A NOTE ON THE DUAL OF AN UNCONSTRAINED (GENERALIZED) GEOMETRIC PROGRAMMING PROBLEM

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A note on the dual of an unconstrained (generalized) geometric programming problem.

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Abstract

In this note we show that the strong duality theorem of an unconstrained (generalized) geometric programming problem as defined by Peterson (cf.[1]) is actually a special case of a Lagrangian duality result. Contrary to [1] we also consider the case that the set C is compact and convex and in this case we do not need to assume the standard regularity condition.

Keywords: Generalized geometric programming, Lagrangian dual, regularity conditions.

1 Introduction.

In [1] a dual problem is introduced for a so-called unconstrained (generalized) geometric programming problem. To introduce this optimization problem we will use the same notation as in [1]. Although this notation is not standard for a (convex) cone in convex analysis it is done to facilitate comparing the result in this note and Theorem 1 in [1]. Let $X \subseteq \mathbb{R}^n$ be a nonempty proper cone, $C \subseteq \mathbb{R}^n$ a nonempty set and $g : C \to \mathbb{R}$ a real-valued function. An unconstrained (generalized) geometric programming problem is denoted in [1] by problem A and given by

$$\varphi := \inf_{x \in X \cap C} g(x). \tag{A}$$

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It is argued in [1] that the importance of the above formulation with a proper cone $X \subseteq \mathbb{R}^n$ is due to the possibility to reformulate a nonseparable optimization problem into an instance of problem A with a separable objective function g and a "separable" proper cone. This procedure is then carried out for a quadratic optimization problem with linear constraints. Introducing as in [1] the function $h : \mathbb{R}^n \to (-\infty, \infty]$ given by

$$h(y) := \sup\{y^\top x - g(x) : x \in C\}. \tag{1}$$

and the set

$$D := \{y \in \mathbb{R}^n : h(y) < \infty\}$$

the dual problem of problem A is denoted by problem B and this problem has the form

$$\psi := \inf\{h(y) : y \in Y \cap D\} \tag{B}$$

with Y the so-called dual cone of X given by

$$Y = \{y \in \mathbb{R}^n : y^\top x \geq 0 \text{ for every } x \in X\}.$$

In Theorem 1 of [1] a strong duality result between problem A and problem B (under some regularity condition) is given for the case that C is a closed convex set, X a closed convex cone and g a proper lower semicontinuous convex function on C. In this note we will show that a similar result holds without any regularity condition if additionally C is compact and $g : U \to \mathbb{R}$ a real-valued convex function on an open set U containing C. To verify this we first give a Lagrangian dual interpretation of problem B. In our opinion the Lagrangian dual approach in combination with the natural concept of a relaxation of a problem is a pedagogically good approach teaching duality theory to students in Operations Research and Engineering. At the same time our proof can also be easily adapted to the noncompact case as considered in [1]. To do this we first let problem A resemble a standard nonlinear programming problem (see for example [2]) by introducing the binary relation $\leq_X$given by $x_1 \leq_X x_2$ if and only if $x_2 - x_1 \in X$. Observe this binary relation is transitive if X is a convex cone. Now problem A is the same as

$$\inf\{g(x) : x \in C, x \geq_X 0\}. \tag{A}$$

Inserting now the restriction $x \geq_X 0$ into the objective function and penalizing it with a Lagrangian dual variable $y \in Y$ we introduce the Lagrangian function $\theta : Y \to (-\infty, \infty)$ given by

$$\theta(y) := \inf\{g(x) - y^\top x : x \in C\}.$$

It follows immediately for every $y \in Y$ that

$$\theta(y) \leq \inf\{g(x) - y^\top x : x \in C, x \geq_X 0\} \leq \varphi. \quad (3)$$

Since we like to achieve the value φ it is now obvious by relation (3) to consider the Lagrangian problem

$$\vartheta := \sup_{y \in Y} \theta(y)$$

By relation (1) it follows that $\sup_{y \in Y} \theta(y) = -\inf_{y \in Y} h(y)$ and this is the dual problem considered in [1]. Moreover, it follows by (3) that

$$-\psi = \vartheta \leq \varphi.$$

We will now show the following extension of Theorem 1 in [1] for C a compact convex set. Observe the proof of this result is an adaptation of the proof in subsection 10.1.1 in [2] given for the Lagrangian dual of optimization problem $\inf\{g(x) : x \in C, x \geq 0\}$. A similar result also appeared on page 150 of [7]. To keep the paper selfcontained we first mention part 2 of Theorem 1 of Peterson (cf.[1]). Actually the formulation given below is a slight extension of part 2 of Theorem 1 in [1]. In the next formulation we do not assume contrary to [1] that the function $g : C \to \mathbb{R}$ is a lower semicontinuous function.

Theorem 1 *If X is a closed convex cone, C a convex set, $ri(C) \cap ri(X)$ is nonempty and $g : C \to \mathbb{R}$ is a real-valued convex function on C, then it follows that problem B has an optimal solution and $\varphi = -\psi$.*

The condition $ri(C) \cap ri(X) \neq \varnothing$ is a so-called regularity condition and this condition trivially implies that the feasible set

$$\mathcal{F} := \{x \in C : x \geq_X 0\}$$

of problem A is nonempty. In this note we will show in the next proof if additionally C is a compact convex set and of course $\mathcal{F}$ is nonempty, then the same result holds for any real-valued convex function $g : U \to \mathbb{R}$ with U an open set containing X. We also observe that it is easy to adapt the next proof to verify Theorem 1 (see Remark 2).

Proof. We already showed $\vartheta \leq \varphi$. To prove that $\vartheta = \varphi$ we assume by contradiction that $\vartheta < \varphi$ and so there exists some $v \in \mathbb{R}$ satisfying

$$\vartheta < v < \varphi. \quad (4)$$

Introduce now the vector valued function $h : C \to \mathbb{R}^{n+1}$ given by $h(x) := (x, g(x))^\top$. If $h(C) := \{h(x) : x \in C\}$ denotes the range of the function h and $\mathbb{R}_+ = [0, \infty)$ we first observe that

$$h(C) + ((-X) \times \mathbb{R}_+) = \{(x - k, r) : r \geq g(x), x \in C, k \in X\} \tag{5}$$

This shows by the convexity of the function g on U and X a convex cone that the set $h(C) + ((-X) \times \mathbb{R}_+)$ is convex. Moreover, since $g : U \to \mathbb{R}$ is a real-valued convex function on the open set U the function g is continuous on U (cf.[2]). By the compactness of $C \subseteq U$ this implies that the set $h(C)$ is compact and using X is a closed convex cone we obtain that the set $h(C) + ((-X) \times \mathbb{R}_+)$ is also closed. It is now easy to verify by contradiction that the vector $(0, v)$ does not belong to $h(C) + ((-X) \times \mathbb{R}_+)$ and so we can strictly separate the closed convex set $h(C) + ((-X) \times \mathbb{R}_+)$ and the point $(0, v)$ (see for example [6], [4] or [5]). Hence there exists some nonzero vector $(y_*, \alpha) \in \mathbb{R}^{n+1}$ such that

$$\alpha v < -y_*^\top (x - k) + \alpha(g(x) + \beta) \tag{6}$$

for every $x \in C$, $\beta \geq 0$ and $k \in X$. Since $\beta \geq 0$ and $k \in X$ in relation (6) it must follow by contradiction that $\alpha \geq 0$ and $y_* \in Y$. If we assume by contradiction that $\alpha = 0$ we obtain by relation (6) and $0 \in -X$ ($-X$ is a closed convex cone!) that $y_* \in Y$ is nonzero and

$$y_*^\top x < 0 \tag{7}$$

for every $x \in C$. Since by assumption the feasible region $\mathcal{F} = \{x \in C : x \geq_X 0\}$ is nonempty and $y_* \in Y$ one can find some $x_0 \in \mathcal{F} \subseteq C$ satisfying $y_*^\top x \geq 0$ and this contradict relation (7). Hence $\alpha > 0$ and by relation (6) we obtain

$$v < g(x) - \alpha^{-1} y_*^\top x$$

for every $x \in C$. Since $\alpha^{-1} y_* \in Y$ this implies by relation (4) that $v < \theta(\alpha^{-1} y_*) \leq \vartheta < v$ and we have a contradiction. This proves the desired result. □

Although we did not give a proof of Theorem 1 it is easy to adapt the above proof to show this result. How to change this proof is discussed in the next remark.

Remark 2 *If the convex set C is not compact it still follows that the set $H(C) + (\mathbb{R}_+ \times (-X))$ is convex for $g : C \to \mathbb{R}$ a real-valued convex function on the convex set C and $(0, v)$ does not belong to $H(C) + ((-X) \times \mathbb{R}_+)$. However,* *the set $H(C) + ((-X) \times \mathbb{R}_+)$ might not be closed and so we cannot apply the strict separation result. However, we can apply in this case a slight extension of the strict separation result given by the proper separation result between a convex set and a point outside this set (see for example Theorem 1.2 of [4] or [6]). Since in the proper separation result it follows that (y_*, α) is nonzero and y_* can be selected from the affine hull of the set $C - X$ and it is well-known ([6]) that $ri(C - X) = ri(C) - ri(X)$ the above proof can be easily adapted to verify Theorem 1. Actually one can use this proof to show a strong duality result for the Lagrangian dual of a more general optimization problem (cf.[3]).*

To extend Theorem 1 to nonconvex functions g and C not convex we first introduce the m-dimensional unit simplex

$$\Delta_m := \{\lambda \in \mathbb{R}^m : \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1\}.$$

It follows for every x belonging to $conv(C)$ that there exists by definition some $\{x_1, ..., x_m\} \subseteq C, m \in \mathbb{N}$ satisfying

$$x = \sum_{i=1}^m \lambda_i x_i, \lambda \in \Delta_m. \tag{8}$$

By the definition of the Lagrangian function θ and relation (8) this yields

$$\theta(y) \leq \inf_{1 \leq i \leq m}\{g(x_i) - y^\top x_i\} \leq \sum_{i=1}^m \lambda_i g(x_i) - y^\top x \tag{9}$$

for every $y \in Y$. If the function $g : C \to \mathbb{R}$ is bounded below by a affine function on C it is shown in Proposition $2.5.1$ of [5] that the biggest convex function $co(g) : conv(C) \to \mathbb{R}$ majorized by g on C is given by

$$co(g)(x) := \inf \left\{ \sum_{i=1}^m \lambda_i g(x_i) : \begin{array}{c} x = \sum_{i=1}^m \lambda_i x_i \\ \lambda \in \Delta_m, x_i \in C \end{array}, m \in \mathbb{N} \right\} \tag{10}$$

This implies by relation (9) that

$$\theta(y) \leq co(g)(x) - y^\top x$$

for every $x \in conv(C)$ and $y \in Y$. Hence by the definition of the Lagrangian function θ and $co(g)$ we obtain

$$\theta(y) = \inf\{co(g)(x) - y^\top x : x \in conv(C)\} \tag{11}$$

for every $y \in Y$ and using Theorem 1 the next result follows.

Theorem 3 *If X is a closed convex cone, $ri(conv(C)) \cap ri(X)$ is nonempty and $g : C \to \mathbb{R}$ is bounded below by an affine function on C, then it follows that problem B has an optimal solution and*

$$\sup_{y \in Y} \theta(y) = \inf\{co(g)(x) : x \in conv(C), x \geq_X 0\}.$$

If the set $C \subseteq \mathbb{R}^n$ is compact we observe by Lemma 1.8 of [4] that the set $conv(C)$ is also compact. To formulate now Theorem 3 for C compact we assume additionally that the real-valued function g is defined on an open bounded set U containing C and that $g : U \to \mathbb{R}$ is bounded below (on this open set) by an affine function. This implies by relation (10) that the convex function $co(g) : conv(U) \to \mathbb{R}$ is bounded below by an affine function on the open convex set $conv(U) \subseteq conv(C)$. Hence the convex function $co(g)$ is real-valued on $conv(U)$ and so it is continuous on $conv(U) \subseteq conv(C)$. Replacing now C by $conv(C)$ and g by $co(g)$ in the previous proof and applying relation (11) it follows that Theorem 3 also holds for C compact, the real-valued function $g : U \to \mathbb{R}$ is bounded below by an affine function on an open bounded set U containing C and $conv(C) \cap X$ is nonempty.

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