Linearization and Decomposition Methods for Large Scale Stochastic Inventory Routing Problem with Service Level Constraints

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ABSTRACT AND KEYWORDS	
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Linearization and decomposition methods for large scale stochastic inventory routing problem with service level constraints

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Abstract:

A stochastic inventory routing problem (**SIRP)** is typically the combination of stochastic inventory control problems and NP-hard vehicle routing problems, for a depot to determine delivery volumes to its customers in each period, and vehicle routes to distribute the delivery volumes. This paper aims to solve a large scale multi-period SIRP with split delivery (**SIRPSD**) where a customer's delivery in each period can be split and satisfied by multiple vehicles if necessary. The objective of the problem is to minimize the total inventory and transportation cost while some constraints are given to satisfy other criteria, such as the service level to limit the stockout probability at each customer and the service level to limit the overfilling probability of the warehouse of each customer. In order to tackle the SIRPSD with notorious computational complexity, we propose for it an approximate model, which significantly reduces the number of decision variables compared to its corresponding exact model. We develop a hybrid approach that combines the linearization of nonlinear constraints, the decomposition of the model into sub-models with Lagrangian relaxation, and a partial linearization approach for a sub model. A near optimal solution of the model can be found by the approach, and then be used to construct a near optimal solution of the SIRPSD. Numerical examples show that, for an instance of the problem with 200 customers and 5 periods that contains about 400 thousands decision variables where half of them are integer, our approach can obtain high quality near optimal solutions with a reasonable computational time on an ordinary PC.

Keywords: Inventory routing problem, Stochastic demand, Split delivery. Vehicle routing

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problem, Lagrangian relaxation

1. Introduction

An inventory routing problem (IRP) is frequently found in a vendor managed inventory (VMI) system with one central vendor (depot) and multiple geographically dispersed customers. The depot operates vehicles with limited capacity for distributing products to its customers. The IRP aims to determine the delivery volume for every customer and a set of feasible vehicle routes for the delivery volumes in each period so that a system wide total inventory and transportation cost is minimized. Such problems are common in VMI systems that have been adopted in many firms like P&G, Dell, HP, Barilla, Wal-Mart (Yu et al., 2009a; Yu et al., 2009b) and Air Products for gas distribution (Adelman, 2004; Bell et al., 1983).

This paper focuses on a multiple-period stochastic inventory routing problem with split delivery (**SIRPSD**) where the depot has a fleet of homogenous capacitated vehicles, and customers' demands are stochastic in each time period (e.g. every day). In such a stochastic setting, except for the minimization of the total inventory and transportation cost, some other criteria, due to the stochastic demands, have to be satisfied, such as the service level to limit the stockout probability at each customer and the service level to limit the overfilling probability of the warehouse of each customer in each period. Moreover, split delivery, which allows the delivery volume to satisfy a customer's demand in each period to be split and served by multiple vehicles, is taken into consideration in this problem because it is common in practice.

SIRP is a class of notoriously difficult problems and related literature can be classified in three categories: those who study a single period problem, those who study an infinite-period problem, and those who study a finite multi-period problem. A single period problem is firstly studied in Federgruen and Zipkin (1984) which considers the corresponding inventory control problem as a newsvendor problem and the corresponding routing problem as a TSP. Dror and Ball (1987) develop a heuristic technique to reduce a long-run average problem to a single period problem. The infinite-period SIRP is mainly stemmed from Kleywegt et al.(2002; 2004) and Adelman (2004). They formulate the problem as a Markov decision problem (MDP) over an infinite horizon where dynamic programming can be applied to solve the problem. To make the MDP solvable, they assume that the state of the system can be observed at the beginning of each period so that an action can be taken, and that customer demands are observed after any action has been made. The state of the system is described through the inventory levels of all the customers. They aim to determine a policy for the MDP to minimize the expected total inventory and transportation costs plus possible revenue gained for each delivery over an infinite horizon. Further extensions of those researches can be found in Hvattum et al.(2009), Lejeune and Ruszczynsk (2007).

For multi-period SIRP, Trudeau and Dror (1992) and Dror and Trudeau (1996) consider stochastic demands over a rolling horizon. Both papers solve a slightly different model with a specific application to the distribution of oil and gas. In these models, a product has to be delivered from one depot to many customers whose demand is different in each period. Trudeau and Dror (1992) develop heuristics to solve their problems by minimizing the long-run average transportation costs, and Dror and Trudeau (1996) focus on maximizing operational efficiency (average number of units delivered in one hour of operation) and minimizing the average number of stockout in each period. Similar literature can be found in Jaillet et al.(2002) and Schwarz et al (2006). The main differences between the above cited papers and our paper lie in:

1) We consider split delivery, that is, one customer’s demand can be satisfied by multiple vehicles. Although split delivery is extensively studied in VRP literature, it is rarely considered in inventory routing problems, especially in stochastic inventory routing problems. As demonstrated by Dror et al. (1994) and Dror and Trudeau (1990), considering split delivery (SDVRP) in a VRP makes it much more difficult to solve. Exact methods can only deal with small instances of the problem with few customers, such as the shortest path approach proposed by Lee et al. (2006) solves only an instance with 7 customers. If a large number of customers are considered, their methods can not guarantee the optimality of solutions, like a tabu search heuristic proposed by Ho and Haugland (2004). For an algorithm that can give the gap between the lower and upper bounds of the optimal value of the problem to evaluate the performance of the algorithm, Belenguer et al. (2000) only consider instances with up to 48 customers with reported gaps between 0% and 12%. As the consideration of split delivery in SIRP significantly increases its complexity, it is a challenge to develop an efficient and effective algorithm to solve SIRPSD.

2) We consider the service levels of customer demands and warehouses, which are rarely treated in SIRP related literature.

3) We consider stochastic demands of any distribution with the help of model simplification and a nonlinear objective function. The demands of each customer in multiple periods can be correlated.

This paper is a stochastic version of our previous work Yu et al. (2008) on a deterministic IRP, but they have distinct differences in their models and solution approaches: 1) the objective function is nonlinear in this paper but linear in Yu et al. (2008); 2) the constraints of the models are different; some service level related constraints are considered in this paper and they are also nonlinear. Therefore, new approaches have to be developed to deal with those stochastic and nonlinear components. We borrow some ideas about how to construct a near optimal solution of the SIRPSD from its model's solution from our previous paper, but the construction approach has to be adapted to the new features of the SIRPSD and its model.

The contributions of the paper include: 1) study a new SIRPSD where the service levels of customers' demands and warehouses are considered, 2) propose a hybrid approach to find near-optimal solutions of the SIRPSD for large instances (i.e., with 200 customers). In order to efficiently solve such kind of large instances of the problem, we propose the following approaches:

Firstly, we propose an approximate stochastic IRP model instead of an exact stochastic model. The approximate model allows us not to dedicate decision variables to individual vehicles since the vehicles considered are homogeneous. This can significantly reduce the number of decision variables. For example, if the vehicle fleet size is 20, our modeling only requires 1/20 vehicle related decision variables compared with an exact model where decision variables are dedicated to individual vehicles. Although the solution of such an approximate model might not be a feasible solution of the studied SIRPSD, the infeasibility can be effectively repaired without affecting solution quality. Moreover the optimal solution of the approximate model provides a lower bound of the optimal cost of our studied SIRPSD.

Secondly, we transform the approximate stochastic model into a simplified deterministic model which is easier to solve. Meanwhile some constraints are eliminated and the feasible domains of some decision variables are reduced without losing optimal solutions (see subsection 2.2).

Thirdly, we develop a Lagrangian relaxation approach to decompose the model into sub-models, which are an inventory problem and a vehicle routing problem, respectively. The inventory problem is nonlinear and is solved by a partial linearization approach. The routing problem is further decomposed into many smaller subproblems which can be quickly solved.

Finally, assignment problems are introduced to construct feasible solutions of the SIRPSD. Some local search improvements are also proposed to improve the quality of the found feasible solutions of the SIRPSD.

Besides, our approach can provide a tight lower bound of the optimal cost of the studied SIRPSD for evaluating the quality of a feasible solution of the SIRPSD; a lower bound of the SIRPSD is provided by the optimal solution of approximate model. The lower bound of the approximate model can be obtained from the optimal Lagrangian dual value provided by the Lagrangian relaxation approach. Therefore the dual value is a lower bound of the SIRPSD. The quality of the feasible solution of the SIRPSD can therefore be evaluated by the gap between the cost of the found solution of the SIRPSD and the dual value. The smaller the gap is, the better the solution is.

The rest of this paper is organized as follows: In next section, the approximate model is proposed and simplified. The near optimal solution of the approximate model is found in Section 3, based on which the near optimal solution of the studied SIRPSD is found in Section 4. In Section 5, a special case of customer demand probability distribution is analyzed and the performance of our proposed approach is evaluated. Section 7 concludes the paper.

2. Approximate generic model and it simplification

The studied multiple-period SIRPSD consists of multiple customers, a central depot, and a fleet of vehicles, where

(1) Each customer's demand is stochastic in each period, and the customers require the depot to satisfy their demands with a certain service level by limiting the possibility of stockout within a given value. The stockout of one period can not be compensated by that of its immediate next period.

(2) The depot is responsible for distributing a product to satisfy the requirements of its customers on demand and service levels by a fleet of homogeneous and capacitated vehicles. Note that assuming homogeneous vehicles is common in literature (Fumero and Vercellis, 1999;

Hvattum et al., 2009; Yu et al., 2008).

(3) A multi-period horizon is considered. Periodically, the depot has to make a planning for the next time horizon about when and how much every customer should be replenished. Because the demands of each customer are stochastic but its delivery volumes over the time horizon have to be determined at the beginning of the time horizon, the customer's warehouse may be overfilled in the next period if the demand in the current period is low but the delivery volume for the next period is high. We therefore have to consider the service levels of warehouses in order to model the limit of the overfilling possibility.

(4) Split delivery is allowed. In practice, if a customer's demand is large, the delivery volume of the customer is most likely to be served by multiple vehicles.

(5) The objective is to minimize the total inventory and transportation cost over a given time horizon subject to given service level constraints. The inventory cost depends on the inventory level of each customer at the end of each period. The transportation cost includes not only fixed usage cost which is related to vehicle insurance, depreciation, and drivers' rewards, but also a variable cost, which depends both on transported quantity and traveled distance. This transportation cost structure, adopted by Fumero and Vercellis (1999), can not only model purely distance proportional cost components (such as fuel costs) in classical VRP but also model the transportation cost in the third party logistics where the transportation cost charged is usually proportional to the shipped volume.

The related notations are given as follows.

Indices

$i, j = 0, 1, ..., N$ Index of customer or depot, where $i, j = 1, ..., N$ are customer indexes, and 0 is the depot index,

t=1,…, T	Period index,
Parameters	
C	Vehicle capacity in volume,
c_{ij}	Variable shipping cost per unit of product along arc (i, j) where $c_{ij} = c_{ji}$ and triangle inequality holds ($c_{ij} + c_{jk} \geq c_{ik}$),
c_{i0}^{b}	traveling cost of an empty vehicle from customer i back directly to depot,
f_t	Fixed vehicle cost per tour in period *t*,
h_{it}	Holding cost per unit product for customer i in period t,
I_{i0}	Initial inventory level at beginning of period 1,
I_{it}	Inventory level of customer i at the end of period t,
I_{it}^{+} =max(0, I_{it})	On-hand inventory of customer i at the end of period t,
V_i	The inventory capacity for customer i,
α_{it}	Service level for customer i in period t (probability in which customer *i*'s demand is satisfied in period *t*.),
β_{it}	The service level of customer i 's warehouse in period *t* (probability in which customer i 's warehouse is not overfilled in period *t*),
ζ_{it}	Stochastic demand for customer i in period t,
$\zeta_{i,(1,t)}$	$\sum_{s=1}^{t} \zeta_{is}$ cumulative stochastic demand from period 1 to t,
$F_{i,(1,t)}(.)$	Accumulative probability distribution function of stochastic demand $\zeta_{i,(1,t)}$,
Variables	
d_{it}	Delivery volume to customer i in period t,
q_{ijt}	Demand quantity transported on directed arc (i, j) in period t,
x_{ijt}	The number of the times that customer j is visited directly after customer i in period t.

2.1 Approximate model

The approximate model for the SIRPSD (denoted by **P**) can therefore be formulated as:

Model **P**:

$$Z = \min \sum_{t=1}^{T} E(\sum_{i=1}^{N} h_{it} I_{it}^{+}) + \sum_{t=1}^{T} \sum_{\substack{j=0 \\ j \neq i}}^{N} \sum_{i=0}^{N} c_{ij} q_{ijt} + \sum_{t=1}^{T} \sum_{i=1}^{N} f_t x_{i0t} + \sum_{t=1}^{T} \sum_{i=1}^{N} c_{i0}^{b} x_{i0t} \tag{1}$$

Subject to

$$I_{it} = I_{i,0} + \sum_{s=1}^{t} d_{is} - \sum_{s=1}^{t} \zeta_{is} \quad i = 1,...,N, \ t = 1,...,T \tag{2}$$

$$\text{Prob}(I_{it} \geq 0) \geq \alpha_{it} \quad i = 1,2,...,N, \ t = 1,...,T \tag{3}$$

$$\text{Prob}(I_{i,t-1} + d_{it} \leq V_i) \geq \beta_{it} \quad i = 1,2,...,N, \ t = 2,....,T \tag{4}$$

$$I_{i,0} + d_{i1} \leq V_i \quad i = 1,...,N \tag{5}$$

$$\sum_{\substack{j=0 \\ j \neq i}}^{N} x_{ijt} = \sum_{\substack{j=0 \\ j \neq i}}^{N} x_{jit} \quad i = 0,...,N, \ t = 1,...,T \tag{6}$$

$$\sum_{\substack{j=0 \\ j \neq i}}^{N} q_{jit} - \sum_{\substack{j=0 \\ j \neq i}}^{N} q_{ijt} = d_{it} \quad i = 1,...,N, \ t = 1,...,T \tag{7}$$

$$\sum_{i=1}^{N} q_{0it} = \sum_{i=1}^{N} d_{it} \quad t = 1,...,T \tag{8}$$

$$q_{ijt} \leq C \cdot x_{ijt} \quad i = 0,...,N, \ j = 1,...,N \ i \neq j, \ t = 1,...,T \tag{9}$$

$$d_{it} \geq 0 \ \ i = 1,...,N, \ q_{ijt} \geq 0 \ \ i = 0,...,N \ \ j = 1,...,N \ \ j \neq i \ \ t = 1,...,T \tag{10}$$

$$x_{ijt} \geq 0 \text{ and integer} \quad i, j = 0,...,N, \ i \neq j, \ t = 1,...,T. \tag{11}$$

Equation (1) gives the total cost including both expected inventory cost for all customers ($\sum_{t=1}^{T} E(\sum_{i=1}^{N} h_{it} I_{it}^{+})$), variable transportation cost ($\sum_{t=1}^{T} \sum_{\substack{j=0 \\ j \neq i}}^{N} \sum_{i=0}^{N} c_{ij} q_{ijt}$), fixed transportation cost ($\sum_{t=1}^{T} \sum_{i=1}^{N} f_t x_{i0t}$) and transportation cost with empty vehicles ($\sum_{t=1}^{T} \sum_{i=1}^{N} c_{i0}^{b} x_{i0t}$). It is a stochastic version of the cost structure in Yu et al. (2008) and Fumero and Vercellis (1999). Constraints (2) are the inventory balance constraints for individual customers. Constraints (3) ensure that the probability for customer i's demand satisfied in period t is no less than α_{it} for period t = $1,\ldots,T$, which represents the service levels of the depot to satisfy customer demand in each period. Constraints (4) describe the service levels related to the capacities of the customers' warehouses and guarantee that the probability of customer i's warehouse capacity being able to accommodate its maximum inventory level is no less than β_{it} at period $t = 2,..,T$. Constraints (5) ensure that every customer's warehouse inventory capacity should be no less than its maximum inventory level in period 1. Constraints (6) ensure that the number of vehicles leaving from a customer or the depot is equal to that of arriving vehicles. Constraints (7) are the product flow conservation equations, ensuring flow balance at each customer and eliminating possible subtours. Constraints (8) assure the total volume shipped from the depot equals the total delivery volume of all the customers in each period. Constraints (9) model the vehicle capacity and logical relationship between q_{ijt} and x_{ijt}.

Model P defines some necessary conditions of a feasible solution of the SIRPSD and therefore its optimal solution provides a lower bound of the optimal cost of the SIRPSD. However, in the decision variables q_{ijt} and x_{ijt}, no subscripts are dedicated to individual vehicles, and feasible q_{ijt} and x_{ijt} of model P may be infeasible for the SIRPSD. q_{ijt} and x_{ijt} have to be split and assigned to individual vehicles in order to make them feasible for the studied SIRPSD. In the following, we firstly simplify the model (see subsection 2.2), and find its near optimal solutions (see section 3). How to construct a near optimal solution of the studied SIRPSD will be given later in Section 4.

2.2 Model Simplification

Model P can be simplified from three aspects: 1) transforming the stochastic terms in Model P (i.e., Equations (1), (2), (3) and (4)) into deterministic ones, 2) adding some valid constraints to reduce the feasible domains of the decision variables, 3) simplifying some decision variables.

Transformation of the stochastic terms. The stochastic terms are in (1), (2), (3) and (4). For the objective function (1), by substituting Equation (2) into the objective function (1), $E(\sum_{i=1}^{N} h_{it} I_{it}^{+})$ can be reformulated as

$$\sum_{i=1}^{N} h_{it} E(I_{it}^{+}) = \sum_{i=1}^{N} h_{it} \int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x) dF_{i,(1,t)}(x) .$$

For Constraints (3), correspondingly, substituting (2) into Constraints (3), we have

$\text{Prob}(I_{it} \ge 0) = \text{Prob}(I_{i,0} + \sum_{s=1}^{t} d_{is} \ge \sum_{s=1}^{t} \zeta_{is}) = \int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} dF_{i,(1,t)}(x) = F_{i,(1,t)}(I_{i,0} + \sum_{s=1}^{t} d_{is})$ $-F_{i,(1,t)}(0) = F_{i,(1,t)}(I_{i,0} + \sum_{s=1}^{t} d_{is})$ because $F_{i,(1,t)}(0)$ =0 for all practical purposes. Consequently, Constraint (3) can be reformulated as $F_{i,(1,t)}(I_{i,0} + \sum_{s=1}^{t} d_{is}) \ge \alpha_{it}$, or equivalently as

$$\sum_{s=1}^{t} d_{is} \ge F_{i,(1,t)}^{-1}(\alpha_{it}) - I_{i,0} \qquad i = 1,2,...,N \ , \ t = 1,...,T \ . \tag{3'}$$

For Constraints (4), substituting (2) into Constraints (4), $\text{Prob}(I_{i,t-1} + d_{it} \le V_i) =$ $\text{Prob}(I_{i,0} + \sum_{s=1}^{t-1} d_{is} - \sum_{s=1}^{t-1} \zeta_{is} + d_{it} \le V_i) = \text{Prob}(I_{i,0} + \sum_{s=1}^{t} d_{is} - V_i \le \sum_{s=1}^{t-1} \zeta_{is}) = \int_{I_{i,0}+\sum_{s=1}^{t} d_{is} - V_i}^{+\infty} dF_{i,(1,t-1)}(x)$ $=1- F_{i,(1,t-1)}(I_{i,0} + \sum_{s=1}^{t} d_{is} - V_i)$. Thus Constraints (4) can be formulated as $F_{i,(1,t-1)}(I_{i,0} + \sum_{s=1}^{t} d_{is} - V_i)$ $\le 1- \beta_{it}$, or equivalently as:

$$\sum_{s=1}^{t} d_{is} \le V_i + F_{i,(1,t-1)}^{-1}(1 - \beta_{it}) - I_{i,0} \qquad i = 1,2,...,N \quad t = 2,....,T \ . \tag{4'}$$

As the result of the above transformation, constraints (2) are removed simultaneously.

Once $F_{i,(1,t-1)}^{-1}(\cdot)$ is known, constraints (3') and (4') become linear now.

Addition of valid constraints. Without proof, the optimal solution of Model P must satisfy Constraints (12) below:

$$q_{i0t} = 0 \qquad i = 1,...,N \quad t = 1,...,T \tag{12}$$

The constraints imply that each vehicle must be empty when it returns to the depot.

Simplification of decision variables. With Theorem 1 below, x_{ijt} for $i, j = 1,..., N$ as integer can be simplified as binary variables, $x_{ijt} \in \{0,1\}$.

Theorem 1. If model **P** is feasible, and c_{ij} $i, j = 1,...N$ satisfy the triangle inequality, then the model has an optimal solution where no two routes with the same direction have more than one common customer, i.e., $x_{ijt} \in \{0,1\}$ for $i, j = 1,...N$.

Here Theorem 1, taken from Yu et al. (2008), is given directly without proof. The theorem 1 is proved by Dror and Trudeau (1990) in case of VRP with split delivery.

Therefore Constraints (11) can be replaced by

$$x_{ijt} \in \{0,1\} \quad j \neq i, \; x_{i0t}, x_{0jt} \text{ integer } i, j = 1,..., N \tag{11'}$$

According to the above analysis, Model **P** can be simplified as the following equivalent model, (denoted by **P'**).

Model **P'**:

$$\min J = \sum_{t=1}^{T}\sum_{i=1}^{N} h_{it}\int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x) dF_{i,(1,t)}(x) + \sum_{t=1}^{T}\sum_{\substack{j=1 \\ j\neq i}}^{N}\sum_{i=0}^{N} c_{ij} q_{ijt} + \sum_{t=1}^{T}\sum_{i=1}^{N} f_t x_{i0t}$$

$$+ \sum_{t=1}^{T}\sum_{i=1}^{N} c_{i0}^{b} x_{i0t} \tag{1'}$$

Subject to Constraints (5)-(10), (3'), (4'), (11'), and (12).

3. Solution methodology of Model P'

Model P' is obviously NP-hard as its simplified single period problem without considering inventory is SDVRP that is NP-hard. This motivates us to seek for approximate approaches to solve the problem and Lagrangian relaxation (LR) approach is selected since it can decompose our model into easily solvable sub-problems.

In this section the Lagrangian relaxation (LR) approach to find a near optimal solution of model P' will be presented. The solution will then be used to construct a feasible near optimal solution of the SIRPSD using a heuristic approach. The optimal dual value obtained by the Lagrangian relaxation approach provides a lower bound of the optimal cost of the studied SIRPSD for evaluating the quality of the feasible solution of SIRPSD.

3.1 Lagrangian relaxation

In Model P', the constraints that complicate the resolution of this problem are constraints (9) which couple q_{ijt} and x_{ijt}. They are relaxed by introducing non-negative Lagrange multipliers

$\lambda = (\lambda_{ijt})_{(N+1)\times N\times T}$ with a penalty term added to the objective function (1’). The corresponding Lagrangian relaxed problem (denoted by RP) can be formulated as:

Model **RP**:

$$Z_\lambda(d,q,x) = \sum_{t=1}^{T}\sum_{i=1}^{N} h_{it}\int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x)dF_{i,(1,t)}(x) + \sum_{t=1}^{T}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{i=0}^{N} c_{ij}q_{ijt} + \sum_{t=1}^{T}\sum_{i=1}^{N} f_t x_{i0t} +$$

$$\sum_{t=1}^{T}\sum_{i=1}^{N} c_{i0}^{b} x_{i0t} + \sum_{t=1}^{T}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{i=0}^{N} \lambda_{ijt}(q_{ijt} - C\cdot x_{ijt}) = \sum_{t=1}^{T}\sum_{i=1}^{N} h_{it}\int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x)dF_{i,(1,t)}(x)$$

$$+\sum_{t=1}^{T}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{i=0}^{N} (\lambda_{ijt} + c_{ij})q_{ijt} + \sum_{t=1}^{T}\sum_{i=1}^{N} (c_{i0}^{b} + f_t)x_{i0t} \text{ - } C\sum_{t=1}^{T}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{i=0}^{N} \lambda_{ijt}x_{ijt} \tag{13}$$

subject to $\lambda \geq 0$, (3’), (4’), (5)-(8), (10), (11’), and (12).

The problem can therefore be decomposed into the following two independent subproblems while global minimization is reserved.

The inventory subproblem (denoted by **INV**), which determines the d, q values, can be formulated as:

$$Z_\lambda^1(d,q) = \min \sum_{t=1}^{T}\sum_{i=1}^{N} h_{it}\int_0^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x)dF_{i,(1,t)}(x) + \sum_{t=1}^{T}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{i=0}^{N} (\lambda_{ijt} + c_{ij})q_{ijt} \tag{14}$$

subject to Constraints (3’), (4’), (5), (7), (8), (10) and (12).

The routing subproblem (denoted by **ROU**), which determines the x values, can be formulated as:

$$Z_\lambda^2(x) = \min \sum_{t=1}^{T}\sum_{i=1}^{N} (c_{i0}^{b} + f_t)x_{i0t} \text{ - } C\sum_{t=1}^{T}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{i=0}^{N} \lambda_{ijt}x_{ijt} \tag{15}$$

subject to (6) and (11’).

For **ROU**, it can be further decomposed into T independent subproblems, one for each period, given by:

$$Z_\lambda(x_{(t)}) = \min \sum_{i=1}^{N} (c_{i0}^{b} + f_t)x_{i0t} \text{ - } C\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{i=0}^{N} \lambda_{ijt}x_{ijt} \tag{16}$$

subject to Constraints (11’) and

$$\sum_{\substack{j=0\\ j\neq i}}^{N} x_{ijt} = \sum_{\substack{j=0\\ j\neq i}}^{N} x_{jit} \quad i = 0,...,N \,. \tag{17}$$

Denote $D(\lambda)$ be the function of the optimal objective value of RP for any given Lagrange multipliers $\{\lambda_{ijt}\}_{(N+1)\times N\times T}$. The Lagrangian dual problem (denoted by **DP**) is

Model **DP**:

$$\max_{\lambda} D(\lambda) \tag{18}$$

where $\max D(\lambda)$=max{ $Z_{\lambda}(d,q,x)$ | s.t. $\lambda \geq 0$, (3'), (4'), (5)-(8), (10), (11'), and (12)}.

For each given $\{\lambda_{ijt}\}_{(N+1)\times N\times T}$ >0, we have:

$Z_{\lambda}(d,q,x) = Z_{\lambda}^{1}(d,q) + Z_{\lambda}^{2}(x) = Z_{\lambda}^{1}(d,q) + \sum_{t\in T} Z_{\lambda}^{2t}(x_{(t)})$ and $Z_{\lambda}(d,q,x) \leq Z^{*}$, where Z^{*} is the optimal value of model **P'**.

3.2 Partial linearization for subproblem INV

The subproblem **INV** is a nonlinear programming problem and can not be easily solved by using a commercial software such as Lingo. Fortunately, the objective function of **INV** is obviously a convex and increasing function of d_{it} $t = 1,...,T$, and all the constraints of **INV** are linear. It can be solved by using a partial linearization method proposed by Patriksson (1993) as follows.

Defining $G_{it}(d_{i1},...,d_{it}) = h_{it}\int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x)F_{i,(1,t)}^{'}(x)dx$, we have

$$\sum_{t=1}^{T}\sum_{i=1}^{N} h_{it}\int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} (I_{i,0} + \sum_{s=1}^{t} d_{is} - x)dF_{i,(1,t)}(x) = \sum_{t=1}^{T}\sum_{i=1}^{N} G_{it}(d_{i1},...,d_{it}).$$

The gradient of function $G_{it}(d_{i1},...,d_{it})$ with respect to d_{iv} $v = 1,...,t$ is

$$\frac{\partial G_{it}(d_{i1},...,d_{iv},...,d_{it})}{\partial d_{iv}} = h_{it}\int_{0}^{I_{i,0}+\sum_{s=1}^{t} d_{is}} F_{i,(1,t)}^{'}(x)dx = h_{it}F_{i,(1,t)}(I_{i,0} + \sum_{s=1}^{t} d_{is}),\ \ v = 1,...,t\,.$$

At any point d_{it}^{k}, $t = 1,...,T$, $i = 1,...,N$, the linearization of

$\sum_{t=1}^{T}\sum_{i=1}^{N}h_{it}\int_0^{I_{i,0}+\sum_{s=1}^{t}d_{is}}(I_{i,0}+\sum_{s=1}^{t}d_{is}-x)dF_{i,(1,t)}(x)$ is thus $\sum_{t=1}^{T}\sum_{i=1}^{N}h_{it}\int_0^{I_{i,0}+\sum_{s=1}^{t}d_{is}^k}(I_{i,0}+\sum_{s=1}^{t}d_{is}^k-x)dF_{i,(1,t)}(x)$

$+\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{s=1}^{t}(d_{is}-d_{is}^k)h_{it}F_{i,(1,t)}(I_{i,0}+\sum_{s=1}^{t}d_{is}^k))$.

The linearization of $Z_\lambda^1(d,q)$ at the point, denoted by $\bar{Z}^1((d,q),(d^k,q^k))$, is thus

$$\bar{Z}^1((d,q),(d^k,q^k))=\sum_{t=1}^{T}\sum_{i=1}^{N}h_{it}\int_0^{I_{i,0}+\sum_{s=1}^{t}d_{is}^k}(I_{i,0}+\sum_{s=1}^{t}d_{is}^k-x)dF_{i,(1,t)}(x)$$

$$+\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{s=1}^{t}(d_{is}-d_{is}^k)h_{it}F_{i,(1,t)}(I_{i,0}+\sum_{s=1}^{t}d_{is}^k))+\sum_{t=1}^{T}\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{i=0}^{N}(\lambda_{ijt}+c_{ij})q_{ijt} \qquad (19).$$

The partial linearization method solves a linear programming problem and performs a line search at each iteration. For our problem, at each iteration k , the method solves the following linear programming problem (denoted by IP^k):

IP^k :

$$\min\ \bar{Z}^1((d,q),(d^k,q^k)) \qquad (20)$$

subject to Constraints (3’), (4’), (5), (7), (8), (10) and (12).

and performs a line search to minimize $Z_\lambda^1(d,q)$ for subproblem **INV**, i.e.,

$$\operatorname*{Min}_{\rho}\{Z_\lambda^1(d,q)\,|\,(d,q)=\rho(d^k,q^k)+(1-\rho)(\bar{d}^k,\bar{q}^k),0\le\rho\le1\}, \qquad (21)$$

where $(\bar{d}^k,\bar{q}^k)$ is an optimal solution of IP^k . The starting point (d^{k+1},q^{k+1}) of the iteration $k+1$ is taken as the solution of the line search at the iteration k .

The iterative procedure continues until (d^k,q^k) also solves IP^k . That is, $\bar{Z}^1((d,q),(d^k,q^k))=\bar{Z}^1((d,q),((\bar{d}^k,\bar{q}^k)))$. Initially at k = 0, (d^k,q^k) is taken as a feasible solution of subproblem **INV.**

3.3 Minimum cost flow for ROU subproblem

The constraint matrix of subproblem **ROU** is totally unimodular and the right-hand side are integers, so every basic feasible solution is integral (see Wolsey (1998). In other words, solving the problem as a linear program using the simplex method always yields an integral solution.

Moreover, subproblem **ROU** can be transformed into a minimum cost flow (MCF) problem that can be solved by the out-of-kilter algorithm, Klein, Jewell, Busacker & Gowan's method etc. These algorithms run in polynomial time, and have a lower complexity than the fastest linear programming algorithms. In our **ROU** subproblem, $x_{ijt} \in \{0,1\}$ thus is relaxed to $0 \le x_{ijt} \le 1$ and **ROU** subproblem be solved by minimum cost flow method.

3.4 Subgradient method for the dual problem

The Lagrangian relaxation approach maximizes the dual objective (18) by using subgradient method. We use adaptive step sizing strategy to set the step size of the method in each iteration. The algorithm steps for the Lagrangian relaxation is given as follows.

Step 0. Give an initial value λ^0=0, $\theta^0 = 1$ and k=0

Step 1. Calculate subproblem **INV** and subproblem **ROU**.

Step 2. Calculate step size s^k in iteration k by

$$s^k = \beta(L^* - \tilde{L}^k)/\left\|g^k\right\|^2$$

where β is a parameter with $0 < \beta < 1$, $\tilde{L}^k$ is current lower bound, $\tilde{L}^{[k]}$ is the best dual obtained prior to iteration k, L^* is estimated by $(1+\dfrac{\omega}{\theta^\rho})\tilde{L}^{[k]}$, where $\omega \in [0.1, 1.0]$, $\rho \in [1.1, 1.5]$, $\theta^{k+1} = \max(1, \theta^k - 1)$ (θ^k is value of θ in iteration k if $L^k > \tilde{L}^{[k]}$, otherwise $\theta^{k+1} = \theta^k + 1$), and $\left\|g^k\right\|^2 = \sum_{t=1}^{T}\sum_{\substack{j=1 \\ j\ne i}}^{N}\sum_{i=0}^{N}(q_{ijt}^k - C\cdot x_{ijt}^k)^2$.

Step 3. Calculate $\lambda_{ijt}^{k+1} = \max\{\lambda_{ijt}^k + s^k(q_{ijt}^k - C\cdot x_{ijt}^k), 0\}$.

Step 4. Check stopping criteria. The criteria may be given by

(1) $\sum_{ijt}\left|\lambda_{ijt}^{k+1} - \lambda_{ijt}^k\right| \le \varepsilon_1$ or $\left\|\lambda^{k+1} - \lambda^k\right\| \le \varepsilon_2$, or

(2) a given maximal iteration time reached; where ε_1 and ε_2 are given little positive numbers.

If the criterion is met, stop and output all required results. Otherwise, set k=k+1 and go to Step 1.

The solution of the Lagrangian relaxation problem is not only provide a lower bound of Model P', then the SIRPSD, but also can be used to construct a near optimal feasible solution of Model P'.

3.5 Feasible solution construction for the Model P'

Based on d, q obtained by solving the Lagrangian relaxed problem, similar to Yu et al. (2008), a feasible solution Model P' can be constructed by solving the following problem, denoted by **FP**.

Model **FP**:

$$Z_{\lambda}(x) = \min \sum_{t=1}^{T}\sum_{i=1}^{N}(c_{i0}^{b} + f_t)x_{i0t} \tag{22}$$

Subject to

$$\sum_{\substack{j=0 \\ j\neq i}}^{N} x_{ijt} = \sum_{\substack{j=0 \\ j\neq i}}^{N} x_{jit} \quad i = 0,...,N \quad t = 1,...,T \tag{23}$$

$$\left\lceil \frac{q_{ijt}}{C} \right\rceil \leq x_{ijt} \qquad i = 0,...,N\,, \quad j = 1,...,N \;\; i \neq j \quad t = 1,...,T \tag{24}$$

$$x_{ijt} \in \{0,1\} \quad j \neq i\,, \;\; x_{i0t}, x_{0jt} \;\; \textbf{integer} \;\; i, j = 1,...,N \quad t = 1,...,T \tag{25}$$

where q_{ijt} $i = 0,...,N$, $j = 1,...,N$ $i \neq j$ is obtained from the solution of the relaxed problem solved by subgradient method in subsection 3.4. The problem FP can be reformulated as a minimal cost flow problem again by relaxing *x* to:

$$\left\lceil \frac{q_{ijt}}{C} \right\rceil \leq x_{ijt} \leq 1 \quad i = 1,...,N\,, \quad j = 1,...,N \;\; i \neq j \quad t = 1,...,T \tag{26}$$

$$x_{0jt} \geq \left\lceil \frac{q_{0jt}}{C} \right\rceil, \quad j = 1,...,N \quad t = 1,...,T \tag{27}$$

The problem can be decomposed into T sub-problems, one for each period.

In order to obtain a good feasible solution of Model P', the feasible solution is constructed based on every solution obtain in every iteration of the subgradient method in subsection 3.4. The best one (with the smallest total cost measured by Equation (1')) is selected as the final one.

4. Feasible solution construction and improvement for the SIRPSD

This section provides an approach to repair a obtained feasible solution of Model P' in subsection

3.5 to a feasible solution of the studied SIRPSD, and improve them with some local search improvements.

4.1 Repair a solution of P' to a feasible solution of SIRPSD

In subsection 3.5, a feasible solution, d, q, x of Model P', is obtained, but this solution is not implementable since *q* has not been dedicated to individual vehicles as feasible routes. Moreover, the solution may not define a feasible solution of the original problem SIRPSD. (shown later in Figure 2).

In order to obtain a feasible solution of the SIRPSD, a method is required to trace a set of feasible routes in every period based on the solution of model P'. Because the method is the same for every period, for simplification we omit the subscript *t* of the corresponding variables and parameters in the following discussion. Similar to the approach in (Yu et al., 2008), the method can be represented with the following steps.

Step 1. Build a directed transportation graph. With x values of the feasible solution of Model P', a directed **transportation graph,** as exemplified in Figure 1(a), can be defined in each period t where two customer nodes (or a customer and the depot nodes) i and j are connected x_{ij} times by directed arcs (i, j) if $x_{ij} \geq 1$. The depot is split into two virtual ones: an outgoing depot (numbered as 0) and an incoming depot (numbered as 0'). The directed arcs associated with $\{x_{ji} \mid x_{ji} \geq 1, j = 0,..., N\}$ are called **incoming** arcs of customer node i, and the directed arcs associated with $\{x_{ij} \mid x_{ij} \geq 1, j = 0,..., N\}$ are called **outgoing** arcs of customer node i. The customer nodes are virtually numbered here by the rule: an unnumbered customer node *i* can be numbered as next if and only if all the notes pointed to *i* have been numbered. The numbering will give the sequence to assign arcs into vehicle routes.

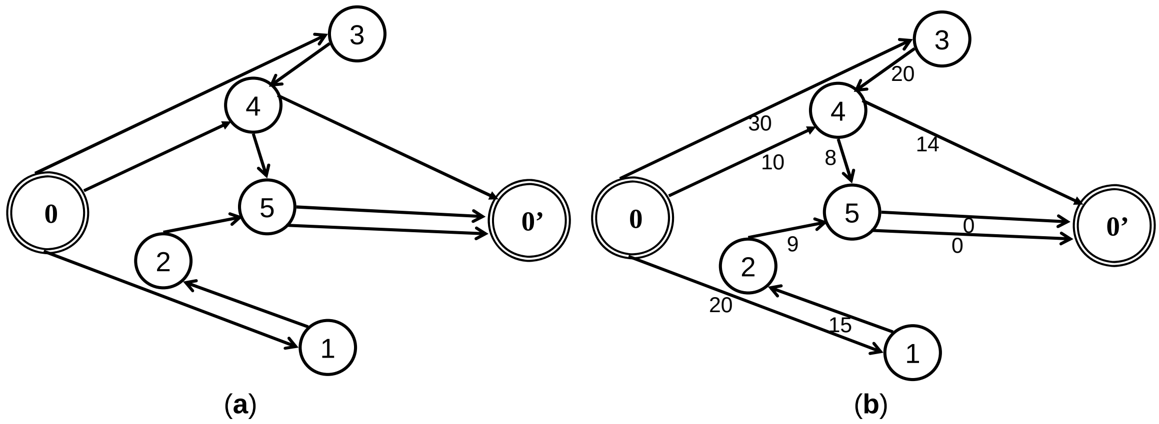

Figure 1. A directed transportation graph

Step 2. Evaluate directed arcs. In the feasible solution of model P', for each $x_{ij} \geq 1$ corresponds a q_{ij}. We evaluate all directed arcs with q_{ij} to obtain evaluated graph as shown in Figure 1(b). For customer node i, $\{q_{ji} \mid x_{ji} \geq 1, j = 0, ..., N\}$ form its **inflows**, and $\{q_{ij} \mid x_{ij} \geq 1, j = 0, ..., N\}$ form its **outflows**.

Step 3. Assign q on directed arcs into individual vehicle routes. Starting from numbered customer 1, we check customers 2,…,*n* successively. If the number of incoming arcs and the number of outgoing arcs of every customer node is equal to 1, that is, each customer's delivery is realized by only a single vehicle, and then a set of feasible routes can be naturally traced from a feasible solution d, q, x of Model P'. The solution of Model P' is also a feasible solution of the SIRPSD. Otherwise, there exists at least one customer node that has more than one incoming and outgoing arcs. In this case, the assignment problem is recalled to match incoming flows with outgoing flows to construct individual vehicle routes. The objective function of the assignment problem is given by: if an inflow of customer *i* is assigned to a larger outflow, the match of the inflow and outflow is **feasible** and the assignment does not occur a cost in the objective function. Otherwise, the match is **infeasible** and a penalty is given as the assignment cost. Besides, because x_{0i} can be an integer larger than 1 (corresponding to multiple arcs), q_{0i} has to be split to multiple quantities to give every arc a corresponding inflow. This can be realized by letting each inflow equaling an outflow of customer *i* respectively and then assign the rest of unassigned quality to each arc considering vehicle capacity.

If the objective function of the assignment problem is 0, the adjustment of the matches to be feasible is not necessary, otherwise go to next step.

Step 4. Adjust an infeasible match into a feasible one.

When the objective value of the assignment problem for a customer is not equal to zero, $x\quad q$, related to the customer node need to be **adjusted** so that the tracing of feasible routes can continue. The typical example for the infeasibility is shown in Figure 2(a); no matter how the inflows and incoming arcs and outgoing arcs of customer node 3 are matched, it is impossible to find an outflow (i.e., 14 on arc 3->5 or12) on arc 3->4) feasibly matching with an inflow (10 on arc 1->3).. In this case, the value adjustment is required based on the solution of the assignment problem. The solution of the corresponding assignment problem for customer node 3 may suggest that the inflow on arc (1, 3) is matched with outgoing arc (3, 4) to trace a route, and another outgoing arc of the customer node is arc (3, 5). In this case, the values x, q in Figure 2(a) are adjusted to those in Figure 2(b). The value adjustment is important because it makes the matching of incoming arcs and outgoing arcs of customer node 3 feasible, and then **the procedure of tracing feasible routes can proceed** to the customer node 3's immediate successor-customer node 4.

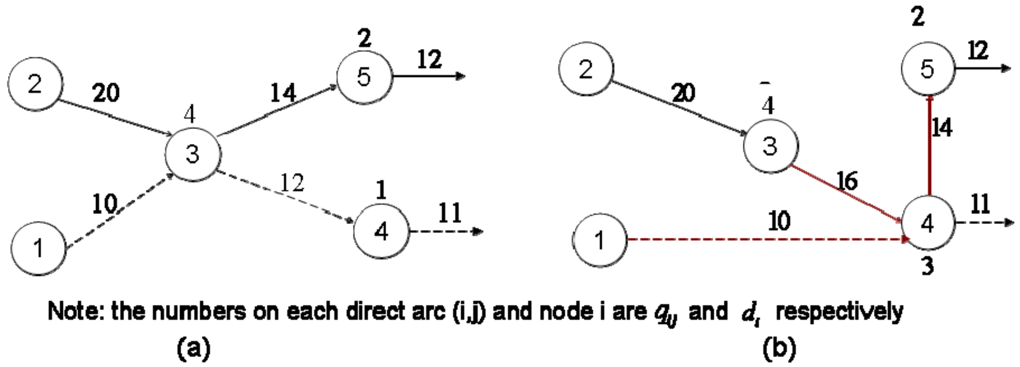

Figure 2. An example for value adjustment

When the procedure of the matching and value adjustment for all customer nodes in every period is finished, a feasible solution of the studied SIRPSD is obtained.

4.2 Local search improvement of the feasible solution of the SIRPSD

A feasible solution of the SIRPSD can be improved further by local search. This subsection presents several local search heuristics to reduce the total cost of the SIRPSD as follows.

4.2.1 Relocation of customer delivery between two routes

When multiple vehicle routes serve a common customer, as in Figure 3(a), where customer 1 is a common customer served by two routes, the feasible routes may be improved by using the methods of relocating a customer's delivery from one route to another, as illustrated in Figure 3(b)-(c) where the delivery form customer 1 is relocated from route 2 to route 1 in Figure 3(b) and it is relocated from route 1 to route 2 in Figure 3(c). The variable transportation costs are reduced by $70(c_{01}+c_{13}-c_{03})=70(2+2-3)=70$ and $40(c_{01}+c_{12}-c_{02})=40(2+2-2)=80$ for those two relocations respectively because 80>70, the relocation in Figure 2(c) is selected.

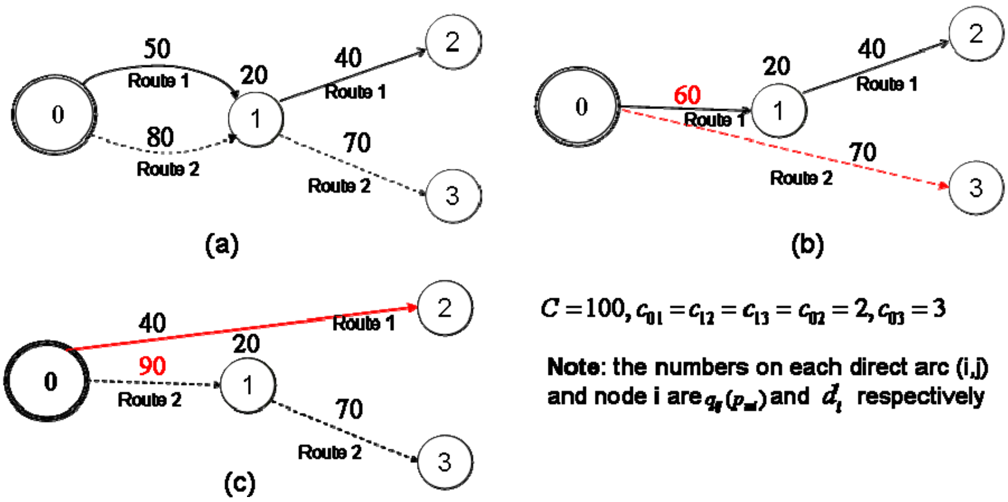

Figure 3. Example 1 for local search improvement

As an exceptional case, it is possible that multiple routes visit a common customer with one route may visit the common customer with null delivery as route 1 visiting common customer 3 shown in Figure 4(a). In this case, by deleting the common customer from the route, the variable transportation cost can be reduced by $100(c_{01}+c_{13}-c_{03})=100(2+2-3)=100$ according to the triangle inequalities for c_{ij}.

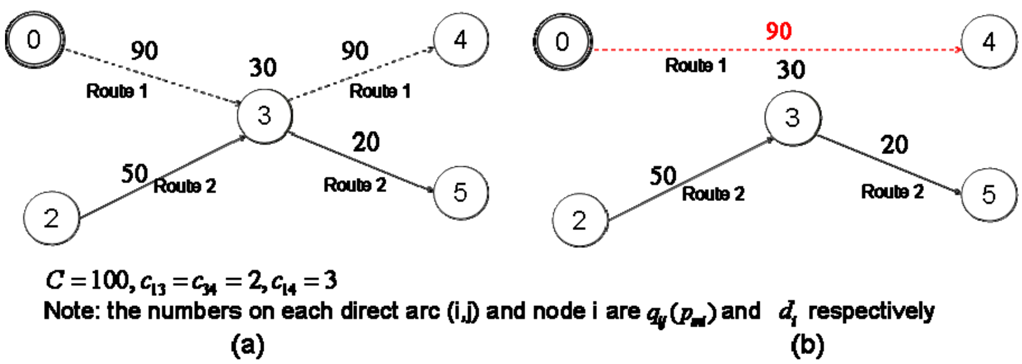

Figure 4. Example 2 for local search improvement

Note that the situation like in Figure 3 **only occurs** where the common customer (e.g. customer 1 in Figure 3) is the first served customer of multiple routes since x_{ij} $i, j = 1, \ldots, N$ are binary variables in Model P**'.** However, for the situation like in Figure 4, the common customer of multiple routes may not be the first served customer of the multiple routes.

4.2.2 Reduce the number of routes

The number of routes can be reduced by merging two vehicles not fully loaded to make all routes as fully loaded as possible if the merging reduces the total cost. With the reduction of route number, it is possible to increase the total vehicle utilization, and reduce the fixed cost $\sum_{t=1}^{T}\sum_{i=1}^{N} f_t x_{i0t}$. In this local search, every pair of vehicles is checked for possible merging, and two vehicle routes are merged if the merged route is feasible and the total transportation cost is reduced. Figure 5(b)-(c) illustrates that two vehicle routes in Figure 5(a) can be merged together to reduce the total cost. In Figure 5(b), a new route is obtained by first following the route with dotted line and then the route with solid line and the transportation cost is reduced by $f_t + 50c_{02} + c_{40}^b - 50(c_{01} + c_{14} - c_{42}) = 100 + 50 + 3 - 50(1+1+1) = 3$. In Figure 5(c), the route with solid line is followed firstly and the fixed transportation cost is reduced by $f_t + 30c_{02} + c_{30}^b - 30(c_{01} + c_{14} + c_{42}) = 100 + 30 + 3 - 30(1+1+1) = 43$. As a result, the route merging in Figure 5(b) is selected.

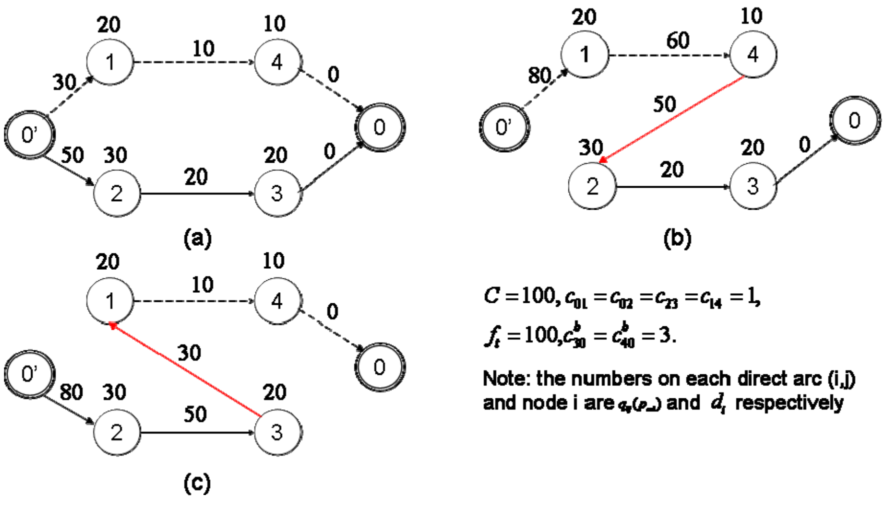

Figure 5. Example 3 for local search improvement

5. Normal distribution for demands and solution evaluation

In Model P or P’, the stochastic demand in each period is in a generic form that is applicable to different stochastic demand distribution functions, like normal distribution function, Weibull Distribution, Log Normal Distribution, etc. In order to apply our proposed model and solution approaches to numerical examples, we have to specify the distributions of the customers’ stochastic demands.

For a given distribution of ζ_{it} in our Model P, we need to obtain $F_{i,(1,t)}(I_{i,0}+\sum_{s=1}^{t}d_{is})$ value for Equation (1’), constraints (3’) and (4’) and update these constraints. We show here the update process for the normal distribution of ζ_{it} since it is mostly used in theory and practice.

5.1 Normal distribution for stochastic demands

Supposing that ζ_{it} is a random variable subject to a normal distribution with mean u_i and standard deviation σ_i, that is,

$$\zeta_{it} \sim N(u_i,\sigma_i^2). \tag{28}$$

We have a probability density function : $f_{it}(\zeta_{it})=\dfrac{1}{\sigma_i\sqrt{2\pi}}e^{-\frac{(\zeta_{it}-u_i)^2}{2\sigma_i^2}}$ $\quad -\infty \le \zeta_{it} \le +\infty$

The accumulative customer demand $\zeta_{i,(1,t)}=\sum_{s=1}^{t}\zeta_{is}$ obeys $\zeta_{i,(1,t)} \sim N(tu_i,t\sigma_i^2)$

Defining $\Phi(x)=\dfrac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}dt$, we have

$$F_{i,(1,t)}(r)=P(\zeta_{i,(1,t)}\le r)=P(\frac{\zeta_{i,(1,t)}-tu_i}{\sqrt{t}\sigma_i}\le\frac{r-tu_i}{\sqrt{t}\sigma_i})=\Phi(\frac{r-tu_i}{\sqrt{t}\sigma_i}). \tag{29}$$

The stochastic part $\sum_{t=1}^{T}E(\sum_{i=1}^{N}h_{it}I_{it}^{+})$ in the objective function (1’) can therefore be reformulated as:

$$\int_0^{I_{i,0}+\sum_{s=1}^{t}d_{is}}(I_{i,0}+\sum_{s=1}^{t}d_{is}-r)dF_{i,(1,t)}(r)=\frac{1}{2}((e^{-(I_{i,0}+\sum_{s=1}^{t}d_{is}-tu_i)^2/(2t\sigma_i^2)}-e^{-tu_i^2/(2\sigma_i^2)})\sigma_i\sqrt{\frac{2t}{\pi}}$$

$$+2(I_{i,0}+\sum_{s=1}^{t} d_{is} \text{ - } tu_i)(\Phi(\frac{u_i\sqrt{t}}{\sigma_i}) \text{ - } \Phi(\frac{(tu_i \text{ - } (I_{i,0}+\sum_{s=1}^{t} d_{is}))}{\sigma_i\sqrt{t}}))) .$$

For Constraints (3'), we have $F_{i,(1,t)}(I_{i,0}+\sum_{s=1}^{t} d_{is}) - F_{i,(1,t)}(0) \geq \alpha_{it}$, that is,

$$\Phi(\frac{((I_{i,0}+\sum_{s=1}^{t} d_{is})-tu_i)}{\sigma_i\sqrt{t}}) - \Phi(\frac{\text{-}u_i\sqrt{t}}{\sigma_i}) \geq \alpha_{it} \quad \text{and} \quad \Phi(\frac{(tu_i \text{ - } (I_{i,0}+\sum_{s=1}^{t} d_{is}))}{\sigma_i\sqrt{t}}) \leq \Phi(\frac{u_i\sqrt{t}}{\sigma_i}) - \alpha_{it} .$$

After reformulating, we have $\sum_{s=1}^{t} d_{is} \geq tu_i \text{ - } \sigma_i\sqrt{t}\Phi^{-1}(\Phi(\frac{u_i\sqrt{t}}{\sigma_i}) - \alpha_{it}) - I_{i,0}$. in practice, if $\sigma_i \leq 30\% u_i$, we can omit $F_{i,(1,t)}(0)$ since $F_{i,(1,t)}(0) \leq F_{i,1}(0)$ =4.3×10^{-4}, and the constraints can be reformulated as $\sum_{s=1}^{t} d_{is} \geq tu_i \text{ - } \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha_{it}) - I_{i,0}$.

Similarly, Constraints (4) or (4') can be reformulated as

$$\sum_{s=1}^{t} d_{is} \leq V_i + (t-1)u_i + \sigma_i\sqrt{t}\Phi^{-1}(\sqrt{\frac{t-1}{t}}(1-\beta_{it})) - I_{i,0} .$$

5.2 Solution evaluation

In Section 4, a feasible solution of the SIRPSD is obtained and its corresponding total cost provides an upper bound of the minimum total cost of the studied SIRPSD. The quality of the solution can then be evaluated by the relative gap between the lower and upper bounds, e.g., (the upper bound-the lower bound)/the upper bound ×100%. The smaller the gap is, the better the solution is. A lower bound of the minimum total cost is already available as the result of Subsection 3.4 for the evaluation in this paper.

6. Numerical examples

This section aims to evaluate the performance of our developed approach. The stochastic demand ζ_{it} is subject to a normal distribution with $\zeta_{it} \sim N(u_i, \sigma_i^2)$. We first consider a base example with the total number of customers and the depot, N_0, being 100. In the example, the length of the time horizon is taken as T=5, which corresponds to five working days every week. Parameters u_{it}, C, f_t , h_{it}, I_{i0}, and V_i are randomly and uniformly generated from the intervals [50, 400], [100, 300], [400, 700], [0.5, 2] , [50, 400], and [600, 1000] respectively; For c_{ij}, to ensure that the triangle inequality condition is satisfied, we first generate the coordinates of all customers and the central depot from a 10×10 square, and then calculate c_{ij} as the geometrical distance between customers i and j. $\sigma_{it} = 0.2u_{it}$, $\alpha_{it} = \beta_{it} = 95\%$, and c_{i0}^b = $10 \times c_{i0}$. The domains of the above parameters are mainly taken from Yu et al (2008). We generate 10 random examples for the base example with corresponding results presented in Table 2. The notations used for presenting the results are shown in Table 1.

In order to evaluate the robustness of our approach, based on the parameters of the base example, we generate 10 instances for each of other four scenarios: a) the service levels of customer demands α_{it}, and the customers' warehouses β_{it} are both changed from 95% to 99% (see results in Table 3), b) σ_{it} is changed from $0.2u_{it}$ to $0.3u_{it}$ (the results shown in Table 4), c) T is changed from 5 to 10 (the results shown in Table 5), and d) N_0 is changed from 100 to 200 (the results shown in Table 6).

The approach is coded in C++ using callable library of Lingo 6.0. To obtain a high quality solution of the SIRPSD, we construct a feasible solution of the Model P' based on the solution of its relaxed problem in every iteration. The best feasible solution of Model P' is repaired to be the final feasible solution of the SIRPSD. The test is conducted on an Intel (R) Core (TM) Due CPU 2.4GHz notebook PC with 2 G RAM and the termination condition of each instance for the Lagrangian relaxation approach is 150 iterations.

Table 1: Notations used in numerical results

UB	Upper bound of the SIRPSD (given by the near optimal solution of the SIRPSD)
LB	Lower bound of the SIRPSD (found by the Lagrangian relaxation approach in Section 3)
Gap	Value of (UB-LB)/UB $\times 100\%$
CT	Computational time (minutes: seconds)

Table 2: Computational results for the base example

Instance	***LB(10^5)***	***UB(10^5)***	***Gap(%)***	***CT***

1	10.36	10.69	3.14	12:43
2	8.06	8.51	5.28	11:59
3	9.14	9.94	8.05	10:25
4	7.41	7.90	6.19	11:13
5	12.27	12.36	0.78	13:01
6	11.89	12.03	1.15	12:54
7	10.24	10.61	3.50	11:22
8	7.00	7.59	7.70	11:10
9	8.52	9.05	5.84	11:18
10	10.25	10.51	2.43	11:05
Average	**9.51**	**9.85**	**4.41**	**11:47**

Table 3: Results for the example with service levels (α_{it} and β_{it}) no less than 99%

Instance	***LB(10^5^)***	***UB(10^5^)***	***Gap(%)***	***CT***
1	11.80	12.01	1.75	12:40
2	12.78	13.04	2.00	13:11
3	11.03	11.31	2.48	12:37
4	11.73	12.16	3.55	11:04
5	8.49	8.95	5.14	12:39
6	8.42	8.81	4.46	11:27
7	9.12	9.59	4.93	12:13
8	8.70	9.33	6.74	11:52
9	9.66	10.20	5.22	11:17
10	7.59	8.33	8.87	11:10
Average	**9.93**	**10.60**	**4.51**	**12:07**

Table 4: The results for the example with $\sigma_{it} = 0.3u_{it}$

Instance	***LB(10^5)***	***UB(10^5)***	***Gap(%)***	***CT***
1	9.90	10.38	4.62	10:33
2	9.29	9.74	4.62	10:04
3	9.48	10.00	5.16	11:31
4	13.89	14.07	1.25	10:46
5	8.80	9.44	6.87	12:17
6	9.42	9.75	3.34	10:15
7	10.14	10.59	4.24	11:01
8	9.59	10.04	4.44	10:33
9	8.79	9.36	6.09	11:09
10	9.09	9.49	4.17	11:23
Average	**9.84**	**10.37**	**4.48**	**10:54**

Table 5: The results for the example with T=10

Instance	***LB(10^5)***	***UB(10^5)***	***Gap(%)***	***CT***
1	21.20	21.59	1.80	29:52
2	17.33	18.31	5.35	30:27
3	18.18	18.95	4.06	27:44
4	20.03	20.83	3.83	28:45
5	16.93	18.45	8.20	28:54
6	18.01	18.64	3.39	31:02
7	17.84	19.08	6.49	27:37
8	19.84	20.70	4.16	26:60
9	15.77	17.10	7.78	28:52
10	17.56	18.44	4.77	31:23
Average	**18.27**	**19.29**	**4.98**	**28:55**

Table 6: The results for the example with N_0=200

Instance	***LB(10^5)***	***UB(10^5)***	***Gap(%)***	***CT***
1	19.61	20.23	3.05	75:27
2	14.98	15.93	5.99	67:41
3	18.55	19.85	6.55	63:18
4	17.01	17.49	2.76	73:36
5	15.83	16.86	6.06	65:09
6	21.51	22.11	2.70	77:34
7	15.34	16.19	5.29	65:21
8	16.27	17.34	6.16	68:42
9	22.17	22.65	2.14	67:57
10	17.27	18.32	5.70	68:03
Average	**16.23**	**18.74**	**4.64**	**69:25**

From Tables 2-6, we obtain the following observations:

1) Our algorithm can obtain high quality near-optimal solutions to the studied SIRPSD with the average gap between the upper bound and the lower bound of the problem less than 5%for all considered scenarios.

2) For all the scenarios, the largest gap is 8.87% (in

3) Table 3) and the smallest gap is 0.78% (in Table 2). This shows that our approach is robust since the gap for a scenario does not change much with the change of parameter values.

4) Our approach can solve large instances of the SIRPSD in a reasonable computational time on an ordinary PC, with the average computational time of the instances of the base example being only 11 minutes and 47 seconds. With the increase of the problem size from

N_0 =100 to 200, our approach can obtain near optimal solutions within 70 minutes on the average. Although the number of decision variables increases from about 99 thousands to about 400 thousands and the number of integer variables from about 50 thousands to about 200 thousands for each instance, similar results can be obtained when T is changed from 5 to 10.

5) With the increase of the problem size, the average gap increases slightly. For instance, with the increase of N_0 from 100 to 200 or the increase of T from 5 to 10, the average gap only increases 0.23% and 0.57% respectively from Tables 5-6.

6) With the increase of the service level β_{it} for each customer *i*'s warehouse, and the increase of the service level α_{it} for each customer *i*, the average total cost for a scenario increases but the average gap between the upper and lower bounds is rarely changed which can be seen from the comparison between Table 2 and 3.

7) With the increase of the uncertainty of the customers' demands by changing $\sigma_{it} = 0.2u_{it}$ to $\sigma_{it} = 0.3u_{it}$, we can see that the average gap increases slightly by 0.07%. This may be because meeting the required service levels becomes more difficult with the increase of the demand uncertainty.

7. Conclusion

This paper studies a stochastic inventory routing problem with split delivery where the service level to satisfy each customer's demand measured in stockout probability and the service level to each customer's warehouse measured in its overfilling probability are considered. The complexity of the SIRPSD with service levels motivates us to develop a hybrid approach which uses techniques such as the transformation of stochastic components of a model of the SIRPSD into deterministic ones, the use of Lagrangian relaxation to decompose the model into submodels, the partial linearization of the nonlinear objective function of the model, and local search improvement of feasible solutions of the studied SIRPSD to solve it. The numerical examples demonstrate that our proposed approach can obtain high quality solutions in a reasonable computational time on an ordinary personal computer.

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