You can call it a clan, or a network, or a family, or a group of friends. The way you call it is not relevant. What matters is that it exists and often you will need one. A large body of empirical work shows that networks are pervasive in social and economic interactions. This book contains four essays on the economics of networks, using a non-cooperative game theoretical approach. The first two essays study what are the structural properties of social networks when heterogeneous individuals have the discretion to form social ties. The last two essays analyse how networks influence the strategic decision making of individuals in games of conflict and in market-regulated settings. I argue that networks are likely to exhibit very central structure and short distances across individuals. Furthermore, the presence of networks alters the incentives of individual players, sometimes creating inefficient outcomes.

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Contents

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This thesis is the result of my research on the economics of networks. Each of the essays is self-contained and can be read independently. I am of course thrilled to have reached the end of my project. During the Ph. D. the difficulty was not so much the work load, but my own ups and downs, a mindset that only now I am getting use to live with. I was ecstatic at least ten times a day- thrilled every time I thought I had proved something. I was also upset ten times a day- every time I understood that my proofs were wrong. I knew it would be a tough period. I certainly wasn't wrong there. How did I ever get through those days? Well, not alone of course, but with the help of many people. Each of the persons mentioned below has transformed a professional experience as a Ph. D. in a beautiful experience of life. Time and space constraints make the list neither complete nor adequate, but each word comes from my heart.

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Introducing social and economic networks

We are caught in an inescapable network of mutuality, tied in a single garment of destiny. Whatever affects one directly, affects all indirectly.
Martin Luther King Jr.

1 Introduction

If you search in Econlit for papers published in economics that contain the word "network" in their title you would receive only 18 observations between 1974 and 1983, such number would increase to 195 between 1984 and 1993 and you would be overloaded by 1127 observations in the last decade. Similar results can be obtained using different search methods. This suggests that the study of networks is a growing field and leads us to ask why is it so? What is a network and why should we study networks? What are the new insights we could obtain by explicitly modelling network relationships? In the rest of this brief introduction of the thesis, we shall show that those models explicitly taking into account networks are tractable and capable of providing interesting and somewhat surprising insights.

Economic agents generally operate in environments of imperfect information. When choosing where and what to buy consumers do not fully know which products are available, their qualities and their prices. When applying for a job a candidate would like to be aware about the various openings and how many other applicants are competing for the same vacancies. Doctors have to decide on new treatments without having a complete knowledge of their efficacy. In each one of these situations, before choosing, we all try to gather some information. Our friends, colleagues, acquaintances generally represent our main providers. Hence, what we choose depends on how much informed we are, which partly depends on our social and economic relationships.

But this is just a part of the story. Indeed, as connections are valuable to an individual, it is natural to think that an individual will strategically decide to whom to be link with. Neighborhood segregation and ghetto formation are widely studied empirically. Schelling (1975) shows that even if members of a group are not inclined to segregation, it can still emerge as a result of the self-organization of the group. In this sense, segregation is a mere consequence of the structural properties of the pattern of interactions across players.

These examples illustrate the abundance of situations in which networks play a role and the crucial importance of networks in shaping the final outcome of these interactions. However, the above mentioned examples deal with many different subjects. What a network is remains unclear. One useful way of explicitly defining a network is by using a mathematical object: the graph. A graph is defined as a set of nodes connected via links. Depending on our specific interest, we may name the nodes as individuals, firms, consumers, countries, ideas, languages, scientific papers, and the links among them as relationships. In this way we move from a mathematical object to its ontological counterpart: the network. The use of graphs as a device to model network relationships allows to develop a systematic theory, which examines the interplay between the structure of relationships and the individual incentives, in a variety of strategic situations.

In what follows we will survey some recent works on network formation, networks and games of conflict and networks and markets. We will proceed by elaborating on the empirical evidence of each subject considered. We will then describe the related main theoretical findings to then conclude by introducing the contribution that the present thesis provides in each of the aforementioned issues.

2 A theory of network formation

Networks in many instances share some robust structural properties. Whatever network you map, let it be the World Wide Web network, scientific collaboration networks, sexual connections, communication networks, phone call networks and so on, you will notice that most agents in the network have only few links and only few players have many direct neighbors. The few players with many links are central in the network and they reduce dramatically the average distance among players. This leads to a second intriguing property of social networks: the small-world property. In words, even if two individuals do not hold a direct relationship, they are relatively close in the social structure as there exists a short chain of intermediaries connecting them. For example, Rogers and Kincaid (1981) present communication networks from rural areas in different parts of the world. They report that in these networks there are only few very well connected players (central players). Albert and Barabasi (2002) report similar evidence for a number of large-scale networks. For example, the World Wide Web exhibits high centrality and short average distances (within the core set of nodes). This high centrality arises because some nodes have very high number of outgoing links, while some others have very high number of incoming links. Phone calls networks display similar properties.[1]

This widespread regularity of networks motivates the development of a systematic theory of how networks form. This theory aims at predicting which networks of relationships we should observe when the individuals have the discretion to strategically choose their connections. Doing so, we would also be able to describe the role that networks have in determining the aggregate performance of the system, which allows for a normative analyses to be implemented.

The study of socioeconomic networks is well established in so-

[1] See also Goyal, van der Leij and Moraga-Gonzalez (2003) for coauthorship networks in economics and Newman (2001) for coauthorship networks in other fields.

ciology.[2] This literature provides useful insights on the role networks play in different settings. It focuses mainly on understanding the role that different types of connections play. However, a study of how such connections come about and their implication for efficiency is missed. In this perspective the economic theory of network formation should be seen as complementary to the sociological theory.

In what follows we will discuss the *connections model*, which has been extensively studied in the literature. The basic idea of this model is that social networks are the result of the choice of individual players who trade-off the cost of investing in links and their potential rewards.

2.1 The connections model

A network, g, is a list of pair of players, i and j, linked to each other and it is the result of the decision of each player belonging to a finite population, N. The basic element of a network game is the network formation process. In a static game, the network formation process is a set of rules, which specifies when a connection will be formed. In a dynamic setting, the evolving process of networks must be specified as well. The natural way of analysing how networks shape the individuals' incentives to invest in connections is one where links are costly and they generate externalities. Forming a link with another individual requires to make some costly effort and it allows access, in part and in due course, to the benefits available to the latter via her own links.

There are two versions of the connections model. The case where individual players can form links unilaterally was introduced in Bala and Goyal (2000a) and Goyal (1993), while the case where links are formed on the basis of bilateral agreements was introduced in Jackson and Wolinsky (1996).[3] Both

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[2] See, among others, Coleman (1988), Burt (1992), Granovetter (1973, 1974). See Wasserman and Faust (1994) for an introductory book on socioeconomic networks in sociology.

[3] The term connections model is due to Jackson and Wolinsky (1996).

these versions have been extensively studied in the literature.[4] Heuristically, the requirement of bilateral agreements is more adequate when we model network relationships such as friendships, co-authorships, family relationships and firms alliances. Conversely, the unilateral formation process is better when we focus on connections forming the World Wide Web, the networks of quotations in referee journals, telephone calls and, more generally, investments in social ties which bring benefits to both parties. The distinction between bilateral and unilateral agreement is not simply a conceptual one; indeed, with respect to the formation process we specify, we need to consider a different equilibrium notion. For this reason we proceed first by reporting the main findings of Bala and Goyal (2000a) and then we will turn to the model of Jackson and Wolinsky (1996)

- **Bala and Goyal (2000a)**

The unilateral formation process allows us to analyse the game using standard tools from non-cooperative game theory. For our purposes, it is enough to focus on the static version of the model.[5] The strategy of each player, say g_i, is a vector describing with whom player i wants to form a link. The cost of each link, say c, is paid by the player who sponsors the link. Each player is endowed with some non-rival good, which has a value v, and the benefit to each player is increasing in the amount of information accessed directly or indirectly in the network. Bala and Goyal (2000a) examine two versions of this model: the case where the link formed by player 1 with player 2 creates benefits for both parties, the *"two-way flow model"*, and the other where only the player who sponsors the link obtains benefits, the *"one-way flow model"*. For simplicity, we provide here a discussion of

[4] McBride (2003) studies a connections model where players have imperfect information about the structure of the networks. Bala and Goyal (2000b) and Haller and Sarangi (2001) examine a connection model where links are not fully reliable. Johnson and Gilles (2000) introduce players' heterogeneity in the framework of Jackson and Wolinsky (1996).

[5] Bala and Goyal (2000a) also examine the evolving of networks under a myopic-best response dynamics. See Goyal (2003) for a survey on theoretical models on learning in networks.

the two models in the case where the length of the path does not matter in defining the benefits (i.e. there is no decay).[6]

The starting point is to examine the structural properties of Nash equilibria: a network g is Nash when no player has a strict incentive to deviate, given the strategies of the other players as fixed. Not surprisingly, Bala and Goyal (2000a) show that both in the one-way and two-way flow model there is a huge number of Nash networks: any minimally connected network and the empty network are Nash equilibria for some range of parameters.[7]

The huge multiplicity of Nash equilibria leads the authors to refine the equilibrium concept in order to obtain sharper predictions on the architecture of the equilibrium networks. A natural way of doing so is to consider the notion of strict Nash network: a Nash network where each player is playing his unique best response. The refinement of strict equilibrium is a quite useful one. In the one-way flow model a strict equilibrium is either the empty network or the wheel. In the two-way flow model a strict equilibrium is either the empty network or the center-sponsored star network. Figure 1.1 illustrates these architectures in a society composed of 5 players.[8]

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[6] See Bala and Goyal (2000a) for a partial characterization of equilibrium networks when there is decay in the information flow. See Feri (2004a, 2004b) for an analysis of stochastically stable networks in presence of decay.

[7] A connected network is one where each player can access all other players. A minimally connected network is a connected network where it is enough to delete an arbitrary link to break connectedness.

[8] In the center-sponsored star network a bold line on a link next to a player indicates that this player has formed the link and pays for the link. In the wheel network we represent a link $g_{i,j} = 1$ as an edge starting at j with the arrowhead pointing at i.

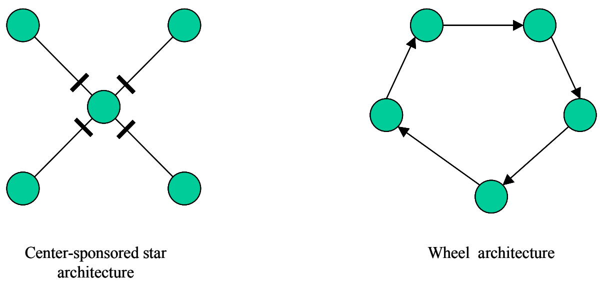

Figure 1.1. The star and the wheel architecture.

The star is a very asymmetric network with two main structural properties. The first is that there is one player, the center, who plays an important role in connecting the others, who would be otherwise disconnected. This underlines the relevant role centrality plays in information networks. The second property is that there are short distances among players. In contrast, the wheel is a symmetric network in which each player forms and receives one and only one link.

In order to answer to the question of whether these networks are efficient, Bala and Goyal (2000a) consider a notion of aggregate efficiency: a network g is efficient if it maximizes the sum of the utilities to each player. The most important result here is that, for intermediate levels of the costs of linking, there exists a trade-off between individual and social incentives. Players do not fully internalize network externalities. Such conflict is clear in the center-sponsored star. The benefit the central player obtains by maintaining a link with a spoke player is simply a unit of information, while the network externalities such link produces are huge as each player connected in the network benefits from accessing the spoke player. It is clear that when the costs of linking are moderately high, the central player will not be willing to keep this link, even if it enhances efficiency in the society as a whole.

- **Jackson and Wolinsky (1996)**

The symmetric connections model introduced in Jackson and Wolinsky (1996) shares the same features as the 2-way flow model in Bala and Goyal (2000a), but differs in that each player pays a cost c for every link formed. Moreover, the requirement of mutual consensus for the formation of a social tie leads to consider some sort of coalition equilibrium concept. Elegantly, Jackson and Wolinsky (1996) introduce the notion of pairwise stability in the attempt to model directly what a stable network is. More precisely, a network g is pairwise stable when each individual player does not wish to delete his links and every pair of players does not find profitable to form a new link.[9] Clearly, there are limitations with respect to this notion, as only pairwise deviations are taken into account. Nevertheless, the notion of pairwise stability has been extensively used in the literature and turns out to provide interesting insights.

Let us suppose that information flows through paths with some decay factor. This being the case, when the costs of linking are sufficiently low, a network where every pair of players forms a link is uniquely stable. More interestingly, when the costs of linking are moderate, the star network emerges and when costs are too high the unique stable network is the empty one. Similarly to Bala and Goyal (2000a) when the formation of a link requires mutual consensus there exists a trade-off between individual and social incentives, for some range of parameters. This last result underlines, together with the results mentioned above, that there exists an inherent trade-off between individual and social incentives in the formation of network relationships.[10]

Summarising, the connections model is a natural framework for studying the formation of network relationships. The predictions provided are sharp and replicate the main properties of real social networks. The analysis of the formation of social ties from a strategic point of view, as well as the focus on its efficiency properties are the two main features that distinguish the economic literature of network formation from that of other

[9] More precisely in Jackson and Wolinsky (1996) only the deviation of deleting a link is considered; Goyal and Joshi (2003) modify the notion of pairwise stability allowing to delete any link a player has.

[10] See Jackson (2003) for a more detail and general survey on the trade-off between individual and social incentives in network games.

fields, as sociology and physics, among others.

The models presented, and indeed most of the existing literature, focus on homogeneous players network games. However, ex-ante players' asymmetries arise quite naturally in several contexts. For instance, in the context of information networks, it is often the case that some individuals are very much interested in particular issues (such as computer software) and, therefore, better informed, fact that makes them more valuable as contacts. Similarly, individuals differ in their communication and social skills. Finally, individuals can often be classified according to distinct groups (based on geographical and for cultural characteristics). It is natural to think that forming links within a group is cheaper than forming links across groups. This reasoning motivates the analysis of strategic link formation when players differ with respect to values as well as costs of linking. Such analysis represents the focus of chapter 2 and 3 of the thesis. Let us now proceed to illustrate our main results.

In chapter 2 we study the two-way flow model.[11] We begin by analysing a general model of heterogeneity and show that value heterogeneity determines the level of connectedness of the network, but not the equilibrium network's architectures. By contrasts, costs of linking heterogeneity shape both the level of connectedness and the architecture of equilibrium networks. This leads to ask: does strategic link formation have something to say in settings with particular type of cost heterogeneity? To address this question we examine a society which is divided into distinct groups, where intra-group links are cheaper with respect to inter-group links. We find that inter-connected stars with locally central players are prominent in equilibrium.[12] Such finding suggests that centrality and short-distance are robust features of social networks.

[11] This chapter subsumes two joint papers, 'Equilibrium networks with heterogeneous players', by Galeotti and Goyal, and 'Stable equilibrium networks with heterogeneous players' by Galeotti and Kamphorst.

[12] For example, in a society composed of two groups an inter-connected stars architecture is a network where each group forms a star architecture and there exists a link between the two centers of the two groups.

In chapter 3 we focus on the one-way flow model. As in the two-way flow model, both values and costs asymmetries are crucial in determining the level of connectedness of the network. Interestingly, we find unconnected equilibria to be asymmetric and the possibility for central players to emerge. Furthermore, as far as the costs of linking are not partner-specific, equilibrium networks are wheels (and its variants). Otherwise, different architectures, such as the flower or its variants, constitute an equilibrium.[13] These findings suggest that in the 1-way flow model with heterogeneous players centrality is a distinctive feature of equilibrium networks.

The findings of chapters 2 and 3 show that even in settings with substantial players heterogeneity the theory of strategic links formation delivers sharp results, which are in line with the empirical evidence. Centrality and short distances continue to be distinctive features of equilibrium networks in settings where benefits flow in both directions (chapter 2). When players' heterogeneity is taken into account, centrality emerges also in settings where benefits flow only toward the investor (chapter 3).

3 Networks and games of conflict

Even if cooperation behaviors occur in many situations, traditional economics fails to explain it. The classical example provided by economists is the one-shot Prisoners' Dilemma where the unique Nash equilibrium is defection. A way of explaining the emergence of norms which are able to sustain cooperation is to consider infinitely repeated interactions. Mechanisms of punishment and, more in general, of social control are necessary conditions for this to occur. Few theoretical works, though, examine the influence of the social structure on the individual incentives to cooperate, whereas, a large body of empirical work suggests that network of relationships matter.

[13] The flower network is a bunch of wheels (petals) which have in common a single player. Such player, the center, connects the different petals.

For example, Coleman (1988) shows that gains from interactions are easier to obtain in networks that are clustered. In a recent experiment, Rield and Ule (2002) examine the role of endogenous network formation in the way rational players play a repeated prisoners' dilemma game. Comparing a treatment where the network is exogenously given with treatments where the network is formed endogenously, they observe that in the latter case cooperation rates are significantly higher than in the former case. Finally, Cassar (2002) analyses the cooperative behavior in three classes of networks: random networks, small world networks and regular networks. Her main result is that small world networks exhibit a lower level of cooperation than the one appearing in the other network structures considered.

The examples above show the importance of developing a structural analysis of individual incentives in games of conflict. In this respect, there exists an extensive literature on the spatial evolution of social norms which spans the fields of biology, computer science and physics, in addition to economics.[14] In economics, the main contribution to this literature is the work of Eshel, Samuelson and Shaked (1998). They study a dynamic setting where players are exogenously arranged in a circle and learn by imitating their neighborhoods. Players choose to be either altruistic or egoistic after comparing the average payoffs from the two actions. The authors' main finding is that altruistic behavior arises when altruistic agents are grouped together. This being the case, they primarily benefit by reciprocating altruism. Differently, egoists survive only if they are few and strewn. Given that the society is large enough and that the players' initial strategies are determined by i.i.d. variables, these mixed configurations where the society is composed of a majority of altruistic individuals and a minority of egoistic players are the only absorbing states.

A different and natural approach to examine the interplay between networks and incentives to free-ride is to consider the

[14] See, e.g. Ullman-Margalitt (1977), Eshel and Cavalli-Sforza (1982), Wynne-Edwards (1986), Nowak and May (1992), Axelrod (1997).

scope of cooperation in repeated games when they are played on networks. It is surprising to notice that there are few works which have taken this direction. In a recent paper, Haag and Lagunoff (2000) analyse a prisoners' dilemma game where players are ex-ante heterogeneous with respect to their discount factor and they play with their immediate neighbors. They mainly focus on the network architecture which supports high cooperation. Their main result is that, when players are restricted to play the same action with respect to each player they are interacting with, the network architecture which best sustains cooperation requires that a set of players are fully connected and that each of these players is linked with some other players outside the clique. The players belonging to the clique must be substantially patient and they always cooperate, while the other players will be quite impatient and they will always free-ride.

The purpose of the analysis of Haag and Lagunoff (2000) is to select the architecture which is most desirable to enhance cooperation. However, we do not know whether these networks are strategically viable. A recent theoretical paper, which considers endogenous cooperation networks is Vega-Redondo (2002). In this paper, agents play a collection of infinitely repeated prisoner's dilemmas on the current social network. The payoffs in the prisoners' dilemma game fluctuate over time and players can communicate via their links information regarding the behaviour of their acquaintances. The analysis addresses two main questions. The first attains the structural properties of endogenous networks that may sustain cooperative behavior. The second issue tackled is how these structural properties vary when the level of uncertainty (payoffs uncertainty) in the society is altered. Vega-Redondo's main result is that, for a sufficiently low level of payoff volatility, cooperation is sustained in dense networks. The intuition is that network density, together with the possibility for players to communicate behavioral information along links, makes monitoring more effective and this mitigates the individuals' incentives to free-ride. In line with this finding, Vega-Redondo (2002) shows that in more uncertain environments cooperative networks exhibit lower average distance among players.

In Vega-Redondo (2002) the effect of networks on cooperative behavior is due to the possibility of players to communicate behavioral information along links. Many are the situations in which connections serve to share non-rival goods, such as information and knowledge.[15] In these cases, the externalities produced by the network are fully realized only if relationships are stable over time. However, free-riding problems, among others, may undermine the stability of (more or less) informal relationships. Chapter 4 of the thesis presents a model by means of which we study the effect of endogenous network externalities on cooperative behavior.[16] Players first invest in connections and this then results in a network of relationships specifying the interaction pattern among players. Once the network is formed, each pair of linked players plays an infinitely repeated game consisting of two games. On the one hand, each player plays a prisoners' dilemma game with each one of his immediate neighbors. If two linked players cooperate (defect) they share the cost of the link at the cooperative (defection) level, while if one player cooperates and the other defects the former must entirely pay the cost of that link (at the exploitative level). On the other hand, each player, endowed with some valuable information, decides whether to provide or withhold such information to each of his acquaintances. Networks with different architectures will allow for more or less sever punishments and this allows for a systematic analysis of the interplay between stable network architectures and individual incentives in games of conflict.

We show that when network externalities are taken into account players may sustain efficient interactions in situations that would not be possible otherwise. We also observe that the architecture of a network is crucial in determining the strategical viability of efficient interactions. More precisely, when an efficient interaction requires players to mutually cooperate, efficient

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[15] Examples are scientific collaborations, collaborations for the development of new products, friendship relationships, agreement among different cities to built up common infrastructure.

[16] This Chapter is based on a joint work with Miguel Melendez.

social norms are best sustained in symmetric networks, i.e. the line network. By contrast, when an efficient interaction requires players to play asymmetrically (one cooperates and the other free-rides), efficient social norms are best sustained in fully centralized architectures, i.e. the star network. These results illustrate the importance of carrying out a structural analysis of the individuals' incentives, in many strategic situations.

4 Networks and markets

We are used to think of markets as made of two sides, the buyers and the sellers, freely interacting with each other. Prices coordinate these interactions. However, a large body of the empirical work suggests that interactions within one side of the market and across the two sides are built up on a variety of networks relationships. R&D collaborations among firms are an example. Hagedoorn (2002) shows that, in recent years, joint R&D and technology exchange agreements are more prominent than joint ventures (which were instead intensively used in the early 60ies). Therefore, firms undertake research projects with shared resources and joint product development agreements and then compete in the market.

Network of relationships also play an important role on the consumers side. In the marketing literature it is well established that consumers obtain much of their information by interacting with their social contacts (Feick and Price (1986,1987)). In relation to this, firms have increasingly recognized the need for using informal channels as a way to market their product. The practice of consumers referral is an example.[17] According to the Direct Selling Association (1999), annual sales of firms that rely entirely on consumer referral grew from 13 billion to nearly 23 billion dollars, between 1991 and 1998.[18]

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[17] Firms provide different sort of benefits such as discounts to clients who bring new costumers.

[18] We provide another example. Granovetter (1974) shows that in the process of finding a job, people rely intensively on their social connections. Therefore, phenomena such as unemployment, wages distribution and inequality are also affected by the interactions

Also the relationships between buyers and sellers play an important role. For example, in the process of producing and selling goods, firms must discover from whom they can purchase inputs, the potential demand they will encounter and who is interested in distributing their products. To this end, when transaction costs are too high, informal relationships may be of help. Nishiguchi (1994) shows how, in the Japanese electronics and automobiles sectors, firms always rely on specific subsets of suppliers with whom they maintain close business relationships. These examples motivate the development of a theory explaining the role network relationships play in the way markets function. We now discuss the main theoretical contributions in this respect.[19]

Kranton and Minehart (2001) analyse what drives buyers and sellers to form links with multiple partners and whether these networks are efficient or not. The natural way of modelling these relationships is to consider that trade between a buyer and a seller may occur only if a link is in place. The buyer's and seller's network thus specifies with whom each player may transact. In such a framework, buyers have independently and identically distributed utilities for the object. They form costly links with sellers, each one of which has a single unit to sell. The price of each transaction is the result of an ascending-bid auction. Each buyer knows his own evaluation but not the evaluation of other buyers. Buyers drop-out of the bidding as the price exceeds their evaluation. This process continues until demand equals supply. The main results the authors obtain are the following. First, competition generates an efficient allocation of goods in the network. Second, the network connecting buyers and sellers is crucial in determining the price at which transactions take place. In particular, the utility to a buyer equals the marginal social value of his participation in the network.[20]

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structure of the society. See also Calvo and Jackson (2004a, 2004b) and Montgomery (1991).

[19] See Goyal and Moraga-Gonzalez (2003) for a survey of models on firms, markets and networks.

[20] Other works on this issue are Kranton and Minehart (2000), Corominas-Bosch (1999)

Goyal and Moraga-Gonzalez (2001) study the incentives of competing firms to form collaborative agreements with each other. More precisely, there is a finite number of firms, which, prior to competing in market, form pair-wise collaborations with other firms. Once the network has been formed each firm chooses a level of costly R&D effort.[21] Thus, for each link connecting two firms the R&D output is determined, which results in a lower cost of production for the two partners. Finally, firms compete in the market taking as given the costs of production. Goyal and Moraga-Gonzalez (2001) find that when firms compete by setting quantities, firms' R&D effort is declining in the level of collaborative activity. Social welfare is maximised under an intermediate level of collaboration. Since firms in some cases can gain market power by increasing the number of suitable agreements, firms may have an incentive to form too many links and as a consequence inefficiency may emerge in the market.

In Chapter 5 we develop a theoretical model to analyse the interplay between network relationships among consumers and the functioning of the market. We examine a search model a la' Burdett and Judd (1983). Consumers are embedded in a consumers network and they may search at a cost for price quotations. The information thus gathered is non-excludable along direct links. To maintain symmetry on the consumers' side we assume that each consumer holds the same number of connections. Varying the number of connections that each consumer has allows us to investigate the effect of network relationships on consumers and firms' incentives, as well as on market competitiveness.

The first result is a full characterization of the equilibria of the game. When the network is empty (there are no connections across players), in equilibrium consumers randomize between searching for one price and for two prices (high search intensity equilibrium).[22] By contrast, when network externalities are

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and Wang and Watts (2002).

[21] Goyal and Joshi (2003) analyse a similar model where firms do not choose the effort level and therefore the extent of costs reduction is exogenously given.

[22] In this case the model degenerates to the duopolistic version of Burdett and Judd (1983).

taken into account this equilibrium exists only for sufficiently low search costs, otherwise a new equilibrium where consumers randomize between searching for one price and not searching at all emerges (low search intensity equilibrium). The second finding is that, in both equilibria, consumers search less frequently in denser networks. An increase in the density of the network leads consumers to free-ride more on each other. Finally, we show that this free-riding effect may have somewhat surprising consequences on equilibrium pricing and social welfare. In particular, in the high search intensity equilibrium the more connections consumers have the higher is the expected price in equilibrium as well as the social welfare, while the lower is the consumer surplus. These results are reversed in the low search intensity equilibrium.

Network formation with heterogeneous players

1 Introduction

The role of social and economic networks in shaping individual behavior and aggregate phenomena has received increasing attention in recent years. This work has been accompanied by research of sociologists, economists and physicists into the character of actual networks. This research shows that communication networks, scientific collaboration networks, social networks and the web exhibit high levels of centrality and small average distances.[1] This widespread stability of centrality and small distances has led researchers to develop theories of network formation which can explain these features.

The connections model proposed in Bala and Goyal (2000a) offers a simple framework for the study of network formation.[2] In this model there is a set of players who each gain from accessing other players. Player 1 can access player 2 directly by forming a link; this link also allows player 1 access to other players that player 2 is accessing on his own. We will suppose that the link formed by 1 with 2 creates a similar flow of benefits to

[1] See Rogers and Kincaid (1981) for networks of communication, Goyal, van der Leij and Moraga-Gonzalez (2003) for co-authorship networks in economics and Newman (2001) for co-authorship networks in other subjects, Burt (1992) for work on social networks, Albert and Barabasi (2002) for evidence on the architecture of the World Wide Web.

[2] There are two versions of the connections model: the case where individual players can form links unilaterally was introduced in Bala and Goyal (2000a) and Goyal (1993), while the case where links are formed based on bilateral agreement was introduced in Jackson and Wolinsky (1996). The term *connections model* is due to Jackson and Wolinsky (1996). Both the versions have been extensively studied in the literature. Theoretical work on this model includes Bala and Goyal (2000b), Deroian (2003), Dutta and Jackson (2000), Feri (2004a, 2004b), Haller and Sarangi (2001) and Watts (2001a, 2001b). There have also been several experimental tests of the predictions of the connections model; see e.g., Falk and Kosfeld (2003), Callander and Plott (2004) and Goeree, Riedl and Ule (2004).

2.[3] Bala and Goyal (2000a) show that if a player's payoffs are increasing in the number of other players accessed and decreasing in the number of links formed, then an equilibrium network can have only one of two possible structures: it is either a center-sponsored star (a network in which one player, the center, forms links with all the other players) or the empty network (which has no links). We note that a star exhibits high centrality and short distances between individuals. In this paper we examine the impact of ex-ante player heterogeneity on these findings.

Ex-ante asymmetries arise quite naturally in many contexts. For instance, in the context of information networks it is often the case that some individuals are more interested in particular issues (such as computer software) and therefore better informed which makes them more valuable as contacts. Similarly, individuals differ in communication and social skills. Finally, individuals can often be classified into distinct groups (based on geographical or cultural reasons) and forming links within a group is cheaper as compared to forming links across groups.

We start with a general model of heterogeneous players: the costs to player i of a link with player j as well as the benefits of such a link are allowed to depend on both i and j. In addition, we assume that the length of the path does not matter in defining the benefits (there is no decay). We first consider a particular form of cost heterogeneity: for any player i the costs of forming links with every other player are c_i but we allow this cost to vary across players. In this setting we find that if benefits are homogeneous then a strict equilibrium is either an empty network or a center-sponsored star. By contrast, if values are heterogeneous then partially connected networks can also arise, though each (non-singleton) component constitutes a center-sponsored star (Proposition 2.1). These results suggest that heterogeneity in benefits is important in determining the

[3] Examples which can be interpreted in this spirit are telephone calls in which people exchange information, investments in personal relationships which create a social tie yielding value to both partners, and the creation of blogs (short for web log). When a blog user i enters another blog user j, he or she can leave a comment and this automatically creates a link from j to i.

level of connectedness of a network. We then move to a model with general cost heterogeneity where costs of forming links vary across individuals and in addition for the same individual the costs of forming links are sensitive to the identity of the potential partner. In this setting we obtain the following equivalence result: a strict equilibrium network is minimal and conversely every minimal network is a strict equilibrium for suitable costs and benefits. We also find that this equivalence obtains even if benefits are restricted to be homogeneous (Proposition 2.2). Taken together these results suggest that cost heterogeneity is important in shaping the level of connectedness of networks as well as the architecture of individual components. These results also clarify the role of different forms of cost heterogeneity and in particular imply that the 'everything is possible' nature of our equivalence result is closely related to cost heterogeneity which arises when the costs of linking vary for the same player.

This last finding on the impact of cost heterogeneity leads us to ask: Does strategic link formation have something to say in settings with restricted types of cost heterogeneity? This question is the motivation behind the *insider-outsider model* where the society is composed of distinct groups. The cost of forming a link between two players is (weakly) increasing in the distance between the groups to which the two players belong. Thus, the distance among groups may be interpreted as the degree of heterogeneity across players.

We start with a study of a static model with no decay. In this setting, we obtain *two* main results. Our *first* result is a complete characterization of strict Nash equilibrium networks. It shows that an equilibrium network is either a center-sponsored star or a variation of this architecture (Proposition 2.3).[4] Figure 2.1 depicts all the strict Nash architectures in a society composed

[4] The following phenomenon which is widely observed corresponds to center-sponsorship: one friend acts as a host to a social gathering in which friends are invited. The host (center) sponsors the invitations (center-sponsorship) and the social gathering offers an opportunity for sharing information and goods (two-way flow of benefits). A specific example of this is mentioned in Rappaport (1968); he points out that in the Maring tribe of New Guinea periodically one clan acts as a host (center-sponsorship) to a big feast in which all neighboring clans participate (two-way flow of benefits) .

of two groups.

Our *second* result is about efficient networks. In the insider-outsider model, it is clear that an efficient network must minimize the number of outsider links since they are costlier as compared to insider links. Thus in a society with 2 groups an efficient connected network has each group entirely internally linked and 1 outside link (Proposition 2.4). By contrast, a (connected) strict equilibrium network is a generalized center-sponsored star,[5] with $n - n_l$ outsider links (where n_l is the number of players in the core group). If there are 2 groups and 50 players in each group then an efficient network has 98 insider links and 1 outsider link, while a strict equilibrium network has 49 insider links and 50 outsider links! The relative abundance of across group links is a reflection of the center-sponsorship property of the network. This leads us to examine the robustness of the equilibrium predictions.

We do this by examining the role of a small amount of decay. We show that a strict equilibrium always exists and stars (and variants of stars) are prominent in equilibrium networks. However, we also find that center-sponsorship is not the only way a star can arise in equilibrium; there is a much wider range of parameters for which periphery-sponsorship prevails in equilibrium (Proposition 2.5).[6] Periphery-sponsorship is intimately related to another feature of equilibrium networks: the existence of stars constituted of members of a single group. Group-based stars minimize inter-group links and this suggests that there is

[5] This network is formally defined in section 4 of this Chapter.

[6] Periphery-sponsored centrality is widely observed empirically. We present three examples to illustrate this. The first example is rural communication networks. Rogers and Kincaid (1981) present communication networks from rural areas in different parts of the world. One of the distinctive features of these networks is the presence of a few very well connected people (stars). In these examples the average person connects with these well connected people (periphery-sponsorship). The second example is the World-Wide Web. Albert and Barabasi (2002) report that the Web exhibits high centrality and short average distances (within the core set of nodes). This high centrality arises because some nodes have very high number of outgoing links (center-sponsorship) while some nodes have very high number of incoming links (periphery-sponsorship). A third example is the network of telephone calls. Albert and Barabasi (2002) report that these networks also exhibit high centrality with some nodes having a very large number of periphery sponsored links.

considerable overlap between equilibrium and efficient networks, in the presence of a small amount of decay.

Finally, we consider the dynamic stability of equilibrium networks. We show that a dynamic process based on individual myopic best responses converges to a minimal curb set and we provide a full characterization of these sets. We find that a minimal curb set is either a strict equilibrium identified in the static model or is an interlinked center-sponsored stars network (in which each group constitutes a center-sponsored star and there exists a single link between the center-sponsored stars (Proposition 2.6)). Therefore, local centrality can arise in the long run; this reduces across group links and brings about a closer alignment between social and individual incentives.

We summarize our findings as follows: even in settings with considerable heterogeneity, strategic models of network formation yield sharp predictions and equilibrium networks exhibit high centrality and small average distances.

The theory of network formation is a very active area of research currently.[7] Most of the existing literature focuses on homogeneous player models; we now briefly discuss three other papers which examine heterogeneity. Johnson and Gilles (2000) considers two-sided link formation in a model where individuals are located around a circle and costs of links are increasing in the distance between players. McBride (2003) focuses on value heterogeneity and partial information about network structure. In a new paper Hojman and Szeidl (2003) develop a general model of decay and show that periphery-sponsorship is a robust feature of equilibrium networks. In contrast to these papers, the focus of the present paper is on the impact of different forms of heterogeneity on the architecture of equilibrium networks. We first show that value heterogeneity is not important while cost heterogeneity is critical in shaping equilibrium network architectures. This motivates the study of network formation in a model of restricted cost heterogeneity, the insider-outsider model. We

[7] See e.g. Aumann and Myerson (1988), Jackson and Watts (2002), Kranton and Minehart (2001), Slikker and van den Nouweland (2001).

find that centrality and short-average distances are features of equilibrium networks in the insider-outsider model.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 presents results on equilibrium networks under general cost and value heterogeneity. Section 4 analyzes an insider-outsider model, while section 5 concludes

2 The model

Let $N = \{1, ..., n\}$ be a set of players and let i and j be typical members of this set. We shall assume throughout that the number of players $n \geq 3$. Each player is assumed to possess some information of value to himself and to other players. He can augment his information by communicating with other people; this communication takes resources, time and effort and is made possible via *pair-wise* links.

A strategy of player $i \in N$ is a (row) vector $g_i = (g_{i,j})_{j \in N \setminus \{i\}}$ where $g_{i,j} \in \{0, 1\}$ for each $j \in N \setminus \{i\}$. We say that player i has a link with j if $g_{i,j} = 1$. A link between player i and j can allow for either one-way (asymmetric) or two-way (symmetric) flow of information. We assume throughout the paper that a link $g_{i,j} = 1$ allows both players to access each other's information. The set of strategies of player i is denoted by $\mathcal{G}_i$. Throughout the paper we restrict our attention to pure strategies. Since player i has the option of forming or not forming a link with each player of the remaining $n-1$ players, the number of strategies of player i is clearly $|\mathcal{G}_i| = 2^{n-1}$. The set $\mathcal{G} = \mathcal{G}_1 \times ... \times \mathcal{G}_n$ is the space of pure strategies of all the players.

A strategy profile $g = (g_1, ..., g_n)$ can be represented as a directed network. Let $g \in \mathcal{G}$. We use $g - g_{i,j}$ to refer to the network obtained when a link $g_{i,j} = 1$ is deleted from g. To describe information flows, it is useful to define the closure of g: this is a non-directed network denoted $\overline{g}$ =cl(g), and define by $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for each i and j in N.[8] Pictorially, the

[8]Note that $\overline{g}_{i,j} = \overline{g}_{j,i}$ so that the order of players is irrelevant.

closure of a network simply means replacing every directed edge of g by a non-directed one. We say there is a path in g between i and j if either $\overline{g}_{i,j} = 1$ or there exist players $j_1, ..., j_m$ distinct from each other and i and j such that $\left\{\overline{g}_{i,j_1} = ... = \overline{g}_{j_m,j} = 1\right\}$. We write $i \overset{\overline{g}}{\longleftrightarrow} j$ to indicate a path between i and j in g. Furthermore, a path between i and j is said to be $i - oriented$ if either $g_{i,j} = 1$ or there is a sequence of distinct players $i_1, i_2, ..., i_n$ with the property that: $\{g_{i,i_1} = g_{i_1,i_2} =, ..., g_{i_n,j} = 1\}$. Define $N^d(i; g) = \{k \in N \,| g_{i,k} = 1\}$ as the set of players with whom i maintains a link and let $\mu_i^d(g) = \left|N^d(i; g)\right|$ be the cardinality of the set. The set $N(i; \overline{g}) = \left\{k \in N \middle| i \overset{\overline{g}}{\longleftrightarrow} k\right\} \cup \{i\}$ consists of players that i accesses in g, while $\mu_i(g) = |N(i; \overline{g})|$ is its cardinality.

Given a network g, we define a component as a set $C(g) \subset N$ such that $\forall i, j \in C(g)$ there exists a path between them and there does not exist a path between $\forall i \in C(g)$ and an player $k \in N \diagdown C(g)$. Given a network g, let $\#C(g)$ be the number of components in g. A network g is said to be minimal if $\#C(g) < \#C(g - g_{i,j})$, for any $g_{i,j} = 1$. Moreover a network g is said to be connected if it is composed of only one component, i.e. $\#C(g) = 1$. If this component is minimal, then g is said to be minimally connected. Finally, network g is partially connected if it is neither empty nor connected.

We note that center-sponsored star, g^{css}, is a network architecture in which one player forms links with each of the other $(n-1)$ players and there are no other links.

To complete the definition of a normal-form game of network formation, we specify the payoffs. Let $V_{i,j}$ denote the benefits that player i derives from accessing player j. Similarly, let $c_{i,j}$ denote the cost for player i of forming a link with player j. The payoff to player i in a network g can be written as follows:

$$\Pi_i(g) = \sum_{j \in N(i;\overline{g})} V_{i,j} - \sum_{j \in N^d(i;g)} c_{i,j} \tag{2.1}$$

We shall assume that $c_{i,j} > 0$ and $V_{i,j} > 0$ for all $i, j \in N$. Given a network $g \in \mathcal{G}$, let g_{-i} denote the network obtained when all of player i's links are removed. Note that the network g_{-i} can be regarded as the strategy profile where i chooses not to form a link with anyone. The network g can be written as $g = g_i \otimes g_{-i}$ where the '$\otimes$' indicates that g is formed as the union of the links in g_i and g_{-i}. The strategy g_i is said to be a *best response* of player i to g_{-i} if:

$$\Pi_i (g_i \otimes g_{-i}) \geqslant \Pi_i (g_i' \otimes g_{-i}) \text{ for all } g_i' \in \mathcal{G}_i. \qquad (2.2)$$

The set of all of player $i's$ best responses to g_{-i} is denoted by $\mathcal{BR}_i (g_{-i})$. Furthermore, a network $g = (g_1, ..., g_n)$ is said to be a *Nash network* if $g_i \in \mathcal{BR}_i (g_{-i})$ for each i, i.e. players are playing a Nash equilibrium. If a player has multiple best responses to the equilibrium strategies of the other players then this could make the network less stable as the player can switch to a payoff equivalent strategy. This switching possibility in non-strict Nash networks has been exploited and has been shown to be important in refining the set of equilibrium networks in earlier work (see e.g. Bala and Goyal (2000a)). So we will focus on strict Nash equilibria in the present paper. A *strict* Nash equilibrium is a Nash equilibrium where each player gets a strictly higher payoff from his current strategy than he would with any other alternative strategy.

We now define social welfare and efficiency of a network. There are different ways of measuring efficiency; we follow the convention in this literature and focus on the sum of payoffs of all players. Formally, given a network g, its welfare, $W : \mathcal{G} \rightarrow R$, can be stated as follows:

$$W (g) = \sum_{i=1}^{n} \Pi_i (g) \text{ for } g \in \mathcal{G}. \qquad (2.3)$$

A network is said to be efficient if $W (g) \geqslant W (g')$ for any $g' \in \mathcal{G}$. Our notion of efficiency is equivalent to the concept of strong efficiency in Jackson and Wolinsky (1996).[9]

[9] An alternative definition would be in terms of Pareto dominance. In settings where utility is not transferable, efficient networks are always Pareto-efficient, but the converse is generally not true. However if payoffs are transferable across players then clearly the two notions are equivalent.

3 General heterogeneity

In this section we shall study the scope of individual incentives in restricting network architectures in a setting of general costs and value heterogeneity. Our main finding is that value heterogeneity is important in determining the connectedness of a network while heterogeneity in costs matters both for the level of connectedness as well as for the architecture of individual components of a network.

We start with a consideration of a setting in which players may differ in their costs of forming links but the costs of forming links for an individual are independent of the potential partner. Our first result establishes an equivalence between the set of center-sponsored star networks and the set of strict equilibrium networks if values are homogeneous. On the other hand, if values are allowed to vary freely then we find an equivalence between the set of minimal networks in which non-singleton components are center-sponsored stars and the set of strict equilibrium networks.

Proposition 2.1. *Let payoffs satisfy (2.1) and suppose* $c_{i,j} = c_i$, $\forall j \in N$. *If* $V_{i,j} = V, \forall i, j \in N$, *then a strict equilibrium is either empty or a center-sponsored star; conversely any such network is a strict equilibrium for some* $\{c_i, V\}$. *If values vary freely then a strict equilibrium is either empty or a minimal network in which every (non-singleton) component is a center-sponsored star; conversely any such network is a strict equilibrium for some* $\{c_i, V_{i,j}\}$.

Proof: We note first that any equilibrium network is minimal; this follows from the no decay assumption. We next show that if $c_{i,j} = c_i$, $\forall j \in N$ then any non-singleton component $C(g)$ in a strict equilibrium network g must be a center-sponsored star. If there are two players in this component then the claim is obviously true. So let us consider a component with 3 or more players. Without loss of generality there is a pair of players i and j such that $g_{i,j} = 1$. We note that player i cannot access any other player k via this link with player j. If there were such a player then since $c_{i,j} = c_i$, $\forall j \in N$, player i would be indifferent between linking with j and k and g would not be a strict equilibrium. We next note that no such player k forms a link with i. If k formed a link with i then k would in turn be indifferent between linking with i and j. Combining these observations it follows that player i must be forming links with all players in the component and so it constitutes a center-sponsored star. We next take up the cases of homogenous and heterogeneous values, respectively.

First, we consider the case of homogeneous values. Suppose g is a non-empty (strict) equilibrium network. We will show that it is connected. Let $C_1(g)$ be a non-singleton component in g and let $j \notin C_1(g)$. From above it follows that there exists a player $i \in C_1(g)$ who is central and sponsors all links in $C_1(g)$. Since g is a strict equilibrium this implies that $c_i < V$. The marginal payoff to forming a link with j is at least V, and so player i can increase his payoff by forming an additional link, contradicting the hypothesis that g is an equilibrium. Thus g is connected and we have proved that if values are homogeneous then an equilibrium network is either empty or a center-sponsored star. We now take up the converse case. The empty network is a (strict) equilibrium if $c_i > V$ for all i, while a center-sponsored star with i at the center is a (strict) equilibrium if $c_i < V$.

Second, we consider the case of heterogeneous values. From the above arguments it follows that any component in a non-empty (strict) equilibrium network must be a center-sponsored star. We now prove the converse. Fix some minimal network g in which every (non-singleton) component is a center-sponsored star. Let there be m components in this network, $C_1(g), ..., C_m(g)$. Let $i \in C_1(g)$ be the central player in $C_1(g)$. For any link $g_{i,j} = 1$, set $c_i < V_{i,j}$, while for every component $C_k(g)$, $k = 2, ..., m$, set $\sum_{j \in C_k(g)} V_{x,j} < c_x$,for all $x \in C_1(g)$. It follows that the links of i are optimal while no additional links are profitable for any player $x \in C_1(g)$. Since $C_1(g)$ was arbitrary, the proof follows.■

The above result illustrates the role of value heterogeneity in defining the level of connectedness of networks: homogeneous values ensure connectedness of networks, while heterogeneity can generate partially connected networks. We next note that the introduction of costs heterogeneity decreases the multiplicity of equilibria. Indeed, only players who have a sufficiently low cost of linking can be at the center of a center-sponsored star network. We finally note that $c_i = c$ is a special case of the above result. This tells us that the results on equilibrium networks with homogeneous costs and values obtained in Bala and Goyal (2000a) can in fact be generalized to allow for heterogeneity in costs of forming links across individuals. Is this also true if costs of forming links are different for the same individual, depending on the potential partner? The following proposition shows that matters are quite complicated in this case.

Proposition 2.2. *Let payoffs satisfy (2.1) and suppose costs vary freely. Then a strict equilibrium is minimal; conversely, any minimal network is a strict equilibrium for some* $\{c_{i,j}, V_{i,j}\}$.

Proof: Minimality follows directly from the no decay assumption. We now prove the converse. Fix some minimal network g. We set the costs and values as follows: $V_{i,j} = V, \forall i, j \in N$ and for any link $g_{i,j} = 1$, let the corresponding cost $c_{i,j} = \epsilon < V$, while for any link $g_{i,j} = 0$, set the corresponding cost $c_{i,j} > (n-1)\,V$. The proof follows. ■

This result shows that if costs of forming links for an individual vary across partners and costs of forming links are different for different players then strategic interaction imposes no restrictions on network architecture. We also note that the proof of the second part of the result actually uses homogeneous values to support arbitrary minimal networks. This shows that, in case of general cost heterogeneity, the level of value heterogeneity plays no important role in determining network architecture. We summarize our analysis of the general heterogeneity model in the following table.

Costs \ Values	Homogeneous	Heterogeneous
Homogeneous	g^e, g^{css}	g^e, minimal networks in which every non-singleton component is a center-sponsored star
$c_{ij}=c_i$	g^e, g^{css}	g^e, minimal networks in which every non-singleton component is a center-sponsored star
Heterogeneous	Minimal networks	Minimal networks

Table 2.1. The role of general heterogeneity

This table tells us that value heterogeneity is important in determining the level of connectedness of networks. We also observe that cost heterogeneity is important in shaping both the level of connectedness as well as the architecture of individual components. Finally, this table also highlights the significance of different forms of cost heterogeneity in shaping networks. In particular it implies that the 'everything is possible' nature of our equivalence result is closely related to cost heterogeneity which arises when the costs of linking vary for the same player. This finding motivates an examination of settings with specific types of cost-heterogeneity.

4 An insider-outsider model

In this section we consider a society in which individuals are divided into pre-specified groups, and the costs of forming links within the groups is lower as compared to costs of forming links across groups. This leads to a model in which costs of linking are partner specific. We start with a basic static model with no decay and provide a complete characterization of equilibrium and efficient networks. We then examine the robustness of the findings to decay and dynamics. Our main finding is that centrality and small distance are robust features of equilibrium networks.

We consider a society composed of m groups. Let $n_l = |N_l|$ be the size of group l, with $l = 1, 2, 3, ..., m$. The set of players is then $N \equiv \cup_{l=1}^{m} N_l$. We assume perfect symmetry in value across individuals and we normalize it to one, i.e. $V_{i,j} = 1$ for all $i, j \in N$.[10] To allow for cost heterogeneity we consider a spatial cost structure: groups can be ordered in a line according to some well defined characteristics. The distance between two groups can be interpreted as a measure of the heterogeneity that distinguishes them. Given two players $i \in N_l$ and $j \in N_k$, the cost of forming a link $g_{i,j}$, is:

$$c_{i,j} = c_{j,i} = f\left(|l - k|\right) \tag{2.4}$$

If i and j belong to the same group we let: $c_{i,j} = c_{j,i} = f(0) = c_L$. We shall assume that $f(\cdot)$ is (weakly) increasing in its argument and $c_L > 0$.

We note two interesting cases of our specification. First, when $f(0) = f(1) = ... = f(m-1) = c$ the insider-outsider model degenerates in the linear payoff model presented in Bala and Goyal (2000a). Second, if we assume that $f(d) = c_H$, $\forall d \geqslant 1$, and $f(0) = c_L < c_H$, we then have a two-cost levels model: the cost of creating an outside link across groups, c_H, is higher than the cost of creating an inside link within a group, c_L.

Let $N^{d,k}(i; g) = \{j \in N_k | g_{i,j} = 1\}$, for $k = 1, ..., m,$; then define $N^d(i; g) \equiv \cup_{k=1}^{m} N^{d,k}(i; g)$. Furthermore, let $\mu_i^{d,k}(g)$ be the cardinality of $N^{d,k}(i; g)$. In other words, $\mu_i^{d,k}(g)$ represents the number of links initiated by i with members of group k. Hence, given a network g and a player $i \in N_l$, the payoff function

[10] This normalization simplifies the statement of our results; on occasion this normalization can create some confusion between the notions of component value and component size. For instance, our statements relating costs of forming links with specific networks are clearly restrictions on component value and not on component size alone.

described by (2.1) can be rewritten as follows:

$$\Pi_i(g) = \mu_i(g) - \sum_{k=1}^{m} \mu_i^{d,k} f(|l-k|) \tag{2.5}$$

We now develop some additional notation. Given a network g, we say that two players $i, i' \in N_l$ are internally linked if either $g_{i,i'} = 1$ or there exists a group of distinct players $\{i_1, i_2, ..i_k\}$ where $i_x \in N_l$ for any $x \in \{1, ..., k\}$ such that $\bar{g}_{i,i_1} = \bar{g}_{i_1,i_2} = ... = \bar{g}_{i_k,i'} = 1$. A group N_l is entirely internally linked if every pair of players $i, i' \in N_l$ is internally linked. Similarly, a pair of players i, i' is externally linked if $g_{i,i'} = 0$ and there exists a group of distinct players $\{j_1, j_2, ..j_k\}$ where $j_x \notin N_l$ for any $x \in \{1, ..., k\}$ such that $\bar{g}_{i,j_1} = \bar{g}_{j_1,j_2} = ... = \bar{g}_{j_k,i'} = 1$. A group N_l is entirely externally linked if every pair of players $i, i' \in N_l$ is externally linked. Finally, let the diameter of a non-singleton component $C(g)$ be defined as the length of the largest geodesic distance between any pair of players belonging to it, *i.e.* $D(C(g)) = \max_{i,j \in C(g)} d(i, j; C(g))$.[11] We now define some network architectures that arise in this model.

Definition 2.1. *A generalized center-sponsored star is a minimally connected network which satisfies the following conditions:*

- (i) *There is a group l_0 and a player $i_0 \in N_{l_0}$, such that $g_{i_0,j} = 1, \forall j \in N_{l_0} \setminus \{i_0\}$.*
- (ii) *For any $j \in N$, $i_0 \overset{\bar{g}}{\longleftrightarrow} j$, is an i_0 − oriented path.*
- (iii) *Consider an i_0 − oriented path, where $i_0, i_1, i_2, ..., i_n$ with $\{g_{i_0,i_1} = ... = g_{i_{n-1},i_n} = 1\}$. Let $i_k \in N_{l_k}$, for $k \in \{0, ..., n-1\}$, then $f(|l_k - l_{k+1}|) < f(|l_k - l_x|)$ for $x \in \{k+2, k+3, ..., n\}$.*

We note that a generalized center-sponsored star will have the feature that along any path starting from the central player

[11] Given two players i and j in g, the geodesic distance, $d(i, j; g)$, is defined as the length of the shortest path between them.

there can be at most m players. Thus the diameter of any such network is at most $2m$, which is independent of the size of the society and only depends on the number of groups. We shall use g^{gcs} to refer to any generalized center-sponsored star network. A network in which each group constitutes a distinct center-sponsored (periphery-sponsored) star and there are no links across groups has the unconnected center-sponsored (periphery-sponsored) stars architecture. We shall use g^{ucs} (g^{ups}) to refer to any network with this architecture.

Our first result provides a complete characterization of strict Nash networks in the insider-outsider model.

Proposition 2.3. *Suppose (2.4) and (2.5) hold. Assume that* $n_l \geqslant 2$, $\forall l = 1, ..., m$.

1. *If* $c_L > 1$ *then the empty network is the unique strict equilibrium.*

2. *Suppose* $c_L \in (0,1)$*, then there are three cases: (2a) if* $f(1) \in (c_L, 1)$*, then a generalized center-sponsored star is the unique strict equilibrium network. (2b) If* $f(1) \in (1, \max[n_1, ..., n_m])$*, then a strict equilibrium does not exist. (2c) If* $f(1) > \max[n_1, ..., n_m]$ *, then a network with unconnected center-sponsored stars is the unique equilibrium network.*[12]

Figure 2.1 illustrates the different strict Nash architectures for a society with two groups of three players each $(n_1 = n_2 = 3)$.[13] We note that strict equilibrium networks have very specific architectures and thus strictness is a useful refinement. We discuss some aspects of this characterization result. The *first* remark is about insider and outsider links. Our result shows that in connected equilibria there is one group, the *core* group, which is entirely internally linked, while all other groups are entirely

[12] We note that weak equilibrium always exist in the insider-outsider model: in case (2b) a network in which each group forms a star and the centers of the stars are linked is a weak equilibrium. Moreover, all equilibria are weak in this case.

[13] In this figure a bold line on a link next to a player indicates that this player has formed the link and pays for the link. We are assuming here that $f(1) > f(0)$.

externally linked. In other words, the formation of local connections is not allowed in equilibrium (except for one group). *Two*, we note that the diameter of connected strict equilibrium networks is independent of the number of players, and depends only on the number of groups. Thus we expect equilibrium networks to have a relatively short diameter.

The *third* observation concerns the centrality and center sponsorship properties. If the strict Nash network is connected, there is a player i such that all paths are oriented toward him. Hence, this player plays a particularly central role in the network. Furthermore, if the strict Nash network is non-empty but unconnected, then each component consists of members of one group and it has the center-sponsored star structure. Therefore, centrality and center-sponsorship are prominent properties of equilibrium networks.

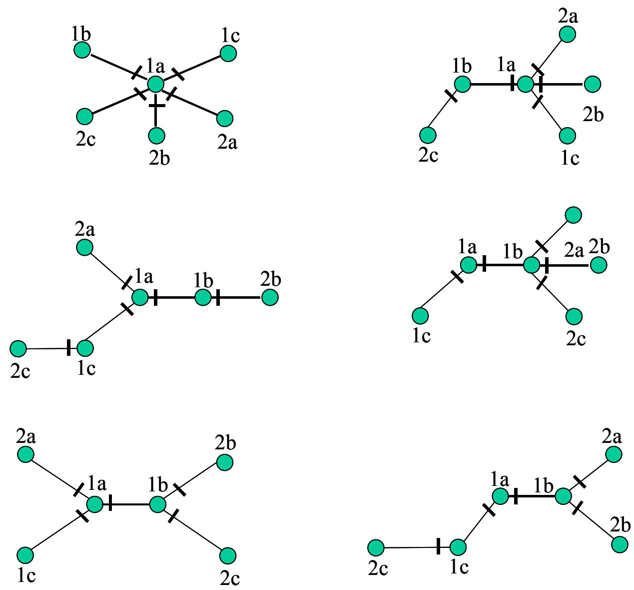

Figure 2.1. Strict Nash Architectures.

We now turn to the issue of efficiency. We first introduce some new terminology that will be used in the proposition below. Let g^{mc} refer to a minimally connected network with each group

N_l forming a minimally connected component with $n_l - 1$ inside links and $(m-1)$ outside links of distance one. Finally, a partially connected network with each group generating a minimally connected component will be denoted as g_m^{pc}.

The following result provides a complete characterization of efficient networks for the case of equal group sizes.[14] Let $n_l = \overline{n}$ for all $l = 1, 2, .., m$; moreover, we define $c_1 = m\overline{n}^2$ and $c_2 = [m\overline{n}(m\overline{n} - 1) - (m\overline{n} - m)c_L]/(m-1)$.

Proposition 2.4. *Suppose (2.4) and (2.5) hold. In addition suppose that $n_l = \overline{n}$, $\forall\ l = 1, 2, .., m$.*

1. *Suppose $c_L \in (0, \overline{n})$. If $f(1) \in (c_L, c_1)$ the network g^{mc} is uniquely efficient, while if $f(1) > c_1$ then the network g_m^{pc} is uniquely efficient.*
2. *Suppose $c_L \in (\overline{n}, m\overline{n})$. If $f(1) \in (c_L, c_2)$ then the network g^{mc} is uniquely efficient, while if $f(1) > c_2$ then the empty network is uniquely efficient.*
3. *If $c_L > m\overline{n}$ then the empty network is uniquely efficient.*

Figure 2.2 illustrates an efficient architecture for a society composed of three groups and three players each. We have showed that if g^{mc} is efficient the corresponding set of strict Nash networks does not contain any architectures compatible with the efficient one. This conflict persists until the level of $f(1)$ is such that any outside link is not beneficial both from an individual and social point of view. When this is the case, our problem degenerates in a sum of independent homogeneous problems leading to unconnected center-sponsored stars networks. It follows that the trade-off between efficiency and stability fades in this case.

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[14] If we allow groups having different sizes a variety of efficient networks arise. However, the architectural properties of these networks are qualitatively the same as in the case of equal size. To illustrate this consider the two-cost levels case in a society composed of a small group and two large groups. Let g be an efficient network. When c_L is low enough each group is entirely internally linked in g. Furthermore, when c_H is low enough g is connected. For moderate level of c_H the two large groups are connected while the small group is left isolated. Finally, for sufficiently high level of c_H each group is isolated.

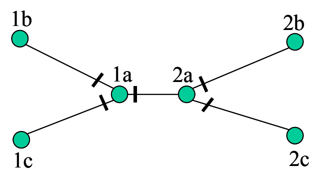

Figure 2.2. Efficiency.

4.1 The role of decay

In the basic model, we assume that the transmission of value is independent of the length of the path between players. In this section we examine the robustness of our findings to the presence of decay. A general analysis of decay is outside the scope of the present paper. We will consider the case of small levels of decay and we will focus on the case of two groups. Our principal finding is that centrality and small distances are salient properties of equilibrium networks while center-sponsorship is not a robust feature of equilibrium networks in the presence of decay.

We measure the level of decay by a parameter $\delta \in (0,1)$. Given a network g it is assumed that if the shortest path between agent i and j has $q \geq 1$ links, then the value of j's information to i is δ^q. The costs of forming links still take the form (2.4) and the payoff (2.5) to player $i \in N_l$ in a network g can be rewritten as follows:

$$\Pi_i(g) = \sum_{j \in N(i;g)} \delta^{d(i,j;g)} - \mu_i^{d,l} c_L - \mu_i^{d,k} c_H \tag{2.6}$$

where $l, k = 1, 2$ and $l \neq k$

A network in which each group constitutes a star and a single player i of group l forms a link with the central player j of group l', $l' \neq l$, is referred to as an interlinked stars network. If each star is center-sponsored (periphery-sponsored) we will say that the network is an interlinked center-sponsored (periphery-sponsored) stars. A one group periphery-sponsored star is a partially connected network where one group forms a periphery-sponsored star, while the other group is empty.

Proposition 2.5. *Suppose (2.4) and (2.6) hold. In addition suppose that there are two groups and that* $n_l = \overline{n} \geq 3$, $\forall\ l = 1, 2$.

1. *Suppose* $c_L \in (0, 1)$. *There exists a* $\tilde{\delta}(c_L, c_H) < 1$ *such that for any* $\delta \in (\tilde{\delta}(c_L, c_H), 1)$ *the following is true: (1a) if* $c_H \in (c_L, \overline{n})$ *any interlinked stars network is a strict equilibrium. (1b) if* $c_H > \overline{n}$ *any unconnected stars network is a strict equilibrium.*

2. *Suppose* $c_L \in (1, \overline{n})$. *There exists a* $\tilde{\delta}(c_L, c_H) < 1$ *such that for any* $\delta \in (\tilde{\delta}(c_L, c_H), 1)$ *the following holds: (2a) if* $c_H \in (c_L, \overline{n})$ *the interlinked periphery-sponsored stars network and the empty network are the only strict equilibria. (2b) if* $c_H > \overline{n}$ *the unconnected periphery-sponsored stars network, the one group periphery-sponsored star network and the empty network are the only strict equilibria.*

3. *if* $c_L > \overline{n}$ *the empty network is the unique strict equilibrium.*

Figure 2.3 illustrates the strict equilibria presented in part 2 of the above proposition. We first observe that the introduction of a small amount of decay does not undermine the structural properties such as centrality and short diameter, which were derived in case of perfect information flow. However, in contrast with the perfect information flow case, here we note that local connections are allowed for any group. This is closely related to the idea that peripheral players may invest in connections while in the model without decay only central players invest in connections. Secondly, we note that since the efficiency results derived in Proposition 2.4 are strict, they also hold when a small amount of decay is introduced. Thus, our analysis suggests that some decay in flow of benefits can potentially serve to enhance social efficiency.

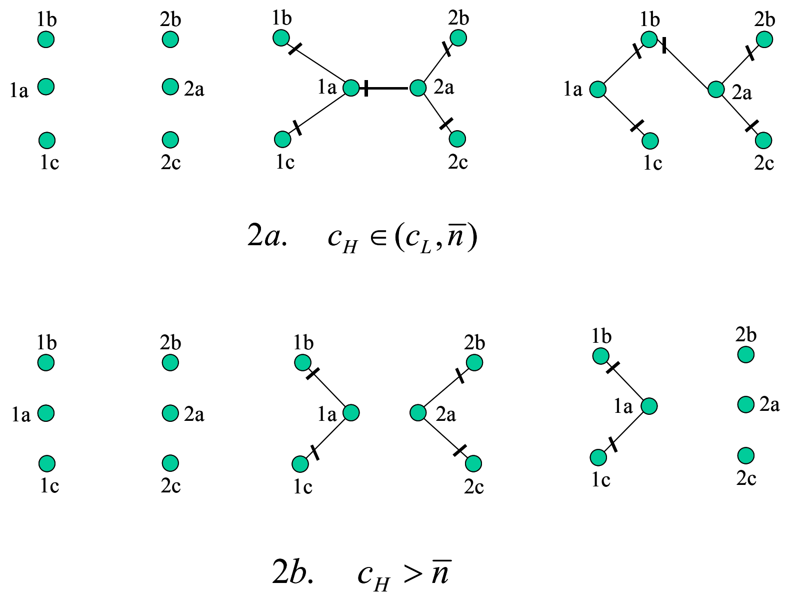

Figure 2.3. Strict Nash and decay.

4.2 Dynamics and local stars

In this section we shall examine a dynamic model of network formation based on myopic best response decision making by individuals. Our results establish that the dynamic process always converges and provide a characterization of the networks that arise in the long run.

For a given set A, let $\Delta(A)$ denote the set of probability distributions on A. We suppose that $\forall i \in N$ there exists a number $p_i \in (0,1)$ and a function $\phi_i : \mathcal{G} \rightarrow \Delta(\mathcal{G}_i)$ where ϕ_i satisfies, for all $g \in \mathcal{G} : \phi_i(g) \in$ Interior $\Delta(\mathcal{BR}_i(g_{-i}))$. For $\hat{g}_i$ in the support of $\phi_i(g)$, the notation $\phi_i(g)(\hat{g})$ denotes the probability assigned to $\hat{g}_i$ by the probability measure $\phi_i(g)$. If the network at time $t \geq 1$ is $g' = g'_i \otimes g'_{-i}$, the strategy of agent i at time $t+1$ is assumed to be given by:

$$g_i^{t+1} = \begin{cases} \hat{g}_i \in \text{support } \phi_i(g), & \text{with probability } p_i \times \phi_i(g)(\hat{g}_i) \\ g'_i, & \text{with probability } 1 - p_i \end{cases}$$

We assume that the choice of inertia as well as the randomization over best responses by different agents is independent across agents. Thus, our decision rules induce a transition matrix T mapping the state space $\mathcal{G}$ to the set of all probability distributions $\Delta(\mathcal{G})$ on $\mathcal{G}$. Let $\{X_t\}$ be the stationary Markow chain starting from the initial network $g \in \mathcal{G}$ with the above transition matrix.

We will use the notion of curb sets in our analysis. A strategy profile set, $\tilde{\mathcal{G}} \subseteq \mathcal{G}$ is closed under rational behavior (*curb*) if $\mathcal{BR}(g) \subseteq \tilde{\mathcal{G}}$ for any $g \in \tilde{\mathcal{G}}$. A curb set $\tilde{\mathcal{G}}$ is minimal if there not exist a proper subset which is a curb set.[15] To provide a full characterization of minimal curb sets we use a refinement of this notion: super tight curb set. A set of networks $\mathcal{G}^* \subseteq \mathcal{G}$ is a super tight curb set if $\mathcal{BR}(g) = \mathcal{G}^*$ for any $g \in \mathcal{G}^*$. This set may be considered a generalization of the strict Nash notion as the best response set of a player i is invariant inside the set $\mathcal{G}^*$. Our result establishes an equivalence between minimal curb sets and super tight curb sets and provides a full characterization of these sets. Consider an interlinked center-sponsored stars network where a single player i of group l forms a link with some player j of group l', $l' \neq l$. Note that as player i varies his links across players of group l', distinct interlinked center-sponsored stars networks arise. We shall use $\mathcal{G}_i^{ilcss}$ to denote the set of these networks. We shall use $\mathcal{G}^{ilcss}$ to refer to sets with this property in general. The next result considers a society composed of two groups.

Proposition 2.6. *Suppose (2.4) and (2.5) hold and there are 2 groups. For generic values of c_H and c_L, the dynamic process converges to a super tight curb set, $\mathcal{G}^*$, with probability* 1. *Furthermore*

1. *Suppose $c_L \in (0,1)$: if $c_H \in (c_L, \max[n_1, n_2])$ then $\mathcal{G}^* \in \{g^{gcs}, \mathcal{G}^{ilccs}\}$, while if $c_H > \max[n_1, n_2]$ then $\mathcal{G}^* = \{g^{ucs}\}$.*
2. *Suppose $c_L > 1$: then $\mathcal{G}^* = \{g^e\}$.*

[15] A game with a compact strategy set and payoffs that are continuous with respect to the strategies of players has at least one minimal curb set; however, a game may have several such sets, each of them containing networks with different architectures.

We would like to emphasize two aspects of the above result. First, that myopic individual learning leads over time to a stable architecture of networks, which have a specific architectural forms. This is a strong result given the very large number of possible network architectures. Second, it shows how dynamics can complement the static analysis nicely. In the static model we noted that for some parameter ranges no limiting state exists and that there is a sharp conflict between strict Nash and efficient networks. The study of dynamics shows us that in the case where no strict Nash exist the process is still very well behaved and we can pin down precisely the long run outcomes: interlinked center-sponsored stars networks.[16] In this architecture local connections are allowed within groups and the center-sponsorship property holds only at the local level. Finally, we find that these new long-run outcomes attain higher social welfare than the strict Nash networks derived in the static analysis. Thus dynamics may help resolve some of the tension between individual and social incentives.

5 Conclusion

We have studied a connections model of network formation in which players are heterogeneous with respect to benefits as well as the costs of forming links. We start by showing that value heterogeneity across players is crucial in determining the connectedness of a network, while differences in costs of linking across players are crucial in shaping both the level of connectedness as well as the architecture of individual components in a network. We then explore an insider-outsider model in which it is cheaper to form intra-group links as compared to inter-group links. Our main finding here is that properties such as centrality and short distances are robust features of equilibrium networks. Moreover, we find that equilibrium networks are also socially efficient in many instances.

[16] We note that such networks are weak Nash equilibria in case 2b of Proposition 2.3; Moreover, all equilibria are weak in that case.

6 Appendix chapter 2

Proof of Proposition 2.3. We recall some definitions that will be used in the proof. In a network g, a path between i and j is said to be $i-$*oriented* if either $g_{i,j}=1$ or there is a sequence of distinct players $\{i_1,i_2,...,i_n\}$ with the property that: $\{g_{i,i_1}=g_{i_1,i_2}=1,...,g_{i_n,j}=1\}$. The proof consists of a sequence of steps, which are covered in the following lemmas.

Lemma 2.1. *Suppose g is a strict Nash network. If $g_{i,j}=1$, where $i\in N_l$ and $j\in N_{l'}$, $l\neq l'$, then i does not access any player j' via the link $g_{i,j}=1$ where $j'\in N_k$ and k is such that $|l-k|\leq|l-l'|$.*

Proof: Consider a strict Nash network g. Choose $i\in N_l$ and $j\in N_{l'}$, $l\neq l'$, such that $g_{i,j}=1$. Let $j'\in N_k$ where k is such that $|l-k|\leq|l-l'|$. Suppose i accesses j' via the link $g_{i,j}=1$. The spatial cost structure implies that i can do at least as well by deleting his link with j and forming a link with j'. This contradicts strict Nash. ■

Lemma 2.2. *Suppose g is a strict Nash network. Assume $g_{i,j_0}=1$, $i\in N_l$, $j\in N_{l_0}$, $l\neq l_0$ and let $\{j_0,j_1,...,j_k\}$ where $j_x\in N_{l_x}$ for any $x\in\{0,...,k\}$, be the set of players who agent i accesses via the link $g_{i,j_0}=1$, then $g_{j',i}=0$, $\forall j'\in N_k$ such that $|k-l|\geq|k-l_x|$ for some $x\in\{0,...,k\}$.*

Proof: Suppose $g_{j',i}=1$. Since the cost of forming links is non-decreasing in the distance between players' groups, j' can do at least as well by deleting his link with i and forming a link with j_x. This contradicts strict Nash. ■

Lemma 2.3. *Suppose $n_l\geqslant 2$, $\forall\ l=1,...,m$ and that g is a strict Nash network, then in any non-singleton component there exists a pair of players who belong to the same group (this group will differ across components) and have a direct link.*

Proof: Consider a non-singleton component $C(g)$. There exists $g_{i,j}=1$, $i\in N_l$ and $j\in N\diagdown\{i\}$. Suppose that $j\in N_{l'}$, $l\neq l'$. We first note that, given $g_{i,j}=1$, it must be true that $N_l\subset C(g)$. This follows by noting that the returns to a player

$k \in N_l$ from linking with component $C(g)$ are strictly greater than the returns to player i, while the costs are strictly smaller (since k forms a link with i). Hence every player $k \in N_l$ must belong to $C(g)$. Therefore $i \in N_l$ must access every $i' \in N_l$ in g. Lemma 2.1 implies that i cannot access i' via j; thus, i accesses i' via a player j', where $g_{j',i} = 1$. Because each group consists of at least 2 players and player i was chosen arbitrarily it follows that every player belonging to $C(g)$ receives at least one link. Therefore, there are at least $|C(g)|$ links sponsored in $C(g)$, which implies that $C(g)$ is not minimal. This contradicts that g is Nash. Hence, the proof follows. ■

Lemma 2.4. *Assume* $n_l \geqslant 2$, $\forall\ l = 1, ..., m$. *Suppose* g *is a non-empty strict Nash network. If* $g_{i,i'} = 1$, $i, i' \in N_l$, *then* $g_{i,i''} = 1$, $\forall i'' \in N_l \diagdown \{i\}$.

Proof: Consider a non-singleton component, $C(g)$. Given the argument in Lemma 2.3, if $g_{i,i'} = 1$, for $i, i' \in N_l$, then $N_l \subset C(g)$. We first note that, if $g_{i,i'} = 1$, then $g_{i'',i} = 0$, $\forall\ i'' \in N_l \diagdown \{i\}$. This follows from the standard switching argument: if $g_{i'',i} = 1$ then player i'' is indifferent between linking with i and i', and g is therefore not a strict Nash network. We now have two possible configurations. First, suppose that $N_l \equiv C(g)$. Then an application of the switching argument immediately implies that $g_{i,i''} = 1$, for all $i'' \in N_l$. Second, suppose $N_l \subsetneq C(g)$. Since $C(g)$ is connected, there is a path between i and i'', and $d(i, i'') \geq 2$. Then there is some player $j \neq i''$ such that $\bar{g}_{i,j} = 1$. Suppose that $j \in N_l$. If $g_{i,j} = 1$ then a simple switching argument applies with regard to player i and this contradicts the hypothesis that g is strict Nash. If $g_{j,i} = 1$ then the switching argument applies to player j, who is indifferent between the link with i and the link with i'. This contradicts the hypothesis that g is strict Nash. Similar arguments can be used in the case that $j \notin N_l$ to complete the proof of this lemma. ■

Lemma 2.5. *Assume* $n_l \geqslant 2$, $\forall\ l = 1, ..., m$. *Suppose* g *is a connected strict Nash network and let* $i \in N_l$ *be the player identified by Lemma* 4. *Then any path* $i \overset{g}{\longleftrightarrow} j$, $\forall j \in N \diagdown \{i\}$, *is* $i - oriented$.

Proof: Let g be a strict Nash network which is connected. Since g is minimal, every path starting at i ends with a well defined end-player. The proof proceeds by contradiction. Suppose there is a path ending with player j, which is not i-oriented. If $\bar{g}_{i,j} = 1$ and j is not i-oriented then $g_{j,i} = 1$. From Lemma 2.4 we infer then that $j \in N_{l'}$ where $l' \neq l$. Next, since $n_l \geq 2$, we can apply a switching argument for player j with respect to some $i' \in N_l$, and that contradicts the hypothesis that g is a strict Nash network.

Suppose next that $\bar{g}_{i,j} = 0$. Let $\{i_1, i_2, i_3, ..., i_n\}$, be the players on the path between i and j, with $\bar{g}_{i,i_1} = ... = \bar{g}_{i_n,j} = 1$. We first take up the case $g_{j,i_n} = 1$. Let $j \in N_x$; if $i_n \notin N_x$ then a simple switching argument with regard to player j and some member of group x implies that g is not a strict Nash network. If $i_n \in N_x$, there are two possibilities: (i) $g_{i_{n-1},i_n} = 1$ and (ii) $g_{i_n,i_{n-1}} = 1$. In the first case, player i_{n-1} is indifferent between a link with player i_n and a link with player j. This contradicts the hypothesis that g is a strict Nash network. In the second case, there are two sub-cases: suppose i_n and i_{n-1} belong to the same group; then a switching argument applies to player j, with respect to players i_n and i_{n-1}. If i_n and i_{n-1} belong to different groups then a switching argument applies to player i_n with regard to members of the group of i_{n-1} (given that $n_l \geq 2$, for all $l = 1, 2, ..., m$).

Consider finally the case $g_{i_n,j} = 1$. Let k be the first player along the path $\{i_1, i_2, ..., i_n\}$, such that $g_{k,k-1} = 1$. Let $i_{k-1} \in N_y$. Since $g_{k-2,k-1} = 1$ by hypothesis, Lemma 1 implies that $i_k, i_{k+1}, ..., i_n \notin N_y$. By hypothesis, $n_y \geq 2$, and so there is a player $p \in N_y$, $p \neq i_{k-1}$, and we know that $p \notin \{i_k, i_{k+1}, ..i, j\}$. This is true because otherwise i_{k-2} can switch from i_{k-1} to p. Thus, $p \in N \backslash \{i_{k-1}, i_k, ..., i_n, j\}$. In this case however, a switching argument would apply to player i_k with regard to p. Hence g is a not a strict Nash network. This contradiction completes the proof of the lemma.■

Lemma 2.6. *Assume* $n_l \geqslant 2$, $\forall\ l = 1, ..., m$. *Suppose* g *is a* *connected strict Nash network. Then* $D(g) \leq 2m$.

Proof: This follows directly by Lemma 2.1, 2.3, 2.4 and 2.5 ■

We now complete the proof of Proposition 2.3.

1. Consider a strict Nash network g and suppose $c_L > 1$. We claim that the only strict Nash network is the empty one. Suppose that there exists a non-singleton component $C(g)$. Using arguments from Lemma 2.3 it follows that if $i \in N_l$, and $g_{i,j} = 1$, then $N_l \subset C(g)$. If $N_l \equiv C(g)$, then it is easy to show by applying the switching argument that $C(g)$ is a center-sponsored star. However, this is impossible given the hypothesis that $c_L > 1$. If on the other hand, $C(g)$ contains players from more than one group then it follows that g is a connected network. Lemma 2.5 now implies that there is central player and that all paths are oriented towards this player. However, given that $f(1) \geq c_L > 1$, this is not sustainable in equilibrium. This contradicts the hypothesis that g is a strict Nash equilibrium. Hence the empty network is the only possible strict Nash network.

2a. Suppose $c_L \in (0,1)$ and $f(1) \in (c_L, 1)$. Suppose g is a strict Nash network; given the parameter restrictions, it is immediate that g must be connected. Lemma 2.3 and Lemma 2.4 imply that g satisfies property (i) of definition 2.1. Since g is connected, Lemma 2.5 holds and that implies property (ii) of definition 2.1. Considering the restrictions imposed by Lemma 2.1, property (iii) of definition 2.1 follows by verification.

2b. Suppose $c_L \in (0,1)$ and $f(1) \in (1, \max[n_1, ..., n_m])$. Suppose g is a strict Nash network; first we note that it must be connected. Lemma 2.5 implies that g has a central player i, and that all paths are i-oriented. However, $f(1) > 1$, g cannot be sustained in equilibrium, leading to a contradiction. Hence, there does not exist a strict Nash network.

2c. Suppose $c_L \in (0,1)$ and $f(1) > \max[n_1, ..., n_m]$. Consider a strict Nash network g. From Lemmas 2.3 and 2.4 it follows that either g has m components corresponding to each of the groups or it is connected. In the former case, Lemmas 2.3 and 2.4 imply that each of the components is a center-sponsored star. In the latter case, Lemma 2.5 implies that g has a central player and all the paths are oriented towards this player. But then the argument from Part 2b applies and such a network cannot arise in equilibrium given that $f(1) > \max[n_1, ..., n_m]$. ■

Proof of Proposition 2.4. In this proposition we assume equal group size, *i.e.* $n_l = \overline{n}$ for any $l = 1, ..., m$. We first start with two observations: (a) The no-decay assumption implies that each non-singleton component part of an efficient architecture is minimal; (b) If g is efficient and non-empty then it is either minimally connected with $m-1$ outside links of 'length' one and $m\overline{n} - m$ inside links, or partially connected with each group generating a minimally connected component. This observation follows by the assumption of equal group size and by the definition of efficiency concept. If a link between two members of the same group is socially efficient, then, from a societal point of view, each group should be internally linked. Furthermore, the assumption of equal group sizes implies that each group internally linked contributes equally to the total social welfare produced by the network. It follows that if an outside link is social enhancing, then an efficient network should be minimally connected. Moreover, since the definition of efficiency requires the minimization of the total cost of information flow, a connected efficient network should have $m-1$ outside links, each of them which requires a cost equal to $f(1)$. Using these observations we compare three different architectures:

1) The social welfare from g^{mc}, is given by:

$$W(g^{mc}) = (m\overline{n})^2 - m(\overline{n} - 1)c_L - (m-1)f(1) \qquad (2.7)$$

2) The social welfare from g_m^{pc}, is given by:

$$W(g_m^{pc}) = m(\overline{n})^2 - m(\overline{n} - 1)c_L \qquad (2.8)$$

3) The social welfare from g^e is given by:

$$W\left(g^{e}\right)=m\overline{n} \tag{2.9}$$

First, we compare g_m^{pc} with g^e. It is easily checked that $W\left(g_m^{pc}\right) \geqslant W\left(g^e\right)$ if and only if $c_L \leq \overline{n}$.

Second, suppose $c_L \in (0, \overline{n}]$ and compare g^{mc} with g_m^{pc}. Simple computations show that $W\left(g^{mc}\right) \geqslant W\left(g^{pc}\right)$ if and only if $f\left(1\right) \leq m\overline{n}^2 = c_1$. It follows that given $c_L \in (0, \overline{n}]$ if $f\left(1\right) \in (c_L, c_1]$ the only efficient network is g^{mc}, while if $f\left(1\right) > c_1$ the only efficient network is g_m^{pc}. This proves part $\left(1\right)$.

Third, suppose $c_L > \overline{n}$ and compare g^{mc} with g^e. Again, simple computations show that $W\left(g^{mc}\right) \geqslant W\left(g^e\right)$ if and only if $f\left(1\right) \leq \frac{m\overline{n}(m\overline{n}-1)-(m\overline{n}-m)c_L}{m-1} = c_2$. We note that c_2 is a decreasing function of c_L and attains the value $m\overline{n}$ when $c_L = m\overline{n}$. Suppose therefore that $c_L \in (\overline{n}, m\overline{n})$. If $f\left(1\right) \in (c_L, c_2]$ then g^{mc} is uniquely efficient, while if $f(1) > c_2$ then g^e is uniquely efficient. Finally, if $c_L \geqslant m\overline{n}$ then $c_2 \leq c_L$. Given our hypothesis that $f\left(1\right) > c_L$ it follows that empty network is uniquely efficient. This proves parts (2) and (3). ■

Proof of Proposition 2.5. The proof of parts 1 and 3 is straightforward and omitted. We provide a proof of part 2. We first observe that as δ is close 1 an equilibrium network is minimal. Second, we observe that if $g_{i,j} = 1$, for some $i \in N_l$, $j \in N \backslash \{i\}$, then group N_l is connected. Suppose not, then the payoff to a player $i' \in N_l \backslash \{i\}$ from sponsoring a link with player i is strictly higher than the payoff obtained by player i. Third, it is immediate that the empty network is always a strict equilibrium if $c_L > 1$. In what follows we focus on non-empty strict equilibrium networks, g. Here, we have two possibilities, which we analyse in turn.

(I.) There are no links across groups, *i.e.* $g_{i,j} = 0$, $\forall i \in N_x$, $j \in N_y$, $x \neq y$. The second observation above implies that in an equilibrium either a group is connected or disconnected. Next note that since there are no links across groups the problem for each group is analogous to the homogeneous case studied by Bala and Goyal (2000a). Then it follows from Proposition

5.4 of Bala and Goyal (2000a) that if a group is connected then it forms a periphery-sponsored star. Hence, g is either an unconnected periphery-sponsored stars network or a one-group periphery-sponsored star network. It is clear that such networks are strict equilibria only if $c_H > \overline{n}$.

(II). There are links across groups, *i.e.* $g_{i,j} = 1$ for some $i \in N_x, j \in N_y$, $x \neq y$. It is now easy to see that g must be connected. We now prove the following: If $c_L \in (1, \bar{n})$ then there exists a $\tilde{\delta} < 1$ such that for any $\delta \in [\tilde{\delta}, 1)$ if $g_{i,j} = 1$ for some $i \in N_l$ and $j \in N_l'$, $l \neq l'$, then $g_{j',j} = 1$ for any $j' \in N_{l'}$.

We first note that since $c_L > 1$ any end agent (say) $\hat{\jmath}$ who is accessed by player i via $g_{i,j} = 1$ sponsors his link; let $g_{\hat{\jmath},y_1} = 1$. (We note that since $c_H > 1$ there exists at least one such end-player distinct from j.) Second, we show that i only accesses players in $N_{l'}$ via the link $g_{i,j} = 1$. Suppose not; then there exists a player $i' \in N_l$ accessed by i via the link $g_{i,j} = 1$. Let $g' = g - g_{i,j} + g_{i,i'}$, it is easy to see that $N_i(g) = N_i(g')$. Thus, $\Pi_i(g) - \Pi_i(g') = \sum_{j \in N(i;\bar{g})} \delta^{d(i.j;\bar{g})} - \delta^{d(i,j;\bar{g}')} - (c_H - c_L) < 0$, as $\delta \to 1$. This contradicts Nash. Third, we show that i accesses every player in $N_{l'}$ via $g_{i,j} = 1$. Suppose not; then there exists some player $N_{l'}$ accessed by i via some player $k \neq j$. Among such players let j' be the player closest to player i and assume j' accesses i via the link $\bar{g}_{j',i'} = 1$. By construction $i' \in N_l$; the previous argument implies that $g_{i',j'} = 1$ and that any player accessed by i' via the link $g_{i',j'} = 1$ belongs to $N_{l'}$. Select one of the end players, say j_1, who player i' accesses via the link $g_{i',j'} = 1$. Since $c_L > 1$, player j_1 sponsors his link, say $g_{j_1,y_2} = 1$. Using a variant of the switching argument it is now easy to see that either $\hat{\jmath}$ or j_1 strictly gains by deleting the link with y_1 or y_2 and creating a new link with y_2 or y_1, respectively. This contradicts Nash. Fourth, we note that since group $N_{l'}$ is entirely internally linked, we can use Proposition 5.4 in Bala and Goyal (2000a) to conclude that $N_{l'}$ forms a periphery-sponsored star. Finally, it is easy to see that player j must be the center of the periphery-sponsored star; for otherwise player i strictly gains by switching from j to the central player of group $N_{l'}$.

Finally, we note that also group N_l is entirely internally linked and therefore forms a periphery-sponsored star. Thus, if g is a strict equilibrium it is an interlinked periphery-sponsored star. The proof follows. ■

Proof of Proposition 2.6.

(I). $c_L < 1$: The *first* step shows that from any initial network g_0 there is a positive probability of transiting to a minimal network g_1. Fix a network g_0. Number the players $1, 2, ..., n$. Consider this sequence of players moving one at a time starting with 1. We claim that after player n has moved the network g_1 is minimal. Suppose not and there is a cycle of players. In that case consider players in the cycle who initiate links. Within this set of players fix the player who moved last. Clearly, this player did not choose a best response, as deleting one of his links in the cycle would have increased his net payoffs. This contradiction completes the argument.

The *second* step shows that starting from a minimal network g_1 there is a positive probability that the process transits to a minimal network g_2 in which there is at least one group with one internal link. We focus on the case where g_1 is connected and both groups are entirely externally linked. Then there exist players $i, i' \in N_1$ with $\bar{g}_{i,j} = 1$ where $j \in N_2$, and player i accesses i' via j. If $g_{i,j} = 1$ then (since $c_L < 1$) there exists a best response for player i in which he will disconnect from j and instead link with i' and this will yield a hybrid group. The other possibility is that $g_{j,i} = 1$. Since $n_l \geq 2$, for $l = 1, 2$, there is a player $j' \in N_2$ who is accessed by j. If this player is accessed via i the above argument leads to N_2 being a hybrid group. Since g_1 is minimal and we only let one player update at a time, the network g_2 must be minimal. The other possibility is that j' is accessed via some other player i''. In this case again variants of the above argument apply and the process transits to a network with one group having at least one internal link. Similar arguments apply if the initial network is not connected.

The *third* step shows that starting from a minimal network g_2 in which group N_1 is hybrid there is a positive probability that the process transits to a minimal network g_3 in which group N_1 is entirely internally linked. Let $\sigma_1(g)$ be the number of links between pairs of players in group N_1. By hypothesis, $\sigma_1(g) \in [1, n_1 - 1)$. Let $g_{i,i'} = 1$, for some pair of players $i, i' \in N_1$. We distinguish between two cases. The first case arises if players of N_1 are spread over more than one component. Pick some player $i \in N_1$ and get him to choose a best response. It is straightforward to verify that since $c_L < 1$ any best response of i, g_i', has the property that he accesses all players in own group. Let $g' = g_i' \otimes g_{-i}$ be the new network. It follows that $\sigma_1(g') \geq \sigma_1(g) + 1$. We note that since g is a minimal network and g_i' is a best response, it follows that g' is a minimal network as well. The second case is one in which all members of group N_1 belong to a single component. Since N_1 is hybrid it follows that there exists a pair of players $x, y \in N_1$ such that $x \overset{\bar{g}}{\leftrightarrow} y$ contains only players belong to N_2. This implies in turn that there is at least one player $i'' \in N_1$ who is not internally linked with i. We will focus on the case where the path $i \overset{\bar{g}}{\leftrightarrow} i''$ contains only players $j_1, j_2, ..j_n \in N_2$.[17] There are two sub-cases to consider.

(2a). If $g_{i,j_1} = 1$, then allow player i to play a best response. It follows from the hypothesis $c_L < 1$ that there is a best response in which player i will maintain all his current links with players in own group (since network is minimal); in addition in any best response, he will delete the link $g_{i,j_1} = 1$ and replace it with a link with some player of his own group along the path. We can suppose without loss of generality that the link $g_{i,i''} = 1$ is formed. Define $g' = g_i' \otimes g_{-i}$. It follows that $\sigma_1(g') > \sigma_1(g)$; again note that g' is a minimal network. A similar argument applies if $g_{i'',j_n} = 1$.

(2b). $g_{j_1,i} = g_{j_n,i''} = 1$: There are two possibilities here. (i). $j_1 \in N_2$ does not access any player $j' \in N_2$ via the link $g_{j_1,i} = 1$ and (ii) j_1 does access some $j' \in N_2$ via this link $g_{j_1,i} = 1$. We take these cases up in turn.

2b(i). We first allow player j_1 to choose a best response; he

[17] It is possible that for instance player i' lies along this path; the arguments given below can be adapted to deal with this complication easily.

is indifferent between linking with i and i'. If he does not link with the component that contains i then we arrive a network in which i does not access i'' and we get i to choose a best response. This leads clearly to a network g' in which $\sigma_1(g') \geq \sigma_1(g) + 1$, and we are done. The other possibility is that j_1's best response g'_i involves a link with i's component and in that case let us suppose that he forms a link with i' and this yields a new network $g' = g'_i \otimes g_{-i}$. In the new network g', player i is indifferent between linking with i' or i''. Given g' let j_1 and i move simultaneously. There is a best response in which player i switches from i' to i'', while player j_1 switches from i' to i, yielding the network g''. We note that in g'', player i' will be isolated and that g'' will not be minimal. Now allow player i' to choose a best response. Any best response will involve a link with the component containing i and we can suppose without loss of generality that he forms a link with player i. We now get player j_1 to move and any best response will involve deletion of the link $g_{j_1,i}$. We have reached a minimal network g''' in which $\sigma_1(g''') \geq \sigma_1(g) + 1$.

2b(ii). Let j' be the first player of group N_2 along the path $j_1, i, i_1, ., i_n...$ in g. We first consider the case that $g_{j',i_n} = 1$. Allow players j_1 and j' to choose a best response. In any best response player j_1 will delete the link $g_{j_1,i} = 1$ and instead link with some player of his own group such as j'. Suppose this is the case. Similarly, in any best response player j' will delete the link $g_{j',i_n} = 1$ and instead link with someone of own group such as j_1. Denote by g' the resulting network. Now consider player i: in any best response he will want to form a link with someone such as i''. Allow player i to choose a best response. Finally, let player j_1 move and the resulting network g'' is minimal as well. It follows that $\sigma_1(g'') \geq \sigma_1(g) + 1$. Next we take up the case $g_{i_n,j'} = 1$. Let j_1 choose a best response. It follows that in any best response he will delete the link with i and switch to a player of own group such as j'. Denote the resulting network as g'. We now note that g' is minimal and in g', agent i_n observes i'' via the link $g_{i_n.j'}$. Thus, we are in case 2(a) above, and the argument follows. We have thus shown that starting from a minimal network g

with N_1 as a hybrid group there exists a path which leads to a minimal network g' in which $\sigma_1(g') \geq \sigma_1(g) + 1$. Since the minimal network g is arbitrary we can repeat this step to arrive at a minimal network in which group 1 is entirely internally linked.

The *fourth* step shows that starting from network g_3 the process transits with positive probability to a network g_4 in which group N_1 is a center-sponsored star. Moreover, g_4 is minimal. Suppose that N_1 is entirely internally linked. Now assume that all players $j \in N_2$ exhibit inertia. We note that the process is analogous to a process with only homogenous players choosing links starting at a minimally connected network. So the arguments in Theorem 4.1 in Bala and Goyal (2000) can be applied to show that there exists a sequence of best responses leading to a network g' in which N_1 is a center-sponsored star.

We now complete the proof for $c_L \in (0, 1)$ and $c_H < \max\{n_1, n_2\}$: First suppose g_4 consists of two center-sponsored stars one for each group. If the network is connected and minimal then it must be the case that there is a single link between the two stars. If this network is Nash then it is easily verified that the process has entered a set of networks in which the player i initiating this single cross-group link is indifferent between forming this link with any of the players in the other star and the set of networks generated by this switching of links by the player i constitutes a super-tight curb set. Suppose the network is connected but not Nash. Since $c_L < 1$ and $c_H < \max\{n_1, n_2\}$, this must mean that there is a player $j \in N_l$, $l = 1, 2$ who wishes to delete the cross group link. Allow this player j to move. He will delete this link and retain any internal links he has in g_4 (since $c_L < 1$). Next choose the central player in the other group $N_{l'}$, with $l' \neq l$ and get him to choose a best response. From $c_L < 1$ and $c_H < \max\{n_1, n_2\}$ it follows that he will retain all his current links with own group members and in addition form a link with some player in n_1. We have now reached a Nash network and the first part of the argument can now be applied. We note that if g_4 contains two center-sponsored stars and the network is not connected then allowing any player in the smaller group to move will lead to a Nash network as above.

Second, we examine the case where g_4 has only one center-sponsored star and let it consist of N_1. Given that $c_L < 1$ we can assume that N_2 is connected as well. Here we have two possibilities. One, group N_2 is a hybrid group. Using the arguments presented in steps 3-4 it follows that there exists a sequence of best responses which leads to a network where group N_2 constitutes a center-sponsored star as well. We can then apply the arguments presented above. Two, suppose group N_2 is entirely externally linked. Then it has to be the case that g is minimally connected. If all the links have the appropriate orientation then g is a generalized center-sponsored star. Then if $c_H \in (0, 1)$ it follows that g is strict Nash and the proof follows. If $1 < c_H < \max\{n_1, n_2\}$ then g is not Nash. In particular, no outside links with isolated players are profitable. Let all $i' \in N_1$ move while all $j \in N_2$ exhibit inertia. Denote the resulting network by g'. Note that in g' group N_1 is a center-sponsored star, while each $j \in N_2$ is a singleton. Now have a player $j \in N_2$ move and any best response by him will yield a network with 2 center-sponsored stars. If in addition there is a single link across the groups initiated by j then we are done. Otherwise, get a player $i \in N_1$ to move and he will form a link with some $j \in N_2$ (because $c_H < \max\{n_1, n_2\}$. Finally, assume one of the links is not suitably oriented. Since group N_2 is entirely externally linked, and N_1 constitutes a center-sponsored star, there exists some player $j \in N_2$ who forms a link with some $i' \in N_1$. Let player j update. It follows from $c_L < 1$ and $c_L < c_H$ that the link to i' will be replaced by a link to some $j' \in N_2$. In the resulting minimal network group 2 is a hybrid group, while the architecture of group 1 is unchanged. We then apply arguments in steps 3-4 to arrive at two center-sponsored stars and the above arguments in this step to complete the proof.

We next complete the proof for $c_L \in (0, 1)$ and $c_H > \max\{n_1, n_2\}$: If there are two center-sponsored stars in g_4 then this network is a strict Nash network and we are done. If there is only one center-sponsored star consisting of group N_1 then consider the other group. Suppose as before that it is connected. If it is hybrid then we first use arguments in steps 3-4 to get this group to form a center-sponsored network and then follow with the arguments above. The other case is that this group is entirely externally linked. We get players from group 1 to move and since $c_H > 1$, they will all delete links with players in N_2. Now get a player in N_2 to move and this player will link with all players in own group. We have arrived at a network with two center-sponsored stars and we are done.

(II). $c_L > 1$: First, we note that the empty network is the unique strict Nash network in this parameter range. We will argue that there is a positive probability of transiting from any network g to the empty network g^e. The *first* step constructs a path of transition to a minimal network g_1. This is similar to what we did in step 1 in part (I) above. The *second* step checks if there is any player who wants to form a link. If the answer is *no* then we have all players move at the same time and they all delete any links they have and form no new links, which yields the empty network and we are done. If the answer is *yes* then we suppose that this player i belongs to N_1, without loss of generality. We then construct a path of transition such that the process reaches a minimal network in which players in N_1 are directly or indirectly connected. Here, we first get player i to move and let g' be the resulting network. We then choose a player $i' \in N_1$ who is not a member of the same component as i in g' to move and so on. The resulting network is denoted by g_2.

The *third* step is the main part of the proof: here we construct a path which leads to a network g_3 in which all members of N_1 are isolated. Let C_1 be the component that contains all members of N_1 and let $i_n \in N_1$ who has the maximum internal links. Since g_2 is minimal it follows that for each path leading away from i_n, we can define a player who is furthest away from i_n and call him an end-player. Let $E_k(g_2)$ for $k = 1, 2$ be the set of end players belonging to groups 1 and 2, respectively. We first take up end-players $i \in E_1(g_2)$ who have initiated links. We let them move one at a time. If they have a best response in which they form no links then we allow them to delete their links and they become isolated. Note that they will not form links with any other component if they do not form a link with C_1. If they have a best response which involves forming links then surely they have a best response in which they form a single link with player i_n. Let them all link with i_n and continue this process so long as there is any end-player of group 1, who initiates links with some player other than i_n. This process thus leads to a network g' in which if an end-player belongs to group 1 then he either does not initiate a link or initiates a link with i_n. Moreover if $x \in N_1$ but $x \notin C_1(g')$ then x is isolated.

We now take an end-player $i \in E_1(g')$ who does not initiate a link. If there is some such player x then there exists $y \in N_2$ such that $g_{y,x} = 1$. Let player y move. Since $1 < c_L < c_H$, any best response of y must have $g_{y,x} = 0$, and player x will then be isolated. We repeat this step until all end-players not initiating a link have been isolated, and so all end-players in group 1 are initiating a link with i_n. Now consider an end-player $j \in N_2$ and look at the path $j \leftrightarrow i_n$. If there is no such player then we have arrived at a periphery-sponsored star and we can proceed to the last part of this argument. If there is such an end-player and he initiates the link then check whether this player wants to remain linked with this component. If not then allow the player to move and delink from the component, and the end-players as above. If this player wishes to remain linked with the component, then using arguments above we arrive at a network in which all players in group 2 are connected. Now define $j_n \in N_2$ the player who has the maximum internal links and it follows that player j has a best response in which he forms a link with player j_n. We now repeat the steps above but for end-players in group 2 and arrive at a network in which all end-players of group 2 are initiating links with j_n or isolated.

We have now defined two central players one for each group 1 and 2, respectively. We repeat the above argument in tandem to proceed with the agglomeration process with regard to each of the groups. This process leads to a network g''' with one the following structures: there is a single component which is an inter-linked periphery-sponsored star with members of N_1

forming one star and members of N_2 forming the other star, it is two distinct periphery-sponsored stars, it is one periphery-sponsored star with members of group 1 or group 2 and the other group has disintegrated or the network is empty. In the last case we are done. In the first three cases we use the following transition path: we number the periphery-players of a star from 1 to m, and get player 1 to switch his link from the center to a link with player 2 and player 2 to link with 3 and so on, until m links with 1. This leads to the central player becoming isolated. We now get all players in the circle to move and their unique best response is to delete their single link in the circle. We have thus reached a network in which no pair of players in the group are connected to each other. It is now easy to repeat the argument with the other group and we arrive at the empty network. ■

One-way flow networks: the role of heterogeneity

1 Introduction

The role of social and economic networks in shaping individual behaviors and aggregate phenomena has been widely documented in recent years.[1] Such evidence provides the main motivation for developing a theory which aims at understanding the process of networks' formation and what are the networks' architectural properties. The most popular model of network formation is the "connections model".[2] Although variants of this model have been proposed, in order to analyse different social and economic situations, much of the work has explored settings with homogeneous players. However, it is also true that ex-ante asymmetries across players arise quite naturally in reality. For instance, in the context of information networks, it is often the case that some individuals are more interested in particular issues and, therefore, better informed than other individuals, fact that makes them more valuable contacts. Similarly, individuals differ in their communication and social skills and, therefore, forming links is cheaper for some individuals as compared to others.[3] In this paper, we analyse the role played by ex-ante asymmetries across players in shaping network architectures. To do this, we consider a version of the connections model where individuals unilaterally invest in social ties (one-sided network)

[1] There is a large body of work on this subject. See e.g., Burt (1992) on the career of professional managers, Montgomery (1991) on wage inequality in labour markets, Granovetter (1974) on the flow of job information, and Coleman (1966) on the diffusion of medical drugs.

[2] This model has been extensively studied in the literature; see e.g., Bala and Goyal (2000a, 2000b), Dutta and Jackson (2000), Goyal (1993), Jackson and Wolinsky (1996), and Watts (2001a, 2001b).

[3] In other settings players can be classified in terms of the cost of accessing them. For example, on the web the terminology user-friendly web site is used to describe home pages which are easier to access as compared to others.

and the flow of benefits is frictionless and directed only towards the investor (one-way flow network).

Bala and Goyal (2000a) analyse this model with homogenous players (i.e. the costs of forming links and values of accessing players are homogenous). They show that if players' payoffs are increasing in the number of the other players accessed and decreasing in the number of links formed, a strict Nash network is either a wheel (i.e a connected network in which each player creates and receives one link) or the empty network (with no links). The intuition for this result is as follows. Consider a minimally connected network where player 1 initiates a link with player 2 and 3, and each of these players has a link with player 1. Whenever players are homogeneous, this network is not a strict equilibrium: player 2 is indifferent between maintaining the link with 1 and switching to player 3. A generalization of this argument implies that a connected strict equilibrium is symmetric and has a wheel architecture. It is worth noting that this result depends crucially on the assumption of homogenous values and costs. To see this, assume that player 1 is just slightly cheaper to be linked with than player 2 and 3, ceteris paribus. The introduction of such small heterogeneity implies that the network described above becomes a strict equilibrium. In the present paper, we study the role played by heterogeneous players in shaping equilibrium networks. Players are heterogeneous in terms of the costs of linking and the values of accessing other players.

We start with a setting where values and costs of linking are heterogeneous across players but such heterogeneity is not partner specific: the cost for player i to invest in a social tie is c_i, and the benefit to player i of accessing another player is V_i. We show that a connected equilibrium is a wheel network and that an unconnected equilibrium network is either a center-sponsored star, a wheel with singletons, a wheel with local center-sponsored stars or empty (Proposition 3.1). Figure 3.1 illustrates all strict equilibria in a society with four players. This result shows that players' heterogeneity alters the level of connectedness of the equilibrium networks. In any non-empty unconnected equilibrium there is a set of players sharing a maximum amount of information while the remaining players are socially isolated (they do not access any information). In sharp contrast with the homogeneous setting, these equilibria are asymmetric and central players may emerge: (i) the players maximally informed are connected in a wheel component and the players socially isolated are either (ii_a) singletons or (ii_b) spokes of center-sponsored stars. We finally note that the wheel is robust to asymmetries that are independent from the potential partner.

We then turn to settings where heterogeneity also depends on the potential partner. We show that the wheel architecture is still prominent if costs are not partner specific; otherwise any minimally connected network is a strict equilibrium for some costs and values (Proposition 3.2). This leads us to conclude that costs heterogeneity is responsible for shaping the architecture of the equilibrium networks.

To investigate the role heterogeneity plays in shaping the architecture of equilibrium networks we impose some restriction on the cost of forming links. To this end, we study a *targeted-partner model:* the cost of forming a link with a player i is symmetric across players, but each player has a different cost of being accessed. We show that a connected equilibrium is either a wheel or a flower, in which case the player with the lowest access cost occupies the central position. Furthermore, an unconnected strict equilibrium is either a wheel with a local periphery-sponsored star or a flower with a local periphery-sponsored star (Proposition 3.3).[4] Figure 3.2 illustrates these architectures in a society composed of 4 players.

Let us now comment on three aspects of these results. Firstly, the unique asymmetric connected equilibrium has a flower architecture and the center of the flower is the player with the lowest access cost. The center is the only player in the network which promotes and receives more than one link. His function is to connect sets of players which would be otherwise discon-

[4] The strict equilibria with only singleton components are a limit case of the architectures described in Proposition 3.3 and they are described in the appendix (Proposition 3.4).

nected. Secondly, unconnected strict equilibria have well defined architectures. A set of individuals shares information with each other (the core group), while the remaining players (the periphery group) access the information of the core group directly from the players with the lowest access cost in the population. We finally note that the flower (and its variants) is a less efficient equilibrium as compared to the wheel equilibrium (and its variants). The inefficiency inherent in asymmetric equilibria arises from the over-investment carried out by the central player.

This paper represents a contribution to the theory of network formation, which, at present, is a very active area of research. Whereas most of the existing literature focuses on homogeneous player models, our analysis elaborates on the role of heterogeneity, with respect to both values and costs of forming links, in shaping the equilibrium architectures of a one-way flow connections model. The work that comes closest to ours is Kim and Wong (2003). They study a one-sided connections model with heterogeneous players where agents form two-flow connections but where basic links are only one-flow. In other words, this implies that a player i accesses player j only if there exists a sequence of basic links connecting i to j and vice versa. The current paper departs from Kim and Wong (2003) in two directions. First, we analyse several forms of players' asymmetries, while Kim and Wong (2003) focus exclusively on settings where asymmetries are not partner specific. Second, we do not distinguish between basic links and flow connections, which implies that in this framework a player can access another individual, without the reverse being necessarily true.[5]

Finally, we relate the findings to a recent experimental paper by Falk and Kosfeld (2003). This paper shows that the predictions based on Nash and Strict Nash equilibria for the one-way flow model are consistent with the experimental results, while they generally fail in the two-way flow model.[6] The

[5] For example, the web is characterized by one-way link and one-way flow connections.

[6] Bala and Goyal (2000a) show that with homogeneous players and when information flow is bidirectional a strict equilibrium is either a center-sponsored star (only one player, the center, promotes all the links) or empty (no links).

authors argue that the success of the one-way flow model relies, among other things, on the strategic symmetry (symmetric distribution of links) which characterises equilibrium networks under the one-way flow assumption. The analysis developed in the present paper shows that the property of symmetric distribution of links depends on the assumption of homogeneous players. An experiment, which takes into account ex-ante asymmetries in the costs of forming links, may help to understand the role played by strategic symmetry in the process of network formation.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the results on equilibrium networks under general cost and value heterogeneity. Section 4 analyses the targeted-partner model. Section 5 concludes. Proofs are provided in the Appendix.

2 The model

Let $N = \{1, ..., n\}$ be a set of players and let i and j be typical members of this set. We shall assume throughout the model that the number of players is $n \geq 3$. Each player is assumed to possess some information which are of value to himself as well as to other players. He can augment his information by communicating with other people. This communication takes resources, time and effort and is made possible via *pair-wise* links.

A strategy of player $i \in N$ is a (row) vector $g_i = (g_{i,j})_{j \in N \setminus \{i\}}$ where $g_{i,j} \in \{0, 1\}$ for each $j \in N \setminus \{i\}$. We say that player i has a link with j if $g_{i,j} = 1$. We assume throughout the paper that a link between i and j allows player i to access j's information. The set of strategies of player i is denoted by $\mathcal{G}_i$. Throughout the paper we restrict our attention to pure strategies. Since player i has the option of forming or not forming a link with each of the remaining $n - 1$ players, the number of strategies of player i is clearly $|\mathcal{G}_i| = 2^{n-1}$. The set $\mathcal{G} = \mathcal{G}_1 \times ... \times \mathcal{G}_n$ is the space of pure strategies of all the players.

A strategy profile $g = (g_1, ..., g_n)$ in $\mathcal{G}$ can be represented as a directed network. Let $g \in \mathcal{G}$, we say that there is a path in g from i to j if either $g_{i,j} = 1$ or there exist players $j_1, ..., j_m$, distinct from each other, and i and j such that $\{g_{i,j_1} = ... = g_{j_m,j} = 1\}$. We write $i \xrightarrow{g} j$ to indicate a path from i to j in g. Given two players i and j in g, the geodesic distance, $d_{i,j}(g)$, is defined as the length of the shortest path from i to j. Furthermore, we define $N^d(i;g) = \{k \in N \,|g_{i,k} = 1\}$ as the set of players with whom i maintains a link, whereas we refer to $N(i;g) = \left\{k \in N \middle| i \xrightarrow{g} k\right\} \cup \{i\}$ as the set of players that i observes in g. Let $\mu_i^d : \mathcal{G} \to \{1, ..., n\}$ and $\mu_i : \mathcal{G} \to \{1, ..., n\}$ be defined as $\mu_i^d(g) = \left|N^d(i;g)\right|$ and $\mu_i(g) = |N(i;\overline{g})|$.

Given a network g, a non-singleton component of g is a non-singleton set $C(g) \subset N$ where $\forall i, j \in C(g)$, there exists a path between them but there is not a path between $\forall i \in C(g)$ and a player $k \in N \diagdown C(g)$. A component $C(g)$ of a network g is minimal if, ceteris paribus, $C(g)$ is no longer a component upon replacement of a link $g_{i,j} = 1$ between two agents $i, j \in C(g)$ by $g_{i,j} = 0$. A network g is minimal if every component of g is minimal. A network g is connected if it has a unique component containing all players. If the unique component is minimal the network g is minimally connected. A network which is not connected is unconnected. Given a network g, a player i is a singleton player if $g_{i,j} = g_{j,i} = 0$ for any $j \in N$. Finally, the empty network, denoted as g^e, is an unconnected network where no links are formed.

To complete the definition of a normal-form game of network formation, we specify the payoffs. Let $V_{i,j}$ denote the benefits to player i from accessing player j. Similarly, let $c_{i,j}$ denote the cost for player i of forming a link with player j. Player $i's$ payoff in a network g can be written as:

$$\Pi_i(g) = \sum_{j \in N(i;g)} V_{i,j} - \sum_{j \in N^d(i;g)} c_{i,j} \tag{3.1}$$

We shall assume that $c_{i,j} > 0$ and $V_{i,j} > 0$ for all $i, j \in N$.[7]

[7] The results developed further qualitatively carry on when relaxing the linearity

Given a network $g \in \mathcal{G}$, let g_{-i} denote the network obtained when all player i's links are removed. Note that network g_{-i} can be regarded as the strategy profile where i chooses not to form a link with anyone. Network g can be written as $g = g_i \otimes g_{-i}$ where '$\otimes$' indicates that g is formed as the union of the links in g_i and g_{-i}. The strategy g_i is said to be a *best response* of player i to g_{-i} if:

$$\Pi_i\left(g_i \otimes g_{-i}\right) \geqslant \Pi_i\left(g_i' \otimes g_{-i}\right) \text{ for all } g_i' \in \mathcal{G}_i. \tag{3.2}$$

The set of all of player $i's$ best responses to g_{-i} is denoted by $\mathcal{BR}_i\left(g_{-i}\right)$. Furthermore, a network $g = (g_1, ..., g_n)$ is said to be a *Nash network* if $g_i \in \mathcal{BR}_i\left(g_{-i}\right)$ for each i, i.e. players are playing a Nash equilibrium. If a player has multiple best responses to the equilibrium strategies of the other players this could make the network less stable, as the player can switch to a payoff equivalent strategy. This switching possibility in non-strict Nash networks has been exploited and has been shown to be important in refining the set of equilibrium networks in earlier work (see e.g., Bala and Goyal (2000a)). In the present paper we will therefore focus on strict Nash equilibria only. A *strict* Nash equilibrium is a Nash equilibrium where each player gets a strictly higher payoff from his current strategy than he, otherwise, would, using any other alternative strategy.

3 General Heterogeneity

In this section we investigate the effects of values and costs of linking heterogeneity on the level of connectedness and the architecture of strict equilibria. We shall show that values' and costs' heterogeneity are equally important in determining the level of connectedness of the equilibrium networks, while only cost heterogeneity shapes the architecture of non-singleton components. We shall also show that when players are heterogeneous equilibrium networks are asymmetric.

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assumption of the payoffs functions.

We start by considering a setting in which players have distinct costs of linking, as well as distinct benefits from accessing other players. While these costs and values vary across players, they are independent from the identity of the partner, i.e. $V_{i,j} = V_i$ and $c_{i,j} = c_i$, for any $i, j \in N$. For example, some individuals are more expert in surfing the net as compared to others; ceteris paribus, this allows them to access other internet members at a lower cost.[8] Before carrying out our analysis, let us introduce some architectures.

A star architecture is an unconnected network where there exists a player i, the center, such that either $g_{i,j} = 1$ or $g_{j,i} = 1$ for any $j \in N \backslash \{i\}$, and no other links are formed. The network g has a center-sponsored star architecture if g is a star and the center forms all the links. A non-singleton component has a wheel architecture if players within the component are arranged as $\{i_1, ..., i_n\}$ with $g_{i_2,i_1} = ... = g_{i_n,i_{n-1}} = g_{i_1,i_n}$ and there are no other links between players within the component. A wheel architecture is a connected network with the unique component being a wheel. A wheel network with local center-sponsored stars is an unconnected network with a unique wheel component, say $C(g)$, and where $\forall j \notin C(g)$, $\exists! i \in C(g)$ such that $g_{i,j} = 1$. Finally, a wheel network with singleton players is an unconnected network with a unique wheel component made up of at least three players and where $g_{i,j} = g_{j,i} = 0$ for any $i \notin C(g)$ and for any $j \in N$.

The next result shows which networks can be sustained in equilibrium.

Proposition 3.1. *Let payoffs satisfy (3.1) and assume that* $c_{i,j} = c_i$ *and* $V_{i,j} = V_i$ $\forall j \in N \backslash \{i\}$. *A connected strict equilibrium is a wheel. Otherwise, a strict equilibrium is either the empty network, or the wheel with singletons, or the wheel with local center-sponsored stars or the center-sponsored star. Con-*

[8] Similarly, some individuals may value more information provided on the web as compared to others. In general, individuals differ in communication and social skills and it seems natural that the costs of establishing links as well as the values of accessing information vary across individuals.

versely, any of such networks is a strict equilibrium for some $\{c_i, V_i\}_{i \in N}$.

Figure 3.1 illustrates all the strict equilibria in a society composed of 4 players. We represent a link $g_{i,j} = 1$ as an edge starting at j with the arrowhead pointing at i. The proof of Proposition 3.1 proceeds as a sequence of Lemmas. We sketch here its main steps. We firstly show that a strict Nash network is minimal. This follows from the no-decay assumption. Secondly, using a standard switching argument, we show that each player receives at most one link. Thirdly, using this equilibrium property, it follows that each non-singleton component has a wheel architecture. Therefore, a connected strict equilibrium is a wheel. Fourthly, we take up the case of non-empty unconnected equilibria in which each component is composed of a single player. Using the finiteness of the set of players, we show that an equilibrium is a center-sponsored star network. Finally, an elaboration of the arguments used in the previous lemmas establishes the result for those unconnected equilibria having at least a non-singleton component.

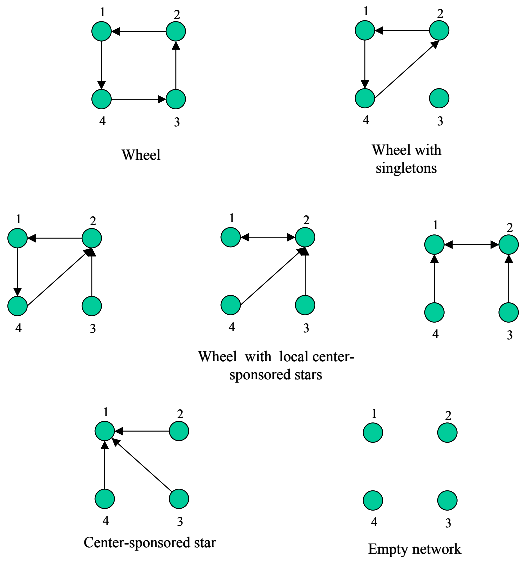

Figure 3.1. Proposition 3.1.

Proposition 3.1 provides some interesting insights. As in the homogenous setting, the unique connected equilibrium is the wheel. Therefore, the wheel architecture is prominent also in settings where costs and values asymmetries are partner independent. Next, values and costs heterogeneity alters the level of connectedness of strict equilibria. In any unconnected (and non-empty) equilibrium there is a set of players accessing a maximum amount of information while all the other players are socially isolated (they do not access any information). Furthermore, the maximally informed players are connected in a wheel, while the isolated players are either singletons or spokes of center-sponsored stars. Thus, unconnected equilibria are generally asymmetric and central players may emerge. Finally, we note that the results presented in Proposition 3.1 carry on in settings with homogenous values (costs of linking) and heterogeneous costs of linking (values). This implies that, as far as heterogeneity is independent from the partner, costs and values asymmetries have equivalent effects on strategic interaction.

We now ask under which conditions ex-ante asymmetries across players alter the architecture of equilibrium networks. The next result establishes that the wheel architecture is prominent as far as the costs of linking are partner independent.

Proposition 3.2. *Let payoffs satisfy (3.1). First, assume* $c_{i,j} = c_i$ *while values vary freely, then a connected equilibrium is a wheel. Conversely, the wheel is an equilibrium for some* $\{c_i, V_{i,j}\}_{i,j \in N}$*. Second, assume* $V_{i,j} = V_i$ *while costs vary freely, then a connected equilibrium is minimal; conversely, any minimally connected network is a strict equilibrium for some* $\{c_{i,j}, V_i\}_{i,j \in N}$.

We note that when values vary freely while costs asymmetries are partner independent connected equilibria have (still) a wheel architecture. This result also holds with homogenous costs of linking. Differently, when the costs of linking are allowed to vary freely across players, the only restriction imposed by the equilibrium notion to connected network is minimality. This results holds regardless of values asymmetries.

The analysis of this section can be summarised as follows. *Firstly*, the level of connectivity of a network is equally sensitive both to values and costs heterogeneity. Interestingly, unconnected equilibria are asymmetric and central players may emerge. *Secondly*, asymmetries in values do not alter the architecture of non-singleton components, as compared to the homogeneous setting. The same observation applies when cost heterogeneity is independent from the potential partner. *Thirdly*, when the costs of linking are allowed to be partner specific, social interactions lead to a *'everything is possible'* type of result. Hence, we can conclude that it is heterogeneity in the cost of linking which is mainly responsible in shaping the architecture of equilibrium networks. The next section explores the possibility to set plausible conditions on the cost parameters, in order to obtain further restrictions on the architecture of equilibrium networks.

4 Targeted-partner Model

In this section we consider a setting where values are homogeneous, while the costs of linking are exclusively partner specific. In particular, each player i has a different cost of being accessed, which is, however, homogenous with respect to the players who initiate a link with i. For example, some web sites are user-friendlier than others, feature that allows players to access the information provided more easily.[9] Formally, let $V_{i,j} = V$, for any $i, j \in N$ and assume the following cost structure:

$$c_{i,j} = c_j \text{ for any } i \in N \tag{3.3}$$

We shall assume that $c_1 > 0$ and, without loss of generality, that $c_j < c_x$ whenever $j < x$.[10] Given a network g and (3.3),

[9] We provide some other examples. Individuals have different opportunity costs. It seems natural to consider that to access players with higher opportunity costs is more costly as compared to others. To apply for some jobs is less costly than for others. Different countries have different immigration policies. Countries which implement more strict immigration policy are more difficult to be accessed as compared to countries with "soft" immigration policies.

[10] Further, we shall discuss the implication of allowing groups of players having the same accessibility cost.

the payoff to player i can be rewritten as follow:

$$\Pi_i(g) = \mu_i V - \sum_{j \in N^d(i;g)} c_j \tag{3.4}$$

We introduce some additional notation. An unconnected network with a unique wheel component, say $C(g)$, where $g_{j,i} = 1$ for any $j \notin C(g)$ and a unique player $i \in C(g)$, is called a wheel with a local periphery-sponsored star with center i. A flower component, $C(g)$, of a network g partitions the set of players belonging to $C(g)$ into a central player, say i, and a collection of $\mathcal{P} = \{P_1, ..., P_q\}$, where each $P \in \mathcal{P}$ is non-empty. A set P of agents is referred to as petal. Denote the agents in P as $\{j_1, ..., j_n\}$. A flower component is then defined by setting $\{g_{i,j_1} = g_{j_1,j_2} = ... = g_{j_n,i} = 1\}$ for any $P \in \mathcal{P}$ and $g_{i,j} = 0$ otherwise. If g is connected with a flower component and i is the center, then g is a flower network with center i. An unconnected network with a unique flower component where player i is the center and where $g_{j,i} = 1$ for any $j \notin C(g)$ is called a flower with a local periphery-sponsored star with center i. Figure 3.2 depicts all the aforementioned possible architectures in a society composed of four players.

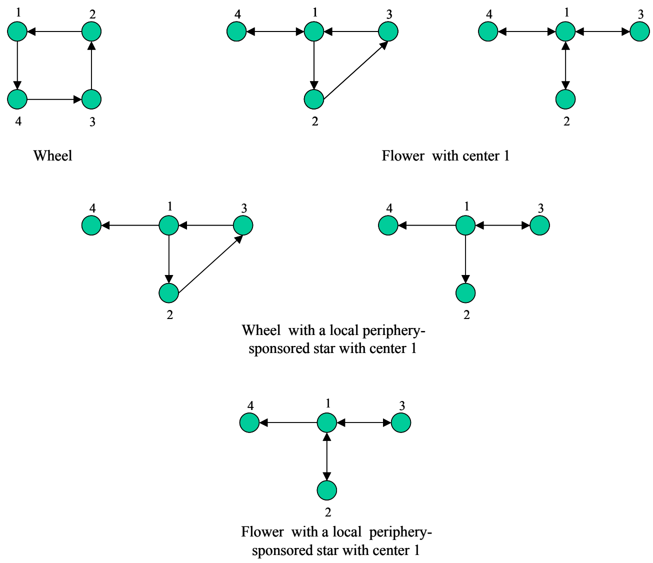

Figure 3.2.

4.1 Strict equilibrium networks

We start by introducing some necessary conditions to characterize strict Nash equilibria. The role of these conditions is to constrain the arrangement of players belonging to the architectures we introduced above. We start by defining the ordered condition for a wheel network with a local periphery sponsored star.

Definition 3.1. *A wheel with a local periphery-sponsored star network, say g, where player h is the center, is ordered if for any $g_{i,j} = 1$, $i, j \in C(g)$ then (i) $c_j - c_y < d_{j,y}(C(g))V$ for any $y \in C(g)$ and (ii) $c_j - c_y < d_{j,h}(C(g))V$ for any $y \notin C(g)$.*

Condition (i) requires any player i linked to player j in a wheel component not to find profitable to switch to another player belonging to the same wheel. Condition (ii) takes care of the switching possibilities of a player belonging to a component with players outside the component. Note that a wheel network is ordered if condition (i) in Definition 1 is satisfied. We now turn to define the ordered conditions for networks with a flower component.

Definition 3.2. *A flower with a local periphery sponsored star network, say g, where player h is the center, is ordered if for any $P \in P$ and for any $g_{i,j} = 1$, $i, j \in P$, then (i) $c_j - c_y < d_{j,y}(P(g))V$, $\forall\, y \in P$ and (ii) $c_j - c_{y'} < d_{j,h}(P(g))V$, $\forall y' \notin P$.*

The interpretation of definition 3.2 is analogous to the one in definition 3.1. As $c_n - c_1 < V$ the ordered conditions in definition 3.1 and 3.2 are always satisfied. The next proposition provides the set of strict equilibria of this model. We focus on equilibria in which there exists at least one non-singleton component.[11]

Proposition 3.3. *Let (3.3)-(3.4) be satisfied. A connected strict equilibrium is either an ordered wheel or an ordered flower where player 1 is the center. An unconnected strict equilibrium*

[11] The analysis of equilibria in which each component is composed of a single player is provided in the appendix (see proposition 3.4). The architecture of these equilibria are a limit case of the architectures of equilibria with a non-singleton component.

with at least a non-singleton component is either an ordered wheel or an ordered flower with a local periphery sponsored star where player 1 is the center. Conversely any such network is a strict equilibrium for some $\{c_i, V\}_{i \in N}$.

The proof of Proposition 3.3 proceeds as a sequence of Lemmas. We sketch here its main steps. First, the assumption of no-decay in the information flow implies that a strict equilibrium is minimal. Second, using a standard switching argument, we show that in a non-singleton component it is only the player with the lowest access cost that can receive more than one link. This implies that a connected equilibrium is either a wheel or a flower where player 1 is the center. Third, we show that if a player $j \notin C(g)$ promotes a link with a player $i \in C(g)$, then j does not receive any link. Suppose player j' forms a link with j, then the player linked with i in $C(g)$ gains by switching to player j. Finally, using the fact that the set of players is finite, we show that players within a component do not promote links with players outside that component. These last two observations together imply that any two players belonging to two different components access two distinct sets of agents. A simple switching argument establishes that in a strict equilibrium at most one non-singleton component exists.

Let us now discuss some aspects of this result. The *first* remark is about the flower architecture. When players are heterogeneous in terms of cost of being accessed, asymmetric connected networks are strategically viable and a coordinator emerges, i.e. the player with the lowest access cost. The coordinator connects sets of players who would otherwise be disconnected. It is worth noticing that the flower network also arises in homogeneous settings when a small amount of decay in the information flow is introduced. In particular, Bala and Goyal (2000a) show that, for a sufficiently small amount of decay, the flower architecture is the only strict Nash network.[12] In that case the role of the center is to decrease the distance between the players.

[12] See Bala and Goyal (2000a) for a detailed discussion on this issue.

Differently, in this model the flower arises because the center is the more profitable player to be linked with. Bala and Goyal's (2000a) result on decay is reinforced in our setting where the advantage deriving from linking with the center is not only to have short information channels but also to obtain a decrease in the investment cost.

The *second* remark is about unconnected equilibria. In these equilibria players can be divided in two groups. On the one hand, a core group composed of players belonging to a non-singleton component; on the other hand, a periphery group composed of all the remaining players. Interestingly, all players in the society access the players in the core group, but no player accesses those individuals belonging to the periphery group.

Thirdly, we would like to make some considerations on the assumption that the ranking of players, in term of their costs of being accessed, is strictly increasing.[13] Suppose that the players in the society can be grouped in m distinct and different groups, $N = \cup_{i=1}^{m} N_i$ and that the players belonging to the same group have the same cost of being accessed. If N_1 is composed of a single player, then an ordered flower network, where player 1 is the center, can be sustained as a strict equilibrium. Indeed, in this case, the ordered conditions are enough to take care of switching possibilities. By contrast, when also group N_1 is composed of more than one player the problem becomes more delicate. The following example clarifies this point. Consider three groups, say $1, 2$ and 3, each composed of three players, say a, b and c. Let g be the network depicted in figure 3.3 and let $c_3 < (n-1)V$. The network g is not strict Nash. On the one hand, player 2_a (3_a) is indifferent between retaining a link with 1_a or switching either to 1_b or 1_c; on the other hand, all other players have a unique best response. However, it has to be noticed that each player's best responses are insensitive to such changes made by either player 2_a or 3_a. This implies that the considered network g, along with the possible best responses of player 2_a and 3_a,

[13] In other words, we discuss the effect of allowing groups of players to have the same accessibility cost.

constitutes a minimal curb set of the game.[14] Using standard results on best response dynamics it follows that this set is absorbing, in the sense that once a best response dynamic enters such a set it will cycle within the set forever.

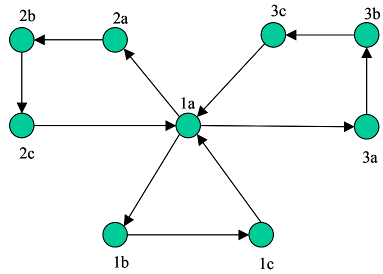

Figure 3.3.

Finally, we compare the efficiency properties of the wheel architecture with respect to the flower architecture.[15] Consider a wheel and a flower network. It is easy to see that both architectures generate the same amount of network externalities, but the former, the wheel, requires a lower level of total investment.[16] It follows that the flower network is less efficient than the wheel as the flower's center over-invests in social ties. The same argument holds along all the variants of the flower and wheel architectures. The reasoning above implies that if we move away from an homogenous setting and we go towards an heterogeneous environment, in such a way as to make social interactions less costly, then the strategic interactions lead to the emergence of inefficient equilibria. For example, let's assume that players are fully homogeneous and that $V > c$. In this case, the unique equilibrium is the wheel architecture. Furthermore, the wheel is uniquely efficient. Assume now that a player becomes slightly cheaper to be linked to, as compared to the others. The introduction of such a small heterogeneity implies that the flower

[14] For a formal definition of supertight curb set see section 2.4.2 of chapter 2. See Basu and Weibull (1991) for a discussion of the notion of minimal curb set.

[15] We consider the social welfare of a network g as the sum of payoffs of all players.

[16] It is easy to see that the wheel architecture is the more efficient architeture in the class of connected architectures.

architecture is also an equilibrium. However, this equilibrium is inefficient.

5 Conclusion

We have studied a connection model where heterogeneous players decide unilaterally to invest in social ties that imply a direct return for the investor only. The main results obtained can be summarized as follows. Firstly, the level of connectedness of a network is equally sensitive to value and cost heterogeneity. Furthermore, non-empty unconnected equilibria have an asymmetric distribution of links and central players may emerge in equilibrium. Secondly, the wheel architecture (along with its variants) is robust to players' asymmetries, as far as cost heterogeneity is independent from the potential partner. Conversely, when the costs of linking are allowed to vary freely, the only restriction imposed by the strategic interaction on the architecture of non-singleton components is minimality. We also explored the role played by partner specific cost asymmetries, in the targeted-partner model, i.e. each player has a distinct cost of being accessed. Here, non-singleton components are either wheels or flowers. The flower network has an asymmetric architecture where the central player connects sets of players who would otherwise be disconnected. Finally, note that asymmetric architectures are less efficient than symmetric ones. We interpret these results as saying that, in heterogeneous settings, equilibrium networks are asymmetric and central players emerge.

6 Appendix chapter 3

Proof of Proposition 3.1.

First part. We first note that an equilibrium network is minimal. This follows from the assumption of no-decay in the information flow. The proof now proceeds as a sequence of Lemmas. The next result shows that in equilibrium each player receives at most one link.

Lemma 3.1. *Let g be a strict equilibrium. If $g_{i,j} = 1$ then $g_{k,j} = 0$ for any $k \in N \backslash \{i\}$.*

Proof: Suppose, for a contradiction that $g_{i,j} = 1$ and $g_{k,j} = 1$. Since g is minimal, i does not access player k in g; however, in this case player i strictly prefers to delete the link with player j and linking up with player k, instead. This is a contradiction.■

Using this result we show that each non-singleton component part of a strict equilibrium is a wheel.

Lemma 3.2. *Let $C(g)$ be a non-singleton component part of a strict equilibrium g. Then $C(g)$ has a wheel architecture.*

Proof: We note that if a player i belongs to a non-singleton component, say $C(g)$, then $g_{i,j} = 1$ for at least one player $j \in C(g)$ and $g_{k,i} = 1$ for at least one player $k \in C(g)$. These two observations and Lemma 3.1 imply that each player $i \in C(g)$ receives one and only one link from the players belonging to $C(g)$. We now claim that for any player $i \in C(g)$, then $g_{i,j} = 1$ for only one player $j \in C(g)$. Suppose, for a contradiction, that for some $i \in C(g)$, $g_{i,j} = g_{i,k} = 1$ for some $j, k \in C(g)$ and $j \neq k$. Since $j, k \in C(g)$ then j and k must access player i; therefore, there exist two paths $\{g_{i,k} = g_{k,k_1} = ...g_{k_{n-1},k_n} = g_{k_n,i} = 1\}$ and $\{g_{i,j} = g_{j,j_1} = ...g_{j_{n-1},j_n} = g_{j_n,i} = 1\}$. Since i receives only one link it must be the case that $j_n = k_n$. However, the same argument applies for player $j_n(= k_n)$, and therefore it must be the case that $j_{n-1} = k_{n-1}$. By induction, it follows that $k_1 = j_1$; since $k \neq j$, it follows that $k_1(= j_1)$ must receive more than one link. This constitutes a contradiction. These observations altogether implies that $C(g)$ is minimally connected and it has a symmetric architecture. It is readily seen that the unique directed graph which satisfies these properties is the wheel. This proves the Lemma.■

Lemma 3.1 and 3.2 implies that a connected equilibrium network is a wheel. We now take up the case of non-empty unconnected strict equilibria in which each component is a singleton. The next Lemma proves the result.

Lemma 3.3. *A non-empty unconnected strict equilibrium where each component is a singleton has a center-sponsored star architecture.*

Proof: Since g is non-empty there exists some $g_{i,j} = 1$. There are two cases. First, suppose $g_{j,j\prime} = 0$ for any $j' \in N$. Since g is strict Nash it must hold that $V_i - c_i > 0$; this implies that $\exists i \overset{g}{\to} j'$ for any $j' \in N$. Select player k which is at the maximum distance from i in g, i.e. $k = \arg\max_{j' \in N} d_{i,j'}(g)$. If $d_{i,k}(g) = 1$ player i accesses each other player directly and the proof trivially follows. If $d_{i,k}(g) > 1$, it must be the case that $\{g_{i,j_1} = g_{j_1,j_2} = ... = g_{j_m,k} = 1\}$ and $g_{k,s} = 0$ for any $s \in N$. Since g is strict Nash then $V_{j_m} - c_{j_m} > 0$ and therefore player j_m accesses any player in g. This implies that player i and j_m belongs to a non-singleton component, which constitutes a contradiction. Second, suppose $g_{j,j\prime} = 1$ for some $j' \in N$. Since g has only singleton components it follows $j' \in N \backslash \{i\}$. Therefore, if $g_{j\prime,k} = 0$ for any $k \in N$, the previous argument applies and we end-up with a contradiction. If $g_{j',k} = 1$ for some k, then it must be the case that $k \in N \backslash \{i, j\}$. Since the number of players is finite, it must exist a player h which is accessed by player i via the link $g_{i,j} = 1$ and such that $g_{h,h'} = 1$ and $g_{h',h''} = 0$ for any $h'' \in N$. However, also in this case the fact that g is strict Nash implies that $V_h - c_h > 0$ and therefore player h must access player i in g. This constitutes a contradiction. Hence the proof follows.■

We now turn to unconnected strict equilibria where at least a non-singleton component exists. Let $C_1(g), C_2(g), ..., C_m(g)$ be the components of an unconnected strict equilibrium g. Lemma

3.1 and 3.2 implies that: (a) $C_x(g)$ is a wheel $\forall x = 1, ..., m$; (b) $g_{j,i} = 0$, $\forall i \in C_x(g)$ and $\forall\ j \in N \backslash \{C_x(g)\}, \forall\ x \in \{1, ..., m\}$.

Lemma 3.4. *Let g be a strict equilibrium and let $i \in C_x(g)$. If $g_{i,j} = 1$ where $j \notin \cup_{y=1}^{m} C_y(g)$, then $g_{j,k} = 0$ for any $k \in N$.*

Proof: Suppose not, i.e. $g_{i,j} = g_{j,k} = 1$. Lemma 3.1 implies that $k \notin \cup_{y=1}^{m} C_y(g) \cup \{j\}$; moreover, it also implies that if $g_{k,h} = 1$ then $h \notin \cup_{y=1}^{m} C_y(g) \cup \{j, k\}$. Suppose that $g_{k,h} = 0$ for any $h \notin \cup_{y=1}^{m} C_y(g) \cup \{j, k\}$; since g is a strict Nash it follows that $V_j > c_j$. In this case player j strictly gains by forming a link with player i. This constitutes a contradiction. If $g_{k,h} = 1$ for some $h \notin \cup_{y=1}^{m} C_y(g) \cup \{j, k\}$ we can iterate the argument above and since the number of players is finite the proof follows.■

Lemma 3.1, 3.3 and 3.4 implies that any pair of players, say i and j, belonging to two different components, say $C_x(g)$ and $C_y(g)$, access two distinct set of players, i.e. if $i \in C_x(g)$ and $j \in C_y(g)$, with $x \neq y$, then $N_i(g) \cap N_j(g) = \Phi$. The next Lemma uses this observation to prove that a strict equilibrium network has at most one non-singleton component.

Lemma 3.5. *A strict equilibrium has at most one non-singleton component.*

Proof: Suppose not and let, without loss of generality, $|N_i(g)| \geq |N_j(g)|$, where $i \in C_x(g)$ and $j \in C_y(g)$, and $x \neq y$. Since g is strict Nash it follows that $|N_j(g)| V_j - c_j > 0$; however, if this is the case, player j is weakly better off by deleting his link in $C_y(g)$ and linking up with player i, *i.e.* $|N_i(g)| V_j - c_j \geq |N_j(g)| V_j - c_j > 0$. This contradiction proves the lemma.■

The next lemma completes the analyses of unconnected strict equilibria which have a non-singleton component.

Lemma 3.6. *Let g an unconnected strict equilibrium with a non-singleton components. Then $g_{j,j'} = 0$* for any $j, j' \notin C(g)$.

Proof: Suppose, for a contradiction, that $g_{j,j'} = 1$. Lemma 3.1 implies that each player outside the component does not access players belonging to the component. Therefore, Lemma 3.5 applies to the set of players $N \backslash \{C(g)\}$, i.e. $g_{j,j'} = 1$ for some $j' \notin C(g)$. However, in this case player j strictly gains by creating a link with a player $i \in C(g)$. This constitutes a contradiction. Hence, Lemma 3.6 follows.■

The combination of Lemma 3.2, 3.4, 3.5 and 3.6 implies that an unconnected strict equilibrium with some non-singleton components is either a wheel with local center-sponsored stars, a wheel with singleton players or a wheel with some local center-sponsored star and some singleton player. It is immediate to see that this last architecture cannot be sustained as a strict equilibrium. This completes the proof of the first part of the proposition.■

Second part. First, let g be the empty network and let $c_i > V_i$ for any $i \in N$; it follows that g is a strict equilibrium. Second, let g be a wheel and set $c_i < V_i$ for any $i \in N$; it follows that g is a strict equilibrium. Third, let g be a center-sponsored star network where player i is the center. For any $j \in N \setminus \{i\}$ let $c_j > (n-1)\, V_j$, while for the central player i let $c_i < V_i$. It follows that g is a strict equilibrium. Fourth, let g be a wheel network with singleton players and let $C(g)$ be the wheel component in g. For any player $i \in C\,(g)$ set V_i and c_i such that $c_i \in (V_i, (|C(g)|-1)V_i)$, while for any other player $j \notin C\,(g)$ set V_j and c_j such that $c_j > |C(g)|\, V_j$. It follows that g is a strict equilibrium. Finally, let g be a wheel network with local center-sponsored stars and let $C(g)$ be the wheel component. For any player $i \in C\,(g)$, set c_i and V_i such that $c_i < V_i$ and for any other player $j \notin C\,(g)$ set c_j and V_j such that $c_j > (n-1)V_j$. It follows that, g is a strict equilibrium. This completes the proof of the second part.■

The proof of Proposition 3.1 is completed.■

Proof of Proposition 3.2. We start by assuming that $c_{i,j} = c_i$, for any $j \in N$, while $V_{i,j}$ varies freely. Let g be a connected strict equilibrium network. The no-decay assumption implies that g is minimal. Furthermore, Lemma 3.1 and 3.2 of Proposition 3.1 applies also to this case; hence g has a wheel architecture. We now prove the converse. Let g be a wheel network and for any link $g_{i,j} = 1$ let $V_{i,j} > c_i$. This implies that each player i

finds it optimal to maintain his links and not to form any other links. Hence, the first part of the proof follows.

We now turn to the second case, i.e. $V_{i,j} = V_i$ for any $j \in N$ and $c_{i,j}$ varies freely. Let g be a connected network. The no-decay assumption implies that g is minimal. Conversely, let g be a minimally connected network. For any link $g_{i,j} = 1$ let $c_{i,j} < I_{i,j}V_i$, while for any $g_{i,j} = 0$ let $c_{i,j} > (n-1)V$. These two conditions assure that each player is playing his unique best response. This completes the proof of the Proposition.■

Proof of Proposition 3.3.

First part. We first note that the no-decay assumption implies that an equilibrium network is minimal. The proof now proceeds as a sequence of Lemmas. The first lemma shows that if a player belonging to a non-singleton component receives more than one link, then this player has the lowest access cost across players within that component.

Lemma 3.7. *Let $C(g)$ be a non-singleton component of a strict equilibrium and let $j \in C(g)$. If $g_{i,j} = 1$, for more that one player $i \in C(g)$ then $c_j = \min_{j' \in C(g)} c_{j'}$.*

Proof: Suppose, for a contradiction, that $g_{i,j} = g_{k,j} = 1$, for some $i, k \in C(g)$, $i \neq k$ and $c_h < c_j$ for some $h \in C(g) \setminus \{j\}$. Since g is minimal and $i, k, j \in C(g)$, player i (or k) accesses h via the link $g_{i,j} = 1$ (or $g_{k,j} = 1$). In this case, player i (or k) strictly gains by deleting the link with j and linking up with h, instead. This contradiction proves the lemma.■

Lemma 3.8. *A non-singleton component of a strict equilibrium is either a wheel or a flower.*

Proof: We have two possibilities. First, suppose any player $i \in C(g)$ receives at most one link in $C(g)$. We note that Lemma 3.2 in Proposition 3.2 also applies in this case. Therefore $C(g)$ has a wheel architecture. Second, suppose $i \in C(g)$ receives more than one link, i.e. $g_{j_1,i} = g_{j_2,i} = ... = g_{j_k,i} = 1$. Lemma 3.7 implies that player i is the player with lowest access cost in $C(g)$. We now claim that if player i receives k distinct links, then $C(g)$

is a flower with k petals. Since $\{i, j_1, ..., j_k\} \in C(g)$, there exists a path $i \rightarrow j_x$, for any $x = 1, ..., k$. The same argument presented in Lemma 3.2 of Proposition 3.2 implies that if player h belongs to the path $i \rightarrow j_x$ then h cannot belong to another path $i \rightarrow j_y, x \neq y$ and $x, y = 1, ..., k$. Consider an arbitrary path $i \rightarrow j_x$; using Lemma 3.2, it follows that each player belonging to that path forms one and only one link. It is readily seen that the only possibility left is that $C(g)$ has a flower architecture with k petals and player i is the center. This completes the proof.■

Lemma 3.8 implies that a connected strict equilibrium is either a wheel or a flower with player $j = 1$ the center. We now note that if g has wheel architecture and it is strict Nash then the wheel is ordered. Suppose, for a contradiction, that g is a wheel but it is not ordered, i.e. for some $g_{i,j} = 1$, $i, j \in C(g)$,then $c_j - c_y \geq d_{j,y}(C(g))\,V$ for some $y \in C(g)$. If this is the case, player i (weakly) gains by deleting the link with player j and linking up with y, instead. A similar argument shows that a flower equilibrium network is ordered.

We now turn to consider unconnected networks with some non-singleton components.

Lemma 3.9. *Let g be an unconnected network and let $C(g)$ be a non-singleton component. If $g_{k,i} = 1$, for some $i \in C(g)$ and $k \notin C(g)$ then $g_{k',k} = 0$ for any $k' \in N$.*

Proof: Assume, for a contradiction, that $g_{k',k} = 1$ for some k'. We first note that $k' \notin C(g)$; for otherwise $k \in C(g)$. Then, let us assume that $k' \notin C(g)$ and let player $i' \in C(g)$ be linked with player i, i.e. $g_{i',i} = 1$. We note that if player i' deletes the link with i and creates a new link with player k, he will still observe all the players he was accessing before the switching (via the new link with player k) and in addition he accesses all players that k accesses in g and that are not accessed by i in g, i.e $\tilde{\sigma}_k = \left|\{h : \nexists i \xrightarrow{g} h \ \wedge \exists k \xrightarrow{g} h\}\right|$. Since g is strict Nash it must be the case that $c_k - c_i > \tilde{\sigma}_k$. Next, we note that if player k' deletes the link with player k and creates a new link with player i, then he (player k') will not accessed any-more all players h that player i does not access in g and that k' accesses in g exclusively via the link with the player k, i.e $\sigma_k = \left|\{h : \nexists i \xrightarrow{g} h \ \wedge \ \exists k \xrightarrow{g} h \ \wedge \ \nexists k' \xrightarrow{g'} h\}\right|$. Since g is strict Nash it must be the case that $c_k - c_i < \sigma_k$. It is readily seen that $\sigma_k \leq \tilde{\sigma}_k$; however, this implies that the two conditions are incompatible. This contradiction completes the proof of the lemma.■

Lemma 3.10. *Let g be a strict equilibrium and let $C_1(g), ..., C_m(g)$ be the non-singleton components belonging to g. If $i \in C_x(g)$, then $g_{i,j} = 0$, $\forall j \notin C_x(g)$.*

Proof: For a contradiction assume that $i \in C_x(g)$ and that $g_{i,j} = 1$ for some $j \notin C_x(g)$. Moreover, let player $i' \in C_x(g)$ and such that $g_{i,i'} = 1$. Lemma 3.9 implies that $j \notin \cup_{y=1}^{m} C_y(g)$. Therefore, we have two possibilities. First, assume that $\forall k \in N$, $g_{j,k} = 0$. In this case player j does not access any player. Since player i has a link with player i' and g is strict Nash then player j strictly gains to initiate a link with player i'. This is a contradiction. Second, assume that $g_{j,k} = 1$ for some $k \in N$. Lemma 3.9 implies that $k \notin \cup_{y=1}^{m} C_y(g) \cup \{j\}$. Iterating the previous argument it follows that if $g_{k,h} = 0$ for any $h \in N$, then player k strictly prefers to create a link with player i'. Therefore, it must be the case that $g_{k,h} = 1$ for some h, which, using Lemma 3.9, implies that $h \notin \cup_{y=1}^{m} C_y(g) \cup \{j, k\}$ $\forall y = 1, ..., m$. We can continue to iterate this argument and since the set of players is finite, it must be the case that there exists a player, say l, who does not access any player and that is accessed by player i via a path in g. However, in this case player l strictly prefers to create a link with player i'. Thus the proof follows.■

Lemma 3.10 implies that if $i \in C_x(g)$ and $j \in C_y(g)$, $x \neq y$ $\forall x, y = 1, ..., m$, then $N_i(g) \cap N_j(g) = \Phi$. Using this property we note that only one non-singleton component can be part of a strict equilibrium. For otherwise, one player belonging to a non-singleton component (weakly) gains be switching to another non-singleton component. Summarizing, it follows that a strict equilibrium with at least a non-singleton components must satisfy: (1) there exists a unique non-singleton component, say $C(g)$, which has a wheel architecture or a flower architecture, and (2) each player within the non-singleton component does not promote links with player outside the component (Lemma 3.10). We now note that each player outside the non-singleton component, say $j \notin C(g)$, accesses players within the components. This fact and Lemma 3.10 implies also that each player outside the component is directly linked with the player who has the lowest access cost within the component. It is now immediate to see that the player with the lowest access cost in the society should belong to the non-singleton component. Finally, the ordered conditions are easily verified. This completes the proof of strict equilibria with some non-singleton components. Hence, the proof of the first part of the proposition follows.■

Second part: We now prove the converse. Let g be a wheel and set $c_i < V$ for any $i \in N$. We note that given $c_i < V$ any wheel is ordered (no player wants to switch) and that each player is playing his unique best response. Therefore g is a strict equilibrium. The same set of restrictions in the costs of linking imply that the flower network with player 1 the center is a strict equilibrium. Finally, suppose that g is a wheel with a periphery-sponsored star with player 1 the center. Set $c_i < V$ for any $i \in C(g)$, while $c_j > nV$ for any $j \notin C(g)$. It follows that g is ordered and it is a strict equilibrium. The same conditions applies to the flower network with a periphery-sponsored star with player 1 the center. This completes the proof of the second part of the proposition.■

The proof of the proposition in now completed.■

We conclude by providing the characterization of strict equilibria in which each component is composed of a single player. We first introduce some notations. A generalized periphery sponsored star with center j is an unconnected architecture where each component is a singleton and where there exists a player, the center j, such that $g_{i,j} = 1$ for some (or all) $i \in N$ and the remaining players, say $\{j_1, j_2, ..., j_m\} \subseteq N$, are arranged in the following way $\{g_{j,j_m} = g_{j_m,j_{m-1}} = ... = g_{j_2,j_1} = 1\}$. Note that if $\{j_1, ..., j_m\} = N$ then g is a line where the information flow from j_m to j, from j_{m-1} to j_m and so on. While if $\{j_1, ..., j_m\}$ is empty then g is a periphery-sponsored star with player j the center. The next definition defines the ordered condition for a generalized-periphery sponsored star network.

Definition 3.3. *A generalized periphery-sponsored star network g is ordered if $\forall\ g_{j',j''} = 1$ then $c_{j''} - c_y < d_{j'',y}(g)V$, $\forall\ y$ accessed by j' via the link $g_{j',j''} = 1$.*

Proposition 3.4. *Let (3.3)-(3.4) be satisfied and let g be a strict equilibrium in which each component is composed of a single player. Then g is either the empty network or the ordered generalized periphery sponsored-star network where $g_{1,i} = 0$ for any $i \in N$. Conversely, any such network is a strict equilibrium for some $\{c_i, V\}$.*

Proof.
First Part: The first part of Proposition 3.4 is based on two Lemmas.

Lemma 3.11. *Let g be an unconnected strict equilibrium. If $g_{i,j} = 1$ and $g_{k,j} = 1$ then i and k do not receive any link, i.e. $g_{j',i} = g_{j',k} = 0\ \forall j' \in N$.*

Proof: For a contradiction, suppose first that $g_{j',k} = 1$ for some $j' \in N$. Since g is minimal $j' \neq i$; since g is unconnected $j' \neq j$ and is different from any player k' such that there exists $j \xrightarrow{g} k'$ in g. Since g is strict Nash, the link $g_{i,j} = 1$ implies that $c_k - c_i > V$, and the link $g_{j',k} = 1$ implies that $c_k - c_j < V$. This is a contradiction. The same applies to the case where $g_{j',i} = 1$. This completes the proof of the Lemma.■

Lemma 3.12. *Let g be a non-empty unconnected equilibrium where each component is a singleton. Then each player promotes at most one link.*

Proof: Suppose not, i.e. there exists a player j such that $g_{j,i} = g_{j,k} = 1$ for some distinct k and i. We first note that if $g_{j,i} = 1$, then player j accesses via the link with player i some player, say i', such that $g_{i',i''} = 0$ for any $i'' \in N$. To see this note that if $g_{i,i_1} = 0$ for any $i_1 \in N$, the proof trivially follows. Moreover, if $g_{i,i_1} = 1$, then $i_1 \neq j$; for otherwise i and j would be part of a non-singleton component. Iterating the argument and noting that set of player is finite, the claim follows. Second, this implies that if player j is linked with i and k, then there exist $\{g_{j,i} = g_{i,i_1} = ... = g_{i_n,i'} = 1\}$ and $\{g_{j,k} = g_{k,k_1} = ... = g_{k_n,k'} = 1\}$, where $g_{i',s} = g_{k',s} = 0$ for any $s \in N$. It is easy to see that if $k' = i'$ then Lemma 11 is violated. Furthermore, if $k' \neq i'$ since g is a strict Nash equilibrium, player k_n strictly gains by forming a link with player i'. This constitutes a contradiction. Hence, the proof follows.■

It is easy to see that Lemma 3.11 and 3.12 imply that a strict equilibrium where each component is a singleton has a generalized periphery-sponsored star. Moreover, if the ordered condition does not hold than some player has an incentive to deviate, which contradicts the fact that g is a strict Nash equilibrium. Finally, we note that the agent which does not promote any link is always the player with the lowest access cost in the whole society, i.e. player 1. Suppose, for a contradiction, that player k does not promote any link. Since there exists a player, say j, such that $g_{j,k} = 1$, it follows that $c_k < V$, but then also $c_1 < V$ and therefore player k strictly gains by creating a new link with player 1. This completes the proof of the first part of the proposition.■

Second Part: Let g be the empty network and let $c_i > V$ for any $i \in N$; it is trivial to see that the empty network is a strict equilibrium. We finally consider the case where g has a generalized periphery-sponsored star architecture where player 1 is the agent which does not promote any link. Set the following conditions on the costs of linking: (i) $c_1 < V$, (ii) for any $i \in N$ such that $g_{i,j} = 1$ then $c_i > nV$, (iii) $mV < c_j < (m+1)V$ and (iv) $(x-1)V < c_{j_x} < xV$, for any $x = 2, ..., m$. These conditions imply that: one, each agent who promotes a link obtain a strictly positive utility, two no agent wants to form an additional link. Therefore, given these conditions, whenever g is ordered g is also a strict equilibrium.[17]. This completes the proof of the second part of the proposition.■

The proof of Proposition 3.4 follows.■

[17]Note that if in the generalized periphery sponsored star players are arranged as follow: $g_{y,x} = 1$ for any $y > x$ and $\{g_{x,x-1} = g_{x-1,x-2} = ... = g_{2,1} = 1\}$, then g is always ordered whenever $c_h - c_{h-1} < V$ for any $h = 1, ..., x$. This condition is consistent with conditions (i)-(iii).

Exploitation and cooperation in networks

1 Introduction

Why should I not free-ride on my coauthor in our scientific collaboration? Why should a firm not free-ride on another firm in the collaboration for the development of a new product? What is that drives economic agents not to free-ride in collaboration activities? We show that the presence of network externalities in the individuals interaction process play a crucial role for the formation of stable and efficient collaborations. Individuals invest in connections, taking into account the potential externalities inherent in networks. In turn, externalities shape individuals' incentives to behave efficiently. We also show that the individuals' incentives in their mutual interactions depend on the architecture of the social network, as well as on the position of the individuals within the network.

We examine a framework where the investment in social or economic ties has a long run nature.[1] A collaboration between two players brings benefits and costs to the two parties involved. Benefits result from the potential exchange of some valuable non-rival good, such as information and knowledge. Costs arise because maintaining a collaboration requires exerting effort and spending time. A player may or may not cooperate and free-riding problems characterize the cost side. As an illustration, we consider the following example. Two researchers establish a scientific collaboration. The benefit accruing to each researcher is the possibility of exchanging ideas, opinions and knowledge.

[1] The assumption that social ties have a long run nature captures the idea that the interaction between two acquaintances occurs more frequently than the formation of the relationship itself. For example, once a scientific collaboration is in place, the two parties meet and interact frequently before the project has been completed. Similarly, if two firms form a collaboration for the development of a new product, they typically interact frequently before the collaboration ends.

Part of the knowledge is intrinsic to these two researchers, but part of it is obtained as a result of the interactions with the other agents in the social network. Further, maintaining the relationship is costly, in terms of both effort and time. If the two parties cooperate, the maintenance cost of the link will be lower as compared to the case in which they both free-ride on each other. The increase in the maintenance cost of the link reflects some sort of inefficiency (for example, a delay in the project) which would not occur in case of cooperative behavior. However, given that one of the two parties cooperates, the other would prefer to free ride and save time to develop other projects on his own. In this paper we ask how the individual investments in collaborations may help to overcome free-riding problems.

We start by presenting the main features of the model. There is a finite set of individuals, each of them endowed with some non-rival information. At the beginning of the game players propose collaborations (links) and this generates a network of relationships. Once the network is in place, every pair of linked players interacts for an infinite number of periods. In the interaction phase, each player observes the entire network and the history of actions that each of his social acquaintances has taken in the interaction with him. Neither the actions his acquaintances played with third parties, nor the actions played by non-acquaintances are observable.[2] In each period each pair of acquaintances, say i and j, plays two simultaneous move games: an *accessibility game* and a *prisoner's dilemma game*. In the accessibility game player i (j) either withholds or provides the information he has (his non-rival good) to player j (i). The outcome of this game across all pairs of social contacts determines how information flows in the network and, therefore, defines the benefits accruing to each player in that specific period. Conversely, in the prisoner's dilemma game, player i (j) decides

[2] The assumption that players fully observe the network is realistic when the network represents a physical infrastructure. When a link means a social relationship, it is hard to think that players observe the entire structure of the social network they belong to. We shall show that, for our results to hold, it is enough that each player has local information of the social network.

whether to cooperate or to defect with player j (i). This determines the cost of that particular collaboration and how it is met by the two parties.[3] Hence, a strategy profile specifies the network and the way players act in the interaction phase. An equilibrium is a strategy profile where the proposed network is pairwise stable and the strategy profile is a sequential equilibrium. In the analysis we first characterise efficient outcomes and then focus on efficient equilibria. Let us proceed to discuss our main results.

An efficient outcome is characterised either by the empty network or by any minimally connected network where players provide full accessibility (Theorem 4.1). In this last case two efficient outcomes arise given two distinct ranges of parameters. First, we obtain a *symmetric efficient outcome* where individuals cooperate with their acquaintances to maintain the cost of their relationship. Second, an *asymmetric efficient outcome* where for each pair of linked players, one individual cooperates and bears the entire cost of the link, while the other player free-rides on him. Figure 4.1 depicts two possible efficient configurations in a society with 4 players.

We now turn to explore when symmetric and asymmetric efficient outcomes may arise as a result of strategic considerations. We start by examining the existence of equilibria that sustain the asymmetric efficient outcome (exploitative efficient equilibria). We show that the exploitative efficient equilibrium that exists for the widest range of parameters has the following features. First, the social network has a star architecture. Second, each player i provides information and cooperates with player j if the amount of information player j accesses exclusively from player i is weakly less that the the amount of information player i accesses exclusively from player j (otherwise player i free-rides on player j). In other words, the way a player behaves with respect to his social contact depends on how much valuable for

[3] In particular, whenever two linked players play symmetrically in the prisoner's dilemma game they share evenly the cost of that link (at the defection level if both players defect, or at the cooperative level otherwise); if they play asymmetrically the player who cooperates bears entirely the cost of that link (at the exploitative level).

the former is the relationship with the latter. This equilibrium exists for a discount factor range that becomes wider as the size of the population increases (Theorem 4.2).

Let us comment on the nature of this equilibrium. Firstly, we observe that the asymmetric efficient outcome is strategically best sustained when players are embedded in the star network. The reason is that, in the star architecture, the central player can directly detect any eventual deviation. Furthermore, his structural position allows him to punish any cheater with immediate social isolation, i.e. withholding all the information. Secondly, in the star network, relationships are periphery-sponsored: the central player always free-rides on his social contacts while the spoke players always cooperate. The role of the periphery-sponsored property is to transfer utility from poorly connected players (spoke players) to well connected players (central players). Such a mechanism aligns individuals and social incentives and, therefore, enhances efficiency. Thirdly, we show that the larger the population the more likely that an efficient equilibrium exists. This is due to the fact that, ceteris paribus, the magnitude of the punishments is increasing in the amount of network externalities. These results suggest centrality and periphery-sponsorship to be crucial in order to sustain efficient asymmetric relationships.

We finally explore the existence of equilibria that sustain the symmetric efficient outcome (cooperative efficient equilibria). We show that the cooperative efficient equilibrium that exists for the widest range of parameters has the following features. First, the network has a line architecture. Second, players cooperate and provide information, while they punish deviations in the network stage, as well as in the interaction phase, by defecting and withholding information (cooperative strategy profile). Finally, this equilibrium exists for a wider discount factor range in larger populations (Theorem 4.3).

Let us focus on this last result. First, in sharp contrast with the exploitative case, symmetric efficient relationships are easily sustained in symmetric architectures. Indeed, the line network is the most symmetric network among the class of minimally connected networks. The underlying reason of this result is that individuals' incentives to deviate in the network formation stage are inversely related to the number of connections players have. Second, we observe that, similarly to the exploitative case, when the exchange of information is used strategically, players may credibly threat their social acquaintances by withholding information. This creates more severe punishments as compared to settings where information is not strategic. Therefore, also a relatively impatient society can sustain cooperative efficient equilibria.

In the last few decades a number of empirical studies have shown that network relationships play a crucial role in shaping individuals' behavior in a variety of strategic situations.[4] The present study attempts to examine the interplay between endogenous strategic links formation and stable efficient relationships. Thus, it relates to different strands of the economic literature such as repeated games, theory of networks, and social capital and trust. We will discuss how our paper relates to these different branches of the literature after having presented our main results. The rest of the paper is organised as follows. Section 2 presents the model. Section 3 characterises efficient outcomes. Section 4 and 5 analyze exploitative and cooperative efficient equilibria, respectively. Section 6 reviews the related literature and section 7 concludes. Proofs are shown in the Appendix.

2 The model

There is a finite set of players and each agent is endowed with some non-rival good, to which we will refer as information hereafter. At the beginning of the game, players form an undirected graph (network formation stage). Undirected graphs are used to

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[4] For example, Munshi (2003) shows that network effects are crucial in determining the labor market participation of Mexican migrants in the U.S. Krishnan and Sciubba (2004) shows that informal institutions matters in determining productivity in agriculture in rural Ethiopia. See also Kosfled (2004) for a survey of experimental work on networks.

model the network of relationships among players. A graph is composed by a set of nodes and a set of links. Each node represents a player while each link indicates a bilateral relationship between two players. Once the network is formed, each pair of linked players play an infinitely repeated game, which consists of two simultaneous move games: in game one, players may augment the information they have by means of exchanging it with their social contacts (Accessibility game); in game two, interacting players play a Prisoner's Dilemma game, which defines the cost a player has to pay for each link he has. The strategy of each player is, therefore, two-dimensional, fact that will play a crucial role in our analysis. Let us now introduce the model in a more formal way.

- **Network Formation**

Let $N = \{1, 2, ..., n\}$ be a set of players and let i be a typical member of this set. To avoid trivialities, we shall assume throughout that $n \geq 3$. In period zero, each player i proposes a set of links, i.e. $\omega_i^0 = \left(\omega_{i,1}^0, \omega_{i,2}^0, ..., \omega_{i,n}^0\right)$, where $\omega_{i,j}^0 \in \{0, 1\}$, $\forall j \in N \backslash \{i\}$. If $\omega_{i,j}^0 = 1$ we say that player i wants to form a link with j. A link between two agents, say i and j, is formed if both players agree on it, i.e. $\omega_{i,j}^0 = \omega_{j,i}^0 = 1$. These decisions are summarised in $\omega^0 = (\omega_1^0, \omega_2^0, ..., \omega_n^0)$ and result in an undirected network $g(\omega^0) = (g_1(\omega^0), g_2(\omega^0), ..., g_n(\omega^0))$, where $g_i(\omega^0) = (g_{i,1}(\omega^0), g_{i,2}(\omega^0), ..., g_{i,n}(\omega^0))$, $g_{i,j}(\omega^0) = \omega_{i,j}^0 \cdot \omega_{j,i}^0$ $\forall j \in N \backslash \{i\}$ and $g_{i,i}(\omega^0) = 0$ $\forall i \in N$. When there is no confusion we will use g^0 instead of $g(\omega^0)$. We say that players i and j have a direct link if $g_{i,j}^0 = 1$, otherwise $g_{i,j}^0 = 0$.[5] Let $\mathcal{G}$ be the set of all possible undirected networks on N. For a network $g \in \mathcal{G}$ the set $N_i^d(g) = \{j \in N \diagdown \{i\} : g_{i,j} = 1\}$ defines the bilateral relationships (social contacts) of player i. Let $\mu_i^d(g)$ be its cardinality.

- **Infinitely Repeated Game**

  - Prisoner's Dilemma Game (PDG)

[5] We note that $g_{i,j}^0 = g_{j,i}^0$, $\forall \{i, j\} \in N$.

In any period $t \geq 1$, each pair of acquaintances (i, j) plays a PDG as represented in table 1. Let denote by $\alpha_{i,j}^{t} \in \{C, D\}$ the action chosen by player i in the interaction with j, where C means cooperation and D defection.

$i \backslash j$	C	D
C	c, c	f, e
D	e, f	d, d

Table 4.1

We shall assume, throughout the paper, that $e > c > d > f$, $2d < f$. We normalize, without loss of generality, $e = 0$. We also denote by $\phi_{i,j}^{t}(\alpha_{i,j}, \alpha_{j,i})$ the cost player i faces when interacting with player j (which is represented in table 4.1). In words, each pair of interacting players may either share the cost of the link symmetrically (either at the cooperative level, c, or at the defection level, d) or it is only one of the two players that bears the entire cost of that link at the exploitative level, f.

- Accessibility Game (AG)

In any period $t \geq 1$, each player is endowed with some non-rival information which has a value v. Then, simultaneously to the prisoner's dilemma game, each player $i \in N$ decides either to withhold or provide (at no cost) information to each of the other players $j \in N \backslash \{i\}$ (both his own information and the information he acquires from other agents). We denote by $\lambda_{i,j}^{t} \in \{0, 1\}$ this decision, where $\lambda_{i,j}^{t} = 1$ indicates that player i transmits the information to player j. For example, when the link is a scientific collaboration, an R&D collaboration or a social tie, then providing accessibility means to share ideas, opinions and knowledge. If a link represents a collaboration to construct a physical infrastructure, like a bridge between two cities, then providing accessibility means to allow entry to the city using that bridge.

Let us define $\lambda_{i}^{t} = \left\{\lambda_{i,j}^{t}\right\}_{j \in N \setminus \{i\}}$. The pattern $\lambda^{t} = \Pi_{i \in N} \lambda_{i}^{t}$ determines the flow of information within a network $g \in \mathcal{G}$. The combination of g and λ^{t} results in a directed network $g^{\lambda^{t}}$, that we will refer to as the "flow network", where $g_{i,j}^{\lambda^{t}} = g_{i,j} \cdot \lambda_{j,i}^{t}$. We say that the information flows from j to i if $g_{i,j}^{\lambda^{t}} = 1$; otherwise

$g_{i,j}^{\lambda^t} = 0$. A *flow path* from j to i in g^{λ^t} is denoted by $j \overset{g^{\lambda^t}}{\to} i$, where either $g_{i,j}^{\lambda^t} = 1$ or there exists a sequence of agents $j_1, ..., j_m$, different from i and j, such that $g_{i,j_1}^{\lambda^t} = g_{j_1,j_2}^{\lambda^t} = ... = g_{j_{m-1},j_m}^{\lambda^t} = g_{j_m,j}^{\lambda^t} = 1$. Thus, given a flow network g^{λ^t}, the set of players that i accesses is $N_i\left(g^{\lambda^t}\right) = \left\{ j \in N \backslash \{i\} : j \overset{g^{\lambda^t}}{\to} i \right\}$ with $\mu_i\left(g^{\lambda^t}\right)$ the cardinality of this set. For simplicity, we assume that information flows across links without decay.

The following notation is important to define the strategy profiles of the game. Given g^{λ^t}, for any $g_{i,j}^{\lambda^t} = 1$ the set $\mathcal{I}_{i,j}(g^{\lambda^t}) = \left\{ k \in N : \left(\exists k \overset{g^{\lambda^t}}{\to} i \right) \wedge \left(j \notin k \overset{g^{\lambda^t}}{\to} i \right) \right\}$ indicates the set of players agent j accesses exclusively via a path containing i and $I_{i,j}(g^{\lambda^t}) \cdot v$, where $I_{i,j}(g^{\lambda^t}) \equiv \left| \mathcal{I}_{i,j}(g^{\lambda^t}) \right|$, represents the benefit player j obtains from the specific interaction with player i.[6] The same definition applies to period $t = 0$ once we impose that $g^{\lambda^0} = g$.

Combining the two games, the action space of each player, with respect to each of his social contact, is $A \equiv \{(\alpha, \lambda)\}_{\alpha_j \in \{C,D\}}^{\lambda \in \{0,1\}}$. Let us define $\mathcal{A}_i \equiv A^{n-1}\ \forall i \in N$ and $\mathcal{A} = \Pi_{i \in N} \mathcal{A}_i$. We note that any element in $\mathcal{A}_i$ is a vector of tuples $a_i = (a_{i,j})_{j \in N \backslash \{i\}}$ representing the actions played by agent i with the remaining players in the constituent game of the infinitely repeated game.

- **Strategy Profiles**

We shall focus on pure strategy profiles. We assume that, at each period $t \geq 1$, each player i observes the social network, the past actions of his social contacts in their specific collaboration, and the information received by each of their social contacts, i.e. $a^t(i; g) = \{(a_{j,i}^{\tau}, a_{i,j}^{\tau}), \left(I_{j,i}(g^{\lambda^t}), I_{i,j}(g^{\lambda^t}) \right)\}_{j \in N_i^d(g), \tau \in \{1,...,t-1\}}$. We also assume that players neither observe the behavior of

[6] To illustrate this, in a star network the strategic information of the center with respect to any other player, say j, is $(n-1)v$, while the strategic information of j with respect to the center is v.

their social contacts with third parties, nor the behavior of non-acquaintances. Let $\Psi^t(i;g)$ be the space of observable actions for player $i \in N$.[7] Then, the observed history at period t of player i is $h^t(i) = \{g, a^t(i;g)\}$ and the set of histories of player i at time t is $\mathcal{H}^t(i) \equiv \left\{ \{g, \psi\}_{\psi \in \Psi^t(i;g)} \right\}_{g \in \mathcal{G}}$. We refer to $s = \{\omega^0, \omega^1, ..., \omega^t, ...\}$ as a pure strategy profile of this game; s_i is a pure strategy of player i consisting in a set of link proposals, ω_i^0, and in a sequence of functions, $\omega_i^1, ..., \omega_i^t, ...$, where $\omega_i^t : \mathcal{H}^t(i) \rightarrow \mathcal{A}_i$, $\forall t \geq 1$. Let player $i's$ strategy set be denoted as S_i, and let $S \equiv \prod_{i \in N} S_i$ be the set of pure strategy profiles.
It is important to note that a strategy profile $s = \{\omega^t\}_{t=0}^{\infty}$ results in an undirected network g^0 and in an infinite sequence of directed networks $\{g^{\lambda^t}\}_{t=1}^{\infty}$, one for each period t. In the analysis we will focus on strategy profiles which are *stationary,* i.e. players play the same action on the equilibrium path (at every period).

- **Payoff structure**

We are now ready to define the payoff structure of the game. Given a strategy profile $s = \{\omega^0, \omega^1, ..., \omega^t, ...\}$, the total value generated at each period t, $v^t(s)$, and the utility player i obtains at that period, $u_i^t(s)$, can be written as:

$$\begin{aligned} v^t(s) &= \sum_{i \in N} u_i^t(s), \quad \text{where} \\ u_i^t(s) &= \mu_i\left(g^{\lambda^t}\right) \cdot v + \sum_{j \in N_i^d(g^0)} \phi_{i,j}\left(\alpha_{i,j}^t, \alpha_{j,i}^t\right) \end{aligned}$$

Therefore, the value generated by a strategy profile s in the entire game, $V(s)$, and the utility to player i in the entire game,

[7] With some abuse of notation, we can define in an arbitrary period $t \geq 1$, the space of observable actions for player $i \in N$ as $\Psi^t(i;g) \equiv A^{2(t-1)\mu_i^d(g)} \times \{0, 1, ..., n-1\}^{2(t-1)\mu_i^d(g)}$.

$u_i(s)$, may be represented as

$$V(s) = \sum_{t=1}^{\infty} \delta^{t-1} v(s) = \sum_{i \in N} u_i(s) \tag{4.1}$$

$$u_i(s) = \sum_{t=1}^{\infty} \delta^{t-1} u_i^t(s) \tag{4.2}$$

where $\delta \in (0,1)$ is the common discount factor displayed by all agents. Hence, the utility player i obtains is given by a discounted sum of infinite earnings, derived from the information i accesses, i.e. $\mu_i^t(g^{\lambda^t}) \cdot v$, and the cost player i bears in his interactions, i.e. $\sum_{j \in N_i^d(g^0)} \phi_{i,j}(\alpha_{i,j}, \alpha_{j,i})$.

- **Equilibrium and Efficiency Notions**

We are interested in determining a strategy profile s such that the proposed network $g(\omega^0)$ is stable given the prescriptions of the strategy s in the continuation game, i.e. $\{\omega^t\}_{t=1}^{\infty}$; and that this strategy is a sequential equilibrium (i.e. it prescribes to play an equilibrium for any possible observed network $g' \in \mathcal{G}$ and any history of play in the infinitely repeated game). To determine the stability of the network we use the notion of pairwise stability.[8] Formally, given $\{\omega^t\}_{t=1}^{\infty}$, the network g^0 is pairwise stable if no pair of players wants to form an additional link and no individual player wants to delete any set of his links.[9]

Definition 4.1. *The strategy profile* $s = \{\omega^0, \omega^1, ..., \omega^t, ...\}$ *is an equilibrium if* $\{\omega^1, ..., \omega^t, ...\}$ *is a sequential equilibrium, and* g^0 *is pairwise stable.*

To complete, we define the notion of efficiency.

[8] The concept of pairwise stability has been introduced by Jackson and Wolinsky (1996). In the current paper we will use a modified version of this notion introduced by Goyal and Joshi (2003)

[9] We propose that g^0 must be pairwise stable, instead of requiring ω^0 to be a Nash equilibrium. In the latter case, given $\{\omega^1, ..., \omega^t, ...\}$, we would obtain a multiplicity of equilibria of the type: whenever $\omega_{i,j}^0 = 0 \ \forall \{i,j\} \in N$, then player j is indifferent between setting $\omega_{j,i}^0 = 1$ or $\omega_{j,i}^0 = 0$. The former case avoids this problem and stresses the architecture of the equilibrium network, g^0.

Definition 4.2. *A strategy s is efficient if $V(s) \geq V(\hat{s})$ for any $\hat{s} \in S$.*

We note that if a strategy is socially efficient, it is also Pareto efficient. The reverse does not hold.

3 Efficient outcomes

We start by characterizing the efficient outcomes. We shall then proceed to investigate strategy profiles that sustain efficient equilibria. We will focus on maximal punishment strategy profiles. This will clarify the role networks play on the emergence of efficient social norms, as well as the effect of the latter in shaping the incentives to invest in connections. By considering these effects together, the analysis will also clarify the tension between individual and social incentives.

We first introduce some notation. Given a network g, we say that there is a *path of links* from j to i, denoted as $j \overset{g}{\longleftrightarrow} i$, if either $g_{i,j} = 1$ or if there exists a sequence of players $j_1, ..., j_m$ not including i and j such that $g_{i,j_1} = g_{j_1,j_2} = ... = g_{j_m,j} = 1$. A set $C(g) \subset N$ is a component of g if, for any $i, j \in C(g)$, there is a path between them, and no path exists between any agent in $C(g)$ and $N \backslash C(g)$. A component is *minimal* if there exists only one path between any pair of players $i, j \in C(g)$. A network g is connected if it has a unique component. If a network is connected and its unique component is minimal, we say that it is minimally connected. A player i in a network g is said to be an end-agent if he has a unique link. A network is empty if there are no links across players. A network g has a star architecture and i is the central player if $g_{i,j} = 1, \forall j \in N \backslash \{i\}$, and there are no other links. A line is a minimally connected network where only two end-agents exist. The following result characterises the efficient outcomes.

Theorem 4.1. *Suppose (4.1) and (4.2) hold.* (*a*) *If* $nv + \max\{2c, f\} > 0$ *then* $s = \{\omega^0, \omega^1, ..., \omega^t, ...\}$ is efficient if and only if *the following conditions hold: (i)* g^0 *is minimally connected; (ii) every period each player provides accessibility; (iii)* *every period the total cost for each link is* $\max\{2c, f\}$. (b) *If* $nv + \max\{2c, f\} < 0$, s *is efficient if and only if* g^0 *is the empty network.*

We sketch here the main steps of the proof of Theorem 4.1. First, since exchanging information is costless and it (weakly) increases social welfare, a social planner will prescribe the mutual exchange of information. Second, the no-decay assumption implies that an efficient network is minimal. Third, since linking up two players otherwise disconnected creates positive network externalities an efficient network is either empty or connected. Finally, in any minimally connected network social welfare is maximized whenever the cost for each link is as low as possible.

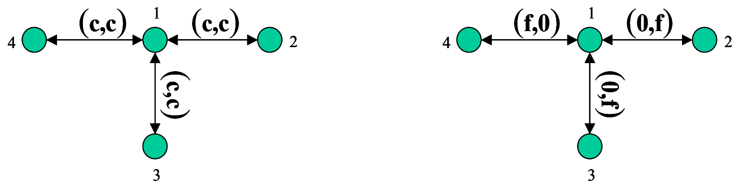
Theorem 4.1 shows that efficiency requires players to form a minimally connected network and to exchange information. Furthermore, two possibilities may arise. First, in some settings, i.e. for $2c > f$, players must mutually cooperate.[10] We will refer to this case as the *symmetric efficient outcome*. It is readily seen that a symmetric efficient outcome generates a social welfare equal to $(n-1)(nv+2c)/(1-\delta)$. Second, in other settings, efficiency requires that, for each link, a player cooperates and bears the entire cost of that link, while the other player free-rides on him. We will refer to this case as the *asymmetric efficient outcome*, which generates a social welfare equal to: $(n-1)(nv+f)/(1-\delta)$.[11]

Figure 4.1 depicts one possible symmetric and asymmetric efficient outcome in a society composed of 4 players. In the figure, a link is represented by an edge connecting two players, whereas an arrowhead pointed to one player indicates that information

[10] For example, assume that in order to maintain the link, agents must exert some effort and that the cost of this effort for a player is convex. In this case, splitting the mainteinance tasks between the two parties is more efficient than letting only one player taking care of them.

[11] For example, when the performance of a task (the maintenance of the link) requires a lot of coordination across players or the opening of different bureaucratic procedures, it may be more efficient to leave the task to be solved unilaterally, than to solve it bilaterally. In the scientific collaboration example, if the two reaserchers belong to two different universities, it is generally more costly for the two researchers to meet at a conference and both of them paying the plane fare, as compared to the case of only one of the two researchers visiting the other one in his own university.

flows in the direction of that player. Finally, the way the cost of a link is met by the two players is indicated with the letters above the edge.



Symmetric Efficient Outcome, 2c>f   Asymmetric Efficient Outcome, 2c<f

Figure 4.1.

In what follows, we shall study the circumstances under which it is possible to strategically sustain the asymmetric and the symmetric efficient outcomes, respectively. We will restrict our analysis to the case in which $\max\{2c, f\} + nv > 0$, which would eventually be the case in large societies, even if the individual cost of a link is high.

4 Exploitative efficient equilibria

We start by examining the existence of equilibria that sustain the asymmetric efficient outcome. We will refer to these equilibria as *exploitative efficient equilibria*. We first analyse a benchmark case where social ties do not play any strategic role in the game. To do this, we assume that the communication of information is not strategic and that the players' behavior in the interaction phase does not depend on the realization of the network stage.[12] The following remark shows that, in this setting, exploitative efficient equilibria do not exist.

Remark 4.1. *Suppose social ties do not play any role and* $2c < f$. *Then an efficient equilibrium does not exist.*

We shall show that, when network externalities are endogenized, this inefficiency fades away and in order to do this we

[12] Recall we are restricting our analysis to stationary strategy profiles.

characterise the efficient exploitative equilibrium existing for the widest range of parameters. This equilibrium is characterised by two features: one, the network has a star architecture; and two, the central player always free rides while the peripheral players always cooperate. Therefore, in equilibrium the player who values more a connection bears its cost.[13] We will first define a strategy profile, called exploitative strategy, which provides the asymmetric efficient outcome. We then show that such strategy sustains the exploitative efficient equilibrium for the widest range of parameters, which is characterised.

The *exploitative strategy* prescribes players to form a set of links, that generates a social network, say g^0. If players observe a network which is different from the prescribed one, they defect and withhold the information. Conversely, if players observe g^0, each agent i provides the information and cooperates with player j if the amount of information player j accesses exclusively from player i is weakly less than the amount of information player i accesses exclusively from player j; otherwise player i free-rides on player j. Furthermore, a deviation in the interaction phase is punished by withholding the information and defecting. It is important to notice that, even if players only observe the behavior of their direct neighbors in their mutual interaction, they can infer deviations of their social contacts with third parties and deviations of non-acquaintances by observing the flow of information they access in their interactions. In other words, even if players have only local information, social punishments may be indirectly implemented. To define the strategy profile formally, we need to introduce some additional notation.

Definition 4.3. *We say that the action taken by player i against j at period t, $a^t_{i,j}$, is well-behaved, WB, with respect to the relative flow of information between i and j, if and only if*

$$a^t_{i,j} = \begin{cases} (C,1) \text{ if } I_{i,j}(g^{\lambda^t}) = I_{i,j}(g^0) \leq I_{i,j}(g^0) = I_{j,i}(g^{\lambda^t}) \\ (D,1) \text{ otherwise} \end{cases}$$

[13] In a different setting, Meléndez-Jiménez (2002) obtains that when two agents bargain on the cost sharing of a link, the agent who values more the link bears a higher part of the cost of the link, and when both value the link equally, they split evenly the cost.

The exploitative strategy profile is then defined as $s^E = \{\omega_i^{E,t}\}_{t=0}^{\infty}$, $\forall i \in N$, where $\omega_i^E = \{\omega_i^{E,t}\}_{t=0}^{\infty}$ is such that for any $\hat{g} \in \mathcal{G}$ and any $j \in N$:

$$\omega_{i,j}^{E,1} = \begin{cases} (D,1) \text{ if } \begin{cases} \hat{g} = g^0 \\ I_{i,j}(g^0) > I_{j,i}(g^0) \end{cases} \\ (C,1) \text{ if } \begin{cases} \hat{g} = g^0 \\ I_{i,j}(g^0) \leq I_{j,i}(g^0) \end{cases} \\ (D,0) \text{ otherwise} \end{cases}, \text{ and } \forall t \geq 2:$$

$$\omega_{i,j}^{E,t} = \begin{cases} (D,1) \text{ if } \begin{cases} a_{k,i}^{t-1} \text{ and } a_{i,k}^{t-1} \text{ are } WB, \ \forall k \in N_i^d(g^0) \\ I_{i,j}(g^0) > I_{j,i}(g^0) \end{cases} \\ (C,1) \text{ if } \begin{cases} a_{k,i}^{t-1} \text{ and } a_{i,k}^{t-1} \text{ are } WB, \forall k \in N_i^d(g^0) \\ I_{i,j}(g^0) \leq I_{j,i}(g^0) \end{cases} \\ (D,0) \text{ otherwise} \end{cases}$$

We are now ready to provide the main result of this section.[14] Let us denote $\bar{n} = \frac{(v-2d)(d-f)}{v^2} + 1$.

Theorem 4.2. *Suppose (4.1) and (4.2) hold and assume* $2c < f$. *An efficient equilibrium exists if and only if* (i) $f + (n-1)v \geq 0$ *and* (ii) $\delta \geq \frac{d-f}{(n-1)v}$. *Furthermore, if* $n > \bar{n}$ *then the unique network which constitutes an efficient equilibrium for the widest range of parameters is the star network.*[15]

The proof of the theorem is based on Lemmas 4.1 and 4.2 which are provided in the appendix. We sketch here the main arguments. In Lemma 4.1 we start by showing that the exploitative strategy sustains the exploitative efficient equilibrium for the widest range of parameters in a star network. The reason is that in the star network the center punishes any deviations directly by withholding all the information; this implies that any cheater would be socially isolated just after one period of his deviation. On the contrary, in any other minimally connected network there exists some player who could deviate and yet enjoy

[14] It is worth noticing that the exploitative strategy profile prescribes to play the Nash equilibrium $(D, 0)$ out-of-equilibrium path. Thus, to define the existence conditions of a sequential equilibrium it is enough to focus on individuals' incentives in the equilibrium path.

[15] We note that even if $n < \bar{n}$, the star network is an efficient equilibrium for the widest range of parameters, but it is not the unique one.

some information for some period after the deviation. Next, conditions (i) and (ii) follow by solving the equilibrium conditions for the exploitative strategy profile when the initial network has a star architecture.

In Lemma 4.2 we first show that both conditions (i) and (ii) are necessary conditions for the existence of an asymmetric efficient equilibrium. It is clear that if condition (i) is violated a minimally connected network cannot be pairwise stable. Suppose condition (ii) is violated; we show that to obtain an asymmetric efficient outcome it must be the case that either there exists an end-agent who entirely bears the cost of its relationship or there exists a non end-agent player who pays for all the links he has. In both cases these players strictly gain by defecting. Finally, we prove that, if the population is large enough $(n > \bar{n})$, when conditions (i) and (ii) bind only the exploitative strategy profile can sustain an asymmetric efficient equilibrium.

On the existence region of exploitative efficient equilibria, two remarks are worth being made. First, both conditions (i) and (ii) are weaker the higher the size of the population is. Asymptotically, when the size of the society tends to infinity, an exploitative efficient equilibrium always exists. This is due to the fact that a large connected population produces a large amount of network externalities, which in turn increase the magnitude of the punishments. Thus the conflict between individual and social incentives is less severe in large societies.

Second, we also note that the pairwise stability notion leads to an equilibrium condition (condition (i)), that is invariant with respect to the architecture of the network, provided that each player pays at most the cost of one link. Therefore, the pairwise stability notion does not undermine the architecture of the network. On the contrary, the requirement of sequential equilibrium crucially depends on the architecture of the network. This implies that individuals' incentives during the interaction phase are embedded in the network of collaborations.

The second set of observations concerns the nature of the efficient exploitative equilibria. Here, we emphasize three remarks. First, the star architecture allows more likely to strategically sustain the asymmetric efficient outcome than other minimally connected networks. The reason is that, in the star network, the central player has the ability to maximally punish deviations. Second, centrality is accompanied by the periphery-sponsored property: the cost of each link is unilaterally met by the player who values it more, i.e. the peripheral player of that specific interaction. Here, the role that periphery-sponsorship plays is to transfer utility from poor connected players to well connected players. These findings suggest that centrality and periphery-sponsorship are crucial structural properties to maintain stable and efficient asymmetric relationships.

Third, we note that exploitative efficient equilibria exhibit hierarchical structures. The following example illustrates this point. Consider a society composed of 9 players, arranged in the network depicted in Figure 4.2, who follow the exploitative strategy profile.

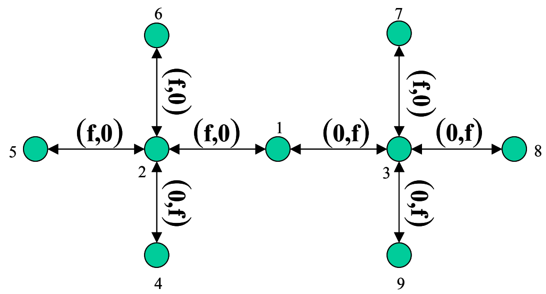


Figure 4.2.

Three types of players emerges. Player 1 is the *exploitative player* in the sense that he free-rides on every of his neighbors (players 2 and 3). The role of the exploitative player is to connect two star components, that would be otherwise disconnected. Players 2 and 3 are hybrid players in the sense that they cooperate with the central player while exploiting all the social contacts they have. The remaining players (end-agents) are always exploited. Note that hybrid players have higher incentives to deviate than exploited players because, as compared to the end-agents, hybrid players could defect with their exploiter (the center), and yet they would enjoy the information from the end-agents for one period after that deviation occurs.

5 Cooperative efficient equilibria

We now explore the existence of equilibria that sustain the symmetric efficient outcome. We shall refer to these equilibria as cooperative efficient equilibria. Similarly to the previous section, we start by analysing a benchmark case where social ties do not play any strategic role. In such cases the trade-off between individual and social incentives is substantial.

Remark 4.2. *Suppose that social ties do not play any role and that* $2c > f$*. An efficient equilibrium exists if and only if* $v + c \geq 0$ *and* $\delta \geq c/d$

Given a minimally connected network, the continuation game degenerates, for each period $t \geq 1$, in a finite number of bilateral prisoner's dilemma games that are strategically independent. As a consequence, the equilibrium condition to sustain mutual cooperation in any link is obtained by applying the standard Folk theorem, i.e. $\delta \geq c/d$. Once players are sufficiently patient, the condition $v + c \geq 0$ is necessary and sufficient so that a minimally connected network is pairwise stable (in this case the critical agent would be the one linked with an end-agent).

Let us now examine the effect of players being allowed to use their information strategically. We shall show that in this case the conflict between individual and social incentives is less severe. As in the previous section, we analyse the cooperative efficient equilibrium existing for the widest range of parameters. This equilibrium is characterised by two features: first, the network has a line architecture; second, every player cooperates.

We start by introducing the cooperative strategy profile. This strategy prescribes players to form a set of links, generating a network of relationships, say g^0. In the interaction phase, each player defects and withholds his information whenever a network, which is different from the prescribed one, is observed. Otherwise, each player cooperates and provides the information to his neighbors, while he punishes any eventual deviation by defecting and withholding the information, in every period onwards, with respect to all his links. Also in this case, each player directly detects the deviations of the social contacts in his interactions, and indirectly uncovers, via the information flow, the deviations of his social contacts with third parties and the deviations of non-acquaintances.

Formally, let us define the cooperative strategy profile as $s^C = \{\omega_i^0, \omega_i^{C,1}, ..., \omega_i^{C,t}, ...\}_{i\in N}$, where $\omega_i^{C,t} = \left\{\omega_{i,1}^{C,t}, ..., \omega_{i,n}^{C,t}\right\}$ is such that for any $\hat{g} \in \mathcal{G}$, any $j \in N$ and any $k \in N_i^d(g)$:

$$\omega_{i,j}^{C,1} = \begin{cases} (C,1) \text{ if } \hat{g} = g^0 \\ (D,0) \text{ otherwise} \end{cases} \text{ and } \forall t \geq 2:$$

$$\omega_{i,j}^{C,t} = \begin{cases} (C,1) \text{ if } \begin{cases} a_{k,i}^{t-1} = a_{i,k}^{t-1} = (C,1) \\ (I_{k,i}(g^{\lambda^{t-1}}), I_{i,k}(g^{\lambda^{t-1}})) = (I_{k,i}(g^0), I_{i,k}(g^0)) \end{cases} \\ (D,0) \text{ otherwise} \end{cases}$$

The Theorem below provides the main result of this section.[16] Let δ^* be the solution of the following equation $\delta^*[v + (n-2)v\delta^* + c - d - d\delta^*] + c = 0$.

Theorem 4.3. *Suppose (4.1) and (4.2) hold and assume* $2c > f$. *An efficient equilibrium exists if and only if* (i) $(n-1)v+2c \geq 0$ *and* (ii) $\delta \geq \delta^*$. *Furthermore, the unique network, which is part of the cooperative efficient equilibrium that exists for the widest range of parameters, is the line network.*

The Theorem follows from Lemmas 4.3 and 4.4 presented in the appendix. In Lemma 4.3 we show that the cooperative strategy profile sustains the cooperative efficient equilibrium which exists for the widest range of parameters when the initial network has a line architecture. To prove this we first observe that, given that players follow the cooperative strategy in the interaction phase, in the network formation stage players' incentives to deviate are increasing in the number of links they have. Furthermore, we show that in any other network different from the

[16] It is worth noticing that the cooperative strategy profile prescribes to play the Nash equilibrium $(D, 0)$ out-of-equilibrium path. Thus, to define the existence conditions of a sequential equilibrium it is enough to focus on individuals' incentives in the equilibrium path.

line, in the interaction phase players have at least the same incentives to deviate as compared to players embedded in a line network. The equilibrium conditions (i) and (ii) are obtained by imposing pairwise stability and sequential rationality, respectively. In Lemma 4.4 we show that for any initial minimally connected network the cooperative strategy profile prescribes maximal punishments, which completes the proof.

Let us now comment on some aspects of the results presented in Theorem 4.3. First, observe that with endogenous network externalities cooperative efficient equilibria exist for parameter ranges where they would not exist, otherwise (see Remark 4.2). The reason is that players may credibly commit to provide the information, conditionally on having inferred that each player has cooperated in his interactions. Similarly to the exploitative case analysed in the previous section, network externalities are higher in a larger society, thus enhancing efficiency. The second observation concerns the nature of the equilibrium existing for the widest range of parameters. Figure 4.3 illustrates this equilibrium in a society composed of 4 players.

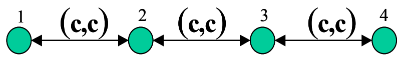


Figure 4.3.

The main feature of this equilibrium is that players are embedded in a very symmetric network: the line network. This suggests that a symmetric distribution of connections across players (fairly compatible networks) is crucial to sustain symmetric efficient collaborations.

6 Related literature

The current paper provides a theoretical account of the effects of endogenous network externalities in the strategic formation of informal relationships and individuals' incentives. Our paper relates to three strands of the economic literature, that is network formation, cooperation in repeated games and social capital and trust. We will refer to each of them in turn.

The first contribution refers to the body of literature which studies the interplay between strategic partnering and economic and social structure, e.g. Goyal and Moraga-Gonzalez (2001), Goyal and Joshi (2003), Bloch and Jackson (2004). The general structure of these games is that first individuals form connections with others and this determines a network of relationships. Then, individuals choose actions and their final payoffs to an individual are shaped by his or her location in the networks. The model we have developed fits in this class of games. However, while the existing literature generally focuses on finite games, we analyse an infinitely repeated games. This allows us to investigate the emergence of efficient social norms. Remarkably, we show that in equilibrium there is a substantial overlap between individual and social incentives.

Secondly, our paper contributes to the literature on cooperation in repeated games. To some extent it relates to the work of Kandori (1992) and Ellison (1994). They analyse a setting where players, belonging to a community, are repeatedly and randomly matched to play a Prisoner's dilemma game. The main result is that cooperation can be sustained if players react to a deviation by punishing subsequent partners. Therefore, the community has a positive effect in the enforcement of cooperative behavior: to free-ride on one player causes sanction by others. The main difference in our approach is that players' interaction takes place in a fixed pattern of play (the community is endogenously structured) and this allows us to investigate the effect of network externalities on the enforcement of efficient long-run and stable relationships.

A paper which shares the same spirit of ours is Haag and Lagunoff (2000), which analyse a repeated prisoners' dilemma game played in a given social networks. Agents have a different discount factor and the authors investigate the network architectures which support a maximum degree of cooperation. Under some restrictions on strategies they show that a desirable structure is one where there is a clique of patient players who cooperates with all their neighbors. Each of these cooperators is linked with a limited set of impatient players, who in turn are free-rides. This pattern can be sustained because of the high level of patience of cooperators. Differently, we consider that players have homogenous discount factor and we analyse the incentive of individuals to cooperate and to free-ride in endogenously formed networks.

Finally, we relate our paper to the theory of social capital and trust.[17] Social capital is a relational concept and its existence is inherent to socioeconomic networks. Social capital affects individuals' behavior as well as aggregate economic phenomena. An individual player can use his social capital, which depends on the nature of his connections, to obtain private economic gains. From a societal perspective, social capital represents the basis of trust in repeated interactions. Sociologists have widely studied this subject. Coleman (1988) emphasizes the role of redundant links for the emergence of trust introducing the so-called *closure* argument. Consider, for simplicity, a society with three players. Coleman (1988) argues that social capital is higher when players are embedded in a cycle network as compared to a star network. The reason is that in a cycle players can monitor rivals' deviations more efficiently than in a star: in a cycle if a friend cheats on me I could communicate this to a common friends (in the star there are no common friends), which would eventually react by punishing the cheater.

By contrast, Burt (1992) emphasizes the importance of non-redundant connections introducing the so-called *structural hole* argument. Structural holes are players who connect networks by linking different components, which would be otherwise disconnected. These players on the one hand integrate additive sources of information, which in turn increases the value generated by the network and on the other hand they may use strategically their structural position to obtain private gains. These two theories should be seen as complementary: the closure argument explains how the benefits of network externalities can be realized in a community, while the structural hole argument ex-

[17] See Sobel (2002) for an extensive discussion on the notion of social capital.

plains how network externalities come about, and the structural properties of players who are crucial for the emergence of these externalities.

In economics, the notion of social capital has been mainly used to study issues related to economic development, criminality and education.[18] However, a theoretical analysis of how social capital emerges is still at a preliminary stage. The first attempt to address this issue is Vega-Redondo (2002), who provides a strategic foundation of the closure argument of Coleman (1988). Agents interact according to a collection of infinitely repeated prisoner's dilemma games played on the current social network. The strategic effects of networks result from the fact that players can communicate via their links behavioral information about their acquaintances. This allows for the formation of stable and dense networks in which players can monitor efficiently other players' behavior and this mitigates the incentives to free-ride. By contrast, in our model social capital emerges because players invest in connections that generates network externalities, and players use these externalities to punish possible deviations. Our paper can be seen as an attempt to provide a microeconomics foundation of the notion of structural holes.[19]

7 Conclusion

Free-riding problems are often solved in many economic and social interactions. We have shown that when we take into account that individuals interact non-anonymously there is a substantial overlap between individuals and social incentives. We now discuss how the results presented in the paper are robust to the main assumptions. We first elaborate on the impact of relaxing the assumption that players observe fully the structure of the network. It is easy to see that for our results to hold it is enough

[18] See Glaeser, Laibson and Sacerdote (2002) for some empirical evidence on the effect of social capital on socioeconomic outcomes. See Dasgupta and Sarageldin (1999) for a discussion of the main contributions on social capital in economics.

[19] See Goyal and Vega-Redondo (2004) for a strategic model on structural-holes.

that players are aware about the connections held by their direct social contacts and the information their social contacts can potentially provide. Indeed, this information is sufficient to allow players to employ the strategy profiles used in our analysis.

Second, we have shown that efficient equilibria are best sustained when the size of the society is large. It is worth noticing that this result holds as far as the payoffs functions are increasing in the amount of information a player accesses. Thus, the fact that we consider linear payoff functions is not crucial. A weaker case can be made if we relax the assumption of frictionless information flow. Suppose we assume that non-direct information is slightly less valuable than direct information (information flow with decay), then an efficient outcome will be a star network where players provide information and the cost of each link is either shared at the cooperative level (when 2c>f) or it is borne unilaterally at the exploitative level (otherwise). In the former case, since the central player bears the cooperative cost for each link, the existence conditions for a cooperative efficient equilibrium will be independent of the size of the population. On the contrary, in the latter case, the results presented in section 4 carry on qualitatively.

Third, we have explored a model where investment in links is sunk and players cannot change their network over time. We note that the strategy profiles use also apply on a repeated game where the network formation and players' interaction occurs simultaneously. Finally, we elaborate on the possibility of time-preference heterogeneous players. Our analyses shows that in equilibrium players having different position in the networks have different incentives. Therefore it is not crucial that players discount the features evenly. A formal analysis of these topics is left for future research.

8 Appendix chapter 4

Efficient Outcomes

We start by proving Theorem 4.1, which characterizes the efficient outcomes of the game.

Proof of Theorem 4.1. We start by proving part (a), i.e. $nv + \max\{2c, f\} > 0$. We first claim that if s is efficient then conditions (i)-(iii) hold. First, the requirement that g^0 is minimal follows from the no-decay assumption. Second, we note that given a minimal network, to provide information strictly increases social welfare. Thus, condition (ii) follows. Third, condition (iii) assures that the cost of each link at any period is minimized. Fourth, we note that g^0 must be connected. For a contradiction let assume that g^0 is minimal but not connected. Consider an end-agent belonging to a component $C(g^0)$ of cardinality $k > 1$; the social welfare produced by the link with the end-agent is $2(k-1)v + \max\{2c, f\} \geq 0$, which is positive since, by assumption, s is efficient. Let us consider a strategy $\hat{s}$ which prescribes a network $\hat{g}^0$, which differs from the original network in the fact that $\hat{g}^0$ has an additional link, say between i and j, where $i \in C(g^0)$, $j \in N \backslash C(g^0)$, and the information is exchanged in the new link. It is readily seen that $V(\hat{s}) - V(s) \geq 2kv + \max\{2c, f\}$. Since $2kv + \max\{2c, f\} > 2(k-1)v + \max\{2c, f\} \geq 0$, it follows that $V(\hat{s}) - V(s) > 0$. This contradicts the fact that s is efficient. Hence, the claim follows. We now observe that any minimally connected network in which conditions (ii) and (iii) are satisfied produces a social welfare equals to $V(s) = \frac{(n-1)(nv + \max\{2c, f\})}{1-\delta} > 0$. This proves the part (a) of the Theorem. Part (b) follows trivially. Hence, the proof is completed.■

Exploitative Efficient Equilibria

Proof of Theorem 4.2. The proof of the Theorem is based on the next two lemmas. Let $\bar{n} = \frac{(v-2d)(d-f)}{v^2} + 1$.

Lemma 4.1 *Suppose (4.1) and (4.2) hold and assume $2c < f$. The strategy profile s^E is an efficient equilibrium if and only if* *(i)* $\delta \geq \frac{d-f}{(n-1)v}$ *and (ii)* $(n-1)v + f \geq 0$. *Further, if* $n > \bar{n}$ *then the strategy* s^E *is an efficient equilibrium for the widest range of parameters when* g^0 *is a star network.*

Proof. Consider the strategy $s^E = \left\{\omega^0, \omega^{E,1}, ..., \omega^{E,t}, ...\right\}$ where g^0 is minimally connected. We first observe that, given s^E, in order to obtain an asymmetric efficient outcome, we need to focus on minimally connected network where $I_{i,j}\left(g^0\right) \neq I_{j,i}\left(g^0\right)$ for any $g_{i,j} = 1$. Next, we observe that, given s^E, in any minimally connected network each player pays at most the cost of one link at the exploitative level. It is readily seen that g^0 is pairwise stable if and only if $(n-1)\,v + f \geq 0$.

Second, we analyse the conditions for the discount factor δ (relative to the interaction stage). We start by noticing that, since as soon as players realize a deviation via the information flow they reverse their behavior to the Nash equilibrium $(D, 0)$, it follows that s^E is optimal, regardless of the players' beliefs. This implies that to determine the parameter conditions for a sequential equilibrium we simply need to focus on the players' incentives on the equilibrium path.

Third, let us assume that g^0 is the star network. Here, we start by noticing that the central player, say j, does not have any incentives to deviate from s^E, since he obtains the maximum achievable payoff in this game, i.e. $u_j^f\left(s^E\right) = \frac{(n-1)v}{1-\delta}$. Furthermore, every agent $i \in N \backslash \{j\}$ faces the same problem; select then an arbitrary player i in this set. Next, we show that i does not deviate if and only if $\delta \geq \frac{d-f}{(n-1)v}$. To see this we note that the utility agent i obtains following the strategy s^E is $u_i\left(s^E\right) = \frac{(n-1)v+f}{1-\delta}$, and the utility if he deviates is $u_i^d\left(s_i^d, s_{-i}^E\right) = (n-1)\,v + \frac{d}{1-\delta}$. Therefore an equilibrium requires that $u_i\left(s^E\right) \geq u_i^d\left(s_i^d, s_{-i}^E\right)$, which is satisfied if and only if $\delta \geq \frac{d-f}{(n-1)v}$. Hence, if $(n-1)\,v + f \geq 0$ and $\delta \geq \frac{d-f}{(n-1)v}$ the strategy s^E where g^0 is a star network is an equilibrium. Differently, if g^0 is not a star network, we note that there always exist at least two end-agents; it is readily seen that an end-agent does not deviate from s^E only if $\delta \geq \frac{d-f}{(n-1)v}$.

Fourth, we show that if $n > \bar{n}$ the strategy s^E is an equilibrium for the widest range of parameters only if g^0 is a star network. Suppose not, then when $n > \bar{n}$, $(n-1)v + f \geq 0$ and $\delta = \frac{d-f}{(n-1)v}$, the strategy s^E is an equilibrium for some minimally connected network g^0 different from the star. We note that in any minimally connected network g^0 it exists an agent, say j, who has k links with k end agents $(k \geq 1)$ and one additional link with a non end-agent, i.e. $\mu_j^d(g^0) = k+1$. The utility this player obtains following the strategy is $u_j(s^E) = \frac{(n-1)v+f}{1-\delta}$. Assume player j deviates with the k end-agents; the utility from such deviation is $u_j^d(s_j^d, s_{-j}^E) = (n-1)v + \frac{d}{1-\delta} + kv\delta + \frac{kd\delta^2}{1-\delta}$. Since g^0 is part of an equilibrium, it must be the case that the incentives to deviate of an arbitrary end-agent i are weakly higher than the incentives of player j, i.e. $u_i^d(s_i^d, s_{-i}^E) \geq u_j^d(s_j^d, s_{-j}^E)$.[20] This is satisfied if and only if $(n-1)v + \frac{d}{1-\delta} \geq (n-1)v + \frac{d}{1-\delta} + kv\delta + \frac{kd\delta^2}{1-\delta}$, which can be rewritten as $v - \delta(v-d) \leq 0$. We now note that when $\delta = \frac{d-f}{(n-1)v}$ the condition $v - \delta(v-d) = v - \frac{(d-f)(v-d)}{(n-1)v} \leq 0$ if and only if $n \leq \frac{(d-f)(v-d)}{v^2} + 1$. Since $\frac{(d-f)(v-d)}{v^2} + 1 < \bar{n}$, this contradicts the assumption that $n > \bar{n}$. Hence if $n > \bar{n}$ the star network uniquely allows the strategy s^E to be an equilibrium for the widest parameter range.

This completes the proof of the Lemma.■

We now prove that, given any strategy profile an efficient equilibrium exists only if the conditions (i) and (ii) in Theorem 4.2 are satisfied and that given that these conditions are binding then a strategy $s = \{\omega^0, \omega^1, ..., \omega^t, ...\}$ different from the exploitative strategy profile is not an efficient equilibrium.

Lemma 4.2 *Suppose (4.1) and (4.2) hold and assume* $2c < f$. *An efficient equilibrium exists only if* (i) $(n-1)v + f \geq 0$ *and* (ii) $\delta \geq \frac{(d-f)}{(n-1)v}$. *Further, given that* $(n-1)v + f = 0$ *and* $\delta = \frac{(d-f)}{(n-1)v}$, *if* $n > \bar{n}$ *every strategy* $s = \{\omega^0, \omega^1, ..., \omega^t ...\}$ *different from the exploitative strategy is not an efficient equilibrium.*

Proof. Let $f > 2c$. Assume that the outcome of $s = \{\omega^0, ..., \omega^t, ...\}$

[20] This is true because the utility to player i and j by following the strategy profile s^E coincides.

is efficient, i.e. g^0 is a minimally connected network, all links are paid at the exploitative level and there is complete flow of information. We first claim that s is pairwise stable only if $(n-1)v + f \geq 0$. In order to get the lower bound condition to attain pairwise stability we shall assume that the strategy s prescribes a maximal punishment in the network formation stage, i.e. if a network $\hat{g} \neq g^0$ is observed then agents play $(D, 0)$ in all the interactions. Then, given that players follow s in the interaction phase the payoff of an agent when the network g^0 is formed is $\frac{(n-1)v+\mu_i^{d,p}(g^0,s)f}{1-\delta}$, where $\mu_i^{d,p}(g^0, s) \leq \mu_i^d(g^0)$ represents the links agent i pays (at the exploitative level) given s. The best deviation of player i in the network formation stage would be to delete all his links and obtain a payoff 0. Hence the condition for pairwise stability is obtained when $(n-1)v + \max_{i \in N} \mu_i^{d,p}(g^0, s) f \geq 0$. Given s, for any g^0, $\min\{\max_{i \in N} \mu_i^{d,p}(g^0, s)\} \geq 1$, thus, s is pairwise stable only if $(n-1)v + f \geq 0$ and the claim follows.

Second, we claim that s is a sequential equilibrium only if $\delta \geq \frac{d-f}{(n-1)v}$. We note that, in order to get a lower bound on δ, we shall consider that the strategy profile s involves maximal punishments in case of any deviation in the interaction stage. Such strategy should prescribe that if at some period t an agent $i \in N$ deviates in his interaction with $j \in N_i^d(g^0)$ then j plays $a_{j,i}^{\tau} = (D, 0)\ \forall \tau \geq t+1$. Therefore, given s, two possibilities may occur: (1) there is at least one end agent, say j, who pays for his link, and (2) no end-agent pays for his link. We start considering case (1); the payoff to the end-agent j, if he follows the strategy is $u_j(s) = \frac{(n-1)v+f}{1-\delta}$, and the payoff if he deviates in his interaction is $u_j^d\left(s_j^d, s_{-j}\right) = (n-1)v + \frac{d}{1-\delta}$. Agent j does not want to deviate whenever $\delta \geq \frac{d-f}{(n-1)v}$. Hence, in this case, our claim follows.

We now consider case (2), i.e. no end-agent pays for his link. Let $E_0(g^0) \in N$ represent the set of end-agents in g^0. We claim that there exists some player $i \in M_0(g^0) \equiv N \backslash E_0(g^0)$ who is paying all his direct links. Assume for a contradiction that no agent pays for all his links, i.e. $\mu_i^{d,p}(g^0, s) < \mu_i^d(g^0)\ \forall i \in N$.

Since the end-agents are not paying for their links, any agent $k_0 \in M_0(g^0)$ linked to an agent $j_0 \in E_0(g^0)$ is paying for the link $\{j_0, k_0\}$. This implies that in case g^0 has a star architecture, the center pays for all his direct links; this is a contradiction. Therefore, let g^0 be a minimally connected network different from the star. Let $g^{0,1}$ be a network obtained by removing from g^0 all agents belonging to $E_0(g^0)$ and their corresponding links. We note that since g^0 is minimally connected, also $g^{0,1}$ is minimally connected. Let $E_1(g^0) \in M_0(g^0)$ be the set of end-agents in $g^{0,1}$. We note that each player $j_1 \in E_1(g^0)$ had some link with some end-agent in g^0 and he was paying for that particular link; since j_1 is an end-agent in $g^{0,1}$ and no agent pay for all his links in g^0, it follows that j_1 does not pay for the link in $g^{0,1}$, i.e. there exists some player $k_1 \in M_1(g^0) \equiv M_0(g^0) \backslash E_1(g^0)$ linked to some agent $j_1 \in E_1(g^0)$ and such that k_1 pays for the link $\{j_1, k_1\}$.

We can proceed with the same reasoning defining the network $g^{0,2}$ as the resultant network from removing from $g^{0,1}$ all agents in $E_1(g^0)$ and their corresponding links. We note that $g^{0,2}$ is also minimally connected. Let $E_2(g^0) \in M_1(g^0)$ be the set of end-agents in $g^{0,2}$. Since each $j_2 \in E_2(g^0)$ had some link with some end-agent in $g^{0,1}$, he was paying for that particular link and this agent just have one link in $g^{0,2}$ it follows that any agent $k_2 \in M_2(g^0) \equiv M_1(g^0) \backslash E_2(g^0)$ linked to an agent $j_2 \in E_2(g^0)$ is paying for that link $\{j_2, k_2\}$. Since the number of players is finite, by induction we obtain that at some finite iteration period τ, the cardinality of the set $M_\tau(g^0)$ is either 1 or 2. Consider the case $M_\tau(g^0) = \{i_1\}$. Then this agent pays for all his links, which contradicts our initial assumption. Now consider the case $M_\tau(g^0) = \{i_1, i_{,2}\}$; note that i_1 and i_2 must be necessarily linked in g^0 and, therefore, one of these agents pays for all his links, a contradiction. This proves the claim.

This claim implies that there exists some player i such that $\mu_i^{d,p}(g^0, s) = \mu_i^d(g^0)$. Since player i is not an end agent in g^0, $\mu_i^d(g^0) \geq 2$. The payoff of such agent from following the strategy s is $u_i(s) = \frac{(n-1)v + \mu_i^d(g^0)f}{1-\delta}$, and the payoff from deviating in all his interactions, s_i^d, is $u_i\left(s_i^d, s_{-i}\right) = (n-1)\,v + \frac{\mu_i^d\left(g^0\right)d}{1-\delta}$. For an equilibrium it must be the case that $u_i\,(s) \geq u_i\left(s_i^d, s_{-i}\right)$, i.e. $\delta \geq \frac{\mu_i^d\left(g^0\right)(d-f)}{(n-1)v}$. We observe that since $\mu_i^d\left(g^0\right) \geq 2$, $\frac{\mu_i^d\left(g^0\right)(d-f)}{(n-1)v} > \frac{d-f}{(n-1)v}$. The argument developed so far shows that conditions (i) and (ii) are necessary for a strategy s to be an efficient equilibrium.

We now prove that if these two conditions are binding and $n > \bar{n}$, any strategy s different from s^E is not an efficient equilibrium. Assume $s = \{\omega^0, ..., \omega^t, ...\}$ is an efficient equilibrium; we start by noting that since $(n-1)\,v + f = 0$ and s is an efficient equilibrium it must be the case that each player $i \in N$ pays at most for one link, i.e. each player cooperates with at most one of his social contacts. Next, consider now an arbitrary pair of players, say i and j, who are directly linked, $g_{i,j}^0 = 1$, and, without loss of generality, let us assume that $I_{i,j}\left(g^0\right) > I_{j,i}\left(g^0\right)$. We have two possibilities, which we analyse in turn.

I.) Suppose the strategy profile s prescribes in the equilibrium path that player j cooperates and player i defects. In this case, since s^E and s are equivalent in the equilibrium path and s^E prescribes maximal punishments for every deviation which eventually occurs, it follows that the incentive of player j (i) to follow s with i (j) cannot be higher than to follow s^E. In this case, we can use Lemma 4.1 to prove the claim.

II.) Suppose that the strategy profile s prescribes in the equilibrium path that player i cooperates and player j defects. The utility of player i to follow s is $u_i\,(s) = \frac{(n-1)v+f}{1-\delta}$. If player i deviates (using his best deviation) against player j at some period t, the utility he obtains in the continuation game is $u_i^d\left(s_i^d, s_{-i}\right) = (n-1)\,v + \left(n-1-I_{j,i}\left(g^0\right)\right)v\delta + \frac{d}{1-\delta} + \frac{\left(\mu_i^d-1\right)d\delta^2}{1-\delta}$. Now we claim that player i have always incentives to deviate (for $\delta = \frac{d-f}{(n-1)v}$). To see this, let us consider the case where player i would have the lowest incentives to deviate,[21] i.e. $I_{j,i}\left(g^0\right) = n/2$

[21] Note that player i's incentives to deviate are decreasing in $I_{j,i}\left(g^0\right)$, because it represents the amount of information he looses when deviating (with a lag of only one period), and these incentives are also decreasing in μ_i^d, since when player i deviates at

and $\mu_i^d = n - 2$.[22] In this case, the utility of the best deviation of player i becomes $\bar{u}_i^d\left(s_i^d, s_{-i}\right) = (n-1)\,v + \left(\frac{n-2}{2}\right) v\delta + \frac{d}{1-\delta} + \frac{(n-2)d\delta^2}{1-\delta}$. We now note that $u_i(s) \geq \bar{u}_i^d\left(s_i^d, s_{-i}\right)$ if and only if $(n-1)\,v\delta - \left(\frac{n-2}{2}\right) v\delta\,(1-\delta) - (n-2)\,d\delta^2 > d - f$. Since $\delta = \frac{d-f}{(n-1)v}$, we can rewrite this condition as $-v + \delta(v-2d) = -v + \frac{d-f}{(n-1)v}(v-2d) > 0$, which is satisfied if and only if $n < \bar{n}$. This contradicts the fact that $n > \bar{n}$ and completes the proof of the Lemma.■

The two lemmas prove the Theorem.■

Cooperative Efficient Equilibria

Proof of Theorem 4.3. We first observe that the cooperative strategy profile prescribes players to play the Nash equilibrium $(D, 0)$ in any possible out-of-equilibrium path. This implies that, to define the existence conditions of a sequential equilibrium, we just need to focus on individuals' incentives in the equilibrium path. Using this fact, the proof of the theorem is based on two Lemmas which are stated and proved below.

Lemma 4.3. *Suppose (4.1) and (4.2) hold and assume* $2c > f$. *The cooperative strategy profile,* $s^C = \left\{\omega^0, \omega^{C,1}, ..., \omega^{C,t}, ...\right\}$ *is an equilibrium for the widest range of parameters when* g^0 *is a line network. In such case,* s^C *is an equilibrium if and only if* $(n-1)\,v + 2c \geq 0$ *and* $\delta \geq \delta^*$, *where* $\delta^*\left[v + (n-2)\,v\delta^* + c - d - d\delta^*\right] + c = 0$.

Proof.

Let us consider the strategy profile $s^C = \{\omega^0, \omega^{C,1}, ..., \omega^{C,t}, ...\}$, where g^0 is a minimally connected network. We first show that the network which is pairwise stable for the widest range of parameters is the line network. The utility a player i obtains from following the cooperative strategy is $u_i\left(s^C\right) = \frac{(n-1)v + \mu_i^d\left(g^0\right)c}{1-\delta}$, where $\mu_i^d\left(g^0\right) \in \{1, 2, ..., n-1\}$. Suppose player i deviates in

some period t he will have to pay a cost d in his relationship with each of his social contacts from period $t+2$ onwards.

[22] Note that this situation is not possible, but we use it to get a lower bound in the incentives to deviate.

the network formation stage. Since players play defection and withhold information in the interaction phase if a network different from g^0 is observed, it is clear that the best deviation of player i is to delete any link he has and, doing so, player i obtains zero utility. Thus, player i follows s^C if and only if $u_i\left(s^C\right) \geq 0$, which is equivalent to $(n-1)\,v + \mu_i^d\left(g^0\right) c \geq 0$. This implies that in any minimally connected network g^0 the player who has the highest incentive to deviate in the network formation stage is player j, such that $\mu_j^d\left(g^0\right) = \max_{i \in N} \mu_i^d\left(g^0\right)$. We now observe that in the line network $\mu_j^d\left(g^{line}\right) = 2 < \mu_j^d\left(g^0\right)$ for any minimally connected network g^0 different from the line. It is readily seen that, given s^C, where g^0 is the line network, the network g^0 is pairwise stable if and only if $(n-1)\,v + 2c \geq 0$.

We now analyse the conditions for the discount factor δ (relative to the interaction stage). Let us assume that g^0 is the line network. We first show that the player who has the highest incentive to deviate in the interaction phase is either an end-agent, say i, or a player linked with an end agent, say j. Consider an end agent i, then the utility this player obtains following the strategy is $u_i\left(s^C\right) = \frac{(n-1)v+c}{1-\delta}$. If player i deviates in the interaction stage, his utility would be $(n-1)v + \frac{c}{1-\delta}$; thus a deviation is not profitable when $\delta \geq \bar{\delta} \equiv \frac{-c}{(n-1)v-d}$. Next, consider a player j linked with an end-agent; we note that player j has two links: one with an end-agent, say i, and one with a non-end agent, say j'. The utility of player j from following the strategy $u_j\left(s^C\right) = \frac{(n-1)v+2c}{1-\delta}$. At any period t, player j has two relevant possible deviations. One, player j may deviate only with player i at period t and deviate with player j' in period $t+1$; let us denote this deviation strategy as s_j^{d1}, then $u_j^{d1}\left(s_j^{d1}, s_{-j}^C\right) = (n-1)\,v + (n-2)\,v\delta + c + \frac{d\delta}{1-\delta} + \frac{d\delta^2}{1-\delta}$. Two, player j may deviate both with player i and j' at period t; let us denote this deviation strategy as s_j^{d2}, then $u_j^{d2}\left(s_j^{d2}, s_{-j}^C\right) = (n-1)\,v + \frac{2d\delta}{1-\delta}$.[23] We

[23] Player j may also deviate only with player i at period t. However, this deviation is strictly dominated by the deviation s_j^{d1}. Finally, player j may deviate with player j' at period t and either deviate with i at period $t+1$ or not. These two possibilities are strictly dominated by s_j^{d2}.

now observe that any other player who is neither an end-agent nor a player linked with an end-agent, say j', has a link with two non end-agents and he may deviate similarly to player j: to deviate only with the agent who is closest to an end agent of the line, say $s_{j'}^{d1}$, and to deviate with both of his social contacts, say $s_{j'}^{d2}$. It is readily seen that $u_j^{d1}(s_j^{d1}, s_{-j}^C) > u_{j'}^{d1}(s_{j'}^{d1}, s_{-j'}^C)$ and $u_j^{d2}(s_j^{d2}, s_{-j}^C) = u_{j'}^{d2}(s_{j'}^{d2}, s_{-j'}^C)$. This proves the claim.

Second, we claim that agent j has a higher incentive to deviate as compared to player i. Above we have shown that player i follows s^C if and only if $\delta > \bar{\delta}$. We now investigate the incentive of player j. Assume $\delta > \bar{\delta}$, then $u_j^{d1} \geq u_j^{d2}$ if and only if $\delta \geq \hat{\delta} \equiv \frac{-c}{(n-2)v-d} > \bar{\delta}$. Player j follows s^C if and only if $u_j\left(s^C\right) \geq u_j^{d1}\left(s_j^{d1}, s_{-j}^C\right)$, which is equivalent to $\delta[v + (n-2)\,v\delta + c - d - d\delta] + c \geq 0$. Let us define the following function: $\Upsilon(\delta) = \delta[v + (n-2)\,v\delta + c - d - d\delta] + c$. We note that $\frac{\partial \Upsilon(\delta)}{\partial \delta} > 0$ and using the fact that $c = d\hat{\delta} - (n-2)\,v\hat{\delta}$. we observe that $\Upsilon\left(\hat{\delta}\right) = -(n-3)\,v\hat{\delta} < 0$. Hence for $\delta \leq \hat{\delta}$ player j has incentives to deviate. This proves the claim. Moreover we can state the condition for player j not to deviate. Since when $\delta > \hat{\delta}$, the best deviation of player j is s_j^{d1}, for an equilibrium we need that $\delta \geq \delta^*$, where $\delta^* > \hat{\delta}$ is such that $\Upsilon(\delta^*) = 0$.

We now claim that in any other minimally connected network g^0 different from the line, say g^0, if $\delta < \delta^*$, s^C is not an equilibrium. We observe that in any minimally connected network it must be the case that there exist a player, say j'' who has k links with k end agents ($k \geq 1$) and one additional link, which may be either with a non end-agent, or with an end-agent (this last case would only be possible with the star network). Here we have two possibilities. One, if $k = 1$ then the incentives to deviate of player j'' are the same that the incentives of a player linked with an end-agent in a line network (player j above); in this case the claim follows. Two, $k \geq 2$; by construction $\mu_{j''}^d(g^0) = k + 1$ and the utility player j'' obtains following s^C is $u_{j'}\left(s^C\right) = \frac{(n-1)v+(k+1)c}{1-\delta}$. Let us assume that player j'' deviates in his interactions with the k end-agents at some period t and with the remaining player at period $t+1$, $s_{j''}^{dk}$. The utility from such deviation is $u_{j''}^{dk}\left(s_{j''}^{dk}, s_{-j''}^{C}\right) = (n-1)v + (n-1-k)v\delta + c + \frac{kd\delta + d\delta^2}{1-\delta}$. Therefore player j'' follows s^C if and only if $u_{j''}\left(s^C\right) \geq u_{j''}^{dk}\left(s_{j''}^{dk}, s_{-j''}^{C}\right)$, which is analogous to $\delta[kv + (n-1-k)v\delta + c - kd - d\delta] \geq -kc$. We now show that when $\delta = \delta^*$, player j'' deviates. To see this we note that $\Upsilon(\delta^*) = 0$ implies that $-c = \delta^*[v + (n-2)v\delta^* + c - d - d\delta^*]$. Assume for a contradiction that player j' does not want to deviate at δ^*, i.e. $\delta^*[kv + (n-1-k)v\delta^* + c - kd - d\delta^*] \geq -kc$. If, in the RHS, we substitute $-c$ from the equation $\Upsilon(\delta^*) = 0$ we obtain $\delta^*[kv + (n-1-k)v\delta^* + c - kd - d\delta^*] \geq -k\delta^*[v + (n-2)v\delta^* + c - d - d\delta^*] \Leftrightarrow \delta^* \leq \frac{-c}{(n-1)v-d}$, which is a contradiction since we have already shown that $\delta^* > \hat{\delta} > \bar{\delta} = \frac{-c}{(n-1)v-d}$. This proves the claim.

This completes the proof of the Lemma.■

Lemma 4.4. *Suppose (4.1) and (4.2) hold and assume $2c > f$. Consider the set of strategies $S^* \subset S$ which result in the asymmetric efficient outcome. If a strategy $s \in S^*$ with $g^0 = g$ is an equilibrium, then s^E with $g^0 = g$ is also an equilibrium.*

Proof.

Assume $2c > f$. To prove this, it is enough to show that s^C is a maximal punishment strategy profile. To see this note that if a player, say i, deviates in the network formation stage, he receives the worst possible outcome from period 1 onwards (because $a_{j,i}^{\tau} = (D,0)$, $\forall j \in N_i^d(g), \tau \geq 1$). If player i deviates in the interaction stage, at some period t then the players with whom player i deviates directly realize the deviation, and hence they play $(D,0)$ from period $t+1$, while the remaining social contacts realize the deviation at $t+1$ and hence they play $(D,0)$ from period $t+2$ onwards. Clearly, given the informational structure, player i receives the maximum punishment when he deviates. This completes the proof.■

Lemmas 4.3 and 4.4 prove the Theorem.■

Consumers networks and search equilibria

1 Introduction

A large body of empirical work shows that in the market a variety of informal relationships complement the price system in coordinating the interaction among buyers and sellers. For example, in marketing it is well established that consumers obtain much of their information via their social contacts (Feick and Price (1986, 1987)). In relation to this, firms have increasingly recognised the need for using informal channels as a way to market their products. The practice of consumers referral is an example;[1] according to the Direct Selling Association (1999), annual sales of firms that rely entirely on consumer referral grew from 13 billion to nearly 23 billion dollars between 1991 and 1998. Similarly, in the process of finding a job people heavily rely on their social contacts in order to obtain information about job opportunities (Granovetter (1974)). In medicine, and other specialised fields, professional networks shape the adoption of new technologies (Coleman 1966).

These examples share a common feature: informal relationships connecting agents transform the information that each individual privately obtains into a public good, and this affects players' incentives as well as aggregate outcomes. This fact represents the primary motivation for the development of a theory studying the interplay between network relationships and market performance. This paper focuses on the role of local information sharing in shaping the information available in the economy, firms' pricing behavior, social welfare and consumer surplus.

We examine a duopolistic version of Burdett and Judd (1983).

[1] Firms provide different sorts of benefits such as discounts to clients who bring new customers.

On the supply side of the market there are two firms producing a homogeneous good. Firms set prices so as to maximize profits. Consumers have a common willingness to pay for the good and buy at most a single unit. The only way for a transaction to take place is that consumers have some information about prices. Consumers may individually search for price quotations and, in this case, they must pay a fixed search cost for each price quotation observed. In addition, consumers are embedded in a social network and they share information freely with their direct neighbors. To maintain symmetry on the consumers side we assume that each consumer holds the same number of connections, say k.[2] Once each consumer has searched, the information observed is freely provided to his direct neighbors and then transactions take place. The game is a one-shot simultaneous move game: firms set prices and consumers decide how many searches to make at the same moment. We focus on symmetric Nash equilibria.[3]

When the network is empty (or inactive), i.e. $k = 0$, we obtain the duopolistic version of Burdett and Judd (1983). By contrast, the possibility of sharing information, i.e. $k > 0$, creates information externalities across consumers. These externalities have two main effects on the functioning of the market. On the one hand, for a given search efforts, they increase the likelihood with which a consumer compare prices, thereby increasing firms' competition. On the other hand, they create incentives for consumers to free-ride on each other. It is exactly the interplay between local information sharing and market competitiveness to be the focus of the present paper. We shall show that, in more dense networks, consumers search less intensively. This may decrease the total information generated in the economy even if consumers are more connected. In such a case, firms charge on average higher prices. Furthermore, we shall show that an increase in the density of the network does not always enhance

[2] Thus, the consumer network is a regular graph with degree k.

[3] We shall discuss strategy profiles where consumers search asymmetrically in Section 6.

social efficiency as well as consumer surplus.

We start by noticing that equilibria exhibit price dispersion. More interestingly, for any positive degree of the network there are two types of price dispersed equilibria. The first is a *high search intensity equilibrium*, where consumers randomize between searching for one price and for two prices. This equilibrium exists for low search costs. The other is a *low search intensity equilibrium,* where consumers randomize between searching for one price and not searching at all, and it exists for moderate search costs. By contrast, when consumers do not share information (the network is empty) only the former equilibrium is strategically viable. The low search intensity equilibrium arises because even a consumer who does not search may observe two price quotations, thereby creating the tension between some consumers over which firms have monopoly power and others consumers over which firms compete for. In what follows we discuss the properties of these equilibria. In particular, we are interested in the effect of an increase in the degree of the network on the consumers' search intensity, expected prices, social welfare and consumer surplus.

Let us first comment on the high search intensity equilibrium. Given that the degree of the network is strictly positive, this equilibrium exists for sufficiently low search costs. Further, as the degree of the network increases, the existence region of this equilibrium shrinks. The intuition is that richer network relationships reduce the marginal gains of searching twice instead of once. Thus, for sufficiently high search costs, consumers cannot be indifferent between the two searching alternatives. Second, we show that *the equilibrium expected price is higher when the network is more dense.* The intuition is based on two considerations. On the one hand, an increase in the degree of the network increases information externalities across consumers, ceteris paribus. Since consumers compare prices more often, firms' competition augments. On the other hand, consumers react to an increase in the number of connections by free-riding more on each other. This decreases information externalities. The result follows because the latter effect dominates the former, a fact which leads firms to compete less frequently and thereby to charge on average higher prices.

Thirdly, we show that social welfare is higher, while consumer surplus is lower, when the consumers network is more dense. The increase in social welfare is due to the strategic substitutability between searching and network degree, and to the fact that consumers are active with probability one. The former effect reduces the waste in search costs, while the second ensures that, in equilibrium, each possible transaction is indeed realized. The decrease in consumer surplus is due to the fact that firms price less aggressively when consumers hold more connections.

We finally turn to discuss the low search intensity equilibrium. This equilibrium exists for moderate search costs; further, for a given regular network, the lowest search costs for which this equilibrium exists equals the highest search cost for which the high search intensity equilibrium exists. Secondly, in sharp contrast with the previous equilibrium, an increase in the degree of the network lowers the expected equilibrium price. The intuition behind this result is that in the low search intensity equilibrium when a consumer free-rides he takes the risk to do not observe even one price. This has a substantial impact on the utility of the consumer and as a consequence it mitigates the temptation of consumers to free-ride on each other. As a result, richer consumers connections make consumers more likely to compare prices and this enhances firms' competition. Even if network relationships are beneficial for consumers, i.e. consumer surplus increases, they *decrease social welfare.* The reason is that the number of realized transactions in equilibrium decreases, an effect which offsets the saving on search costs.

The present model relates to two branches of the economic literature: the theory of networks and the search theory. We start by discussing the contribution of this paper to the theory of networks. The massive empirical documentation of network effects is behind the increasing theoretical attention of the effect of decentralized interactions on a variety of settings such as strategic partnering and competition, variations in criminality activity, local public good problems, income inequality and unemployment.[4] All these studies (including mine) belong to a new general class of games in which the economic activity of players is embedded in a network, which affects non trivially their incentives. In this perspective, the main contribution of the current paper is the study of the interplay between consumers connections and market functioning. The works which come closer to mine are Bramoulle and Kranton (2003) and Goyal and Moraga-Gonzalez (2001). Bramoulle and Kranton (2004) examine a model of social learning where individuals search costly for new information and the results of their searching are non-excludable along links. While in their model the benefit each consumer obtains by searching is exogenously given, in the present paper it is the outcome of firms' competition. Goyal and Moraga-Gonzalez (2001) analyse a game where, prior competition, firms form pairwise agreement for the development of new products and they set an R&D effort which is costly and provide a reduction of the marginal production cost. They find that the R&D effort a firm chooses in each agreement is decreasing in the effort that the partner firm sets and that this free-riding effect may lead to inefficient market outcomes. While Goyal and Moraga-Gonzalez (2001) focus on the impact of network relationships on the supply side of the market, the current paper focuses on information externalities across consumers.

The consumer search literature is well established in economics.[5] We have already discussed above the relation between the present paper and the model of Burdett and Judd (1983). Another paper which comes close to mine is Janssen and Moraga-Gonzalez (2003). They study a version of Burdett and Judd (1983) where consumers are ex-ante heterogeneous: one fraction of consumers are fully informed, while the remaining fraction must search costly to obtain price information. Increasing the

[4] For example, Calvo and Jackson (2004a, 2004b) study of the effect of social networks on employment and inequality. Bala and Goyal (1998) examine the effect of network on learning. Kranton and Minehart (2000,2001) study buyer and seller networks. Goyal and Joshi (2003) investigate the effect of networks of collaboration in oligopoly. Ballester, Calvo and Zenou (2004) study the in impact of networks in criminality activities.

[5] See, among others, Anderson and Renault (2000), Bester (1994), Braverman (1980), Burdett and Coles (1997), Morgan and Manning (1985) and Stahl (1989,1996).

fraction of fully informed consumers creates positive externalities for all consumers by boosting competitiveness and therefore lowering the expected price. The present paper provides a simple way of endogenizing information externalities across consumers using network relationships and it shows that this may create negative consumers externalities which have non trivial effects on the functioning of the market.

The rest of the paper proceeds as follows. In section 2 we define formally the model. Section 3 provides a preliminary equilibrium analysis. Section 4 and 5 characterize equilibria. Section 6 briefly discusses asymmetric equilibria and Section 7 concludes. Proofs are relegated to the appendix.

2 The model

We examine a model of non-sequential search where consumers are embedded in a network of connections. On the supply side there are $N = 2$ firms producing a homogeneous good at constant returns to scale. We normalize their identical unit production cost to zero, without loss of generality.

On the supply side instead, there is a finite number of consumers, which we denote as m. All consumers are identical. They want to buy a single unit of the product and their maximum willingness to pay is $\tilde{p} > 0$. For a transaction to take place, consumers must observe at least one price quotation. A consumer may search simultaneously, the cost for each search being $c > 0$, where $c < \tilde{p}$. In addition, the price information each consumer obtains is freely provided to his neighbors. For the same of symmetry on the consumers side, I assume that the consumer network is a regular graph. Thus, the degree of the network, say k, may vary between 0 to $m - 1$, and it represents the number of connections each consumer holds.[6]

We note that, when $k = 0$, the model is equivalent to a duopolistic version of Burdett and Judd (1983). By contrast, as

[6] A regular graph may not exist when m is odd. Hence, in the paper we assume that m is even.

k becomes positive, consumers strategically choose their search intensity, taking into account that the information their neighbors obtain is non-excludable along direct links. This clearly affects the individual incentives and, therefore, the market equilibrium outcomes. It is exactly on the interplay between the externalities produced by the consumers network and market performance that we focus in the present paper.[7]

Firms and consumers know the architecture of the network and play a simultaneous move game. An individual firm chooses its price, taking the price choices of its rivals, as well as consumers' search behaviors as given. We denote a firm's strategy by the price distribution $F(p)$ defined on a support σ; let $\underline{p}$ and $\bar{p}$ be the lowerbound and the upperbound of σ, respectively. Consumers form conjectures about the firms' price behavior and decide how many price observations to pay for. Once each consumer has searched, information is transmitted to the immediate neighbors. A strategy profile for a consumer is then a probability distribution over the set $\{0, 1, 2\}$.[8] We denote as $q_{i,x}$ the probability of consumer i to search x time; thus a consumer's strategy is $\{q_x\}_{x\in\{0,1,2\}}$. We will consider symmetric Nash equilibria.[9]

3 Preliminary analysis

Let us first analyse the existence and characterization of equilibria in which consumers adopt symmetric pure strategies.

Proposition 5.1. *For any $k \geq 0$ and $c > 0$, the only equilibria in which consumers use symmetric pure strategy take the following form: consumers never search, $q_0 = 1$, and firms charge a price $p \in [\tilde{p} - c, \tilde{p}]$.*

[7] We are assuming that consumers surely provide the information to his neighbours. This represents a situation where local communication across consumers is perfect. More generally, we could relax this assumption by assuming that local information sharing occurs with some probability $\rho \in (0, 1)$. In this new setting the results we shall present further will qualitatively carry on.

[8] It is clear that the restriction of the consumers' strategy set to $\{0, 1, 2\}$ does not affect the equilibrium characterization. Indeed, for a consumer to search more than twice is a strictly dominated strategy.

[9] Asymmetric equilibria are discussed in Section 6.

The proof relies on two facts. One, if consumers search surely for one price quotations, then they either prefer to decrease their search activity (if search cost are sufficiently high) or to search more (otherwise). Two, if consumers search surely for two price quotations, then firms' competition will drive prices to marginal cost. This creates incentive for consumers to search less. This proposition pushes us towards investigating those equilibria in which consumers use a mixed strategy. The next proposition shows the possible candidates for an equilibrium.

Proposition 5.2. *In any equilibrium in which consumers employ a symmetric mixed strategy firms price accordingly to an atomless price distribution,* $F(p)$, *defined on a convex support* σ. *Moreover, if* $k = 0$ *then* $q_1 + q_2 = 1$, $q_1, q_2 \in (0,1)$, *while if* $k > 0$, *then either* $q_1 + q_2 = 1$, $q_1, q_2 \in (0,1)$ *or* $q_0 + q_1 = 1$, $q_0, q_1 \in (0,1)$.

There are two main observations which follow from Proposition 5.2. The first is that, despite the fact that consumers are fully homogenous, price dispersion arises in all equilibria. Since consumers search randomly, some consumers in the market are ex-post more informed than others. In line with Burdett and Judd (1983), this allows firms to extract profits by randomizing their prices. Second and more interestingly, when the network does not play any role, e.g. $k = 0$, consumers must randomize between searching for one price and two prices for an equilibrium to be obtained. We refer to this as high search intensity. However, when local information sharing is taken into account, another equilibrium candidate emerges where consumers randomize between searching once and not searching at all. We call this possibility low search intensity. The intuition for this is that the presence of network relationships allow that, with some probability, even consumers who do not search at all observe both firms' prices. This creates the tension between some consumers over which firms have monopoly power and others consumers over which firms compete for.

We shall now characterize the high search intensity equilibrium and low search intensity equilibrium. For each equilibrium candidate we first characterize firms' behavior, taking consumers' strategy as exogenous. In this way, we illustrate the direct effect that networks have on the strategic way firms price. Next, we endogenize consumers' behavior in order to characterize equilibria. Finally, we analyse the impact of network density on the consumers' search intensity, firms' pricing behavior, social welfare and consumer surplus. Taken together, this analysis will clarify the effect of local information sharing on market competitiveness.

4 High search intensity

Suppose consumers randomize between searching once and searching twice, i.e. $q_1 + q_2 = 1$, $q_1, q_2 > 0$. The expected number of consumers who observe only the price of firm i, say D_i, and the expected number of fully informed consumers, say $D_{i,j}$, can be written as

$$D_i(k, q_1) = \frac{mq_1^{k+1}}{2^{k+1}} \tag{5.1}$$

$$D_{i,j}(k, q_1) = m\left(1 - \frac{q_1^{k+1}}{2^k}\right) \tag{5.2}$$

A consumer obtains only the price of firm i (expression (5.1)) when he and all his neighborhood observe only the price of firm i, $(q_1/2)^{k+1}$. Further, a consumer observes only the price of firm j with the same probability that a consumer observes only the price of firm i, i.e. $D_i = D_j$. Finally, with the remaining probability (expression (5.2)), a consumer observes both prices. It is readily seen that for a given q_1, the more dense the network (i.e. the higher k), the smaller the fraction of partially informed consumers, and the higher the fraction of fully informed consumers.

Using (5.1) and (5.2), the expected profit to firm i is

$$E\pi_i(p_i, p_j; k, q_1) = D_i(k, q_1)\, p_i + D_{i,j}(k, q_1)\, p_i\left[1 - F(p_i; k, q_1)\right] \tag{5.3}$$

The next Proposition summarizes equilibrium pricing by firms, given the consumers' behaviour fixed.

Proposition 5.3. *Assume* $q_1 + q_2 = 1,\ q_x \in (0,1)\,,\ x = 1,2.$ *In equilibrium:*

$$F(p;k,q_1) = 1 - \frac{q_1^{k+1}}{2\left(2^k - q_1^{k+1}\right)} \frac{\tilde{p} - p}{p},\ \forall p \in [\frac{q_1^{k+1}}{2^{k+1} - q_1^{k+1}}\tilde{p}, \tilde{p}]$$

Furthermore, $F(p;k,q_1)$ *dominates in the first order stochastic sense* $F(p;k+1,q_1)\,,\ k = 0,...,m-1.$

Not surprisingly, Proposition 5.3 shows that it is possible to rank the price distributions with respect to k in the first-order stochastic sense: $F(p,k,q_1)$ first order stochastically dominates $F(p;k+1,q_1)$. Therefore, as k increases, firms charge on average lower prices. The intuition is as follows: when consumers hold more connections, information externalities are higher, ceteris paribus. This implies that consumers compare prices more often and thereby firms compete more. Figure 5.1 illustrates the equilibrium price distribution for different levels of network density.

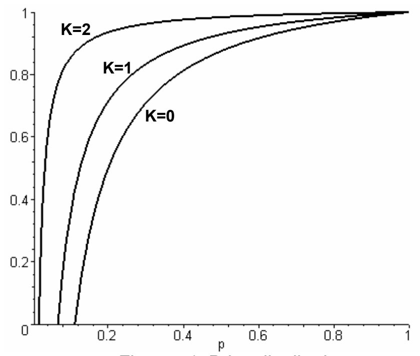


Figure 51. Price distribution

We now endogenize the consumers side. We denote as $E(p)$ the expected price obtained by randomly sampling one price from the distribution of prices F, while $E_{\min}(p)$ indicates the expected minimum price obtained by randomly sampling two prices. The expected utilities to a consumer from the two distinct searching alternatives are:[10]

$$Eu(q_1 = 1) = \tilde{p} - \frac{q_1^k}{2^k} E(p) - \left(1 - \frac{q_1^k}{2^k}\right) E_{\min}(p) - c \quad (5.4)$$

$$Eu(q_2 = 1) = \tilde{p} - E_{\min}(p) - 2c \quad (5.5)$$

In words, an arbitrary consumer j who searches once, expression (5.4), observes only one price quotation when all his social contacts are searching once, q_1^k, and each of them observes the same price quotation that j observes, $1/2^k$. With the remaining probability consumer j is fully informed. In equilibrium a consumer should be indifferent between the two different search alternatives, i.e. $Eu(q_1 = 1) = Eu(q_2 = 1)$. This leads to the following equilibrium condition:

$$\frac{q_1^k}{2^k} \left[E(p) - E_{\min}(p)\right] = c \quad (5.6)$$

Each consumer trades-off the marginal cost of searching once more, c, with its marginal gain. The marginal gain of searching twice instead of once is the difference between buying at the expected price and at the expected minimum price, i.e. $E(p) - E_{\min}(p)$, weighted for the probability with which a consumer who searches for one price will indeed observe only one price quotation, i.e. $q_1^k/2^k$. When the network is empty, i.e. $k = 0$, the marginal gain becomes the difference between the expected price and the expected minimum price.

The next result provides the full characterization of the high intensity search equilibrium for any given $k = 0, ..., m-1$. Let $\bar{c}(k) = \frac{1}{2^k\left(2^{k+1}-2\right)} \left(\frac{2^{k+1}}{2^{k+1}-2} \ln\left(2^{k+1}-1\right) - 2\right)$.

Theorem 5.1. *If $k = 0$ there exists a $\tilde{c} > 0$ such that for any $c \in (0, \tilde{c})$ a stable high search intensity equilibrium exists where firms behave according to Proposition 4.1 and q_1^* is the smallest solution of (5.6). If $k > 0$, there exists $\bar{c}(k) < \tilde{c}$ such*

[10] More precisely expression 5.4 (resp. 5.5) indicates the expected utility to a consumer i who searches for one price quotation (resp. for two price quotations), given that all other consumers are searching for one price quotation with probability q_1, and for two price quotations with the remaining probability, $1 - q_1$.

that for any $c \in (0, \bar{c}(k))$ a high intensity search equilibrium exists where firms behave according to Proposition 4.1 and q_1^* is the unique solution of (5.6). Furthermore, this equilibrium is stable.

We first elaborate on the existence condition of this equilibrium. Figure 5.2a below illustrates the equilibrium condition for different level of k. In the Figure we plot the LHS of expression (5.6) for different level of k as a function of q_1. Figure 5.2b above illustrates that the function $\bar{c}(k)$ decreases in k.

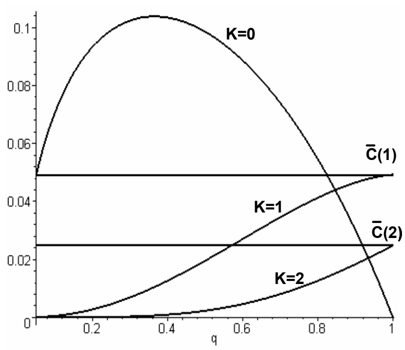


Figure 5.2a.

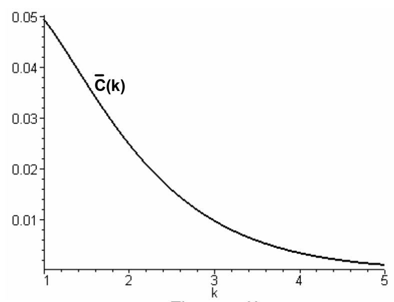


Figure 5.2b.

As already discussed, when $k = 0$ the model is equivalent to the duopolistic version of Burdett and Judd (1983). In this case, for a given c there are at most two equilibria, but only one is stable. Differently, when we introduce information sharing, there is a unique solution of the equilibrium condition (5.6), which is also stable. The first effect of information sharing is that the high intensity search equilibrium exists only when searching is relatively inexpensive. Furthermore, as k increases this equilibrium exists for smaller and smaller search costs. The intuition is as follows. Network externalities reduce the marginal gains of searching twice instead of once, thereby for search costs sufficiently high a consumer cannot be indifferent between the two searching alternatives. The decrease in the marginal gains is due to two effects. The first is that richer network connections increase the probability of a consumer who searches once to compare prices, and the second is that the difference between the expected price and the expected minimum price decreases in k.

We now turn to analyse the effect of consumers network on search incentives, expected prices, consumer surplus and social welfare. The next proposition summarizes the findings.

Proposition 5.4. *Suppose we move from k to $k+1$, $k \in [1, ..., m-2]$ and assume that $c < \bar{c}(k+1)$. Then: (a) consumers search less frequently, i.e. q_2 decreases, (b) expected price increases (c) social welfare increases and (d) consumer surplus decreases.*

We would like to elaborate on three aspects of this local comparative static result. The first is that consumers search intensity decreases as the network becomes denser: an increase in the network degree leads consumers to free-ride more on each other. Secondly, this has a somewhat surprising effect on the equilibrium pricing behavior of firms: *expected price is higher in settings where consumers have more connections*. The intuition is the following. An increase in the degree of the consumers network induces two effects. The first is highlighted in Proposition 5.3 and it tells us that, keeping constant the consumers' behavior, an increase in the number of connections increases the expected number of fully informed consumers. The second is a free-riding effect: more connections lead players to search less intensively and this results in a decrease of the expected number of fully informed consumers. When consumers search intensively, the free-riding effect offsets the former effect and as a consequence firms price less aggressively. In Figure 5.3a below we plot the probability of a consumer who searches once to be fully informed in equilibrium. In line with the intuition above, Figure 5.3a shows that for a given search cost the information in the market decreases when the degree of the network increases. Figure 5.3b shows how the expected price varies with respect to the degree of the network in equilibrium.

Next, we note that consumer surplus decreases. This follows by noting that not only the expected price decreases but the same holds for the expected minimum price. Finally, we show that an increase in the degree of the network enhances social efficiency. This is due to the fact that the free-riding effect leads to saving on the total search cost, yet, since consumers search surely, each possible transaction is realized in equilibrium.

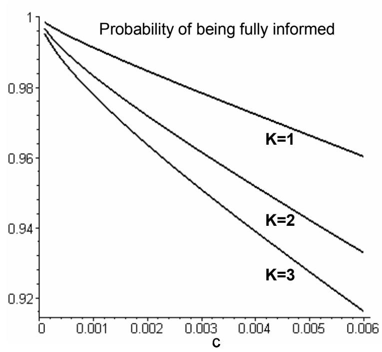


Figure 5.3a.

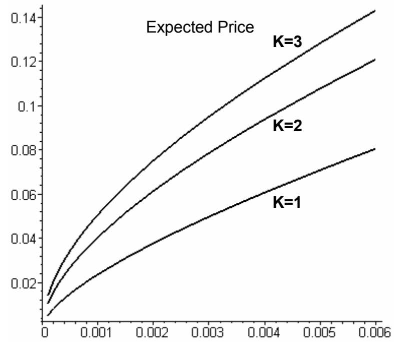


Figure 5.3b.

5 Low search intensity

We now analyse the case in which consumers randomize between searching once and not searching at all, i.e. $q_0 + q_1 = 1$, $q_0, q_1 > 0$. We start by considering consumers' behavior as exogenously given. The expected fraction of consumers who observe only the price of firm i, say D_i, is:

$$D_i(k, q_0) = \frac{m(1-q_0)}{2} \sum_{x=0}^{k} \binom{k}{x} \frac{q_0^{k-x}(1-q_0)^x}{2^x} + mq_0 \sum_{x=1}^{k} \binom{k}{x} \frac{q_0^{k-x}(1-q_0)^x}{2^x} \quad (5.7)$$

The the expected fraction of fully informed consumers, say $D_{i,j}$, is:

$$D_{i,j}(k,q_0) = m(1-q_0)\left(1-\sum_{x=0}^{k}\binom{k}{x}\frac{q_0^{k-x}(1-q_0)^x}{2^x}\right) \tag{5.8}$$
$$+mq\left(1-q_0^k-2\sum_{x=1}^{k}\binom{k}{x}\frac{q_0^{k-x}(1-q_0)^x}{2^x}\right)$$

The interpretation of expression (5.7) is as follows: the first term denotes the fraction of consumers who have searched once on their own and found firm i, i.e. $m(1-q_0)/2$, and that they have either received the same information or no information from their neighbors; the second term indicates the fraction of consumers who did not search, but that have received the price information of firm i from some of their social contacts. Expression (5.8) as a similar interpretation. Expressions (5.7) and (5.8) can be rewritten as follows:

$$D_i(k,q_0) = \frac{m\left[(1+q_0)^{k+1}-2^{k+1}q_0^{k+1}\right]}{2^{k+1}} \tag{5.9}$$

$$D_{i,j}(k,q_0) = \frac{m\left[2^k\left(1+q_0^{k+1}\right)-(1+q_0)^{k+1}\right]}{2^k} \tag{5.10}$$

Thus, the expected profit of firm i is:

$$E\pi(p_i,p_j;k,q_0) = D_i(k,q_0)\,p_i + D_i(k,q_0)\,p_i\left[1-F(p_i;k,q_0)\right] \tag{5.11}$$

The next Proposition summarizes the firms' price behavior in equilibrium.

Proposition 5.5. *Assume* $q_0+q_1=1$, $q_x\in(0,1)$, $x=0,1$. *In equilibrium:*

$$F(p;k,q_0) = 1-\frac{(1+q_0)^{k+1}-2^{k+1}q_0^{k+1}}{2\left(2^k\left(1+q_0^{k+1}\right)-(1+q_0)^{k+1}\right)}\frac{\tilde{p}-p}{p},$$
$$\forall p \in \left[\frac{(1+q_0)^{k+1}-2^{k+1}q_0^{k+1}}{2^{k+1}-(1+q_0)^{k+1}}\tilde{p},\tilde{p}\right]$$

Furthermore, $F(p;k,q_0)$ *dominates in the first order stochastic sense* $F(p;k+1,q_0)$.

As in the high intensity search equilibrium, an increase in the degree of the network has a direct effect on the way firms price: the higher the density of the network, the lower the expected price. This is illustrated in the Figure 5.4 below.

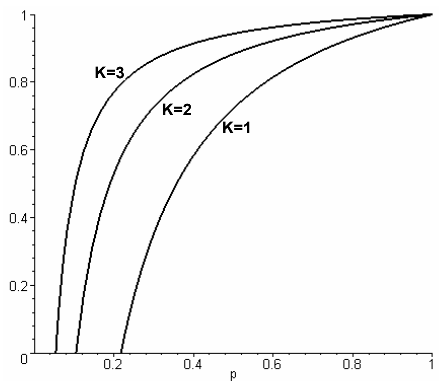


Figure 5.4. Price distribution

We now endogenize the consumers' search behavior. Let $\alpha(q_0,k) = \sum_{x=0}^{k}\binom{k}{x}\frac{q_0^{k-x}(1-q_0)^x}{2^x}$ and $\beta(q_0,k) = \sum_{x=1}^{k}\binom{k}{x}\frac{q_0^{k-x}(1-q_0)^x}{2^{x-1}}$; the utility a consumer gets from the two distinct search alternatives is:

$$Eu(q_1=1) = \tilde{p} - \alpha(q_0,k)E(p) - (1-\alpha(q_0,k))E_{\min}(p) \tag{5.12}$$

$$\begin{aligned} Eu(q_0=1) &= \tilde{p}\left(1-q_0^k\right) - \beta(q_0,k)E(p) - \\ &\quad -(1-q_0^k-\beta(q_0,k))E_{\min}(p) \end{aligned} \tag{5.13}$$

The interpretation of expression (5.12) is the following. Since a consumer searches once on its own he always buys: he buys at the expected price whenever his neighbors provide redundant or no information; otherwise he buys at the expected minimum price. Differently, a consumer who does not search, expression (5.13), buys only when at least one of his social contact searches, $\left(1-q_0^k\right)$. The expressions (5.12) and (5.13) can be rewritten as follows:

$$\begin{aligned} Eu\left(q_1=1\right) &= \tilde{p}-\frac{\left(1+q_0\right)^k}{2^k}E\left(p\right)- \\ &-\left(\frac{2^k-\left(1+q_0\right)^k}{2^k}\right)E_{\min}\left(p\right)-c \end{aligned} \tag{5.14}$$

$$\begin{aligned} Eu\left(q_0=1\right) &= \tilde{p}\left(1-q_0^k\right)-\left(\frac{\left(1+q_0\right)^k-2^k q_0^k}{2^{k-1}}\right)E\left(p\right) \\ &-\left(\frac{2^{k-1}\left(1+{q_0}^k\right)-\left(1+q_0\right)^k}{2^{k-1}}\right)E_{\min}\left(p\right) \end{aligned} \tag{5.15}$$

In equilibrium every consumer must be indifferent between searching once and not searching at all, i.e $Eu\left(q_1=1\right)=Eu\left(q_0=1\right)$. This condition is satisfied if and only if:

$$\frac{\left(1+q_0\right)^k-2^{k+1}q_0^k}{2^k}\left[E\left(p\right)-E_{\min}\left(p\right)\right]+q_0^k\left(\tilde{p}-E_{\min}\left(p\right)\right)=c \tag{5.16}$$

The interpretation of (5.16) is similar to the interpretation of (5.6). The next result shows that for moderate value of search costs there exists at least a stable low intensity search equilibrium.

Theorem 5.2. *For any $k>0$ there exists a $\tilde{c}$ such that for any $c\in\left(\bar{c}\left(k\right),\tilde{c}\right)$ a stable low intensity search equilibrium exists where firms behave according to Proposition 5.5 and q_0^* is the smallest solution of (5.16).*

Theorem 5.2 tells us that for moderate search costs there exists at least a stable solution of the equilibrium condition (5.16). The proof in the appendix also shows that there always exists at least another solution of the equilibrium condition (5.16), which however is not stable. Further, numerical simulations reveal that these are the only two possible solutions.[11] In what follows we

[11] We have run numerical simulations of the equilibrium condition (5.16) for $k=1,...,100$. The simulations reveal that there are at most two solutions, among which only the smaller one is stable.

focus on the stable equilibrium. We start with a discussion of the existence of the low intensity search equilibrium. In Figure 5.5 below we plot the LHS of the equilibrium condition (5.16) with respect to q_0 for different levels of k.

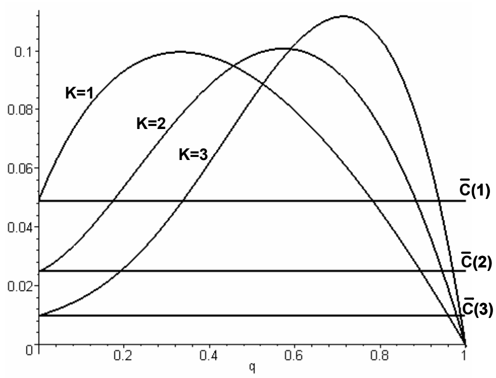


Figure 5.5

We note that for any positive k the stable low intensity search equilibrium exists for search costs which are higher than $\bar{c}(k)$. Moreover, when the search cost is exactly equal to $\bar{c}(k)$ in equilibrium consumers search once with probability one and the low intensity and high intensity search equilibrium coincide.[12]

We now turn to examine the local comparative statics with respect to k. The intractability of the equilibrium condition (5.16), leads us to rely on numerical simulations. The findings are summarized in the following remark. [13]

Remark 5.1. *Suppose we move from k to $k+1$ and assume that $c \in (c(k), \tilde{c})$. Then: (a) consumers search less frequently, i.e. q_0 increases, (b) expected price decreases, (c) social welfare decreases and (d) consumer surplus increases.*

---

[12] It is readily seen that the price distributions in Proposition 4.1 and 5.1 and the equilibrium conditions (5.6) and (5.16) coincide when $q_1 = 1$ (i.e. $q_0 = 1$).

[13] We have run simulations for $k = 1, ..., 100$. For any k we first determine the range of the search costs for which the stable equilibrium exists (the smaller solution of (5.16)), say $[c_1(k), c_2(k)]$. Next, for each $c \in [c_1(k), c_2(k)]$, we derive the stable solution of equation (5.16), $q_0(k, c)$. Finally, using this value we compute the expected price , social welfare and consumer surplus.

The numerical simulations confirm that consumers free-ride more on each other in denser networks. However, in sharp contrast with the high search intensity equilibrium, the effect on the way firms strategically price is reverse: *the higher the density of the network, the lower the expected price.* This difference is due to the fact that in the low search intensity equilibrium if a consumer does not search he may be fully ignorant ex-post. This mitigates the free-riding effects among consumers. Thus, when the degree of the network increases the expected number of fully informed consumers in the economy increases. As firms compete more often for consumers, they charge on average lower prices. Figure 5.6a below shows the expected price for different network degrees in equilibrium. Figure 5.6b illustrates the equilibrium probability of not searching for different network degrees.

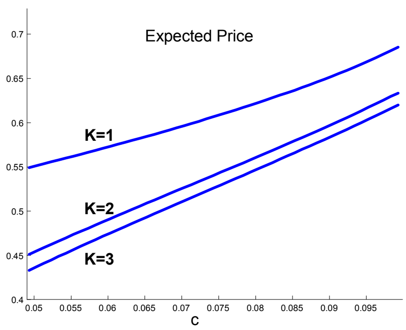


Figure 5.6a.

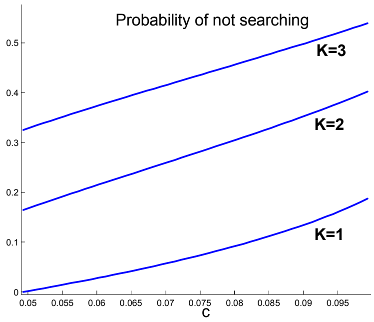


Figure 5.6b.

Next, we note that total social welfare is decreasing in the density of the network. Figure 5.6c plot the social welfare for different degrees of the network. The reason is that when the search cost is moderate, a consumer who completely relies on his connections takes the risk to be ex-post completely ignorant about prices. In such a case a transaction does not take place and this generates a substantial decrease in social welfare. When the degree of the network increases this negative effect dominates the realized savings in search cost and therefore the overall social welfare decreases. Finally, even if social welfare decreases, consumers surplus increases in the density of the network. Figure 5.6d depicts the consumer surplus for different degrees of the network. Two are the reasons behind this result. The expected price and expected minimum price decrease in the density of the network. Further, the overall increase of information shared in the network leads consumers to be more likely to compare prices.

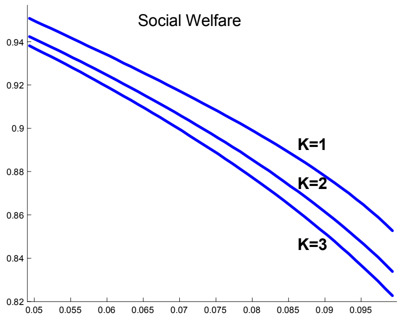


Figure 5.6c.

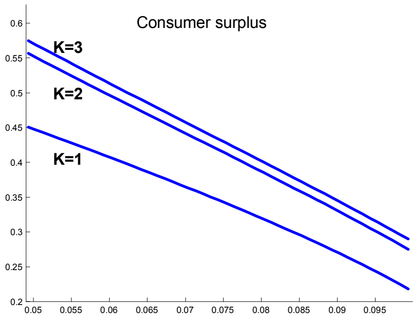


Figure 5.6d.

6 Discussion: asymmetric strategies

The analyses we have developed so far is based on two main restrictions: symmetric strategy and symmetric networks. The main implication of these two restrictions is that for a given regular network of degree of k consumers in different network positions have identical incentives whatever the architecture of the network is. In different words, given two symmetric networks with common degree but different architectures the equilibrium characterization is identical. This is clearly an appealing property as it allows me to perform a standard comparative static analyses to investigate how local information sharing affects market functioning when the network becomes denser. However, a number of interesting questions arise when we allow the network to play a more serious role in the model. A natural way of doing this is to analyse equilibria where consumers employ asymmetric pure strategy.[14] With some abuse of notation we will refer to these strategy profiles as asymmetric strategies. We shall provide an example to illustrate the nature of asymmetric equilibria and a number of interesting questions which arise.[15]

Consider a population composed of 8 consumers. Figure 5.7 depicts two symmetric networks with equal degree 4. The network on the LHS represents a social structure with overlapping neighborhoods, while the graph on the RHS represents a social structure where each consumer has links with their immediate neighborhood as well as more distant links (short-cuts). A black node indicates a consumer who searches once, while a white node an inactive consumer.

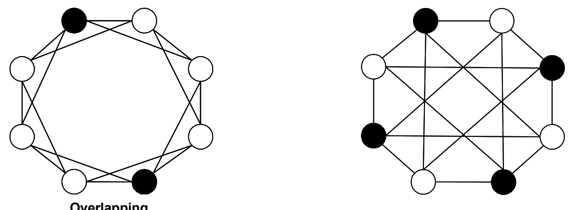


Figure 5.7

It is possible to show that the depicted pattern of consumers' search in each of the network is the only possible pattern for an asymmetric equilibrium to exist. It is also easy to show that an equilibrium exists where firms price randomly in both situations. There are three remarks worth doing. The first is that in both networks only specialized equilibria exist: some consumers search once (expert consumers), while others do not search at all (free-rider consumers). This contrasts with the findings of Bramoulle and Kranton (2004), who show that also distributed (and mixed) equilibria arise.[16] The reason for these differences

[14] Other possibilities are discussed in the conclusion.

[15] A formal and complete analyses of asymmetric equilibria is out of the scope of the current study. A general analyses is in a companion note Galeotti (2004), which is available upon request to the author.

[16] Rougly speaking a distributed configuration in this model is one where there are consumers who search surely but with different search intensities.

is that in the current paper the benefits of searching depend on firms pricing behavior, while in their paper the benefits are exogenously given. This reduces dramatically the multiplicity of equilibria Bramoulle and Kranton (2004) obtain. Second, when the network has overlapping neighborhoods a fewer number of experts arise in the economy as compared to the case where consumers have also short-cuts. This is a consequence of the following equilibrium property: experts can only be linked with free-riders.[17] This suggests that the maximum number of experts which can be sustained in equilibrium is somewhat negatively related to the clustering (the number of trials) of a network. Third, in both networks we have price dispersed equilibria, which again arise because consumers are ex-post asymmetrically informed.

There are a number of interesting questions that can be addressed. A natural one is how strategic pricing, consumers' welfare and social welfare is affected by the different allocation of links across consumers. It is easy to show that the equilibrium price distribution for the short-cuts network first order stochastic dominate the one associated with the overlapping neighbors network.[18] Therefore expected prices are lower in the former case as compared to the latter. The reason is that the presence of short-cuts allow for the emergence of more experts in the economy which in turn increases firms' competition. It is also possible to show that aggregate consumers welfare is higher in the presence of short-cuts as compared to overlapping neighborhoods.[19] This is because the increase on firms' competition due to the higher number of experts dominates the associated increasing in the total search cost. Finally, it is readily seen that social welfare is higher in the network with overlapping neigh-

[17] Formally, the set of experts in equilibrium must form a maximally independent set. Given a graph g an independet set is a set of nodes which are not directly connected. A maximally independet set is an independet set which is not a proper subset of any other independet set.

[18] Formally, the equilibrium price distribution in case of the overlapping neighborhoods network is $F^{O}(p) = 1 - (9/2)(v-p)/p$, $\forall p \in [(9/11)v, v]$. In the case of the short-cuts network $F^{S}(p) = 1 - (17/14)\,(v-p)\,/p$, $\forall p \in [(17/31)v, v]$.

[19] To see this note that the aggregate consumers surplus in the case of overlapping neighbors is $cs^{O} = 9 \ln \frac{11}{9} - 1 - 2c$, which in the case of short-cuts is $cs^{S} = -\frac{1}{2} + \frac{34}{7} \ln \frac{31}{17} - 4c$.

borhoods because the total search cost is minimized and yet each transaction takes place.

7 Conclusion

We have developed a search model which examines the effect of local information sharing among consumers in the consumers' search incentives, firms' price behavior and social welfare. We have shown that consumers search less frequently in denser networks due to a free-riding effect. This has a somewhat surprising implication on the firms' price behavior as well as on the overall performance of the market. In particular, when the search costs are sufficiently low the equilibrium expected price is higher and the consumers welfare is lower in settings where consumers interaction is denser. Furthermore, when search costs are moderate, the market outcome becomes more inefficient as networks density rises.

There are many extensions which may be of interest for further research. The first is to examine the implication of asymmetric connections across players. Even if an analysis for any network architecture may be unfeasible, one could focus on a particular class of networks which matches much empirical evidence on social networks such as the star and variants of this architecture. A second extension would be to endogenize the quality of information sharing. It is natural to think that the quality of each link and therefore how information flow from one consumer to the other depends on the time the two parties are willing to invest on. In this new settings one could investigate the strategic nature between networking investment and private search investment and the implications on market functioning. Finally, it is interesting to generalize the model to an oligopoly. Information sharing introduces two important effects on price. If search effort remains the same, price will reduce as more consumers can make price comparisons. The other effect is that consumers may free ride on others. Clearly, these two effects will depend on the number of firms present in the market.

8 Appendix chapter 5

Proof of Proposition 5.1.

First, it is easy to verify that the strategy profile $\{p, q_0 = 1\}$, where $p \in [\tilde{p} - c, \tilde{p}]$ is a Nash equilibrium. We now prove that these are the only (generic) equilibria in which consumers employ pure strategies. There are two possibilities, which we analyse in turn. First, suppose $q_1 = 1$; if $k = 0$, then each consumer will observe only one price and as a consequence firms will charge $p = \tilde{p}$. However, as far as $c > 0$, a consumer strictly gains by not searching at all. Consider then that $k > 0$; we claim that if this were an equilibrium then firms will price according to an atomless price distribution $F(p)$ defined on a convex support σ. The reason is that since $k > 0$ and $q_1 = 1$ there is a fraction of consumers which will observe both firms' prices with a strictly positive probability. Therefore, if firms charge a price p with a mass point, they will tight at that price with strictly positive probability, but then an individual firm has a strict incentive to undercut the atom. We now show that, given $k > 0$, an equilibrium where consumers search once with probability one, i.e. $q_1 = 1$, exists for a unique value of the search cost (it is not generic). The utility to a consumer is $Eu(q_1 = 1) = \tilde{p} - \frac{1}{2^k} E(p) - \left(1 - \frac{1}{2^k}\right) E_{\min}(p) - c$. In equilibrium it must be the case that $Eu(q_1 = 1) \geq Eu^d(q_x = 1)$, $x = 0, 2$, where $Eu^d(q_0 = 1) = \tilde{p} - \frac{1}{2^{k-1}} E(p) - \left(1 - \frac{1}{2^{k-1}}\right) E_{\min}$ and $Eu^d(q_2 = 1) = \tilde{p} - E_{\min}(p) - 2c$. Solving the two inequality we obtain that:

$$c = \frac{1}{2^k} \left(E(p) - E_{\min}\right)$$

Second, suppose $q_2 = 1$ and $k \geq 0$. It is easy to see that each consumer will observe always two prices. If this were an equilibrium firms would charge the competitive price, $p = 0$. However, a consumer is strictly better-off by searching only once. This completes the proof of the Proposition.■

Proof Proposition 5.2.

We first show that firms price according to an atomless price distribution $F(p)$. If $k = 0$ the model degenerates to the duopoly version of Burdett and Judd (1983) and the claim follows. Next, assume $k > 0$ and suppose there exists some price p^* with a mass point. Since consumers search at least once with some positive probability and $k > 0$, it follows that a fraction of consumers observe two prices with strictly positive probability. Therefore, firms would tie at the price p^* with strictly positive probability; in such a case a firm gains by undercutting p^*. This is a contradiction. We finally show that for any $k \geq 0$ the support σ must be convex. Suppose not, i.e. $\exists \tilde{\sigma} \subsetneq \sigma : F(p) = c \ \forall p \in \tilde{\sigma}$. Let $p^* = \inf \tilde{\sigma}$, then a firm charging p^* gains by increasing such price. This completes the proof of the first part of the Proposition.

We now show that if $k > 0$ in any equilibrium either $q_1 + q_2 = 1,\ q_1, q_2 \in (0,1)$ or $q_0 + q_1 = 1,\ q_0, q_1 \in (0,1)$. We start by claiming that $q_0 + q_2 = 1$ cannot be part of an equilibrium. Suppose it is, then firms would set the competitive price with probability one. The reason is that the expected demand of a firm derives from two sources: consumers who search on their own and consumers who do not search but obtain information from their social contacts. The former would always observe two prices, while the latter either do not observe any price or they also observe two prices. Using a standard undercutting argument it follows that firms must charge the competitive price. Since firms charge the competitive price with probability one a consumer strictly benefits by searching only once.

Next we show that $q_0 + q_1 + q_2 = 1$ cannot be part of an equilibrium. Suppose it is an equilibrium; the same argument above implies that $F(p)$ is atomless and it is defined on a convex support σ. In equilibrium it must be the case that $Eu(q_x = 1) = Eu(q_y = 1),\ x, y = 0, 1, 2$. Let $\alpha(k) = \sum_{x=1}^{k} \binom{k}{x} \frac{q_0^{k-x} q_1^x}{2^x}$, then we obtain that:

$$\begin{aligned}
Eu\left(q_0=1\right) &= \tilde{p}\left(1-q_0^k\right)-\alpha\left(k\right)E\left(p\right)-\left(1-q_0^k-\alpha\left(k\right)\right)E_{\min}\left(p\right) \\
Eu\left(q_1=1\right) &= \tilde{p}-\left(q_0^k+\frac{\alpha\left(k\right)}{2}\right)E\left(p\right)- \\
&\quad -\left(1-q_0^k-\frac{\alpha\left(k\right)}{2}\right)E_{\min}\left(p\right)-c \\
Eu\left(q_2=1\right) &= \tilde{p}-E_{\min}\left(p\right)-2c
\end{aligned}$$

Solving for the equilibrium conditions it follows that:

$$q_0^k\left[\tilde{p}-E\left(p\right)\right]=q_0^k\left[E\left(p\right)-E_{\min}\left(p\right)\right]$$

Given that $q_0>0$ this condition is satisfied if and only if $\tilde{p}-E\left(p\right)=E\left(p\right)-E_{\min}\left(p\right)$. We now show that this is impossible. To see this we note that

$$E_{\min}\left(p\right)=2E\left(p\right)-\int_{\underline{p}}^{\tilde{p}}2pf(p)F(p)dp$$

Therefore:

$$E\left(p\right)-E_{\min}\left(p\right)=\int_{\underline{p}}^{\tilde{p}}2pf(p)F(p)dp-E\left(p\right)=$$

Integrating by parts we can show that:

$$\int_{\underline{p}}^{\tilde{p}}2pf(p)F(p)dp=\tilde{p}-\int_{\underline{p}}^{\tilde{p}}\left[F(p)\right]^2dp$$

which implies that:

$$E\left(p\right)-E_{\min}\left(p\right)=\left[\tilde{p}-E\left(p\right)\right]-\int_{\underline{p}}^{\tilde{p}}\left[F(p)\right]^2dp<\left[\tilde{p}-E\left(p\right)\right] \tag{5.17}$$

This is a contradiction and therefore the claim follows. Hence, the proof for the case $k\geq1$ is complete.

We finally consider the case where $k=0$. The same argument used for $k\geq1$, shows that $q_0+q_2=1$ and $q_0+q_1+q_2=1$ cannot be part of an equilibrium. Therefore, the only possibility left is $q_0 + q_1 = 1$. If this were an equilibrium firms would charge the monopolist price. However, in such a case consumers cannot be indifferent between not searching and searching once, i.e. $Eu\left(q_0 = 1\right) = 0 > -c = Eu\left(q_1 = 1\right)$. This completes the proof of the proposition.■

High Search Intensity Equilibrium

Proof of Proposition 5.3.

We first note that the upper bound of the price distribution must be the reservation price, $\tilde{p}$; for otherwise a firm charging $\overline{p} < \tilde{p}$ strictly gains by increasing it. This implies that the expected equilibrium profit is: $E\pi_i^*\left(\tilde{p}, p_j; k, q_1\right) = \frac{mq_1^{k+1}}{2^{k+1}}\tilde{p}$. In equilibrium a firm i must be indifferent between charging any price in the support σ, i.e. $E\pi_i\left(p_i, p_j; k, q_1\right) = E\pi_i^*\left(\tilde{p}, p_j; k, q_1\right), \forall p \in \sigma$. Solving this condition we obtain the expression of $F\left(p; k, q_1\right)$ and the expression of the lowerbound of the support is obtained by solving for $E\pi_i\left(\underline{p}_i, p_j; k, q_1\right) = E\pi_i^*\left(\tilde{p}, p_j; k, q_1\right)$. Finally, let $\psi = \frac{q_1^{k+1}}{2\left(2^k - q_1^{k+1}\right)}$, then it is easy to see that $\frac{\partial F(p;k,q_1)}{\partial k} > 0$ if and only if $\frac{\partial \psi}{\partial k} = \frac{q_1^{k+1}2^{k-1}}{\left(2^k - q_1^{k+1}\right)^2}\ln\frac{q_1}{2} < 0$. This completes the proof.■

Proof Theorem 5.1.

The proof of the case $k = 0$ is the same as Burdett and Judd(1983) and therefore it is omitted. We focus instead in the case $k > 0$.

Let us define the RHS of expression (5.6) as $\phi\left(p; k, q_1\right) = \frac{q_1^k}{2^k}\left[E\left(p\right) - E_{\min}\left(p\right)\right]$. We start by showing that $\frac{\partial \phi(p;k,q_1)}{\partial q_1} > 0$. Suppose, without loss of generality that $\tilde{p} = 1$. Using the expression of the price distribution $F\left(p; k, q_1\right)$ defined in proposition 5.3. We can invert it to obtain:

$$p\left(z; k, q\right) = \frac{1}{g\left(z; k, q_1\right)} \tag{5.18}$$

where

$$g\left(z; k, q_1\right) = 1 + \frac{2\left(2^k - q^{k+1}\right)}{q^{k+1}}\left(1 - z\right) \tag{5.19}$$

We now note that:

$$\frac{2^k}{q_1^k}\phi\left(p;k,q_1\right) = 2\int_{\underline{p}(k,q_1)}^{1} pf(p;k,q_1)\left(1-F(p;k,q_1)\right)dp - \\ -\int_{\underline{p}(k,q_1)}^{1} pf(p;k,q_1)dp$$

Integrating by parts yields,

$$\frac{2^k}{q_1^k}\phi\left(p;k,q_1\right) = \int_{\underline{p}(k,q_1)}^{1}\left[F(p;k,q_1)\left(1-F(p;k,q_1)\right)\right]dp$$

Using the inverse function $p\left(z;k,q_1\right)$, we can write this expression as:

$$\frac{2^k}{q_1^k}\phi\left(z;k,q_1\right) = \int_0^1\left[p\left(\sqrt{z};k,q_1\right)-p\left(z;k,q_1\right)\right]dz$$

Or,

$$\frac{2^k}{q_1^k}\phi\left(z;k,q_1\right) = \int_0^1 p\left(z;k,q_1\right)\left(2z-1\right)dz$$

Let $a = q^{k+1}$, $b = 2\left(2^k - q^{k+1}\right)$ and $c = 2^{k+1}\left(2k+1\right)$, then,

$$\begin{aligned}\frac{2^k}{q_1^{2k}}\frac{\partial\phi\left(z;k,q_1\right)}{\partial q_1} &= \int_0^1\frac{\left[ka\left(2z-1\right)+c\left(1-z\right)\right]\left(2z-1\right)}{\left[a+b\left(1-z\right)\right]^2}dz \\ &= -\int_0^{\frac{1}{2}}\frac{\left[ka\left(2z-1\right)+c\left(1-z\right)\right]\left(1-2z\right)}{\left[a+b\left(1-z\right)\right]^2}dz + \\ &\quad +\int_{\frac{1}{2}}^1\frac{\left[ka\left(2z-1\right)+c\left(1-z\right)\right]\left(2z-1\right)}{\left[a+b\left(1-z\right)\right]^2}dz\end{aligned}$$

We note that $\frac{\left[ka(2z-1)+c(1-z)\right]}{\left[a+b(1-z)\right]^2}$ is positive and increasing in z for any $z\in(0,1/2)$ and that $\left[a+b\left(1-z\right)\right]^2$ is decreasing in z. Therefore:

$$\begin{aligned}\frac{2^k}{q_1^{2k}}\frac{\partial\phi\left(z;k,q_1\right)}{\partial q_1} &> -\int_0^{\frac{1}{2}}\frac{2^k\left(2k+1\right)\left(1-2z\right)}{2^{2k}}dz + \\ &\quad +\int_{\frac{1}{2}}^1\frac{\left[ka\left(2z-1\right)+c\left(1-z\right)\right]\left(2z-1\right)}{2^{2k}}dz\end{aligned}$$

We now note that $[ka(2z-1)+c(1-z)]$ is positive and it is decreasing in z, which implies that:

$$\begin{aligned}\frac{2^k}{q_1^{2k}}\frac{\partial\phi(z;k,q_1)}{\partial q_1} &> -\int_0^{\frac{1}{2}}\frac{2^k(2k+1)(1-2z)}{2^{2k}}dz+\int_{\frac{1}{2}}^1\frac{q_1^{k+1}k(2z-1)}{2^{2k}}dz\\ &> \left(\frac{2^k(2k+1)+q_1^{k+1}k}{2^{2k}}\right)\int_0^1(2z-1)\,dz\\ &> \left(\frac{2^k(2k+1)+q_1^{k+1}k}{2^{2k}}\right)\left(\frac{(2z-1)^2}{4}\right)_0^1=0\end{aligned}$$

Next, we note that $\lim_{q\to0}\phi(p;k,q_1)=0$ and that $\lim_{q\to1}\phi(p;k,q_1)=\overline{c}(k)$. The facts that $\frac{\partial\phi(z;k,q_1)}{\partial q_1}>0$, $\lim_{q\to0}\phi(p;k,q_1)=0$ and $\lim_{q\to1}\phi(p;k,q_1)=\overline{c}(k)$ imply that for any $c\in(0,\overline{c}(k))$ there exists a unique solution, say $q_1^*\in(0,1)$, of the equilibrium condition (5.6), i.e. $\phi(p;k,q_1^*)=c$.

We finally show that consumers do not want to deviate. Given that all consumers randomize between searching once and twice, the expected utility to a consumer who deviates by not searching at all is:

$$Eu^d(q_0=1)=\tilde{p}-\frac{q_1^k}{2^{k-1}}E(p)-\left(1-\frac{q_1^k}{2^{k-1}}\right)E_{\min}(p)$$

For an equilibrium it must be the case that $Eu^d(q_0=1)\leq Eu(q_1=1)$. Using the expression (5.4) it follows that this deviation is not profitable if and only if:

$$c\leq\frac{q_1^k}{2^k}[E(p)-E_{\min}(p)]$$

This condition is always satisfied because in equilibrium $c=\frac{q_1^k}{2^k}[E(p)-E_{\min}(p)]$. This completes the proof of the Theorem. ■

Proof Proposition 5.4.

We recall that the RHS of the equilibrium condition (5.6) may be written as:

$$\phi(z;k,q_1)=\int_0^1\frac{q_1^k}{2^k}p(z;k,q_1)(2z-1)\,dz$$

First, we show that $\frac{\partial \phi(z;k,q_1)}{\partial k} < 0$. The derivative of $\phi(z;k,q_1)$ with respect to k is:

$$\begin{aligned}\frac{\partial \phi(z;k,q_1)}{\partial k} &= -\frac{q_1^{2k+1}}{2^k}\ln\frac{2}{q_1}\int_0^1\left(\frac{q_1^{k+1}+2\left(2^{k+1}-q_1^{k+1}\right)(1-z)}{\left[q_1^{k+1}+2\left(2^k-q_1^{k+1}\right)(1-z)\right]^2}\right) \\ &\quad (2z-1)\,dz+ \\ &\quad +\frac{q_1^{2k+1}}{2^k}\ln q_1\int_0^1\frac{q_1^{k+1}(2z-1)^2}{\left[q_1^{k+1}+2\left(2^k-q_1^{k+1}\right)(1-z)\right]^2}dz\end{aligned}$$

We note that the second term of this expression is weakly negative and that $-\frac{q_1^{2k+1}}{2^k}\ln\frac{2}{q_1}$ is also weakly negative. Therefore it is sufficient to show that:

$$\xi=\int_0^1\left(\frac{q_1^{k+1}+2\left(2^{k+1}-q_1^{k+1}\right)(1-z)}{\left[q_1^{k+1}+2\left(2^k-q_1^{k+1}\right)(1-z)\right]^2}\right)(2z-1)\,dz>0$$

To see this note that:

$$\begin{aligned}\xi &= -\int_0^{\frac{1}{2}}\left(\frac{q_1^{k+1}+2\left(2^{k+1}-q_1^{k+1}\right)(1-z)}{\left[q_1^{k+1}+2\left(2^k-q_1^{k+1}\right)(1-z)\right]^2}\right)(1-2z)+ \\ &\quad +\int_{\frac{1}{2}}^1\left(\frac{q_1^{k+1}+2\left(2^{k+1}-q_1^{k+1}\right)(1-z)}{\left[q_1^{k+1}+2\left(2^k-q_1^{k+1}\right)(1-z)\right]^2}\right)(1-2z)\end{aligned}$$

Since $\left(\frac{q_1^{k+1}+2\left(2^{k+1}-q_1^{k+1}\right)(1-z)}{\left[q_1^{k+1}+2\left(2^k-q_1^{k+1}\right)(1-z)\right]^2}\right)$ is increasing in z for $z\in(0,1/2)$ and $\left[q_1^{k+1}+2\left(2^k-q_1^{k+1}\right)(1-z)\right]$ is decreasing in z, then:

$$\begin{aligned}\xi &> -\int_0^{\frac{1}{2}}\left(\frac{2^{k+1}}{2^{2k}}\right)(1-2z)+ \\ &\quad +\int_{\frac{1}{2}}^1\left(\frac{q_1^{k+1}+2\left(2^{k+1}-q_1^{k+1}\right)(1-z)}{2^{2k}}\right)(1-2z)\end{aligned}$$

Furthermore, $q_1^{k+1}+2\left(2^{k+1}-q_1^{k+1}\right)(1-z)$ is also decreasing in z, which implies that

$$\begin{aligned}
\xi &> -\int_0^{\frac{1}{2}} \frac{2^{k+1}}{2^{2k}}(1-2z) + \int_{\frac{1}{2}}^{1} \frac{q_1^{k+1}}{2^{2k}}(1-2z) = \\
&= \frac{2^{k+1}+q_1^{k+1}}{2^{2k}} \int_0^1 (2z-1)\,dz \\
&= \frac{2^{k+1}+q_1^{k+1}}{2^{2k}} \left(\frac{(2z-1)^2}{4} \right)_0^1 = 0
\end{aligned}$$

The fact that $\frac{\partial\phi(z;k,q_1)}{\partial k} < 0$ and that $\frac{\partial\phi(z;k,q_1)}{\partial q_1} > 0$ implies that if k increases then q_1 must also increase.

Second, we show that if k increases, then expected prices increase as well. Let $\psi(k,q_1) = \frac{q_1^{k+1}}{2\left(2^k - q_1^{k+1}\right)}$, then the expression of the price distribution defined in Proposition 5.3 can be rewritten as:

$$F(p) = 1 - \psi \frac{\tilde{p}-p}{p}$$

To prove the claim it is enough to show that:

$$\frac{d\psi}{dk} = \frac{\partial\psi}{\partial k} + \psi \frac{\partial q_1}{\partial k} > 0$$

We denote $\phi_k(k,q_1) = \partial\phi(k,q_1)/\partial k$ and $\phi_{q_1}(k,q_1) = \partial\phi(k,q_1)/\partial q_1$. Using the equilibrium condition $\phi(k,q_1) - c = 0$ and applying the implicit function theorem we can derive

$$\frac{\partial q}{\partial k} = -\frac{\phi_k(\cdot)}{\phi_{q_1}(\cdot)}$$

where

$$\begin{aligned}
\phi_k(k,q_1) &= -\frac{q_1^k}{2^k} \ln\frac{2}{q_1} \psi[(1+2\psi)\ln\frac{1+\psi}{\psi} - 2] + \\
&\quad + \frac{q_1^k}{2^k}\psi_k[(1+4\psi)\ln\frac{1+\psi}{\psi} - \frac{3+4\psi}{1+\psi}] \\
\phi_{q_1}(q_1,k) &= \frac{kq_1^{k-1}}{2^k}\psi[(1+2\psi)\ln\frac{1+\psi}{\psi} - 2] + \\
&\quad + \frac{q_1^k}{2^k}\psi_{q_1}[(1+4\psi)\ln\frac{1+\psi}{\psi} - \frac{3+4\psi}{1+\psi}]
\end{aligned}$$

Plugging the expressions $\frac{\partial q_1}{\partial k}$ in $\frac{d\psi}{dk}$, we obtain that

$$\frac{d\psi}{dk} = \frac{\psi}{\phi_{q_1}}\left(\left((1+2\psi)\ln\frac{1+\psi}{\psi}-2\right)\left(\psi_k\frac{kq^{k-1}}{2^k}+\psi_{q_1}\frac{q_1^k}{2^k}\ln\frac{2}{q_1}\right)\right)$$

Since ϕ_{q_1} and $\psi(q_1,k)$ are strictly positive it follows that $\frac{d\psi}{dk} > 0$ if and only if

$$\left((1+2\psi)\ln\frac{1+\psi}{\psi}-2\right)\left(\psi_k\frac{kq_1^{k-1}}{2^k}+\psi_{q_1}\frac{q_1^k}{2^k}\ln\frac{2}{q_1}\right) > 0$$

Computing the derivatives $\psi_k = -\frac{q_1^{k+1}2^k}{2\left(2^k-q_1^{k+1}\right)^2}\ln\frac{2}{q_1}$ and $\psi_{q_1} = \frac{(k+1)q_1^k 2^k}{2\left(2^k-q_1^{k+1}\right)^2}$ it follows that:

$$\left(\psi_k\frac{kq_1^{k-1}}{2^k}+\psi_{q_1}\frac{q_1^k}{2^k}\ln\frac{2}{q_1}\right) = \frac{q_1^{2k}}{2\left(2^k-q_1^{k+1}\right)^2}\ln\frac{2}{q_1} > 0$$

Furthermore, using the expression of $\psi(k,q_1)$ we obtain that:

$$\begin{aligned}(1+2\psi)\ln\frac{1+\psi}{\psi}-2 &= \frac{2^k}{2^k-q_1^{k+1}}\ln\left(\frac{\left(2^{k+1}-q_1^{k+1}\right)}{q^{k+1}}\right)-2 > \\ &> \frac{2^k}{2^k-1}\ln\left(2^{k+1}-1\right)-2 > \\ &> -2+2\ln 3 > 0\end{aligned}$$

This proves the claim.

Third, we show that social welfare increases as k increases. To see this note that for a given k, the social welfare is $SW(k,q_1,c) = \tilde{p}-q_1c-(1-q_1)2c = \tilde{p}-2c+q_1c$; since when k increases, q_1 increases then social welfare increases as well.

Finally, we show that the consumer surplus decreases as k increases. To see this note that the consumer surplus is $CS = Eu(q_1=1) = Eu(q_2=1) = \tilde{p}-E_{\min}(p)-2c$. Given he price distribution $F(p;k,q_1)$, the distribution of the minimum price is $F_{\min}(p;,k,q_1) = F(p;k,q_1)(2-F(p;,k,q_1))$. Using the expression for $F(p;k,q_1)$ illustrated in proposition 5.3. we obtain

$F_{\min}(p;,k,q_1) = 1 - \psi^2 \left(\frac{\tilde{p}-p}{p}\right)^2$. Therefore $\frac{\partial F_{\min}(p;,k,q_1)}{\partial k} < 0$ if and only if $\frac{d\psi}{dk} > 0$, which follows from above. This completes the proof.■

Low Search Intensity Equilibrium

Proof of Proposition 5.5.

We first note that for an equilibrium $\overline{p} = \tilde{p}$; for otherwise a firm charging $\bar{p} < \tilde{p}$ strictly gains by increasing such price. Using expression 5.11 and the fact that $\overline{p} = \tilde{p}$, it follows that the expected equilibrium profit is $E\pi^* = \left(m\left((1+q)^{k+1} - 2^{k+1}q^{k+1}\right)/2^{k+1}\right)\tilde{p}$. In equilibrium it must be the case that $E\pi(p) = E\pi^* \ \forall \ p \in \sigma$. Solving the equilibrium conditions we obtain the expression for $F(p;k)$. Similarly, the expression of the lowerbound is the solution of $E\pi\left(\underline{p}\right) = E\pi^*$. Finally, let $\psi(k,q_0) = \frac{(1+q_0)^{k+1} - 2^{k+1}q_0^{k+1}}{2\left(2^k\left(1+q_0^{k+1}\right)-(1+q_0)^{k+1}\right)}$; then to prove the first order stochastic dominance relation it is enough to see that

$$\frac{\partial\psi(k,q_0)}{\partial k} = \frac{\left[\begin{array}{c} 2^k\left(1-q_0^{k+1}\right)(1+q_0)^{k+1}\left(\ln(1+q_0)-\ln 2\right)+ \\ 2^k q_0^{k+1}\left((1+q_0)^{k+1} - 2^{k+1}q_0^{k+1}\right)\ln(q_0) \end{array}\right]}{2\left(2^k\left(1+q^{(k+1)}\right)-(1+q)^{k+1}\right)^2} < 0$$

■

Proof of Theorem 5.2.

Without loss of generality let $\tilde{p} = 1$. Using the expression of the price distribution $F(p;k,q_0)$ defined in proposition 5.5, we can invert it to obtain:

$$p(z,k,q_0) = \frac{1}{g(z;k,q_0)} \tag{5.20}$$

where

$$g(z;k,q_0) = 1 + \frac{2^{k+1}\left(1+q_0^{k+1}\right) - 2(1+q_0)^{k+1}}{(1+q_0)^{k+1} - 2^{k+1}q_0^{k+1}}(1-z) \tag{5.21}$$

Using (5.20) the equilibrium condition (5.16) can be rewritten as follows:

$$\begin{aligned} c &= \frac{(1+q_0)^k - 2^{k+1}q_0^k}{2^k}\int_0^1 p(z,k,q_0)(2z-1)\,dz + \\ &\quad + q_0^k\left(1-2\int_0^1 p(z,k,q_0)(1-z)\,dz\right) \end{aligned} \tag{5.22}$$

We denote as $\rho(z;k,q_0)$ the LHS of 5.22 and we note that:

$$\begin{aligned} \lim_{q_0\to 0}\rho(z;k,q_0) &= \bar{c}(k) \\ \lim_{q_0\to 1}\rho(z;k,q_0) &= 0 \end{aligned}$$

Furthermore, we note that limit when q_0 goes to zero of the derivative of $\rho(z,k,q)$ is positive:[20]

$$\lim_{q_0\to 0}\frac{\partial\rho(z;,k,q_0)}{\partial q_0} = 1$$

Hence, since $\rho(q_0,k)$ is positive at $q_0 = 0$, increasing in the neighbor of $q_0 = 0$ and it is zero at $q_0 = 1$ it follows that for any $k > 0$ there exists a $\tilde{c} > \bar{c}(k)$ such that for any $c \in [\bar{c}(k), \tilde{c}]$ There exists at least two solutions of the equilibrium condition (5.16). It is easy to see that among these two solutions only the smaller one is stable.

We finally show that a consumer does not have an incentive to deviate. The expected utility to a consumer who deviates by searching twice is:

$$Eu^d(q_2 = 1) = \tilde{p} - E_{\min}(p) - 2c$$

For an equilibrium it must be the case that this deviation is not profitable, i.e. $Eu^d(q_2 = 1) \leq Eu(q_1 = 1)$. Using the expression (5.14) we obtain that $Eu^d(q_2 = 1) \leq Eu(q_1 = 1)$ if and

[20] I develop the result using the program Mathematica. To do this I compute the following transformation. Let $\rho_{q_0}(q_0,k) = \frac{\partial\rho(q_0,k)}{q_0}$, then $\lim_{q_0\to 0}\rho_{q_0}(q_0,k) = e^{\lim_{q_0\to 0}\ln\left(\rho_{q_0}(q_0,k)\right)^{q_0}}$. The computation is avaialable upon request of the author.

only if:

$$c \geq \frac{(1+q_0)^k}{2^k}\left[E(p) - E_{\min}(p)\right]$$

Using the equilibrium condition (5.16), we can rewrite this inequality as

$$E(p) - E_{\min}(p) \leq \tilde{p} - E(p)$$

In the proof of proposition 5.2 we have showed that:

$$E(p) - E_{\min}(p) = [\tilde{p} - E(p)] - \int_{\underline{p}}^{\tilde{p}} [F(p)]^2 \, dp < [\tilde{p} - E(p)]$$

Hence, given that all consumers randomized between searching once and not searching at all, a consumer does not want to deviate by searching twice. This completes the proof.■

Conclusion

You can call it a clan, or a network, or a family, or a group of friends. The way you call it is not relevant. What matters is that it exists and often you will need one. A large body of empirical work shows that networks are pervasive in social and economic interactions. This is the primary motivation to develop a systematic theory of networks.

On the one hand, the theory of network formation attempts to provide a micro foundation for the emergence of social and economic networks. How do networks come about when individuals have the discretion to form informal relationships ? What are the structural properties we are expected to observe ? Are strategic networks efficient ?

On the other hand, we want to examine how networks influence strategic decision making in a variety of settings. This requires the development of a common approach for addressing economic questions from a structural perspective. That is, an approach where the players and the set of relationships across them are not considered as independent but, on the contrary, as strategically connected. Do networks affect the emergence of efficient social norms? How do consumers networks, firms alliances, and buyers and sellers networks influence the functioning of the market?

The first part of this thesis deals with network formation models. Bala and Goyal (2000a) offer a simple framework for the study of this issue. The basic idea of their model is the following: social networks are the result of the choice of individual players who trade-off the costs of investing in links with their potential rewards (which consist in the benefits of accessing other players). One of the crucial assumptions of their model is that both the costs of forming links and the values of accessing other agents are homogeneous across players. Chapter 2 and 3 examine the role that heterogeneity, both in the costs of forming a link and in the benefits arising from it, plays in shaping the architecture of equilibrium networks.

Chapter 2 extends the two-way flow model of Bala and Goyal (2000a) to include ex-ante asymmetries across players. The formation of many networks, such as communication networks, social gathering, phone calls and consumers networks, may be interpreted in the spirit of this model. Empirical work on these networks shows a widespread stability of structural properties such as centrality and short-distances across players. The equilibrium predictions of Bala and Goyal (2000a) are in line with these empirical findings: an equilibrium network is either a center-sponsored star or the empty network. The findings reported in Chapter 2 show that centrality and short-distances are confirmed as distinctive features of equilibrium networks even in settings with substantial heterogeneity.

Chapter 3 focuses on the one-way flow model of Bala and Goyal (2000a). The formation of the World Wide Web can be interpreted in the spirit of this model. Indeed, Web connections are generally formed unilaterally. Furthermore, a link created by user i with user j only allows the former to click-and-go from his home page to the home page of the latter, while the reverse does not hold true. Empirical investigations report that the World Wide Web is fragmented and (within the core component) only few players sponsor and/or receive many links (the so-called central players). These properties contrast with the equilibrium characterization of Bala and Goyal (2000a): an equilibrium network is either a wheel network or the empty network. We find that player heterogeneity breaks the connectedness of equilibrium networks and that central players emerge in unconnected equilibria. Thus, in the one-way flow model players' heterogeneity is crucial to predict properties, such as centrality, which are widely observed in real social networks.

Chapter 4 examines how the endogenous formation of social and economic relationships affects free-riding problems. The model applies to situations in which connections are used to share non-rival goods such as information and knowledge. In these cases, the externalities produced in the network are fully realized only if the relationships are stable over time. Free-riding problems may undermine the stability of social relationships. For example, it is natural to think that both the creation and the maintenance of a relationship require costly investments, and that each of the two parties prefers the other to bear such costs. Henceforth, it is important to investigate the interplay between stable network architectures and individual incentives in games of conflict.

The results show that the stability of efficient norms crucially depends on the network formed by the players. More specifically, in settings where efficiency requires only one of the parties to bear the entire cost of the interaction (one party cooperates and the other one free-rides), efficient social norms are best sustained in the star network. Here, the central player free-rides, while the peripheral players cooperate. The star architecture enhances the stability of efficient norms because it allows the central player to detect any eventual deviator, and to punish him with immediate social isolations. By contrast, when an efficient interaction requires the players to evenly split the interaction costs (this is the case of mutual cooperation), efficient social norms are best sustained in the line network. The reason behind this result is that players' incentives to invest in connections are inversely related to the number of links held by each player. These results indicate that a structural analysis is important in order to understand how individual incentives are shaped in many strategic contexts.

The effects of consumers' networks on consumers' search activity, on firms' price behavior and on social welfare constitute the focus of chapter 5. A large body of empirical work suggests that the interactions between buyers and sellers are built on a variety of network relationships. Examples of these are consumers' networks, professional networks and R&D networks. All these examples share a common feature: informal relationships transform the information privately obtained by each individual to a public good; this affects players' incentives as well as aggregate outcomes.

We study the interplay between market externalities and consumers' network externalities in a search model a' la Burdett and Judd (1983). Consumers are embedded in a network and they may costly search for price quotations; the information gathered is non-excludable along direct links. The first result is that in the absence of the network (Burdett and Judd (1983)) only a high search intensity equilibrium exists. Otherwise, this equilibrium exists for low search costs, while for moderate search costs a low search intensity equilibrium exists. The second result shows that in both equilibria consumers search less frequently in more dense networks. Finally, this equilibrium property has a somewhat surprising effect on firms' price behavior and on social welfare: when search costs are low (moderate) the expected equilibrium price and the social welfare are higher (lower) when consumers hold more connections.

In conclusion, this essay is a contribution to the theory of networks. It addresses a number of different questions such as the formation of networks and the effects networks have on strategic decision making. What makes the research developed in this thesis novel is the explicit consideration of decentralized interaction among individuals to analyse a variety of traditional economic questions. This approach, which consistently accompanies each chapter, delivers tractable models and provides a number of substantive insights which are novel and match empirical findings.

Samenvatting (Summary in Dutch)

U kunt het een clan noemen, of een netwerk, of een familie, of een groepje vrienden; het doet er niet toe hoe u het noemt. Waar het om gaat is dat het bestaat en dat u het vaak nodig heeft. Een overvloed aan empirisch onderzoek laat zien dat netwerken een grote rol spelen bij sociale en economische interacties. Dit is de voornaamste reden om een systematische theorie over netwerken te ontwikkelen.

Enerzijds tracht de theorie van netwerkvorming een microfundering te leggen voor de totstandkoming van sociale en economische netwerken. Wat voor netwerken komen tot stand als individuen de mogelijkheid hebben om zelf informele contacten aan te gaan? Wat voor structurele eigenschappen zouden wij in deze netwerken kunnen waarnemen? Zijn strategisch gevormde netwerken efficiënt?

Anderzijds willen we onderzoeken hoe netwerken van invloed zijn op strategische beslissingen in verschillende omgevingen. Dit vereist de ontwikkeling van een onderzoeksmethode die economische problemen op eenzelfde manier vanuit een structureel perspectief benadert; een methode waarbij de spelers en de relaties tussen hen niet als onafhankelijk worden beschouwd, maar juist als strategisch met elkaar verbonden. Dragen netwerken bij aan de totstandkoming van efficiënte sociale normen? Hoe wordt het functioneren van de markt beïnvloed door netwerken van consumenten, allianties van bedrijven, en netwerken van kopers en verkopers?

Het eerste deel van dit proefschrift behandelt modellen met betrekking tot de vorming van netwerken. Bala en Goyal (2000a) bieden een eenvoudig theoretisch kader om netwerkvorming te analyseren. Het basisidee van hun model is als volgt: sociale netwerken zijn het gevolg van de keuzes van individuele spelers die de kosten om in netwerkrelaties te investeren afwegen tegen de mogelijke opbrengsten (die bestaan uit de mogelijkheid om via het netwerk andere spelers te benaderen.). Een essentiële veronderstelling van hun model is dat zowel de kosten om een relatie op te bouwen als de voordelen om andere agenten te kunnen benaderen, homogeen tussen spelers zijn. Hoofdstukken 2 en 3 onderzoeken hoe heterogeniteit, zowel in de kosten om een relatie te onderhouden als in de voordelen die het gevolg daarvan zijn, gestalte geeft aan de architectuur van netwerken.

Hoofdstuk 2 breidt het 'two-way flow'-model van Bala en Goyal (2000a) uit met het toevoegen van ex-ante verschillen tussen spelers. De vorming van vele netwerken, zoals communicatienetwerken, sociale bijeenkomsten, telefoongesprekken en consumentennetwerken, kan in de trant van dit model worden geïnterpreteerd. Empirisch onderzoek naar dit soort netwerken toont een wijdverspreide stabiliteit van structurele eigenschappen zoals centraliteit en een korte afstand tussen spelers. De evenwichtsvoorspellingen van Bala en Goyal (2000a) stemmen met deze empirische bevindingen overeen: een evenwichtsnetwerk is of een 'center-sponsored star' (een netwerk in de vorm van een ster waarin de centrale agent de kosten van het onderhouden van contacten op zich neemt) of een leeg netwerk. De bevindingen in Hoofdstuk 2 tonen aan dat een centrale ligging en korte afstanden als specifieke eigenschappen van evenwichtsnetwerken ook worden bevestigd in omgevingen met aanzienlijke verschillen tussen spelers.

Hoofdstuk 3 concentreert zich op het 'one-way flow'-model van Bala en Goyal (2000a). De vorming van het World Wide Web kan in de trant van dit model worden geïnterpreteerd. De links tussen websites worden namelijk over het algemeen unilateraal gevormd. Bovendien staat een link die door gebruiker i naar de homepage van gebruiker j wordt aangemaakt, slechts de eerstgenoemde toe om van zijn homepage naar de homepage van de laatstgenoemde te surfen, terwijl het omgekeerde niet mogelijk is. Empirische onderzoeken tonen aan dat het World Wide Web versplinterd is, en (binnen de kerncomponent) slechts weinig spelers vele links onderhouden en ontvangen (de 'centrale spelers'). Deze eigenschappen staan tegenover de evenwichtskarakterisering in Bala en Goyal (2000a): een evenwichtsnetwerk is of een 'wheel' (een netwerk in de vorm van een kring) of een leeg netwerk. Wij vinden dat heterogeniteit tussen spelers de samenhang van evenwichtsnetwerken verbreekt en dat in deze niet-samenhangende evenwichtsnetwerken centrale spelers aanwezig zijn. Dus in het 'one-way flow'-model is de heterogeniteit tussen spelers essentieel om eigenschappen zoals centraliteit te kunnen verklaren die wijdverbreid in echte sociale netwerken worden waargenomen.

Hoofdstuk 4 onderzoekt hoe de endogene vorming van sociale en economische relaties van invloed is op 'free-rider' problemen. Het model is op situaties van toepassing waarin contacten worden aangewend om niet-rivaliserende goederen zoals informatie en kennis te delen. In deze gevallen komen de netwerkexternaliteiten alleen volledig tot stand als de netwerkrelaties stabiel zijn. 'Free-rider'-problemen kunnen de stabiliteit van sociale contacten ondermijnen. U kunt zich bijvoorbeeld indenken dat zowel het aangaan als het onderhouden van een relatie kosten met zich meebrengt, en dat de één graag ziet dat de ander deze kosten op zich neemt. Het is daarom van belang om te onderzoeken hoe in conflictspellen een stabiele netwerkarchitectuur zich verhoudt met individuele prikkels.

De resultaten laten zien dat het voor de stabiliteit van efficiënte normen van cruciaal belang is wat voor netwerk door de spelers gevormd wordt. Om specifieker te zijn, in omgevingen waarin efficiëntie al bereikt wordt als slechts één kant van een relatie de omgangskosten op zich neemt (één kant werkt mee terwijl de andere kant hiervan profiteert) zijn efficiënte sociale normen het beste in een 'ster'-netwerk gediend. Hierin profiteert de centrale speler van de spelers in de periferie terwijl zij al het werk doen. De 'ster'-architectuur bevordert de stabiliteit van efficiënte normen omdat dit het voor de centrale speler mogelijk maakt om iedere afwijkende speler direct op te merken, en hem meteen te straffen door hem in een sociaal isolement te plaatsen. Aan de andere kant, als het voor een efficiënte omgang noodzakelijk is dat de spelers de omgangskosten eerlijk delen (wederzijdse samenwerking), dan zijn efficiënte sociale normen het beste in een 'lijn'-netwerk gediend. De reden achter dit resultaat is dat de prikkels van spelers om in contacten te investeren minder worden als het aantal contacten van iedere speler toeneemt. Deze resultaten wijzen erop dat een structurele analyse belangrijk is om te begrijpen hoe in verscheidene situaties gestalte wordt gegeven aan individuele prikkels.

De effecten van netwerken van consumenten op de zoekactiviteit van consumenten, het prijsgedrag van bedrijven en op het sociale welzijn staan centraal in hoofdstuk 5. Een aanzienlijke hoeveelheid empirisch onderzoek suggereert dat de interacties tussen kopers en verkopers gestoeld zijn op een verscheidenheid aan netwerkrelaties. Voorbeelden daarvan zijn netwerken van consumenten, professionele netwerken en 'R&D'-netwerken. Al deze voorbeelden hebben het volgende overeen; informele contacten zetten informatie dat door ieder individu persoonlijk verkregen is, om in een publiek goed; dit heeft zowel invloed op de prikkels van de spelers als op geaggregeerde uitkomsten.

We bestuderen het samenspel tussen de externaliteiten van de markt en de externaliteiten van de netwerken van consumenten in een zoekmodel à la Burdett en Judd (1983). Consumenten zijn in een netwerk ingebed en ze kunnen naar prijsaanduidingen zoeken, dat zoekkosten met zich meebrengt; de verzamelde informatie is niet uitsluitbaar voor de directe contacten. Het eerste resultaat is dat als een netwerk afwezig is (Burdett en Judd (1983)) er alleen een evenwicht met een hoge zoekintensiteit bestaat. In de aanwezigheid van een netwerk bestaat dit evenwicht voor lage zoekkosten, terwijl voor gematigde zoekkosten een evenwicht met lage zoekintensiteit bestaat. Het tweede resultaat toont aan dat in beide evenwichten consumenten minder vaak zoeken in dichte netwerken. Tot slot heeft deze evenwichtseigenschap een enigszins verrassend effect op het prijsgedrag van bedrijven en op het sociale welzijn; als de zoekkosten laag (gematigd) zijn, dan is de verwachte evenwichtsprijs en sociaal welzijn hoger (lager) indien consumenten meer contacten onderhouden.

Samenvattend, dit proefschrift levert een bijdrage aan de theorie van netwerken. Het richt zich op een aantal verschillende problemen zoals de vorming van netwerken en de effecten die netwerken op het maken van strategische beslissingen hebben. Wat het onderzoek in dit proefschrift zo vernieuwend maakt is de expliciete beschouwing van gedecentraliseerde interacties tussen individuen om een verscheidenheid aan traditionele economische problemen te analyseren. Deze benadering, die steeds in ieder hoofdstuk naar voren komt, levert handelbare modellen en geeft een aantal belangrijke inzichten die vernieuwend zijn en die met empirische bevindingen overeenstemmen.

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