bmatrix}\} \tag{12}$$

where

$$D \equiv \begin{bmatrix} -\pi_1^{-1} & \pi_2^{-1} & 0 & 0 & 0 \\ 0 & -\pi_2^{-1} & \pi_3^{-1} & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -\pi_{m-1}^{-1} & \pi_m^{-1} \\ 0 & 0 & 0 & 0 & -\pi_m^{-1} \end{bmatrix}$$

This test can be equivalently formulated as the following linear program:

$$\max_{\beta\in\mathbb{R}^{m-n},\ \theta\in\mathbb{R}} \left\{ \theta : D\begin{bmatrix} -\left(X_1^T\right)^{-1} X_2^T \\ I_{m-n} \end{bmatrix}\beta + D\begin{bmatrix} \left(X_1^T\right)^{-1} e \\ 0_{m-n} \end{bmatrix} + \theta \le 0 \right\} \tag{13}$$

Portfolio x is SSD efficient if and only if (13) is either unbounded or $\theta^*>0$.[2]

Efficiency test (13) is less computationally demanding than that of Post (2003), since (13) has $m - n + 1$ variables and m constraints, which is n variables and n^2 non-zeros in the constraints matrix less than in Post (2003). By changing variables $\gamma_j = \pi_{j+1}\,\beta_{j+1} - \pi_j\,\beta_j$, $j = 1 \ldots m-1$, and $\gamma_m = \pi_m\,\beta_m$, one can transform (13) to the standard form ($\max\{c^T\gamma\text{: } A\gamma \le b,\ \gamma \ge 0\}$) with an n-by-(m–n) matrix of constraints. The number of non-zeros in this matrix is a good indicator of computational complexity of a linear program (for instance, Performance World (2009) ranks linear programs based on this criterion). The test of Post (2003) in the same standard form will have an n-by-m matrix of constraints all the elements of which are generally non-zeros. The difference of n^2 non-zero elements confirms the computational advantage of (13) relative to Post (2003).

The computational advantage of (13) becomes particularly eminent when n approaches m and in the case of bootstrapping when the efficiency test has to be run many times on multiple data samples generated from the estimated joint distribution of asset returns. However, for large values of n one needs to invert a larger X_1 prior to solving (13). Should X happen to be particularly ill-conditioned, one may use the following equivalent test without decomposing X.

$$\max_{d\in\mathbb{R}^m,\theta\in\mathbb{R}} \left\{ \theta :\ X^T U d = e,\ d \ge 0 \right\}, \tag{14}$$

were U is an upper triangular m-by-m matrix adjusted by the probabilities of the states of nature, d is an m-vector representing the probability-adjusted step differences of vector z, that is $d_j = \pi_{j+1}\, z_{j+1} - \pi_j\, z_j$, $j = 1 \ldots m-1$, and $d_m = \pi_m\, z_m$.

Portfolio x is efficient if and only if $\theta^*>0$.

[2] Equivalently, x is SSD inefficient if (13) is infeasible. The case $\theta^*=0$ also implies inefficiency (non-optimality) by our definition. Some authors consider portfolios corresponding to $\theta^*=0$ efficient as well. In this case the efficiency criterion can be easily adjusted without altering computational complexity of (13).

Program ((14) has m + 1 variables and n equality constraints which is similar to Post (2003) test in terms of computational complexity. However, the two tests are applicable in different circumstances: (13) applies when no short sales restrictions are postulated, whereas the test of Post (2003) requires portfolio possibilities set to be bounded and to contain the subject portfolio x in its interior.

4.2 TSD EFFICIENCY

A portfolio $x^0 \in M_X$ is Third Order SD (TSD) efficient if and only if there exists $u_0 \in U_3$ such that $\mathrm{E}u_0(x^0) = \sup \{\mathrm{E}u_0(x) : x \in M_X\}$, where $U_3 = U_2 \cap \{u\colon u^{(3)}(\mathrm{x})>0\}$.

Employing concavity of the first derivative of any function in U_3, it is straightforward to formulate TSD efficiency criteria:
A portfolio $x^0 \in M_X$ is efficient in U_3 if and only if there exists $z^0 \in \mathbb{R}^m$ such that:

(i) $x^T z^0$ is constant for $x \in M_X$

(ii) $x_i^0 < x_j^0 \Rightarrow \dfrac{z_i^0}{\pi_i} \geq \dfrac{z_j^0}{\pi_j}, \ \forall\, i, j$

(iii) $z^0 > 0$

(iv) $x_i^0 < x_j^0 < x_k^0 \Rightarrow \dfrac{z_j^0}{\pi_j} \leq \dfrac{z_k^0}{\pi_k} + \left(\dfrac{z_i^0}{\pi_i} - \dfrac{z_k^0}{\pi_k} \right) \dfrac{x_k^0 - x_j^0}{x_k^0 - x_i^0}, \quad \forall\, i, j, k$

The corresponding TSD efficiency test for a given portfolio also leads to a linear programming formulation. Indeed, the TSD criteria is nothing else but

$$D\begin{bmatrix} -\left(X_1^T\right)^{-1} X_2^T \\ I_{m-n} \end{bmatrix} \beta \leq -D\begin{bmatrix} \left(X_1^T\right)^{-1} e \\ 0_{m-n} \end{bmatrix}, \text{ where} \tag{15}$$

$$D \equiv \begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & a_{m-2} & b_{m-2} & c_{m-2} \\ 0 & \ddots & 0 & 0 & a_{m-1} & b_{m-1} \\ 0 & 0 & 0 & 0 & 0 & a_m \end{bmatrix}, \text{ with } \begin{cases} \left. \begin{array}{l} a_i \equiv \dfrac{-1}{\pi_i \left(x_{i+2} - x_i\right)} \\ b_i \equiv \dfrac{1}{\pi_{i+1}\left(x_{i+2} - x_{i+1}\right)} \\ c_i \equiv \dfrac{1}{\pi_{i+2}}\left(\dfrac{1}{\left(x_{i+2} - x_i\right)} - \dfrac{1}{\left(x_{i+2} - x_{i+1}\right)} \right) \end{array} \right], \ i = 1 \ldots (m-2) \\ a_{m-1} = -\pi_{m-1}^{-1}, \ a_m = -\pi_m^{-1}, \ b_{m-1} = \pi_m^{-1} \end{cases}$$

System (15) can be solved via the same program (13) that was applied to the SSD test, with redefined matrix D. Therefore, the TST test is a linear program with $m - n + 1$ variables and m constraints.

4.3 SD FOR DECREASING ABSOLUTE RISK AVERSION (DSD)

It is well accepted within expected utility framework that rational individuals possess *decreasing absolute risk aversion* (DARA)[3]. Let us examine Stochastic Dominance efficiency relative to this class of utility functions. Define
$U_{\mathrm{d}} = U_2 \cap \{\mathrm{d}/\mathrm{d}x[-u''(x)/u'(x)]<0, \forall x\}$.
An allocation $x^0 \in M_X$ is efficient in U_{d} (DSD efficient) if and only if there exists $u_0 \in U_{\mathrm{d}}$ such that $\mathrm{E}u_0(x^0) = \sup \{\mathrm{E}u(x^0) : x \in M_X\}$.

To express the risk aversion property in terms of supporting vectors, we have to adapt the efficiency criterion. Let $r(x) \equiv -\dfrac{u''(x)}{u'(x)}$ be the ARA of u(x). We have:
$u'(x) = \exp(-\int r(x)\mathrm{d}x + \mathrm{C})$, and if $x_1 < x_2 < \ldots < x_{\mathrm{n}}$, then

$$u'(x_i) = u'(x_{i-1}) \exp\left(- \int_{x_{i-1}}^{x_i} r(x)dx \right) \tag{16}$$

Therefore, it suffices to require that $r(x) \geq 0$ and $r'(x) < 0$ for all x to ensure that $u \in U_{\mathrm{d}}$. With such $r(x)$, the $\exp\left(- \int_{x_{i-1}}^{x_i} r(x)dx \right)$ is bounded by

$\exp(-r_{i-1}(x_i - x_{i-1})) \leq \exp\left(- \int_{x_{i-1}}^{x_i} r(x)dx \right) \leq \exp(-r_i(x_i - x_{i-1}))$, thus there should hold:

$\exp(-r_{i-1}(x_i - x_{i-1})) \leq \dfrac{u'(x_i)}{u'(x_{i-1})} \leq \exp(-r_i(x_i - x_{i-1}))$, and therefore:

$$r_n \leq \ldots \leq r_{i+1} \leq \frac{-\ln\left(\dfrac{u'(x_{i+1})}{u'(x_i)}\right)}{(x_{i+1} - x_i)} \leq r_i \leq \frac{-\ln\left(\dfrac{u'(x_i)}{u'(x_{i-1})}\right)}{(x_i - x_{i-1})} \leq r_{i-1} \leq \ldots \leq r_1 \tag{17}$$

[3] See e.g. Pratt (1964) for formal derivation and discussion.

As we can see, a decreasing sequence $\{r_i\}$ exists if and only if

$$\frac{\ln(u'(x_i)) - \ln(u'(x_{i+1}))}{(x_{i+1} - x_i)} \leq \frac{\ln(u'(x_{i-1})) - \ln(u'(x_i))}{(x_i - x_{i-1})}, \text{ for all } i=2,\ldots,n-1 \tag{18}$$

We are now ready to adapt the efficiency definition to the class of U_d.

An allocation $x^0 \in M_X$ is efficient in U_d (DSD efficient) if and only if there exists $z^0 \in \mathbb{R}^m$ such that:

(i) $x^T z^0$ is constant for $x \in M_X$ (optimality condition) (19)

(ii) $x_i^0 < x_j^0 \Leftrightarrow \frac{z_i^0}{\pi_i} > \frac{z_j^0}{\pi_j}, \forall\, i, j$ (strict risk aversion)

(iii) $z^0 > 0$ (nonsatiation)

(iv) $x_i^0 < x_j^0 < x_k^0 \Leftrightarrow \frac{\ln\left(\frac{z_j^0}{\pi_j}\right) - \ln\left(\frac{z_k^0}{\pi_k}\right)}{(x_k^0 - x_j^0)} < \frac{\ln\left(\frac{z_i^0}{\pi_i}\right) - \ln\left(\frac{z_j^0}{\pi_j}\right)}{(x_j^0 - x_i^0)}, \forall\, i, j$ (DARA)

Note that DSD efficiency implies TSD efficiency. That follows from the fact that $r'(x) = \frac{(u''(x))^2 - u'(x)u'''(x)}{(u'(x))^2} < 0$. This is also consistent with DSD-TSD criteria: (iv) in DSD implies (iv) in TSD, since the geometric average (in DSD) can never exceed the arithmetic average (in TSD).

We are now ready to formulate a test for DSD efficiency of a given portfolio which will no longer be linear, but still a convex program. Indeed, any $z = \begin{bmatrix} -\left(X_1^T\right)^{-1} X_2^T \\ I_{m-n} \end{bmatrix} \beta$ satisfying the DSD criteria (19) is also a strictly interior point to the set

$$D \ln\left(\begin{bmatrix} \left(X_1^T\right)^{-1}\left(e - X_2^T\right) \\ I_{m-n} \end{bmatrix} \beta \div \pi\right) \leq 0, \text{ where}^4 \tag{20}$$

[4] By $y = \ln(x)$, $x \in \mathbb{R}^m$, we mean element-wise logarithm, that is, $y \in \mathbb{R}^m$ and $y_i = \ln(x_i)$. Similarly, $z = y \div x$, $x,y \in \mathbb{R}^m$, means $z \in \mathbb{R}^m$ s.t. $z_i = y_i / x_i$.

$$D \equiv \begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & a_{m-2} & b_{m-2} & c_{m-2} \\ 0 & \ddots & 0 & 0 & a_{m-1} & b_{m-1} \end{bmatrix}, \text{ with } \begin{cases} \left. \begin{array}{l} a_i \equiv \dfrac{-1}{x_{i+1} - x_i} \\ b_i \equiv \dfrac{1}{x_{i+2} - x_{i+1}} - \dfrac{1}{x_{i+1} - x_i} \\ c_i \equiv \dfrac{-1}{x_{i+2} - x_{i+1}} \end{array} \right], \ i = 1 \ldots (m-2) \\ a_{m-1} = -1, \ \ b_{m-1} = 1 \end{cases}.$$

Even though the constraints on β are no longer linear, they are still convex, and therefore we can find strictly feasible points (or establish that they do not exist) efficiently.

4.4 SD FOR DECREASIND ABSOLUTE AND INCREASING RELATIVE RISK AVERSION (DISD)

In addition to DARA, relative risk aversion is often postulated to be increasing among rational individuals (see e.g. Pratt). In this section we examine optimality conditions in the utility class U_{di} combining the two risk aversion properties:

$U_{\mathrm{di}} = U_{\mathrm{d}} \cap \{\mathrm{d}/\mathrm{d}x[-xu''(x)/u'(x)]>0, \ \forall x>0\}$.

The utility functions under consideration are therefore those having *decreasing absolute* (DARA) and *increasing relative risk aversion* (IRRA).

A portfolio $x^0 \in M_X$ is said to be efficient in U_{di} (DISD efficient) if and only if there exists $u_0 \in U_{\mathrm{di}}$ such that

$\mathrm{E}u_0(x^0) = \sup \{\mathrm{E}u(x^0) : u \in U_{\mathrm{di}}\}$.

Given the ARA values r_{i} and $r_{\mathrm{i+1}}$ (s.t. $r_{\mathrm{i}} \geq r_{\mathrm{i+1}}$) at nodes x_{i} and $x_{\mathrm{i+1}}$, the IRRA requirement restricts $r(x)$ to lie above[5] $f(x) = \left.\dfrac{r_i x_i}{x}\right|_{x=x_i}^{x_{i+1}}$, imposing thereby an extra condition:

$$r_{\mathrm{i}} x_{\mathrm{i}} \leq r_{\mathrm{i+1}} x_{\mathrm{i+1}}, \quad i = 1, \ldots, n. \tag{21}$$

Conversely, if (21) holds, we can always construct $r(x)\big|_{x=x_i}^{x_{i+1}}$ such that $xr(x)$ will be non-decreasing, provided

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[5] *f(x)* is a limiting case of ARA in order for RRA to remain non-decreasing in the interval $[x_{\mathrm{i-1}}, x_{\mathrm{i}}]$. It is the solution of $\{(f(x)x)' = 0, f(x_{\mathrm{i}}) = r_{\mathrm{i}}\}$.

$$r_{i+1}x_{i+1}+r_ix_i\left(\ln\frac{r_i}{r_{i+1}}-1\right)=\int_{x_i}^{x_{i+1}}\max\left\{\frac{r_ix_i}{x},r_{i+1}\right\}dx\le\int_{x_i}^{x_{i+1}}r(x)dx=\ln\frac{u'(x_i)}{u'(x_{i+1})}\le\int_{x_i}^{x_{i+1}}\min\left\{\frac{r_{i+1}x_{i+1}}{x},r_i\right\}dx$$

Therefore

$$r_{i+1}x_{i+1}+r_ix_i\left(\ln\frac{r_i}{r_{i+1}}-1\right)\le\ln\frac{u'(x_i)}{u'(x_{i+1})}\le r_{i+1}x_{i+1}\left(\ln\frac{r_i}{r_{i+1}}+1\right)-r_ix_i \tag{22}$$

This leads to the following DISD efficiency criterion:
An allocation $x^0\in M_X$ is efficient in U_{di} (DISD efficient) if and only if there exist z^0 and $r\in\mathbb{R}^m$ such that:

(i) $x^{\mathrm{T}}z^0$ is constant for $x\in M_X$

(ii) $x_i^0<x_j^0\Leftrightarrow\frac{z_i^0}{\pi_i}>\frac{z_j^0}{\pi_j}$, $\forall\ i,j$

(iii) $z^0>0$

(iv) $x_i^0<x_j^0<x_k^0\Leftrightarrow r_k\le\frac{-\ln\left(\frac{x_k^0\pi_j}{\pi_kx_j^0}\right)}{(x_k^0-x_j^0)}\le r_j\le\frac{-\ln\left(\frac{x_j^0\pi_i}{\pi_jx_i^0}\right)}{(x_j^0-x_i^0)}\le r_i$,

(v) $x_i^0<x_j^0\Leftrightarrow r_jx_j^0+r_ix_i^0\left(\ln\frac{r_i}{r_j}-1\right)\le-\ln\left(\frac{x_j^0\pi_i}{\pi_jx_i^0}\right)\le r_jx_j^0\left(\ln\frac{r_i}{r_j}+1\right)-r_ix_i^0$

The condition above is far less convenient than those for TSD or DSD, as both *r* and *x* are now entering (iv) and (v) in both linear and logarithmic form.

5. CONCLUDING REMARKS

We have pointed out the importance of stochastic dominance efficient sets being convex and further summarized and extended conditions leading to convexity of efficient sets. This property has important practical (passive vs. active investment strategies, efficiency of mutual funds) as well as theoretical value (heterogeneous investors models and asset pricing) and can be analyzed from two different, but interrelated aspects: the returns on underlying assets and the utilities of individual investors. Restricting distributions of returns typically leads to various Factor Models, where complete class of non-satiable and risk-averse investors is assumed. Restricting the set of utilities can also affect efficient sets considerably, as can be seen e.g. in Rubinstein (1974). Unfortunately the extent to which restrictions on sets of utilities affect convexity has not been duly researched.

Based on the efficiency criteria (9), Dybving and Ross (1982) derive the following characterization of SSD efficient sets:

$$E_{SSD} = \bigcup_{z \in Z} \bigcap_{(i,j): \frac{z_i}{\pi_i} < \frac{z_j}{\pi_j}} \left\{ x \in M_X : x_i > x_j \right\} \tag{23}$$

Where the union is taken over all $z \in Z$ having different orderings $\{z_j/\pi_j\}$. Since the dimensionality of z is m, the number of different orderings is at most m! Thus E_{SSD} is a union of a finite number of convex sets.

By analogy, we can explicitly characterize TSD efficient sets:

$$E_{TSD} = \bigcup_{z \in Z} \left(\bigcap_{(i,j): \frac{z_i}{\pi_i} < \frac{z_j}{\pi_j}} \left\{ x \in M_X : x_i > x_j \right\} \bigcap_{(i,j,k): x_i < x_j < x_k} \left\{ x \in M_X : \left(\frac{z_j}{\pi_j} - \frac{z_k}{\pi_k} \right) (x_k - x_i) \le \left(\frac{z_i}{\pi_i} - \frac{z_k}{\pi_k} \right) (x_k - x_j) \right\} \right)$$

Since all restrictions on x are linear, E_{TSD} is again a union of convex sets.

The same applies for DSD:

$$E_{DSD} = \bigcup_{z \in Z} \left(\bigcap_{(i,j): \frac{z_i}{\pi_i} < \frac{z_j}{\pi_j}} \left\{ x \in M_X : x_i > x_j \right\} \bigcap_{(i,j,k): x_i < x_j < x_k} \left\{ x \in M_X : \ln\left(\frac{z_j \pi_k}{\pi_j z_k} \right) (x_j - x_i) < \ln\left(\frac{z_i \pi_j}{\pi_i z_j} \right) (x_k - x_j) \right\} \right)$$

The DISD efficiency characterization is slightly more complex, as r appears along with z:

$E_{DISD} = \bigcup_{\substack{z \in Z \\ r \in R_+^m}} ES_{DI}(z, r)$, where

$$ES_{DI}(z,r) = \bigcap_{(i,j):\frac{z_i}{\pi_i}<\frac{z_j}{\pi_j}} \left\{x \in M_X : x_i > x_j\right\} \bigcap_{(i,j,k):x_i<x_j<x_k} \left\{ r_k \le \frac{-\ln\left(\frac{x_k \pi_j}{\pi_k x_j}\right)}{(x_k - x_j)} \le r_j \le \frac{-\ln\left(\frac{x_j \pi_i}{\pi_j x_i}\right)}{(x_j - x_i)} \le r_i \right\}$$

$$\bigcap_{(i,j):x_i<x_j} \left\{ r_j x_j + r_i x_i \left(\ln\frac{r_i}{r_j} - 1 \right) \le -\ln\left(\frac{x_j \pi_i}{\pi_j x_i}\right) \le r_j x_j \left(\ln\frac{r_i}{r_j} + 1 \right) - r_i x_i \right\}$$

It is not clear if E_{DISD} is necessarily non-convex. On the other hand, it is hard to find a general set of assumptions that would guarantee convexity of a union of convex sets, in contrast to the intersection of convex sets which is automatically convex.

Regarding the link between utility functions and convexity of efficient sets, there are only extreme cases known so far – when investors are nearly homogeneous (like in Rubinstein (1974)), in which case efficient sets are normally rays or lines, and when investors' preferences are spanned unrealistically broadly, like the whole U_2, where efficient sets are too large (even without portfolio restrictions on short sales) and non-convex.

A possible extension of the current research could lie in searching for a reasonable set of well-behaved utility functions for which the efficient sets would be large enough and convex. In analogy to arbitrage pricing theories and factor models for returns, one could try to parameterize investor's preferences. The expo-power utility function of the form $u(x) = \theta - \exp(-\beta x^{\alpha})$, where $\theta > 1$, $\alpha \neq 0$, $\beta \neq 0$, $\alpha\beta > 0$, seems to be a good candidate for that, as it allows all possible combinations of absolute (increasing, decreasing or constant) and relative (increasing or decreasing) risk aversion with just two key parameters.

In addition to the convexity analysis, we have also derived some higher order stochastic dominance efficiency tests in which we incorporate some meaningful restrictions on the set of utilities well recognized in the expected utility framework, such as decreasing absolute and increasing relative risk aversion.

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