Axiomatic characterization of the mean function on trees

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Abstract

A *mean* of a sequence $\pi = (x_1, x_2, \ldots, x_k)$ of elements of a finite metric space (X, d) is an element x for which $\sum_{i=1}^{k} d^2(x, x_i)$ is minimum. The function *Mean* whose domain is the set of all finite sequences on X and is defined by $Mean(\pi) = \{ x \mid x$ is a mean of $\pi\}$ is called the *mean function* on X. In this note, the mean function on finite trees is characterized axiomatically.

Keywords: Location function, mean function, median function, consensus function, tree

1

1 Introduction

Let (X, d) be a finite metric space and set $X^* = \bigcup_{k \geq 1} X^k$, where X^k is the k-fold Cartesian product of X. Elements of X^* are called *profiles* and denoted by $\pi = (x_1, x_2, \ldots, x_k)$ with *length* $|\pi| = k$. A typical problem in location theory and consensus theory is the following: Given a collection of k users (voters, customers, clients, etc.) with each user having a preferred location point in X, find a set of elements of X that jointly satisfy the users with respect to some well-defined criteria. Modelling this situation requires, at the minimum, the notion of a *location function* on X, which is a function $L : X^* \longrightarrow 2^X \backslash \{\emptyset\}$, where 2^X denotes the set of all subsets of X. Because locating points "closest" to a given profile is a natural criterion, three straightforward examples of location functions are:

(a) The *center function Cen* with

$$Cen(\pi) = \{x \in X : e(x, \pi) \text{ is minimum}\}$$

where $e(x, \pi) = \max\{d(x, x_1), d(x, x_2), \ldots, d(x, x_k)\}$ is the *eccentricity* of π.

(b) The *median function Med* with

$$Med(\pi) = \{x \in X : S_\pi(x) \text{ is minimum}\}$$

where $S_\pi(x) = \sum_{i=1}^k d(x, x_1)$ is the *status* of π.

(c) The *mean function Mean* with

$$Mean(\pi) = \{x \in X : SS_\pi(x) \text{ is minimum}\}$$

where $SS_\pi(x) = \sum_{i=1}^k d^2(x, x_1)$ is the *square status* if π.

Location functions can be viewed as a special instance of consensus functions. A *consensus function* is a model to describe a rational process to obtain consensus among a group of agents or clients. The input of the function consists of certain information about the agents, and the output concerns the issue, about which consensus should be reached. The rationality of the process is guaranteed by the fact that the consensus function satisfies certain "rational" rules or "consensus axioms". A typical question in consensus theory is: which set of axioms characterizes a given consensus function. The theory of consensus is widely studied and used, e.g. in Economics and Social Choice Theory, see [1, 2], and in Biomathematics, see [6]. Holzman [8] was the first to study location functions from the axiomatic perspective.

We are concerned with location on finite connected graphs, so let $G = (V, E)$ be such a graph, and let d be the usual distance function on G, where $d(x, y)$ is the length of a shortest path between x and y. Clearly (V, d) is a finite metric space.

Notice that a profile on a graph G is simply a sequence of vertices where repetitions are allowed. Among the location functions given above, the median function has been axiomatically characterized in several contexts, see e.g. [8, 3, 4, 10, 9]. In this note we consider the case when the graph is a tree $T = (V, E)$, i.e. a connected graph without cycles. McMorris, Roberts, and Wang [4] gave an axiomatic characterization for the center function on finite trees, see also [13], but not much is known for more general finite graphs. With respect to the mean function, Holzman [8] has given the first characterization of this function for trees in the "continuous case". Informally, a tree T has an infinite number of elements in the continuous case , its edges are considered to be rectifiable curves, with π and $L(\pi)$ allowed to be located anywhere on edges. In this context Vohra [14] has given an axiomatic characterization for the median function, and also a new characterization for the mean function using a set of axioms different from those used by Holzman. See Foster and Vohra [7] for related results.

Because the optimization criterion for the mean function is just as natural as that for the median, it is surprising that not much is known about the mean function on finite metric spaces from the axiomatic point of view. Biagi [5] finds a set of properties necessarily satisfied by the mean function on finite trees but they are not sufficient to provide a complete axiomatic characterization for the mean function. We will discuss this approach below as we move toward the goal of this paper, which is to give an axiomatic characterization of the mean function on finite trees.

2 Axioms and Preparatory Results

From now on $T = (V, E)$ will be used to represent a finite tree. If π is a profile on T, then from [5] we know that $Mean(\pi)$ is a K_1 or K_2 (one vertex, or two adjacent vertices).

The following axioms could reasonably be expected to be satisfied by a well-behaved location function. Indeed in [5] it is noted that the location function $Mean$ satisfies these properties on a finite tree.

Let L be a location function on $T = (V, E)$.

Faithfulness (F) : $L((x)) = \{x\}$, for all $x \in V$.

Middleness (Mid) : Let $x, y \in V$. If $d(x, y)$ is even, then $L((x, y)) = K_1$ where $d(x, K_1) = d(y, K_1) = d(x, y)/2$. If $d(x, y)$ is odd, then $L((x, y)) = K_2$ where $d(x, K_2) = d(y, K_2)$.

Consistency (C) : If $\pi_1 = (x_1, x_2, \ldots, x_k)$ and $\pi_2 = (y_1, y_2, \ldots, y_s)$ are profiles with $L(\pi_1) \cap L(\pi_2) \neq \emptyset$, then $L(\pi) = L(\pi_1) \cap L(\pi_2)$ with $\pi = (x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_s)$.

Anonymity (A) : Let $\pi = (x_1, x_2, \ldots, x_k)$ be a profile on T and let σ be any permutation of $\{1, 2, \ldots, k\}$, then $L(\pi) = L(\pi^\sigma)$, where $\pi^\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)})$.

These four axioms are not independent: Faithfulness follows from Consistency and Middleness.

Lemma 1 *Let L be a location function on T. If L satisfies axioms (Mid) and (C), then L satisfies axiom (F).*

Proof. Let z be a vertex of T and let $\pi = (z, z)$. Since $d(z, z) = 0$ is even, then by axiom (Mid) we have

$$L(\pi) = L((z, z)) = \{z\} \tag{1}$$

On the other hand, $L((z)) = L((z)) \cap L((z)) \neq \emptyset$, so axiom (C) gives

$$L((z)) \cap L((z)) = L((z, z)) = L(\pi) = \{z\}. \tag{2}$$

and thus $L((z)) = \{z\}$. $\qquad\square$

In light of a similar result for the median function on trees [3], it was natural to conjecture that (C), (Mid), and (A) characterize $Mean$ among all the location functions on trees. However, there are examples of location functions that satisfy axioms (C), (Mid), and (A) that are not equal to the mean function. For instance, let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T and define the *cube status* of a vertex v with respect to π as

$$CS_\pi(v) = \sum_{i=1}^{k} d^3(v, x_i).$$

Now define the location function $Cube$ by

$$Cube(\pi) = \{v \in V \mid CS_\pi(v) \leq CS_\pi(x) \quad \forall x \in V\}.$$

It was shown in [5] that $Cube$ satisfies (C), (Mid), and (A). To show that $Mean(\pi) \neq Cube(\pi)$ consider the path $P(v_1, v_{10}) = v_1 v_2 \ldots v_{10}$ on ten vertices and the profile $\pi = (v_1, v_2, v_{10})$ on $P(v_1, v_{10})$. It is straightforward to verify that $Mean(\pi) = \{v_4\}$ and $Cube(\pi) = \{v_5\}$.

Next, through a series of Lemmas, we establish properties of $Mean$ and location functions that satisfy some of the previously listed axioms. This allows us to propose a new technical axiom that enables us to prove our main result in the next section.

The following lemma was proved in [5].

Lemma 2 *Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. If u, v, w are distinct vertices such that $uw, wv \in E$, then*

$$SS_\pi(w) < Max\{SS_\pi(u), SS_\pi(v)\}.$$

Next we prove a lemma that can be viewed as an extension of Lemma 2.

Lemma 3 *Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. If $m \in Mean(\pi)$ and $\quad P(m, y_s) = m y_1 y_2 \ldots y_s$ is a path such that $y_1 \notin Mean(\pi)$, then*

$$SS_\pi(m) < SS_\pi(y_1) < SS_\pi(y_2) < \cdots < SS_\pi(y_s).$$

4

Proof. We use induction on s. The result is valid for $s = 1$ because $y_1 \notin Mean(\pi)$ implies $SS_\pi(m) < SS_\pi(y_1)$. Assume the lemma is true for any path that contains at most s vertices different than m and let

$$P(m, y_{s+1}) = my_1 y_2 \dots y_s y_{s+1}$$

be a path such that $y_1 \notin Mean(\pi)$. The induction hypothesis implies

$$SS_\pi(y_{s-1}) < SS_\pi(y_s). \tag{3}$$

Since y_s is adjacent to both y_{s-1} and y_{s+1}, then Lemma 2 yields

$$SS_\pi(y_s) < Max\{SS_\pi(y_{s-1}), SS_\pi(y_{s+1})\}. \tag{4}$$

Combining (3) and (4), we obtain $SS_\pi(y_s) < SS_\pi(y_{s+1})$. This proves

$$SS_\pi(m) < SS_\pi(y_1) < SS_\pi(y_2) < \dots < SS_\pi(y_s) < SS_\pi(y_{s+1}),$$

which completes the proof. □

Since the mean of a profile on a tree is K_1 or K_2, the following is obvious.

Lemma 4 *Let π) be a profile on T. Then $Mean(\pi) = \{a, b\}$ if and only if a is adjacent to b and $SS_\pi(a) = SS_\pi(b)$.*

The process of finding a vertex x that belongs to the set $Mean(\pi)$ is a global property, which requires comparing $SS_\pi(x)$ against the square status of every vertex on T. If we know in advance that the mean of π is a set with two elements, then Lemma 4 gives a method that avoids a lot of calculations. For any given vertex x, we first obtain the set of neighbors of x, and if we are lucky to find in this set a vertex y satisfying the condition $SS_\pi(x) = SS_\pi(y)$, then we know that $Mean(\pi) = \{x, y\}$.

We would like to have a similar local procedure that we can apply when $|Mean(\pi)| = 1$. Notice that if $u \in Mean(\pi)$, then

$$SS_\pi(u) \leq SS_\pi(z) \quad \forall z \in V.$$

In particular
$$SS_\pi(u) \leq SS_\pi(z)$$

for any vertex z adjacent to u. This suggests that in trying to discover if a vertex x is in the set $Mean(\pi)$, we could first verify that $SS_\pi(x)$ is less or equal than the square status of every neighbor of x. If that conditions holds, then x is a good candidate to be in the mean of π. This argument motivates the following definition.

Definition 1 *Let π be a profile on T. A vertex u satisfies the mean condition with respect to π if*
$$SS_\pi(u) \leq SS_\pi(z)$$

for any vertex z adjacent to u.

The following lemma shows that the elements of the set $Mean(\pi)$ are precisely the vertices that satisfy the mean condition with respect to π.

Lemma 5 *Let π be a profile on T. Then $u \in Mean(\pi)$ if and only if u satisfies the mean condition with respect to π.*

Proof. It is clear that if $u \in Mean(\pi)$, then u satisfies the mean condition with respect to π. Conversely, let u be a vertex satisfying the mean condition with respect to π and assume $u \notin Mean(\pi)$. Let $x \in Mean(\pi)$ be a vertex such that $d(Mean(\pi), u) = d(x, u)$, and let

$$P(x, u) = x v_1 v_2 \dots v_s u$$

be the path from x to u. By Lemma 3 we obtain

$$SS_\pi(x) < SS_\pi(v_1) < SS_\pi(v_2) < \cdots < SS_\pi(v_s) < SS_\pi(u).$$

We see that v_s is adjacent to u and $SS_\pi(v_s) < SS_\pi(u)$. However, this contradicts that u satisfies the mean condition with respect to π. This shows that $u \in Mean(\pi)$. $\square$

Let $\pi = (x_1, x_2, \dots, x_n)$ be a profile on T, and let $z \in V$. Define the profile $\pi \diamond z$ as follows

$$\pi \diamond z = (x_1, x_2, \dots, x_n, z).$$

Lemma 6 *Let $\pi = (x_1, x_2, \dots, x_n)$ be a profile on T. Let L be a location function satisfying axioms (F) and (C). If $z \in L(\pi)$, then $L(\pi \diamond z) = \{z\}$.*

Proof. Define the profile $\beta = (z)$, and notice that axioms (F) and (C) imply

$$L(\pi \diamond z) = L(\pi) \cap L(\beta) = \{z\},$$

and we are done. $\square$

The next lemma shows that the mean function satisfies also the converse of Lemma 6.

Lemma 7 *Let π be a profile on T. If y is a vertex such that $Mean(\pi \diamond y) = \{y\}$, then $y \in Mean(\pi)$.*

Proof. Let z be a vertex adjacent to y, and observe that $Mean(\pi \diamond y) = \{y\}$ implies

$$SS_{\pi \diamond y}(y) < SS_{\pi \diamond y}(z).$$

Note that

$$SS_{\pi \diamond y}(y) = SS_\pi(y),$$
$$SS_{\pi \diamond y}(z) = SS_\pi(z) + 1,$$

which means

$$SS_\pi(y) \le SS_\pi(z).$$

Since this is true for any vertex z adjacent to y, it follows that y satisfies the mean condition with respect to π. By Lemma 5, we have $y \in Mean(\pi)$. $\qquad\square$

In the search for mean vertices we might proceed as follows. We are at vertex a and compare the square status of a with that of an adjacent vertex b. If the square status of b is smaller than that of a we "improve" our position by moving to b. Now we compare the square status of b with that of a new vertex c adjacent to b. Thus we have to compare the square status of adjacent vertices but also of three consecutive vertices on a path of length 2. Therefore, we introduce the following notation.

Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T and assume $ab, bc \in E$. Denote by π_{ab} the profile that contains the elements x of π (in the same order) that are closer to a than to b, that is, satisfy the condition

$$d(a, x) + 1 = d(b, x).$$

Notice that the vertex x appears in π_{ab} as many times as it appears in π.
Similarly, denote by π_{abc} the profile that contains the elements x of π that satisfy the following conditions

$$d(b, x) + 1 = d(a, x),$$
$$d(b, x) + 1 = d(c, x).$$

We use these profiles to define two quantities. The first could be considered as a *surplus* when comparing the square status of the adjacent vertices a and b. The second one could be considered as an *indifference count* with respect to a and c by computing the distance sum of all the vertices having equal distance to a and c. The use of these quantities will become clear in the following computations.

$$R_\pi(a, b) = \sum_{x \in \pi_{ba}} d(b, x) - \sum_{x \in \pi_{ab}} d(b, x),$$
$$D_\pi(a, b, c) = 2 \sum_{x \in \pi_{abc}} d(b, x).$$

By Anonymity we may write

$$\pi = \pi_{ab}\pi_{ba} \quad \text{as well as} \quad \pi = \pi_{ab}\pi_{abc}\pi_{cb}.$$

The representation of π as $\pi = \pi_{ab}\pi_{ba}$ leads to the following relationship between $R_\pi(a, b)$, $SS_\pi(b)$, and $SS_\pi(a)$.

$$SS_\pi(a) - SS_\pi(b) = 2R_\pi(a, b) + |\pi|. \tag{5}$$

7

Using the identity $SS_\pi(a) - SS_\pi(b) = -[SS_\pi(b) - SS_\pi(a)]$ and (5) we get the following connection between $R_\pi(a, b)$ and $R_\pi(b, a)$:

$$R_\pi(a, b) = -R_\pi(b, a) - |\pi|. \tag{6}$$

Similarly, from $\pi = \pi_{ab}\pi_{abc}\pi_{cb}$ we obtain

$$SS_\pi(a) - SS_\pi(b) = -2 \sum_{x \in \pi_{ab}} d(b, x) + 2 \sum_{x \in \pi_{abc}} d(b, x) + 2 \sum_{x \in \pi_{cb}} d(b, x) + |\pi| \tag{7}$$

$$SS_\pi(c) - SS_\pi(b) = 2 \sum_{x \in \pi_{ab}} d(b, x) + 2 \sum_{x \in \pi_{abc}} d(b, x) - 2 \sum_{x \in \pi_{cb}} d(b, x) + |\pi|. \tag{8}$$

Next we prove two lemmas that we need later.

Lemma 8 *Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. If $ab, bc \in E$, then*

$$R_\pi(a, b) + R_\pi(c, b) = D_\pi(a, b, c).$$

Proof. Adding (7) and (8), we get

$$[SS_\pi(a) - SS_\pi(b)] + [SS_\pi(c) - SS_\pi(b)] = 4 \sum_{x \in \pi_{abc}} d(b, x) + 2|\pi|. \tag{9}$$

From (5) we obtain

$$SS_\pi(a) - SS_\pi(b) = 2R_\pi(a, b) + |\pi|,$$
$$SS_\pi(c) - SS_\pi(b) = 2R_\pi(c, b) + |\pi|.$$

Notice also that

$$4 \sum_{x \in \pi_{abc}} d(b, x) = 2D_\pi(a, b, c).$$

When we substitute these identities in (9), we obtain

$$R_\pi(a, b) + R_\pi(c, b) = D_\pi(a, b, c),$$

which completes the proof. $\square$

Lemma 9 *Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. If $|\pi|$ is odd, then $|Mean(\pi)| = 1$.*

Proof. Assume $Mean(\pi) = \{a, b\}$, then $SS_\pi(b) = SS_\pi(a)$. Since a is adjacent to b, it follows from (5) that

$$SS_\pi(b) - SS_\pi(a) = 2R_\pi(b, a) + |\pi| = 0.$$

This means that $|\pi|$ is even, and contradicts the assumption that $|\pi|$ is odd. Consequently, we have $|Mean(\pi)| = 1$. $\square$

Our next result will be used to define a special property for location functions.

Lemma 10 *Let π be a profile on T. If $ab, bc \in E$ and $SS_\pi(a) < SS_\pi(b) < SS_\pi(c)$, then $R_\pi(b, a) + R_\pi(c, b) > D_\pi(a, b, c)$.*

Proof. From (7) we obtain

$$SS_\pi(b) - SS_\pi(a) = 2 \sum_{x \in \pi_{ab}} d(b, x) - 2 \sum_{x \in \pi_{abc}} d(b, x) - 2 \sum_{x \in \pi_{cb}} d(b, x) - |\pi|. \qquad (10)$$

Adding (10) and (8), we get

$$[SS_\pi(b) - SS_\pi(a)] + [SS_\pi(c) - SS_\pi(b)] = 4 \sum_{x \in \pi_{ab}} d(b, x) - 4 \sum_{x \in \pi_{cb}} d(b, x). \qquad (11)$$

Since $SS_\pi(a) < SS_\pi(b)$, then (10) implies

$$2 \sum_{x \in \pi_{ab}} d(b, x) - 2 \sum_{x \in \pi_{abc}} d(b, x) - 2 \sum_{x \in \pi_{cb}} d(b, x) - |\pi| > 0.$$

$$2 \sum_{x \in \pi_{ab}} d(b, x) - 2 \sum_{x \in \pi_{cb}} d(b, x) > 2 \sum_{x \in \pi_{abc}} d(b, x) + |\pi|.$$

We use this to rewrite (11) in the following way

$$[SS_\pi(b) - SS_\pi(a)] + [SS_\pi(c) - SS_\pi(b)] > 4 \sum_{x \in \pi_{abc}} d(b, x) + 2|\pi|. \qquad (12)$$

From (5) we obtain

$$SS_\pi(b) - SS_\pi(a) = 2R_\pi(b, a) + |\pi|,$$
$$SS_\pi(c) - SS_\pi(b) = 2R_\pi(c, b) + |\pi|.$$

Substituting this in (12), we get

$$[2R_\pi(b, a) + |\pi|] + [2R_\pi(c, b) + |\pi|] > 2D_\pi(a, b, c) + 2|\pi|,$$

so, finally, we get

$$R_\pi(b, a) + R_\pi(c, b) > D_\pi(a, b, c).$$

$\square$

The following result is a simple corollary of Lemma 10.

Corollary 1 *Let π be a profile on T. If $Mean(\pi) = \{a\}$, then $R_\pi(b, a) + R_\pi(c, b) > D_\pi(a, b, c)$ whenever $ab, bc \in E$.*

Corollary 1 is the motivation for the definition of the following property for a location function.
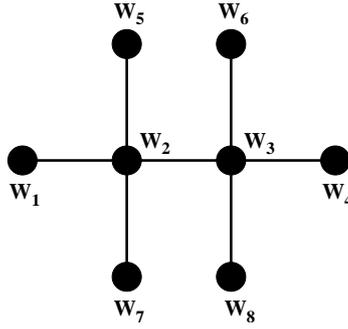
Figure 1: *Mean* does not satisfy converse of (Z)

Property (Z) : Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile and L be a location function on T. If $L(\pi) = \{a\}$, then $R_\pi(b, a) + R_\pi(c, b) > D_\pi(a, b, c)$ whenever $ab, bc \in E$.

To get a feeling for this condition, note that the converse of property (Z) does not necessarily hold when L is the mean function for trees. We will give an example of a tree T and a profile π for which the converse of property (Z) fails for the mean function.

Using the tree T given in Fig. 1, define the profile $\pi = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8)$, and notice that $Mean(\pi) = \{w_2, w_3\}$. For vertex w_2 we have

$$w_2 w_3, w_3 w_4 \in E,$$
$$R_\pi(w_3, w_2) = -4,$$
$$R_\pi(w_4, w_3) = 8,$$
$$D_\pi(w_2, w_3, w_4) = 2,$$

then

$$R_\pi(w_3, w_2) + R_\pi(w_4, w_3) > D_\pi(w_2, w_3, w_4).$$

Notice that

$$R_\pi(w_4, w_3) = R_\pi(w_6, w_3) = R_\pi(w_8, w_3) = 7,$$

this implies

$$R_\pi(w_3, w_2) + R_\pi(w_6, w_3) > D_\pi(w_2, w_3, w_6),$$
$$R_\pi(w_3, w_2) + R_\pi(w_8, w_3) > D_\pi(w_2, w_3, w_8).$$

From this we have that w_2 satisfies the condition:

$$R_\pi(a, w_2) + R_\pi(b, a) > D_\pi(w_2, a, b) \ \text{ whenever } \ w_2 a, ab \in E.$$

If we assume that the converse of property (Z) is true, then $Mean(\pi) = \{w_2\}$, and this contradicts that $Mean(\pi) = \{w_2, w_3\}$.

Notice that a location function L defined on the tree (K_2) cannot get any additional features from property (Z) because this property involves three distinct vertices. The next result shows that if L satisfies axioms (Mid), (C), and (A), then it should be the mean function on K_2.

Lemma 11 *Let T be a tree, and assume $ab \in E$. Let π be a profile containing only vertices in the set $\{a, b\}$. If L is a location function that satisfies axioms (Mid), (C), and (A), then $L(\pi) = Mean(\pi)$.*

Proof. By Lemma 1, L also satisfies axiom (F). Let π be a profile containing only vertices in the set $\{a, b\}$. If π contains an equal number of vertices a and b, then we use the elements of π to define the profile

$$\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$$

where $\alpha_i = (a, b)$. Since L satisfies (C) and (Mid), then

$$L(\alpha) = L(\alpha_1) \cap L(\alpha_2) \cap \ldots \cap L(\alpha_n) = \{a, b\}$$

and we also have

$$Mean(\alpha) = Mean(\alpha_1) \cap Mean(\alpha_2) \cap \ldots \cap Mean(\alpha_n) = \{a, b\}.$$

By axiom (A) we obtain $L(\alpha) = L(\pi) = Mean(\pi)$. On the other hand, if π contains more vertices a that vertices b, we define the profile

$$\alpha = \alpha_1 \alpha_2 \ldots \alpha_n \beta$$

where $\alpha_i = (a, b)$ and $\beta = (a, a, \ldots, a)$. From axioms (C), (Mid), and (F) we have

$$L(\alpha) = L(\alpha_1) \cap L(\alpha_2) \cap \ldots L(\alpha_n) \cap L(\beta) = \{a\}$$

and

$$Mean(\alpha) = Mean(\alpha_1) \cap Mean(\alpha_2) \cap \ldots Mean(\alpha_n) \cap Mean(\beta) = \{a\}.$$

We use now axiom (A) to conclude $L(\alpha) = L(\pi) = Mean(\pi)$. Similarly, we can show that if π contains more vertices b that vertices a, then $L(\pi) = Mean(\pi)$. $\square$

The next objective is to show that if $|L(\pi)| = |Mean(\pi)| = 1$ for some profile π, then $L(\pi) = Mean(\pi)$ whenever L satisfies axioms (C), (Mid), (A), and property (Z). In proving this result, we need to consider the cases when the vertex in $Mean(\pi)$ is a leaf or is not a leaf of T. First we prove that if $Mean(\pi) = \{x\}$ and $L(\pi) = \{y\}$, then x and y are close to each other in T. Property (Z) is the main ingredient in the proof of this result.

Lemma 12 *Let L be a location function on T that satisfies axioms (Mid), (A), and (C). Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. If $Mean(\pi) = \{x\}$, $L(\pi) = \{y\}$, and $x \neq y$, then $d(x, y) \leq 1$.*

Proof. Assume $d(x, y) > 1$, and let $P(x, y) = x v_1 v_2 \ldots v_{s-1} v_s y$ be the path from x to y. By Lemma 3 we have that

$$SS_\pi(v_{s-1}) < SS_\pi(v_s) < SS_\pi(y), \tag{13}$$

Since L satisfies axiom (Z), we have

$$R_\pi(v_s, y) + R_\pi(v_{s-1}, v_s) > D_\pi(y, v_{s-1}, v_s). \tag{14}$$

From (5) we obtain

$$R_\pi(v_s, y) = \frac{SS_\pi(v_s) - SS_\pi(y) - |\pi|}{2},$$
$$R_\pi(v_{s-1}, v_s) = \frac{SS_\pi(v_{s-1}) - SS_\pi(v_s) - |\pi|}{2}.$$

If we substitute this in (14) and simplify we get

$$SS_\pi(v_{s-1}) - SS_\pi(y) > 2D_\pi(y, v_{s-1}, v_s) + 2|\pi| > 0.$$

This implies that $SS_\pi(y) < SS_\pi(v_{s-1})$, and this contradicts (13). So we have $d(x, y) \leq 1$. $\qquad\square$

From the previous result it follows that if we would have $Mean(\pi) = \{v\}$, $L(\pi) = \{u\}$, and $u \neq v$, then u would be adjacent to v. Now, we prove that if v is not a leaf, then $u = v$.

Lemma 13 *Let L be a location function on T that satisfies axioms (Mid), (A), (C), and property (Z), and let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. If v is not leaf, $|L(\pi)| = 1$, and $Mean(\pi) = \{v\}$, then $L(\pi) = \{v\}$.*

Proof. Assume $L(\pi) = \{z\}$ and $v \neq z$, then Lemma 12 implies $d(v, z) \leq 1$. If $d(v, z) = 1$, then there is a vertex u such that $uv, vz \in E$. By Lemma 8 we have

$$R_\pi(u, v) + R_\pi(z, v) = D_\pi(u, v, z). \tag{15}$$

Now L satisfies axiom (Z), hence

$$R_\pi(v, u) + R_\pi(z, v) > D_\pi(u, v, z). \tag{16}$$

Subtracting (16) from (15), we obtain

$$R_\pi(u, v) - R_\pi(v, u) < 0$$
$$R_\pi(u, v) - (-R_\pi(u, v) - |\pi|) < 0 \qquad \text{by (6)}$$
$$2R_\pi(u, v) + |\pi| < 0$$
$$SS_\pi(u) - SS_\pi(v) < 0 \qquad \text{by (5)}.$$

This means that $SS_\pi(u) < SS_\pi(v)$, which contradicts that $Mean(\pi) = \{v\}$. Therefore, we conclude that $L(\pi) = \{v\}$. □

We would like to extend this result to the case when v is a leaf, but this requires us to prove some preparatory results. The next lemma shows that if the mean of a profile π is a vertex that is a leaf, then this vertex is included in π.

Lemma 14 *Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. Let v be a vertex that is a leaf and assume $Mean(\pi) = \{v\}$, then v is an element of π.*

Proof. If v is not an element of π, let w be the vertex adjacent to v. Notice that

$$d(v, x_i) = d(w, x_i) - 1 \quad \forall \ i \in \{1, 2, \ldots n\}.$$

This implies $SS_\pi(w) < SS_\pi(v)$ which contradicts that $Mean(\pi) = \{v\}$. Hence, v is included in π. □

Next we characterize the profiles π having the property that their mean contains two vertices, and one of these vertices is a leaf.

Lemma 15 *Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. If $Mean(\pi) = \{v, x\}$ and v is a leaf of T, then the only vertices in π are x and v. Moreover, π contains an equal number of vertices x and v.*

Proof. If w is a vertex that belongs to π and is different from x and v, then $d(v, w) = d(x, w) + 1$. This implies that $SS_\pi(v) < SS_\pi(x)$, which contradicts $Mean(\pi) = \{v, x\}$. Consequently, π cannot contain any vertex different from x and v. Notice that if π does not contain an equal number of x and v, then $|Mean(\pi)| = 1$, which contradicts $|Mean(\pi)| = 2$. □

If L is a location function that satisfies axioms (Mid), (A), and (C), then Lemmas 11 and 15 provide the following corollary that characterizes the set $L(\pi)$ when this set contains two elements and one of them is a leaf.

Corollary 2 *Let L be a location function on T that satisfies axioms (Mid), (A), and (C). Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T and assume v is a leaf. If $Mean(\pi) = \{v, x\}$, then $L(\pi) = \{v, x\}$.*

The next result tells us that the sets $Mean(\pi)$ and $L(\pi)$ are equal when the set $Mean(\pi)$ contains a leaf of T.

Lemma 16 *Let L be a location function on T that satisfies axioms (Mid), (A), and (C), and let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. If y is a leaf of T such that $y \in Mean(\pi)$, then $Mean(\pi) = L(\pi)$.*

Proof. To prove the result, we use induction on $|\pi|$. If $|\pi| = 1$, then $\pi = (y)$, and in this case $Mean(\pi) = \{y\}$ and, by Faithfulness, we have $L(\pi) = \{y\}$. Assume the result is true for any profile with at most n vertices, and let π be a profile with $n + 1$ vertices such that $y \in Mean(\pi)$ and y is a leaf. Assume $Mean(\pi) = \{y\}$ and denote by α the profile that contains all the vertices of π except for one occurrence of y. Since $Mean(\alpha \diamond y) = \{y\}$, Lemma 7 implies that $y \in Mean(\alpha)$. From Lemma 14 it follows that y occurs in α. Since α contains n vertices, the induction hypothesis yields $Mean(\alpha) = L(\alpha)$. Now we use Lemma 6 to conclude that $L(\pi) = L(\alpha \diamond y) = \{y\} = Mean(\pi)$. On the other hand, it follows from Corollary 2 that if $Mean(\pi) = \{y, x\}$, then $L(\pi) = \{y, x\}$. $\qquad\square$

We are now in position to show that the sets $Mean(\pi)$ and $L(\pi)$ are the same when the mean of π is a leaf, L satisfies property (Z), and $|Mean(\pi)| = |L(\pi)| = 1$.

Lemma 17 *Let L be a location function on T that satisfies axioms (Mid), (A), (C), and property (Z), and let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. If y is a leaf and $|Mean(\pi)| = |L(\pi)| = 1$, then $Mean(\pi) = \{y\}$ if and only if $L(\pi) = \{y\}$.*

Proof. By Lemma 16, we know that $Mean(\pi) = \{y\}$ and implies $L(\pi) = \{y\}$.

On the other hand, assume $L(\pi) = \{y\}$ and $|Mean(\pi)| = 1$. If $Mean(\pi) = \{z\}$ and z is a leaf, then Lemma 16 implies $z = y$. If z is not a leaf then Lemma 13 implies that $z = y$. Therefore we have $Mean(\pi) = \{y\}$. $\qquad\square$

A consequence of Lemmas 13 and 17 is the following generalization of Lemma 17.

Lemma 18 *Let L be a location function on T that satisfies axioms (Mid), (A), (C), and property (Z), and let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T. If $|Mean(\pi)| = |L(\pi)| = 1$, then $Mean(\pi) = \{y\}$ if and only if $L(\pi) = \{y\}$.*

Finally we prove that if π is a profile having an even number of vertices, and L is a location function that satisfies axioms (Mid), (C), (A), and property (Z), then the sets $L(\pi)$ and $Mean(\pi)$ are equal.

Lemma 19 *Let $\pi = (x_1, x_2, \ldots, x_n)$ be a profile on T, and let L be a location function that satisfies axioms (Mid), (C), (A), and property (Z). If $|\pi|$ is even, then $L(\pi) = Mean(\pi)$.*

Proof. Note that if all the elements of π belong to the set $\{a, b\}$ and $ab \in E$, then Lemma 11 implies that $L(\pi) = Mean(\pi)$. Now, assume π contains at least three distinct vertices and also assume $L(\pi) = \{a_1, a_2, \ldots, a_k\}$. By Lemma 6, we get

$$L(\pi \diamond a_1) = \{a_1\},$$
$$L(\pi \diamond a_2) = \{a_2\},$$
$$\vdots$$
$$L(\pi \diamond a_s) = \{a_k\}.$$

Since $|\pi|$ is even, $|\pi \diamond a_i|$ is odd, and, by Lemma 9, we have

$$|Mean(\pi \diamond a_i)| = 1 \quad \forall \, 1 \le i \le k.$$

From Lemma 18 we get

$$L(\pi \diamond a_i) = Mean(\pi \diamond a_i) \quad \forall \, 1 \le i \le k. \tag{17}$$

Observe that (17) and Lemma 7 yield $\{a_1, a_2, \ldots, a_k\} \subseteq Mean(\pi)$. This implies that $k \le 2$, since $|Mean(\pi)| \le 2$. If $L(\pi) = \{a_1, a_2\}$, we have $L(\pi) = Mean(\pi) = \{a_1, a_2\}$. On the other hand, if $|L(\pi)| = |Mean(\pi)| = 1$, then, by Lemma 18, we get $L(\pi) = Mean(\pi)$.

To finish the proof we just need to show that $|L(\pi)| = 1$ forces $|Mean(\pi)| = 1$. So assume $L(\pi) = \{c\}$ and $|Mean(\pi)| = 2$. By Lemma 6 we have $L(\pi \diamond c) = \{c\}$, and, since $|\pi \diamond c|$ is odd, we have $|Mean(\pi \diamond c)| = 1$. Lemma 18 implies that $Mean(\pi \diamond c) = L(\pi \diamond c) = \{c\}$, and Lemma 7 shows that $c \in Mean(\pi)$. This means that $Mean(\pi) = \{b, c\}$, for some vertex b distinct from c, and also that b and c are adjacent. First assume that b is a leaf of T. Then, by Corollary 2, we have $L(\pi) = \{c, b\}$, which contradicts $L(\pi) = \{c\}$. So b cannot be a leaf of T, and we can find a vertex z such that $cb, bz \in E$. Because L satisfies property (Z), we get

$$R_\pi(b, c) + R_\pi(z, b) > D_\pi(c, b, z). \tag{18}$$

Moreover, Lemma 8 shows

$$R_\pi(c, b) + R_\pi(z, b) = D_\pi(c, b, z).$$

If we substitute this in (18) and simplify, we get

$$R_\pi(b, c) > R_\pi(c, b). \tag{19}$$

On the other hand, since $Mean(\pi) = \{c, b\}$, we have $SS_\pi(c) = SS_\pi(b)$. From (5) we obtain

$$SS_\pi(c) - SS_\pi(b) = SS_\pi(b) - SS_\pi(c)$$
$$2R_\pi(c, b) + |\pi| = 2R_\pi(b, c) + |\pi|$$
$$R_\pi(c, b) = R_\pi(b, c). \tag{20}$$

It is clear that (20) contradicts (19). This final contradiction implies that $|Mean(\pi)| = 2$ is not possible. Therefore it follows that if $|L(\pi)| = 1$, then $|Mean(\pi)| = 1$, which concludes the proof. $\square$

3 The Main Result and Independence of Axioms

Finally, we put the above results together in the proof of our main result.

Theorem 1 *Let L be a location function on a finite tree T. Then L is the mean function if and only if L satisfies axioms (A), (Mid), (C), and property (Z).*

Proof. It is immediate that if L is the mean function, then L satisfies axioms (A), (Mid), (C), and property (Z). On the other hand, assume that L satisfies axioms (A), (Mid), (C), and property (Z). By Lemma 19, we have $L(\pi) = Mean(\pi)$ whenever $|\pi|$ is even. In the case that $|\pi|$ is odd we have $|Mean(\pi)| = 1$, by Lemma 9. Hence, by Lemma 18, we have $L(\pi) = Mean(\pi)$, and we are done. $\square$

Next, we establish the independence of the axioms (A), (Mid), (C), and property (Z).

- **(Mid), (A), and (C) are independent from property (Z).**
  The cube function satisfies axioms (Mid), (A), and (C). Consider the path $P(v_1, v_{10}) = v_1 v_2 \ldots v_{10}$ and the profile $\pi = (v_1, v_2, v_{10})$. It is not difficult to verify that $Mean(\pi) = \{v_4\}$ and $Cube(\pi) = \{v_5\}$. Notice that $v_5 v_4$ and $v_4 v_3$ are edges of $P(v_1, v_{10})$ and that $D_\pi(v_5, v_4, v_3) = 0$. Then we have

  $$R_\pi(v_4, v_5) + R_\pi(v_3, v_4) = -2 + 1 = -1 < D_\pi(v_5, v_4, v_3).$$

  Since $R_\pi(v_4, v_5) + R_\pi(v_3, v_4) < D_\pi(v_5, v_4, v_3)$, the cube function does not satisfies property (Z).

- **(C), (A), and property (Z) are independent from (Mid).** Consider the path $P(v_0, v_3) = v_0 v_1 v_2 v_3$, and let π be a profile on $P(v_0, v_3)$. Define the following location function

  $$L(\pi) = \begin{cases} Mean(\pi) & \text{if } |Mean(\pi)| = 1. \\ \{v_1\} & \text{if } Mean(\pi) = \{v_0, v_1\}. \\ \{v_2\} & \text{if } Mean(\pi) = \{v_2, v_3\}. \\ \{v_1, v_2\} & \text{if } Mean(\pi) = \{v_1, v_2\}. \end{cases}$$

  Because the mean function satisfies axioms (A) and (C), function L satisfies these axioms too. Notice also that L satisfies property Z for profiles π for which $|Mean(\pi)| = 1$. If $Mean(\pi) = \{v_0, v_1\}$, then the definition of L indicates $L(\pi) = \{v_1\}$. We need to verify property (Z) in this case. $Mean(\pi) = \{v_0, v_1\}$ implies π contains an equal number k of vertices v_0 and v_1. We can see that v_2 and v_3 are the unique vertices satisfying $v_1 v_2, v_2 v_3 \in E(P(v_0, v_3))$. Furthermore,

  $$R_\pi(v_2, v_1) + R_\pi(v_3, v_2) = k + 3k > D_\pi(v_1, v_2, v_3) = 0,$$

  then L satisfies property (Z) for this π.

  A similar argument can be used to prove that L satisfies property (Z) for profiles π with $Mean(\pi) = \{v_2, v_3\}$. This establishes that L satisfies (A), (C), and property (Z); however, it does not satisfies (Mid) because $L((v_0, v_1)) \neq \{v_0, v_1\}$.

- **(Mid), (A), and property (Z) are independent from (C).** We will use the tree T defined in Fig. 1 above. Denote by β the profile on T having each vertex of T an equal number of times. Let π be a profile on T and define the following location function

$$L(\pi) = \begin{cases} \{w_2\} & \text{if } \pi \text{ contain every vertex of } T \text{ an equal number of times.} \\ Mean(\pi) & \text{otherwise.} \end{cases}$$

We have determined above that w_2 satisfies the condition

$$R_\pi(b, w_2) + R_\pi(c, b) > D_\pi(w_2, b, c) \quad \text{whenever} \quad w_2 b, bc \in E.$$

Consequently, L satisfies property (Z) if π is a profile containing every vertex of T an equal number of times. L satisfies property (Z) for any other profile because the mean function does satisfies this axiom. From the definition of L we obtain

$$Mean((w_1, w_4)) = L((w_1, w_4)) = \{w_2, w_3\},$$
$$Mean((w_5, w_8)) = L((w_5, w_8)) = \{w_2, w_3\},$$
$$Mean((w_7, w_6)) = L((w_7, w_6)) = \{w_2, w_3\},$$
$$Mean((w_2, w_3)) = L((w_2, w_3)) = \{w_2, w_3\}.$$

Assume L satisfies axiom (C), then

$$L((w_1, w_4, w_2, w_3)) = L((w_1, w_4)) \cap L((w_2, w_3)),$$
$$L((w_1, w_4, w_2, w_3, w_7, w_6)) = L((w_1, w_4, w_2, w_3)) \cap L((w_7, w_6)),$$
$$L((w_1, w_4, w_2, w_3, w_7, w_6, w_5, w_8)) = L((w_1, w_4, w_2, w_3, w_7, w_6)) \cap L((w_5, w_8)).$$

Note that all these sets are equal to $\{w_2, w_3\}$. Nevertheless, according to the definition of L we have

$$L((w_1, w_4, w_2, w_3, w_7, w_6, w_5, w_8)) = \{w_2\}.$$

From this we conclude that L does not satisfies (C).

- **(C), (Mid), and property (Z) are independent from (A).**
Consider the tree $T = ab$ with only two vertices. Let π be a profile on T. We say that π satisfies condition (W) if
(i) $|\pi| > 2$,

(ii) π contains an equal number of $a's$ and $b's$,
(iii) if $\pi = \beta_1 \beta_2 \ldots \beta_k$, then $L(\beta_1) \cap L(\beta_2) \cap \cdots \cap L(\beta_k) = \emptyset$.

An example of a profile that satisfies condition (W) is $\pi = (a, a, b, b)$. Define the following location function

$$L(\pi) = \begin{cases} Mean(\pi) & \text{if } \pi \text{ contains more } a's \text{ than } b's \text{ or more } b's \text{ than } a's. \\ \{a, b\} & \text{if } \pi = (a, b) \\ \{a\} & \text{if } \pi \text{ satisfies condition } W. \end{cases}$$

We see that L satisfies axioms (C), (Mid), and property (Z). It does not satisfies (A) and it is not equal to the mean function because
$L((a, a, b, b)) = \{a\} \neq \{a, b\} = Mean((a, a, b, b))$.

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