A note on the paper Fractional Programming with convex quadratic forms and functions by H.P.Benson

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A note on the paper Fractional Programming with convex quadratic forms and functions by H.P.Benson

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Abstract

In this technical note we give a short proof based on standard results in convex analysis of some important characterization results listed in Theorem 3 and 4 of [1]. Actually our result is slightly general since we do not specify the convex set X. For clarity we use the same notation for the different equivalent optimization problems as done in [1].

1 Introduction.

In [1] some important theoretical results are given in Theorems 3 and 4. In this note we will give an alternative short proof of these results. Consider as in [1] optimization problem (P_2) given by

$$\max\{\frac{x^\top Qx}{g(x)} : x \in X\} \tag{P$_2$}$$

with X a compact convex set, Q a symmetric positive semidefinite matrix and g a finite convex and positive function on an open convex set containing X. To avoid the pathological case that (P_2) is a convex program we assume that g is not affine. Since g is a finite convex function on a open set containing X it is well-known that g is continuous on X and hence by Weierstrass theorem (cf.[3])

$$0 < m := \min\{g(x) : x \in X\} \text{ and } M := \max\{g(x) : x \in X\} < \infty$$

Since $x^\top Qx \geq 0$ it follows for every given $x \in X$ that

$$\frac{x^\top Qx}{g(x)} = \max\{\frac{x^\top Qx}{t} : t \geq g(x)\} \tag{1}$$

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and this shows with $p(x,t) := \frac{x^\top Qx}{t}$ and

$$\mathcal{F} := \{(x,t) : x \in X, t \geq g(x), m \leq t \leq M\} \tag{2}$$

that the optimization problem (P_3)

$$\max\{p(x,t) : (x,t) \in \mathcal{F}\} \tag{P$_3$}$$

is in the following sense equivalent to optimization problem (P_2) (see also Proposition 3 of [1]).

Lemma 1 *The vector (x^*, t^*) is an optimal solution of (P_3) if and only if x^* is an optimal solution of (P_2) with optimal objective value $t^* = g(x^*)$.*

We will now investigate the feasible region $\mathcal{F}$. Since g is a continuous convex function on the compact convex set X we obtain that

$$epi(g) := ((x,t) : t \geq g(x), x \in X\}$$

is a closed convex set and by relation (2) the set $\mathcal{F}$ is a compact convex set . For the convex function $h(x) = x^\top Qx$ it is well-known that its so-called perspective

$$(x,t) \rightarrow th(\frac{x}{t}) = \frac{x^\top Qx}{t}$$

of h is again convex (cf.[2]) and so the function $(x,t) \rightarrow p(x,t)$ is convex. We will now further simplify the optimization problem (P_3) using the so-called reduction to principal axes. Since Q is a symmetric positive semidefinite matrix we know that there exists an orthonormal matrix $W = [w_1, ..., w_n]$ with w_j the eigenvector of Q belonging to the nonnegative eigenvalue α_j such that $Q = W^\top DW$. In this case D is a diagonal matrix consisting of the nonnegative eigenvalues $\alpha_j, 1 \leq j \leq n$. By the definition of an orthonormal matrix it follows that $W^\top W = I$. This implies substituting $x = Wy$ in problem (P_3) that we obtain the optimization problem (P_4) given by

$$\max\{p(Wy,t) : (y,t) \in \mathcal{F}_1\} \tag{P$_4$}$$

with the transformed feasible region $\mathcal{F}_1$

$$\mathcal{F}_1 = \{(y,t) : Wy \in X, t - g(Wy) \geq 0, m \leq t \leq M\}.$$

Since X is compact and convex and W is invertible we obtain that $W^{-1}(X) = \{y \in \mathbb{R}^n : Wy \in X\}$ is also compact and convex and so $\mathcal{F}_1$ is a compact and convex set. Also by construction it follows that

$$p(Wy,t) = \frac{\sum_{j=1}^n \alpha_j y_j^2}{t}$$

and since we know that p is convex the objective function of optimization problem (P_4) is also convex. Using now Lemma 1 and the substitution $x = Wy$ with $W^\top = W^{-1}$ we have shown Theorem 3 and 4 of [1].

Lemma 2 *The vector (y^*, t^*) is an optimal solution of (P_4) if and only if $W^\top y^*$ is an optimal solution of (P_2) with optimal objective value $t^* = g(Wy^*)$. Moreover, the function*

$$(y,t) \to t^{-1} \sum\nolimits_{j=1}^{n} \alpha_j y_j^2$$

is convex on the compact and convex region $\mathcal{F}_1$.

If the convex feasible region X equals (cf.[1])

$$X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, 1 \leq i \leq q, L \leq x \leq U\}$$

we obtain that

$$\mathcal{F}_1 := \{(y,t) : g_i(Wy) \leq 0, 1 \leq i \leq q, t - g(Wy) \geq 0, L \leq Wy \leq U\}.$$

In the remainder of the paper by Benson (cf.[1]) a branch and bound procedure is given to solve optimization problem (P_4). Applying that method we can find by Lemma 2 an optimal solution of the original problem (P_2).

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