CHAPTER 4

SEASONALITY

The univariate time series models reviewed in chapter 2 can be applied to economic variables which are measured with any kind of sampling interval, such as daily, monthly or annual time series. Sometimes it may be convenient to restrict model parameterizations, especially in time series which can show recurrent patterns such a seasonal time series. A class of models for seasonal time series which does not necessitate many parameters is proposed in Box and Jenkins (1970, ch.9). A brief discussion of this class, of related models, and of some terminology is given in the first part of section 4.1. In the second part of that section, I propose a selection strategy between simple variants of these models.

The first order differencing filter, as it can be applied in case of nonstationary time series, assumes the presence of a unit root (see also section 3.3). In seasonal time series one may consider the use of a similar annual differencing filter. This annual filter assumes the presence of so-called seasonal unit roots. Hylleberg *et al.* (1990) develop a procedure with which one can test whether there are such roots indeed. Their application is limited to quarterly series, and I extend their results to monthly and bimonthly time series in section 4.2.

The next section is concerned with the effects of misspecification in seasonal time series, in particular when the presence of seasonal unit roots is incorrectly assumed so that the annual differencing filter is inappropriately applied. In this section 4.3, I present some empirical and simulation evidence in which such overdifferencing causes, first, that forecasting performance can deteriorate, second, that dynamic relations between variables may not be found, and finally, that it can introduce nonlinearity. Generally, the empirical models in this chapter have been selected according to the strategy proposed in the previous chapter. To save space however, I have decided not to report every detail.

The final section of this chapter contains a discussion of a new approach to modeling seasonal time series. This method boils down to considering vector autoregressive models for the annual observations per season, and hence treating the seasonal observations as separate series. It turns out that this class of models introduces intermediate cases between currently applied models. Moreover, it appears that selection between such models is largely facilitated and can be done with the Johansen (1988) cointegration approach. Finally, it also introduces a simple device for the graphical representation of seasonal time series. The first part of this section 4.4 considers univariate seasonal time series models, while the second part specifically deals with selection in bivariate quarterly time series models.

4.1 SEASONAL TIME SERIES MODELS

Giving a definition of seasonality appears not to be straightforward. For example, Hylleberg (1986) dedicates a chapter to establishing a definition. His major conclusion is that any definition should not be too restrictive since the phenomenon has many aspects which are reflected in the different modeling and detection devices. For convenience, I will therefore limit the concept of a seasonal time series to the series of observations on a process y_t measured s time a year, of which typical phenomena are that the observations per season s show distinct means and variances, see, e.g., the graphs in section 4.4. In general, s can take values as 2, 3, 4, 6, and 12, and these values correspond to regular calender intervals of a year, such as quarters or months.

Several models

In the literature on seasonality one can find several different views on modeling seasonal time series, see, e.g., Hylleberg (1986). One view is the errors-in-variables or error-components approach which assumes that seasonality is in fact no more than additive noise contaminating the observations. This implies that a series y_t consists of two additive components, a nonseasonal component $y_{ns,t}$ and a seasonal component $y_{s,t}$, each of which may be described by an ARMA time series model. This view forms the basis of seasonal adjustment procedures. A second view assumes that the entire structure of a time series model changes over the seasons. This time-varying parameters approach considers a model for y_t where the parameters depend on $y_{s,t}$. A somewhat restricted form of such a model, i.e. in which the parameters depend on the season but are constant over the sample, is the periodic autoregression model, see, e.g., Osborn (1988).

An alternative view is introduced in Box and Jenkins (1970, ch.9), and it results in a general multiplicative seasonal time series model,

$$\phi_{sP}(B^s)(1-B^s)^D y_t = \theta_{sQ}(B^s)\nu_t, \tag{4.1}$$

where the ν_t is modeled by

$$\phi_p(B)(1-B)^d \nu_t = \theta_q(B)\varepsilon_t. \tag{4.2}$$

Model (4.1) can briefly be denoted as an ARIMA $(p,d,q)\times(P,D,Q)_s$ model. When this model applies to a seasonal series y_t one says that y_t is integrated of order (d,D), or $I(d,D)$, see also Engle, Granger and Hallman (1989) for an extended discussion of such concepts. This model is of course very general, and in practice one mostly considers models in which the orders are relatively small. Examples are the seasonal moving average process of order Q, or ARIMA $(0,0,0)\times(0,0,Q)_s$, or

$$y_t = \varepsilon_t + \theta_s\varepsilon_{t-s} + \ldots + \theta_{sQ}\varepsilon_{t-sQ}, \tag{4.3}$$

and the seasonal autoregressive process of order P, or

$$y_t = \phi_s y_{t-s} + \ldots + \phi_{sP} y_{t-sP} + \varepsilon_t. \tag{4.4}$$

The ACF of these models can be derived along similar lines as dicussed in chapter 2. It can be observed from (4.1), (4.3) and (4.4), that seasonality is modeled using a stochastic process.

When dealing with empirical economic time series one may often observe that the observations in season i, $i = 1,..,s$, are highly dependent on the value of those in the same season in previous years. Furthermore, they may show patterns which seem to be independent from the values in season j, where $j \neq i$. This may indicate that an appropriate value of D in (4.1) can be 1, or that one may apply a seasonal, i.e. an annual, differencing filter $(1-B^s)$. With respect to the nonseasonal part it can often be recognized that the observations at time t are highly dependent on t-1 such that a nonseasonal differencing filter $(1-B)$, i.e. d=1 in (4.2), may be suitable. Therefore, a regularly applied transformation of economic time series is

$$x_t = (1-B)(1-B^s)y_t, \tag{4.5}$$

which assumes that y_t is an $I(1,1)$ variable. When this x_t is modeled with an MA model such that q and Q are 1 in (4.1), one obtains the so-called airline model

$$(1-B)(1-B^s)y_t = (1+\theta_1 B)(1+\theta_s B^s)\varepsilon_t, \tag{4.6}$$

which has been popularized by Box and Jenkins (1970). They also gave it its name since it fits well to their empirical airline passengers series (ibid, p.304). This airline model has already been applied and evaluated in a host of empirical studies, and it seems to be useful in many of these, see, for examples, Nelson (1973, ch.7), Nerlove *et al.* (1979, ch.10), Abraham and Ledolter (1983, ch.6) and Granger and Newbold (1986, ch.3). An explanation for the empirical success of model (4.6) is given in Harvey (1984, p.258). Suppose y_t can be represented by a structural model, which includes a trend, seasonal dummies and an irregular component. When this y_t is doubly differenced, one will end up with model (4.6) where θ_1 as well as θ_s are equal to minus 1. In case, however, the trend and seasonal dummies gradually change over time, an invertible airline model may perform well.

Despite its apparent success, there are also practical occasions in which the doubly differencing filter in (4.5) causes that model (4.6) is noninvertible, i.e., that one or both of its parameters are equal to minus 1. Such situations can emerge when the patterns of the observations per season i are not independent of those of season j, $i \neq j$. The multiplicative character of model (4.1) establishes now that the use of only one of the differencing filters can be sufficient. The choice of the $(1-B)$ or the $(1-B^s)$ filter may be guided by an inspection of the ACFs of the several distinctly differenced variables. A straightforward interpretation of ACFs may however be blurred by the presence of deterministic effects. For example, a time series which has a linear deterministic trend can show an ACF which does not die out after some time. A similar phenomenon may occur in seasonal time series where seasonal dummy variables may account for a part of the seasonal fluctuations. These variables reflect that the observations in season i vary around a constant mean throughout the years. Obviously, the data in season i and in the same season the previous year are now also highly correlated, and hence the values of the ACF at seasonal lags may now die out only slowly. Moreover, the observations in seasons i

and j are related because in principle the differences between these remain constant. This establishes the need for a class of models rival to (4.1) in which seasonality is modeled deterministically,

$$\phi_p(B)(1-B^d)y_t = \alpha_0 + \sum_{i=1}^{s-1}\alpha_i D_{it} + \theta_q(B)\varepsilon_t, \tag{4.7}$$

where the $\phi_p(B)$ and $\theta_q(B)$ can include terms as B^{ks}, where $k = 1,2,..$, and in which intermediate parameters can be set equal to zero. The seasonal dummy variables D_{it} take a value 1 in season i each year, and a value 0 in all other periods. Note that this model essentially can combine stochastic and deterministic seasonality in an additive way. Consider for example the case where d=q=0, p=4, with ϕ_i=0, i=1,2,3, and the α's are unequal to zero.

A model selection test procedure

Given the complicated expressions in (4.1) with (4.2) and (4.7), one can imagine that the selection of an appropriate model for seasonal time series may not be without difficulties. Recently, a number of model selection strategies have been developed, each of which considers one of the following models (see also Ghysels 1990a) to be the null hypothesis,

$$(1-B)(1-B^s)y_t = \eta_{1t}, \tag{4.8}$$

$$(1-B^s)y_t = \eta_{2t}, \tag{4.9}$$

$$(1-B)y_t = \alpha_0 + \sum_{i=1}^{s-1}\alpha_i D_{it} + \eta_{3t}, \tag{4.10}$$

$$y_t = \alpha_0 + \sum_{i=1}^{s-1}\alpha_i D_{it} + \eta_{4t}, \tag{4.11}$$

where the error processes η_{it}, i=1,2,3,4, may be stationary ARMA processes, and hence are not necessarily white noise as ε_t.

The null hypothesis in Dickey, Hasza and Fuller (1984) is (4.9), and it is tested against a stationary AR model,

$$y_t = \phi_s y_{t-s} + \eta_{2t}. \tag{4.12}$$

The null model in Hasza and Fuller (1982) is given by (4.8), and versions of (4.9), (4.10) or (4.11) are the alternatives. The test procedure given in Osborn (1990a), which is equivalent to the one developed in Osborn *et al.* (1988), considers similar models as null and alternative models. The vast literature on testing for unit roots in nonseasonal time series considers (4.10) against (4.11), where the latter model may contain a linear trend. The procedure proposed in Hylleberg *et al.* (1990), which will be discussed in more detail in section 4.2, compares model (4.9) with (4.10), (4.11) and some intermediate cases. From this brief summary it is clear that these test procedures start from distinct nonstationary null models. This implies that the empirical distributions of the test statistics have to be tabulated since these models are nonstationary. Secondly, the practical application of the tests can give rise to conflicting evidence, see also Ghysels (1990a) and Osborn (1990a).

In some occasions there is however a possibility to circumvent these drawbacks. As already said, the airline model in (4.6) appears to be useful in many practical situations, albeit that sometimes the estimated θ_1 and/or θ_s may approach minus 1. This suggests that model selection with respect to the first and annual differencing operator can be carried out by checking the values of θ_1 and θ_s. In particular, assuming that the η_{it} in (4.8) through (4.11) can be restricted to be ε_t, it is clear that when θ_1=θ_s=0 model (4.6) reduces to (4.8). Restricting θ_1 to minus 1, and setting θ_s equal to 0 in (4.6) gives (4.9). At this point, the derivations in Bell (1987) become very useful. There it is shown that when the value of the parameter θ_s is equal to minus 1 in the model

$$(1-B^s)y_t = (1+\theta_s B^s)\varepsilon_t, \tag{4.13}$$

a model with seasonal dummy variables emerges. Of course, similar results apply when y_t is replaced by $x_t = (1-B)y_t$. This implies that with θ_1=0 and θ_s=-1 in (4.6), one obtains a model like (4.10), and with θ_1=θ_s=-1 model (4.11) shows up. Hence it can be seen that, in some specific cases, the airline model is linked with several other seasonal time series models. A natural model selection procedure is now given by testing whether θ_1=-1 and/or θ_s=-1. In case both θ_1 and θ_s are unequal to minus 1, one might consider (4.6) or (4.8). Other outcomes of the test procedure uniquely establish the type of model.

The values of minus 1 for the parameters imply that the airline model becomes noninvertible, which is equivalent to the presence of unit roots in the MA part of the model. This noninvertibility has a downward biasing effect on the estimates of the parameters, see Plosser and Schwert (1977). So, model selection should preferably not be based on checking whether the estimated parameters are equal to minus 1. It may now be more appropriate (cf. also section 3.3) to base inference on the sample autocorrelations of the variable x_t as it is given in (4.5) and when it is modeled as (4.6). The relevant nonzero autocorrelations of the airline model are those at lag 1, s-1, s and s+1 which, when $s > 2$, are given by

$$\rho_1 = \theta_1/(\theta_1^2+1), \tag{4.14}$$

$$\rho_{s-1} = \rho_{s+1} = \theta_1\theta_s/((\theta_1^2+1)(\theta_s^2+1)) = \rho_1\rho_s \tag{4.15}$$

$$\rho_s = \theta_s/(\theta_s^2+1), \tag{4.16}$$

can be estimated by r_1, r_{s-1}, r_s and r_{s+1} using (2.20). These expressions show that it is convenient for the construction of test statistics to consider ρ_1 and ρ_s only, since they are uniquely determined by θ_1 and θ_s, respectively. The estimated autocorrelations of an MA process asymptotically follow normal distributions. This result which is given and proved in Anderson and Walker (1964), has already been given in (3.13) and (3.14). Application of these expressions gives, after some algebraic manipulation, that

$$n^{1/2}\begin{bmatrix} r_1 \\ r_s \end{bmatrix} \sim N\left\{\begin{bmatrix} \rho_1 \\ \rho_s \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix}\right\}, \tag{4.17}$$

where

$$\begin{aligned} A_{11} &= 1 + 6\rho_{s-1}^2 + 8\rho_1^2\rho_{s-1}^2 - 16\rho_1\rho_{s-1}\rho_s + 2\rho_s^2 + 4\rho_1^2\rho_s^2 + 2\rho_1\rho_{s-1} \\ &\quad - 3\rho_1^2 + 4\rho_1^4, \\ A_{12} &= A_{21} = 4\rho_{s-1} + 8\rho_1\rho_{s-1}^2\rho_s - 8\rho_{s-1}\rho_s^2 - 8\rho_1^2\rho_{s-1} + 4\rho_1^3\rho_s \\ &\quad - 2\rho_1\rho_s + 4\rho_1^3\rho_s, \\ A_{22} &= 1 + 6\rho_{s-1}^2 + 8\rho_{s-1}^2\rho_s^2 - 16\rho_1\rho_{s-1}\rho_s + 4\rho_s^4 - 3\rho_s^2 + 2\rho_1^2 + 4\rho_1^2\rho_s^2. \end{aligned}$$

Under the null hypothesis that θ_1=-1 it can easily be verified that $\rho_1=-\frac{1}{2}$, that $\rho_{s-1}=-\frac{1}{2}\rho_s$, and that A_{11} equals $\frac{1}{2} + \rho_s^2 + \frac{1}{2}\rho_s$. The most likely range of values of ρ_s is between $-\frac{1}{2}$ and 0, implying that in this case A_{11}

theoretically varies from $\frac{1}{2}$ to $\frac{7}{16}$. This latter value is the minimum of A_{11} established by $\rho_s=-\frac{1}{4}$. Under the null hypothesis that $\theta_s=-1$ it can similarly be derived that A_{22} becomes equal to $\frac{1}{2}+\rho_1^2$, the range of which is from $\frac{1}{2}$ to $\frac{3}{4}$. Hence, the variance of each estimated autocorrelation under either of the null hypotheses contains a function of the other true but unknown autocorrelation. One might now consider estimates of the latter, and use these to estimate the variances and to construct test statistics as in, e.g., (3.19). The risk involved in this approach is that in small samples the asymptotic results may not be valid. However, considering the ranges of the values for A_{11} and A_{22}, a simple alternative strategy may be to set the variances at their highest values. Note that this is equivalent to testing $\theta_1=-1$ while assuming that $\theta_s=-1$, and vice versa. This may ensure a somewhat conservative procedure in the sense that the null hypothesis may now be accepted more often. Given the forthcoming results in section 4.3, where it is shown that incorrect differencing can have a substantial impact on several types of inference, it is not harmful to be somewhat conservative in this case. Setting $\rho_1=\rho_s=-\frac{1}{2}$ and thus $\rho_{s-1}=\rho_{s+1}$ at $\frac{1}{4}$, (4.7) becomes

$$n^{½}\begin{bmatrix} r_1 \\ r_s \end{bmatrix} \sim N\left\{\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{4} \end{bmatrix}\right\}. \tag{4.18}$$

Simple test statistics for the null hypotheses $\theta_1=-1$, $\theta_s=-1$, and $\theta_1=\theta_s=-1$ are respectively given by

$$T_1 = (2n)^{½}(r_1+\tfrac{1}{2}), \tag{4.19}$$

$$T_s = (\tfrac{4}{3}n)^{½}(r_s+\tfrac{1}{2}), \tag{4.20}$$

$$T_{1,s} = (\tfrac{2}{3}n)^{½}(r_1+r_s+1), \tag{4.21}$$

the marginal distributions of which are $N(0,1)$ distributions. In figure 3.3 it is displayed that the estimated first order autocorrelation of a noninvertible MA(1) process is symmetrically distributed. Hence, the test statistics in (4.19) through (4.21) will be considered in a two-sided test procedure.

The first step in the model selection strategy for a seasonal time series y_t is now to transform it with the $(1-B)(1-B^s)$ filter into x_t. Then, the estimated autocorrelations of x_t at lags k other than 1, s-1, s and s+1 are checked to be zero, otherwise the airline model may not be appropriate. It is easy to derive from (3.14) that for these k it holds that

$$n\text{varr}_k = \sum_{j=-\infty}^{\infty} (\rho_j^2 + \rho_j\rho_{j+2k}). \tag{4.22}$$

Under the null hypothesis $\theta_1 = \theta_s = -1$ it can be derived that the expression in (4.22) is equal to $2\frac{1}{4}$, with some exceptions. For example, for quarterly data $n\text{varr}_2$ is $\frac{3}{4}$, and for monthly data $n\text{varr}_5$, $n\text{varr}_6$ and $n\text{varr}_7$ are equal to 2, $\frac{3}{4}$ and 2, respectively. When this null hypothesis is not valid, (4.22) can not be reduced to such simple expressions, and one strategy may now be to apply the very rough approximation $n^{½} r_k \sim N(0,1)$. The next step for model selection consists of testing whether ρ_1 and/or ρ_s are equal to $-\frac{1}{2}$, and to see which of the seasonal models may be appropriate.

To investigate the performance of the above procedure, some Monte Carlo simulations have been carried out for the case where s=4 and n=100. The 100 replications have been performed with MicroTSP version 6.53 (1989). The rejection frequencies of the three tests are reported in table 4.1. The first four data generating processes are models of the types (4.8) through (4.11),

(i) $\quad y_t = y_{t-1} + y_{t-4} - y_{t-5} + \varepsilon_t, \qquad (\theta_1, \theta_s \neq -1, \quad \text{P,P})$

(ii) $\quad y_t = y_{t-4} + \varepsilon_t, \qquad (\theta_1 = -1, \quad \text{S,P})$

(iii) $\quad y_t = y_{t-1} + D_{1t} - 0.5D_{2t} + 1.5D_{3t} - D_{4t} + \varepsilon_t, \qquad (\theta_s = -1, \quad \text{P,S})$

(iv) $\quad y_t = D_{1t} - 0.5D_{2t} + 1.5D_{3t} - D_{4t} + \varepsilon_t, \qquad (\theta_1, \theta_s = -1, \quad \text{S,S})$

where the null hypothesis, as well as whether the results for the DGP convey information about the empirical size (S) or power (P), are given between parentheses. From the results in table 4.1 it is clear that for these DGPs the size and power of the tests are almost ideal values. To illustrate that the first step of checking the autocorrelations at lags other than 1, s-1, s, s+1 is important, consider the DGP

(v) $\quad y_t = y_{t-4} + \varepsilon_t - 0.6\varepsilon_{t-1}, \qquad (\theta_1 = -1, \quad \text{S,P})$

Doubly differencing results in $x_t=(1-B)(1-\delta B)\varepsilon_t$, which theoretically would show an autocorrelation at lag 2, and so the airline model is not appropriate. When δ equals 0.6, this correlation is about 0.153, and its estimate may often appear to be significant in case $n=100$ (see below (4.22)). From the fifth row of table 4.1 it is clear that now the size is largely affected. Finally, when the DGP is

$$(vi) \qquad y_t = y_{t-4} + \varepsilon_t - 0.6\varepsilon_{t-4}, \qquad (\theta_1=-1,\ \text{S,P})$$

it can be expected that the power of the test for θ_s becomes low. This is reflected by the last row of table 4.1.

Table 4.1

Rejection rates of testing for the noninvertibility of the airline model, based on 100 Monte Carlo replications and sample size 100[1]

DGP	Nominal Size	T_1	T_s	$T_{1,s}$
(i)	0.05	1.00 (P)	1.00 (P)	1.00 (P)
	0.10	1.00 (P)	1.00 (P)	1.00 (P)
(ii)	0.05	0.05 (S)	0.99 (P)	0.99 (P)
	0.10	0.11 (S)	1.00 (P)	0.99 (P)
(iii)	0.05	0.99 (P)	0.01 (S)	0.95 (P)
	0.10	1.00 (P)	0.01 (S)	0.98 (P)
(iv)	0.05	0.06 (S)	0.04 (S)	0.03 (S)
	0.10	0.11 (S)	0.11 (S)	0.09 (S)
(v)	0.05	0.50 (S)	0.99 (P)	0.89 (P)
	0.10	0.70 (S)	1.00 (P)	0.92 (P)
(vi)	0.05	0.08 (S)	0.13 (P)	0.11 (P)
	0.10	0.08 (S)	0.26 (P)	0.17 (P)

(1) The (S) and (P) correspond to the empirical size and power.

To empirically illustrate the above test procedure, I have chosen to consider the seasonal series discussed in Granger and Newbold (1986), the Box and Jenkins (1970) airline data, the Dutch car sales series for the period 1978-1985, see Appendix A.2, the monthly car sales in the U.S. given in Nelson (1973), and the price of steers series for the U.S. in Nerlove *et al.* (1979). The results are summarized in table 4.2. It appears that the airline model may be appropriate, at least in the sense that there is no autocorrelation other than those at lags 1, s–1, s and s+1, for 6 of the 7 series. Only the car sales series shows estimated values for r_2, r_5 and r_9 of –0.28, 0.24 and 0.24, which may be significant under a hypothesis other than $\theta_1=\theta_s=-1$. The GN110 series is clearly doubly overdifferenced, and model (4.11) may be useful. For both the GN108 and GN106 series the double filter can be used, which corresponds to the successful estimation results for both series in Granger and Newbold (1986, p.111).

Table 4.2

Model selection for some empirical seasonal time series via testing for noninvertibility of the airline model

			Autocorrelations		Test statistics		
Series[(1)]	n	s	r_1	r_s	T_1	T_s	$T_{1,s}$
GN110[(2)]	59	4	-0.34	-0.46	1.74	0.35	1.25
GN108[(2)]	95	12	0.17	-0.14	9.24*	4.05*	8.20*
GN106[(2)]	109	12	0.06	-0.25	8.27*	3.01*	6.90*
BJAIR[(3)]	83	12	-0.34	-0.44	2.06*	0.46	1.64
FRCAR[(4)]	71	12	-0.15	-0.28	4.17*	2.14*	3.92*
NECAR[(5)]	253	12	-0.09	-0.41	9.22*	1.65	6.49*
NEPST[(6)]	647	12	0.29	-0.46	28.42*	1.18	17.24*

* Significant at a 5% level.

(1) The series refer to the doubly differenced series.

(2) The sample autocorrelations of these series are given in Granger and Newbold (1986) on pages 110, 108 and 106, respectively.

(3) Monthly airline data given in Box and Jenkins (1970). The last three years were excluded.

(4) Monthly new car sales in the Netherlands, see Appendix A.2.

(5) Monthly new car sales in the U.S., given in Nelson (1973).

(6) Monthly price of steers for the U.S., given in Nerlove *et al.* (1979).

Curiously enough, the airline data themselves are overdifferenced when the airline model is used, and the most suitable model seems to be a model of type (4.10). The FRCAR series illustrates the probable size distortions of the test statistics in case the error process in the airline model is not white noise. In fact, I will later find that a model like (4.10) with ARMA terms may be appropriate (see table 4.6 and 4.9). Although the doubly differenced U.S. car sales series in Nelson (1973) shows a correlation at lag 2 of 0.20 which is even significant under the null hypothesis, there seems to be an indication that the $(1-B^s)$ filter is not suitable. This may also be confirmed by the estimated value of θ_s which is -0.85 (cf. Nelson 1973, p.187). Finally, the price of steers series may also be modeled with a model like (4.10), a result which might have been expected by the estimate of -0.96 for θ_s in Nerlove *et al.* (1979, p.211).

This model selection testing procedure for seasonal time series seems to be appropriate in some occasions, and it also seems to meet its purpose. Advantages of the procedure are that it is a general to simple strategy since it selects a model by testing for restrictions in a general model, and that the test statistics asymptotically follow standard normal distributions. The limitations to the procedure can be found in size distortions because of nonwhite error processes, and in power reduction in case the null and alternative hypotheses are in some sense close to each other. It may therefore be sensible to use the method together with related model selection strategies. Since the proposed procedure necessitates only simple calculations, it might however provide an initial quick check before any other methods are applied.

4.2 SEASONAL UNIT ROOTS

A recent proposal for the selection between models as (4.1) with (4.2) and (4.7) is given in Hylleberg *et al.* (1990). In particular, they consider as their null model a process like (4.9) for quarterly time series, a simple version of which is $(1-B^4)y_t = \varepsilon_t$. They recognize that its differencing filter can be written as

$$(1-B^4) = (1-B)(1+B)(1-iB)(1+iB) \tag{4.23}$$

$$= (1-B)(1+B)(1+B^2) \quad = (1-B)(1+B+B^2+B^3).$$

From this expression it can be seen that the filter $(1-B^4)$ assumes the presence of four unit roots, i.e. ± 1 and $\pm i$, and that it can be written as the product of $(1-B)$ and a term which reflects an annual moving average. The roots -1 and $\pm i$ are called seasonal unit roots, while the root 1 is called the nonseasonal unit root.

From (4.23) it can be seen that transforming a quarterly time series with a $(1-B^4)$ filter is appropriate only in the case of the simultaneous presence of 4 unit roots. However, transforming the series with $(1-B^4)$ yields an overdifferenced series in case only one unit root is present such that, e.g., applying the $(1-B)$ filter is sufficient to make the series stationary and that seasonality can be handled by the inclusion of seasonal dummies. This overdifferencing may cause trouble for the construction of time series models because the (partial) autocorrelation patterns become hard to interpret. Furthermore, one may expect estimation problems because of the introduction of moving average polynomials with roots close to the unit circle. On the other hand, underdifferenced series may yield unit roots in their autoregressive parts, and so classical arguments as those in e.g., Granger and Newbold (1974), for time series containing neglected unit roots apply. So, it is important to test for the presence of (seasonal) unit roots.

The crucial proposition which makes the testing procedure relatively simple is given in Hylleberg *et al.* (1990, pp.221-222). Since I need it in subsequent parts of this section this proposition is given here almost literally, though without proof.

Proposition:

Any (possibly infinite or rational) polynomial, $\varphi(z)$, which is finite valued at the distinct, non-zero, possibly complex points, $\theta_1,\ldots\theta_p$, can be expressed in terms of elementary polynomials and a remainder as follows:

$$\varphi(z) = \sum_{k=1}^{p} \lambda_k \Delta(z)/\delta_k(z) + \Delta(z)\varphi^{**}(z), \tag{4.24}$$

where the λ_k are constants defined by $\lambda_k = \varphi(\theta_k)/\prod_{j\neq k}\delta_j(\theta_k)$, $\varphi^{**}(z)$ is a (possibly infinite or rational) polynomial and

$$\delta_k(z) = 1 - (1/\theta_k)z, \tag{4.25}$$

$$\Delta(z) = \prod_{k=1}^{p} \delta_k(z). \tag{4.26}$$

An alternative form of (4.24) which will be used in the sequel, is

$$\varphi(z) = \sum_{k=1}^{p} \lambda_k \Delta(z)(1-\delta_k(z))/\delta_k(z) + \Delta(z)\varphi^*(z), \tag{4.27}$$

where $\varphi^*(z) = \varphi^{**}(z) + \sum\lambda_k$. From the definition of λ_k it can be seen that the polynomial $\varphi(z)$ will have a root at θ_k if and only if the corresponding λ_k equals zero.

An application of this proposition to testing for (seasonal) unit roots in quarterly time series is discussed in Hylleberg *et al.* (1990). Then using (4.23), (4.27) becomes

$$\varphi(B) = \lambda_1 B\varphi_1(B) + \lambda_2(-B)\varphi_2(B) + \lambda_3(-i-B)B\varphi_3(B) \tag{4.28}$$

$$+ \lambda_4(i-B)B\varphi_3(B) + \varphi^*(B)\varphi_4(B),$$

where

$$\begin{aligned} \varphi_1(B) &= (1+B+B^2+B^3) \\ \varphi_2(B) &= (1-B)(1+B^2) = (1-B+B^2-B^3) \\ \varphi_3(B) &= (1-B^2) \\ \varphi_4(B) &= (1-B^4). \end{aligned} \tag{4.29}$$

Defining

$$\begin{aligned} \lambda_1 &= -\pi_1 \\ \lambda_2 &= -\pi_2 \\ \lambda_3 &= (-\pi_3+i\pi_4)/2 \\ \lambda_4 &= (-\pi_3-i\pi_4)/2, \end{aligned} \tag{4.30}$$

(4.28) can be written as

$$\varphi(B) = -\pi_1 B\varphi_1(B) + \pi_2 B\varphi_2(B) + (\pi_3 B+\pi_4)B\varphi_3(B) + \varphi^*(B)\varphi_4(B). \tag{4.31}$$

Suppose that the quarterly observations are generated by an AR process

$$\varphi(B)y_t = \mu_t + \varepsilon_t, \tag{4.32}$$

where μ_t covers the deterministic elements, and might consist of a constant, seasonal dummies, or a trend. Now, the auxiliary test equation becomes

$$\varphi^*(B)y_{4t} = \mu_t + \pi_1 y_{1,t-1} + \pi_2 y_{2,t-1} + \pi_3 y_{3,t-2} + \pi_4 y_{3,t-1} + \varepsilon_t, \tag{4.33}$$

where

$$\begin{aligned} y_{1t} &= \varphi_1(B)y_t \\ y_{2t} &= -\varphi_2(B)y_t \\ y_{3t} &= -\varphi_3(B)y_t \\ y_{4t} &= \varphi_4(B)y_t. \end{aligned}$$

Applying OLS to (4.33), where the order of $\varphi^*(B)$ is established in an experimental way to whiten the residuals, gives estimates of the π_i. Because the π_i are zero in case the corresponding unit roots are on the unit circle (cf. (4.30)), testing the significance of the estimated π_i implies testing for unit roots. There will be no seasonal unit roots if π_2, and π_3 or π_4 are significantly different from zero. If π_1=0, then the presence of root 1 can not be rejected. This procedure for the nonseasonal unit root is similar to the Dickey and Fuller method when applied to the series y_{1t}. Note that, e.g., in case only π_3=π_4=0 the transformation $(1+B^2)$ can be applied to the raw data series. This kind of test outcomes ensures that the procedure considers also models which are intermediate between (4.1) and (4.7). The alternative hypotheses for the first two unit roots are that the roots are smaller than one in an absolute sense, which implies that the t-tests for π_1 and π_2 are one-sided. A test strategy for π_3 and π_4 may be to test π_4 in a two-sided procedure, and when the insignificance of π_4 is accepted, to check the significance of π_3 with a one-sided t-test. A sensible strategy may also be to jointly test π_3=π_4=0 with an F-test.

To see why it is reasonable to have a two-sided test for π_4 and a one-sided test for π_3, consider for example

$$\varphi(B) = (1+\alpha B^2)\left[\frac{1-B^4}{1+B^2}\right] = (1+\alpha B^2)(1-B^2), \tag{4.34}$$

where $|\alpha| < 1$ in the stationary case. Using (4.24) and (4.25) it can easily be established that for the $\varphi(B)$ in (4.34) it holds that

$$\varphi(-i) = \varphi(i) = 2(1-\alpha)$$

$$\delta_1(-i).\delta_2(-i).\delta_4(-i) = \delta_1(i).\delta_2(i).\delta_3(i) = 4$$

and hence that $\lambda_3 = \lambda_4 = (1-\alpha)/2$, such that $\lambda_3,\lambda_4 > 0$ under the stationary alternative. The transformations in (4.30) give that

$$\pi_3 = -\lambda_3 - \lambda_4$$
$$\pi_4 = i(\lambda_4 - \lambda_3)$$

and hence that tests for π_3 and π_4 may be one- and two-sided, respectively.

The null hypothesis in this test procedure is a nonstationary model, and therefore the critical values for the several test statistics have to be tabulated. Tables with critical values are displayed in Hylleberg *et al.* (1990) for the cases where μ_t can contain several combinations of deterministic elements. The distribution of the t-test statistic for π_1 depends on the inclusion of trend and intercept. The distributions of the t-tests for the π_2, π_3 and π_4 are dependent on the inclusion of seasonal dummy variables. In general it applies that deterministic elements make that the empirical as well as asymptotic distributions shift to the left, or, equivalently, that the critical values become larger in an absolute sense. The tables are based on Monte Carlo replications, and therefore the tabulated critical values are not exact. This naturally implies that it may not be wise to establish the significance of a parameter on the basis of test values which differ from critical values only by their second decimal point.

Monthly time series

An extension of the application of expression (4.27) to monthly data is fairly straightforward. The null model in this case is

$$(1-B^{12})y_t = \varepsilon_t.$$

The equation $1-B^{12}=0$ has 12 solutions lying on the unit circle, see also Abraham and Ledolter (1983), which is clear from

$$\begin{aligned}1-B^{12} = {} & (1-B)(1+B)(1-iB)(1+iB) \qquad (4.35)\\ & (1+(\sqrt{3}+i)B/2)(1+(\sqrt{3}-i)B/2)(1-(\sqrt{3}+i)B/2)(1-(\sqrt{3}-i)B/2)\\ & (1+(i\sqrt{3}+1)B/2)(1-(i\sqrt{3}-1)B/2)(1-(i\sqrt{3}+1)B/2)(1+(i\sqrt{3}-1)B/2),\end{aligned}$$

where all factors other than $(1-B)$ correspond to seasonal unit roots. Collecting two factors at a time in (4.35) yields

$$1-B^{12} = (1-B^2)(1+B^2)(1+B\sqrt{3}+B^2)(1-B\sqrt{3}+B^2)(1+B+B^2)(1-B+B^2) \tag{4.36}$$

$$= (1-B^4)(1-B^2+B^4)(1+B^2+B^4).$$

Note that this is also equal to $(1-B)(1+B+B^2+...+B^{11})$. The product in (4.36) will be useful in the forthcoming test equation. The expression in (4.27) now becomes

$$\begin{aligned}
\varphi(B) = \; & \lambda_1 B\varphi_1(B) + \lambda_2(-B)\varphi_2(B) + \lambda_3(i-B)B\varphi_3(B) + \lambda_4(-i-B)B\varphi_3(B) \\
& + \lambda_5(-(\sqrt{3}+i)/2-B)B\varphi_4(B) + \lambda_6(-(\sqrt{3}-i)/2-B)B\varphi_4(B) \\
& + \lambda_7((\sqrt{3}+i)/2-B)B\varphi_5(B) + \lambda_8((\sqrt{3}-i)/2-B)B\varphi_5(B) \\
& + \lambda_9(-(i\sqrt{3}+1)/2-B)B\varphi_6(B) + \lambda_{10}((i\sqrt{3}-1)/2-B)B\varphi_6(B) \\
& + \lambda_{11}((i\sqrt{3}+1)/2-B)B\varphi_7(B) + \lambda_{12}(-(i\sqrt{3}-1)/2-B)B\varphi_7(B) \\
& + \varphi^*(B)\varphi_8(B),
\end{aligned} \tag{4.37}$$

where

$$\begin{aligned}
\varphi_1(B) &= (1+B)(1+B^2)(1+B^4+B^8) \\
\varphi_2(B) &= (1-B)(1+B^2)(1+B^4+B^8) \\
\varphi_3(B) &= (1-B^2)(1+B^4+B^8) \\
\varphi_4(B) &= (1-B^4)(1-B\sqrt{3}+B^2)(1+B^2+B^4) \\
\varphi_5(B) &= (1-B^4)(1+B\sqrt{3}+B^2)(1+B^2+B^4) \\
\varphi_6(B) &= (1-B^4)(1-B^2+B^4)(1-B+B^2) \\
\varphi_7(B) &= (1-B^4)(1-B^2+B^4)(1+B+B^2) \\
\varphi_8(B) &= (1-B^{12}).
\end{aligned} \tag{4.38}$$

To get rid of the complex terms in (4.37), it is suitable to define

$$
\begin{aligned}
\lambda_1 &= -\pi_1 \\
\lambda_2 &= -\pi_2 \\
\lambda_3 &= -(i\pi_3+\pi_4)/2 \\
\lambda_4 &= -(-i\pi_3+\pi_4)/2 \\
\lambda_5 &= i\pi_5 - ((1+i\sqrt{3})/2)\pi_6 \\
\lambda_6 &= -i\pi_5 - ((1-i\sqrt{3})/2)\pi_6 \\
\lambda_7 &= -i\pi_7 - ((1+i\sqrt{3})/2)\pi_8 \\
\lambda_8 &= i\pi_7 - ((1-i\sqrt{3})/2)\pi_8 \\
\lambda_9 &= i\sqrt{3}\pi_9/3 - ((1+(1/3)i\sqrt{3})/2)\pi_{10} \\
\lambda_{10} &= -i\sqrt{3}\pi_9/3 - ((1-(1/3)i\sqrt{3})/2)\pi_{10} \\
\lambda_{11} &= -i\sqrt{3}\pi_{11}/3 - ((1+(1/3)i\sqrt{3})/2)\pi_{12} \\
\lambda_{12} &= i\sqrt{3}\pi_{11}/3 - ((1-(1/3)i\sqrt{3})/2)\pi_{12}.
\end{aligned} \tag{4.39}
$$

One could of course consider somewhat simpler transformations for the λ_5 to λ_{12}, as is done in Beaulieu and Miron (1991) where independently a similar extension is carried out, but then the final test equation would become more complicated. It is clear that the test procedure for π_1 and π_2 is one-sided. Recently, it has been derived in Beaulieu (1991) along similar lines as (4.34) and below, that the other even π_i should be evaluated in a one-sided test, while for the other odd π_i two-sided tests may be appropriate. Substituting (4.39) into (4.37) gives

$$
\begin{aligned}
\varphi(B) = & -\pi_1 B\varphi_1(B) + \pi_2 B\varphi_2(B) + (\pi_3+\pi_4 B)B\varphi_3(B) + (\pi_5+\pi_6 B)B\varphi_4(B) \\
& + (\pi_7+\pi_8 B)B\varphi_5(B) + (\pi_9+\pi_{10}B)B\varphi_6(B) + (\pi_{11}+\pi_{12}B)B\varphi_7(B) \\
& + \varphi^*(B)\varphi_8(B).
\end{aligned} \tag{4.40}
$$

Assuming again that the data were generated by an autoregression as (4.32), the test equation for the presence of (seasonal) unit roots becomes

$$
\begin{aligned}
\varphi^*(B)y_{8,t} = & \pi_1 y_{1,t-1} + \pi_2 y_{2,t-1} + \pi_3 y_{3,t-1} + \pi_4 y_{3,t-2} + \pi_5 y_{4,t-1} \\
& + \pi_6 y_{4,t-2} + \pi_7 y_{5,t-1} + \pi_8 y_{5,t-2} + \pi_9 y_{6,t-1} + \pi_{10} y_{6,t-2} \\
& + \pi_{11} y_{7,t-1} + \pi_{12} y_{7,t-2} + \mu_t + \varepsilon_t,
\end{aligned} \tag{4.41}
$$

where $\varphi^*(B)$ is some polynomial function of B with all roots outside the unit circle, and with

$$y_{it} = \varphi_i(B)y_t \qquad \text{for } i=1,8, \tag{4.42}$$

and

$$y_{it} = -\varphi_i(B)y_t \qquad \text{for } i=2,\ldots,7,$$

where the φ_i are given in (4.38).

Applying OLS to (4.41) gives estimates of the π_i. There will be no seasonal unit roots if π_2 through π_{12} are significantly different from zero. If $\pi_1=0$, then the presence of root 1 can not be rejected. In case all π_i, $i=1,..,12$ are equal to zero, it is appropriate to apply the $(1-B^{12})$ filter. If they are all unequal to zero, one has encountered a stationary seasonal pattern and one can consider using seasonal dummies. When only some pairs of π's are equal to zero, one might consider the use of the corresponding implied operators, such as $(1-B\sqrt{3}+B^2)$ or $(1-B+B^2)$. In Abraham and Box (1978) it is illustrated that these operators may sometimes be sufficient.

Tables for the critical t-values of the individual π_i are given in Appendix B.1. The tables have been generated by assuming a true data generating process $(1-B^{12})y_t = \varepsilon_t$, where the ε_t have been independently drawn from a standard normal distribution. The number of replications is 5000. Note that the tests for π_i, $i = 4,6,8,10,12$ are in fact one-sided, but I display some more critical values to characterize the distribution of the test statistics. Tables for F-tests of $\pi_3=\pi_4=0$ through $\pi_{11}=\pi_{12}=0$ have also been generated and are displayed in the second part of the Appendix B.1. Finally, to facilitate inference, a table for the F-test for the joint restriction $\pi_3=..=\pi_{12}=0$ is given. A recent application of the procedure in (4.41) can be found in Hylleberg, Jorgenson and Sorensen (1991).

The size of the test has now been established. It is however also of interest to investigate the power of the testing procedure. Attention may then also be given to a comparison of the five auxiliary regressions. It can be expected that when redundant variables enter the auxiliary equation, such as, e.g, when trend and seasonal dummies are unnecessarily included, the power of the tests will decrease. This is because the parameters are inefficiently estimated, and hence t and F values will become lower. The empirical power results for the t-tests for π_1 and π_2, and the joint F-test for the other π_i's, as displayed in the tables 4.3 and 4.4 for the data generating process $(1-\rho B^{12})y_t = \varepsilon_t$ with ρ is 0.9 and 0.5, respectively, confirm these expectations. Similar results are obtained for t-tests for π_3 and π_4 in Beaulieu and Miron (1991), although they only consider one type of auxiliary regressions.

Table 4.3
Empirical powers of test statistics for seasonal unit roots in monthly data based on 1000 Monte Carlo replications.
DGP: $y_t = 0.9y_{t-12} + \varepsilon_t$, $\varepsilon_t \sim$ N(0,1), nominal size 0.05

Regression	T	$t{:}\pi_1$	$t{:}\pi_2$	$F{:}\pi_3,..\pi_{12}$
nc,nd,nt	120	0.091	0.129	0.191
	240	0.156	0.162	0.494
c,nd,nt	120	0.065	0.129	0.196
	240	0.072	0.164	0.505
c,d,nt	120	0.067	0.051	0.093
	240	0.076	0.063	0.185
c,nd,t	120	0.059	0.130	0.210
	240	0.048	0.166	0.509
c,d,t	120	0.061	0.049	0.092
	240	0.056	0.065	0.185

Table 4.4
Empirical powers of test statistics for seasonal unit roots in monthly data based on 1000 Monte Carlo replications.
DGP: $y_t = 0.5y_{t-12} + \varepsilon_t$, $\varepsilon_t \sim$ N(0,1), nominal size 0.05

Regression	T	$t{:}\pi_1$	$t{:}\pi_2$	$F{:}\pi_3,..\pi_{12}$
nc,nd,nt	120	0.500	0.564	0.999
	240	0.928	0.926	1.000
c,nd,nt	120	0.198	0.560	0.999
	240	0.490	0.934	1.000
c,d,nt	120	0.207	0.213	0.809
	240	0.523	0.515	1.000
c,nd,t	120	0.127	0.583	0.999
	240	0.289	0.933	1.000
c,d,t	120	0.119	0.214	0.801
	240	0.312	0.515	1.000

In table 4.5 the results are given in case of deterministic seasonality. The data generating process is now

$$y_t = \sum_{i=1}^{12} \alpha_i D_{it} + \varepsilon_t,$$

with the α_1 through α_{12} are set equal to -1,1,2,3,5,6,8,6,4,2,1,-2, and the ε_t are again drawn from the standard normal distribution. The zero power for π_1 in table 4.5 (first two rows) is because the line of the regression of two almost constant variables on each other is forced through the origin. This implies that the estimated π_1 may often be around zero.

Table 4.5

Empirical powers of test statistics for seasonal unit roots in monthly data based on 1000 Monte Carlo replications. DGP is deterministic seasonal process, nominal size 0.05

Regression	T	$t{:}\pi_1$	$t{:}\pi_2$	$F{:}\pi_3,..\pi_{12}$
nc,nd,nt	120	0.000	0.595	0.999
	240	0.000	0.976	1.000
c,nd,nt	120	0.508	0.652	1.000
	240	0.985	0.984	1.000
c,d,nt	120	0.823	0.831	1.000
	240	0.997	1.000	1.000
c,nd,t	120	0.252	0.664	1.000
	240	0.842	0.984	1.000
c,d,t	120	0.581	0.828	1.000
	240	0.984	1.000	1.000

From these power investigations several conclusions can be drawn. The first is that it may be difficult to distinguish between stationary and integrated seasonality if the ρ equals 0.9 and 0.5. These difficulties mainly concern the test for the presence of the nonseasonal unit root. The test procedure does seem to have reasonable power with respect to the detection of the seasonal unit roots, especially when ρ equals 0.5. Secondly, it can be seen that a clear recognition of the alternative hypothesis, and in particular the elements in μ_t, does have a significant impact on the power. Finally, the power of the tests often increases rather rapidly with the number of observations.

Table 4.6

Testing for (seasonal) unit roots in some empirical monthly series, I

	Variable				
t-statistics	$lnp^{(1)}$	$lnip^{(2)}$	$lnqc^{(3)}$	$lnqt^{(4)}$	$lnu^{(5)}$
π_1	-2.253	-2.471	-2.035	-2.766	-1.586
π_2	-2.984**	-3.360**	-2.638*	-3.232**	-4.186**
π_3	-2.715**	-2.053*	-3.537**	-0.727	-2.137*
π_4	-2.329	-4.800**	-2.943*	-5.008**	-3.245**
π_5	-2.973	-3.786**	-2.861	-3.397**	-4.639**
π_6	-3.881**	-3.825**	-3.292**	-4.256**	-4.374**
π_7	0.933	-0.063*	1.969	3.852**	-0.529**
π_8	-2.086	-1.529	-3.454**	-6.372**	-0.547
π_9	-1.332	-2.338	-1.383	-2.320	-3.341**
π_{10}	-3.626**	-3.789**	-2.880*	-4.324**	-4.003**
π_{11}	-1.331**	-2.577**	-0.265	1.920	-1.389**
π_{12}	-2.085	-3.455**	-3.221**	-6.460**	-1.282
F-statistics					
π_3,π_4	7.028**	14.318**	11.951**	12.806**	7.332**
π_5,π_6	7.895**	7.814**	5.423*	9.286**	10.977**
π_7,π_8	4.940*	5.424*	10.698**	29.587**	2.129
π_9,π_{10}	6.864**	7.329**	4.150	9.350**	9.036**
π_{11},π_{12}	7.206**	22.461**	8.646**	22.502**	3.482
$\pi_3,..\pi_{12}$	15.348**	24.965**	16.083**	19.227**	5.466**

** Significant at a 5% level.

* Significant at a 10% level.

(1) The auxiliary regression contains constant, trend and seasonal dummies, while $\varphi^*(B)$ is $(1-\varphi_1 B^{12})$ and the number of observations equals 84.

(2) The auxiliary regression contains constant, trend and seasonal dummies, while $\varphi^*(B)$ is 1 and the number of observations equals 180.

(3) The auxiliary regression contains constant and seasonal dummies, while $\varphi^*(B)$ is 1 and the number of observations is 84.

(4) The auxiliary regression contains constant, trend and seasonal dummies, while $\varphi^*(B)$ is $(1-\varphi_1 B)$ and the number of observations equals 335.

(5) The auxiliary regression contains constant and seasonal dummies. The $\varphi^*(B)$ is $(1-\varphi_1 B-\varphi_{12}B^{12}-\varphi_{13}B^{13})$, and the number of observations equals 371.

To illustrate the above described procedure for testing for seasonal unit roots in monthly data I consider several time series. The results for some of which will be used in subsequent sections. These series are the first nine years of the Box and Jenkins (1970) airline passengers data (see also table 4.2), *lnp*, the index of industrial production in the Netherlands from 1969 to 1984 given in Appendix A.3, *lnip*, the new car registrations series in Appendix A.2 for 1978 to 1985, *lnqc*, the sales of new trucks from 1960.01 to 1988.12 in the Netherlands in Appendix A.4, *lnqt*, and finally the series of German unemployment for 1948 to 1980, given in Subba Rao and Gabr (1984), *lnu*. To have an idea which deterministic elements may enter the auxiliary regression it can be useful to consider the graphs of the series. The plots of *lnp* and *lnu* can be found in the corresponding references. From these it seems that seasonality in both series may be modeled with seasonal dummies because there are distinct peaks at distinct months. Furthermore, the *lnp* series appears also to be heavily trended. Parts of the graphs of *lnqt*, *lnqc* and *lnip* are given in figures 4.2 and 4.3. Extended plots can be found in Franses (1990d) and Appendix A. For all three series it holds that seasonal dummy variables form a natural alternative hypothesis and that there is little evidence of a trend.

The test results are displayed in table 4.6. From these results the conclusion is that the 5 series may well be modeled with a process like (4.7) where $d = 1$. This is because the nonseasonal unit root can not be rejected for either series, while there appears to be no strong evidence for the presence of seasonal unit roots. This conclusion is based on taking together the outcomes for the t-tests, the F-tests for pairs of π_i, as well as for $\pi_3,..,\pi_{12}$.

Bimonthly time series

The influential study of Clarke (1976) on data interval bias in empirical marketing models has motivated the consideration of bimonthly observations on time series variables in, e.g., market response models which are built according to the econometric and time series analysis approach (ETS), see Hanssens, Parsons and Schultz (1990). Recent examples of studies which consider such bimonthly time series are Leeflang and Reuyl (1985), and Franses (1990b). The latter study investigates the determinants of the primary demand for beer in the Netherlands using an ARMAX specification, proposed in Bierens (1987). Consider

$$1-B^6 = (1-B)(1+B) \tag{4.43}$$
$$(1-(i\sqrt{3}+1)B/2)(1+(i\sqrt{3}-1)B/2)(1+(i\sqrt{3}+1)B/2)(1-(i\sqrt{3}-1)B/2)$$
$$= (1-B^2)(1-B+B^2)(1+B+B^2)$$
$$= (1-B^2)(1+B^2+B^4)$$
$$= (1-B)(1+B+B^2+B^3+B^4+B^5).$$

Application of(4.27) to the first row of (4.43) gives

$$\begin{aligned}\varphi(B) = {} & \lambda_1 B\varphi_1(B) + \lambda_2(-B)\varphi_2(B) + \\ & + \lambda_3((i\sqrt{3}+1)/2-B)B\varphi_3(B) + \lambda_4(-(i\sqrt{3}-1)/2-B)B\varphi_3(B) \\ & + \lambda_5(-(i\sqrt{3}+1)/2-B)B\varphi_4(B) + \lambda_6((i\sqrt{3}-1)/2-B)B\varphi_4(B) \\ & + \varphi^*(B)\varphi_5(B),\end{aligned} \tag{4.44}$$

where

$$\begin{aligned}\varphi_1(B) &= (1+B)(1+B^2+B^4) \\ \varphi_2(B) &= (1-B)(1+B^2+B^4) \\ \varphi_3(B) &= (1-B^2)(1+B+B^2) \\ \varphi_4(B) &= (1-B^2)(1-B+B^2) \\ \varphi_5(B) &= (1-B^6).\end{aligned} \tag{4.45}$$

To get rid of the complex terms in (4.44), it is suitable to define

$$\begin{aligned}\lambda_1 &= -\pi_1 \\ \lambda_2 &= -\pi_2 \\ \lambda_3 &= -i\sqrt{3}\pi_3/3 - ((1+(1/3)i\sqrt{3})/2)\pi_4 \\ \lambda_4 &= i\sqrt{3}\pi_3/3 - ((1-(1/3)i\sqrt{3})/2)\pi_4 \\ \lambda_5 &= i\sqrt{3}\pi_5/3 - ((1+(1/3)i\sqrt{3})/2)\pi_6 \\ \lambda_6 &= -i\sqrt{3}\pi_5/3 - ((1-(1/3)i\sqrt{3})/2)\pi_6.\end{aligned} \tag{4.46}$$

Substituting (4.46) into (4.44) gives

$$\begin{aligned}\varphi(B) = {} & -\pi_1 B\varphi_1(B) + \pi_2 B\varphi_2(B) + (\pi_3+\pi_4 B)B\varphi_3(B) + (\pi_5+\pi_6 B)B\varphi_4(B) \\ & + \varphi^*(B)\varphi_5(B),\end{aligned} \tag{4.47}$$

and the test equation becomes

$$\varphi^*(B)y_{5,t} = \pi_1 y_{1,t-1} + \pi_2 y_{2,t-1} + \pi_3 y_{3,t-1} + \pi_4 y_{3,t-2} + \pi_5 y_{4,t-1} + \pi_6 y_{4,t-2} + \mu_t + \varepsilon_t, \qquad (4.48)$$

with

$$y_{it} = \varphi_i(B)y_t \qquad \text{for} \quad i=1,5$$
$$y_{it} = -\varphi_i(B)y_t \qquad \text{for} \quad i=2,3,4.$$

Tables for the critical t-values of the individual π_i are given in the Appendix B.2. These critical values are generated in the same way as those for monthly data. The tests for π_1 and π_2 are one-sided, and the tests for π_3 and π_5 are two-sided and for π_4 and π_6 are one-sided. Tables for F-tests of $\pi_3=\pi_4=0$ and $\pi_5=\pi_6=0$ are displayed in the second part of Appendix B.2.

Table 4.7

Testing for (seasonal) unit roots in some empirical bimonthly time series

	Variable[1]		
t-statistics	*lnqt*	*lnqb*	*at*
π_1	-2.529	-3.544	-0.775
π_2	-3.432**	-2.667*	-1.394
π_3	1.107	4.135**	0.881
π_4	-5.053**	-3.198*	-3.000*
π_5	-2.653*	-0.893	-0.817
π_6	-4.502**	-3.633**	-2.288
F-statistics			
π_3,π_4	17.660**	10.170**	5.152*
π_5,π_6	10.342**	7.677**	2.753

** Significant at a 5% level.

* Significant at a 10% level.

(1) The auxiliary regressions contain constant and seasonal dummies, while $\varphi^*(B)$ is 1, and the number of observations are 60, 54, and 36.

It can be expected that the power of the test procedure for bimonthly series corresponds to that of monthly data. Furthermore, application to empirical series is also similar, and hence I will limit an illustration to three time series. These are the Dutch truck sales series for the period 1978–1988, and the Dutch series over 1978–1987 for the primary demand for beer, *lnqb*, and over 1978–1984 for total advertising, *at*, the latter two taken from Franses (1990b) in which also the plots of these series can be found. The test results are summarized in table 4.7.

The results for *lnqt* are quite close to those already obtained for monthly data. In Franses (1990b) it has been argued that for *lnqb* as well as for *at* seasonal dummies may be appropriate, and that both series should not be first order differenced. For the primary demand variable this seems to be valid, although the *at* series seems to contain several unit roots. It should however be recognized that the numbers of observations for these variables are rather limited, and it may now be expected that the power of the test procedure is low.

4.3 MISSPECIFICATION EFFECTS

Many modeling strategies for seasonal time series start with the formulation of a suitable model for seasonality. It is current practice to remove seasonality via seasonal adjustment filters or to automatically transform the unadjusted seasonal series with the doubly differencing filter as in (4.5). Some adjustment procedures also use filters as $(1-B^s)$ or $(1+B+B^2+..+B^{s-1})$ and so the presence of several (seasonal) unit roots is often assumed. The selection methods discussed in the previous sections may now be useful to establish the adequacy of these filters. Since the empirical results in tables 4.2, 4.6 and 4.7, and those in Beaulieu and Miron (1991) and Osborn (1990a), indicate that overdifferencing is likely to occur in models as (4.6), it might be interesting to investigate the effects of misspecified seasonal models. In particular, I will consider the use of

$$(1-B)(1-B^{12})y_t = \varepsilon_t + \beta_1\varepsilon_{t-1} + \beta_2\varepsilon_{t-12} + \beta_3\varepsilon_{t-13}, \tag{4.49}$$

which is the multiplicative Box and Jenkins (1970) model (MSBJ), while a more appropriate model is

$$\phi_p(B)(1-B)y_t = \alpha_0 + \sum_{i=1}^{11}\alpha_i D_{it} + \theta_q(B)\varepsilon_t, \tag{4.50}$$

which will be called a first differences seasonal dummies (FDSD) model. The investigation of misspecification effects focuses on forecasting, a part that is largely based on Franses (1990d), on multivariate time series, and on linearity.

Forecasting

The MSBJ model is often used in forecasting exercises. A phenomenon which is sometimes encountered in practice is that its forecasts may all be too low or too high and that the absolute errors increase in time. For example, consider the forecasting of the number of airline passengers in Box and Jenkins (1970), where all 36 monthly forecasts are too high, and the car sales forecasting example in Nelson (1973) where similar results emerge.

It is easy to show with a small experiment that using the MSBJ model while an FDSD model is the data generating process may explain the observed empirical forecast error patterns, and that the usual autocorrelation checks often do not cause alarm. Note however that this is not the only explanation, as we will see below. For an artificial sample, ranging from 1950.01 to 1970.12, observations on y_t are generated from the model

$$y_t = y_{t-1} + \alpha_0 + \sum_{i=1}^{11} \alpha_i D_{it} + 0.3(y_{t-1}-y_{t-2}) + \varepsilon_t - 0.6\varepsilon_{t-1}, \tag{4.51}$$

where in case (a) the α_0 through α_{11} have been set equal to -1, -4, -3, -1, 2, 5, 7, 9, 4, 2, 1, -2, yielding a time series resembling the airline data, and in case (b) the α's are -1, -1, 1, 2, 3, -5, 6, 8, -6, 4, 2, -2. Furthermore, ε_t is drawn from a standard normal distribution, and y_0=0 and y_1=0. From this large sample the first 8 years are deleted to reduce starting-up effects of the MA part of the model, and the last 3 years will be used for out-of-sample forecasting of the level of the series. To the remaining 120 observations the MSBJ model is fitted, after which the residuals are checked for autocorrelation with the usual portmanteau test statistic, see Box and Jenkins (1970) and Granger and Newbold (1986). This exercise has been carried out for 100 replications, where all calculations have been performed with TSP version 6.53 (1989). The results for the autocorrelation tests are summarized in table 4.8.

Table 4.8

Number of times the null hypothesis of no autocorrelation is rejected when an MSBJ model is fitted to observations generated by an FDSD model

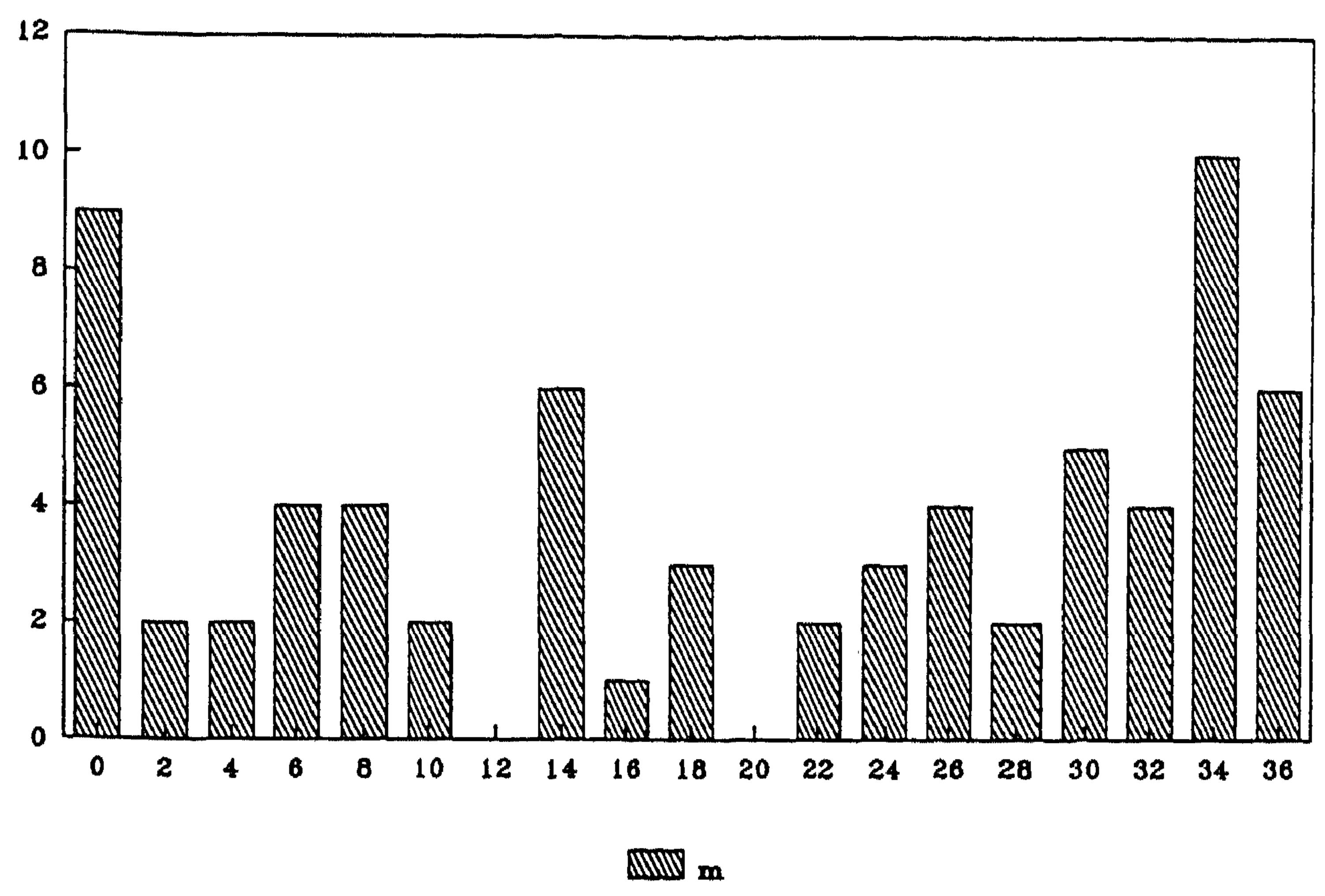
Case	Size	BP(12)[1]	BP(24)[1]
(a)	0.05	26	17
	0.10	31	22
(b)	0.05	26	13
	0.10	36	17

(1) The $\chi^2(9)$ and $\chi^2(21)$ Box-Pierce test statistic for autocorrelation of order 12 and 24. The rejection rate is based on 100 replications.

Suppose that a 10% level of significance is used, and also that the strategy is adopted that models where too much autocorrelation is left in the residuals will not be used in a forecast evaluation for they are already misspecified, then it can be seen that for cases (a) and (b) there remain 69 and 64 replications for forecasting exercises, respectively. For each of these repetitions unconditional forecasts of y_t for 36 months out-of-sample are calculated and compared with the true observations. Let m denote the number of times that the true value exceeds the forecasted value. The distributions of m are given figure 4.1 (a) and (b). In the ideal situation, one would theoretically expect that m is symmetrically distributed with mean 18 and with a standard deviation equal to 3. Or, it would be expected that about 95% of the observations is within the interval 12 to 24. From figure 4.1 it is obvious that this is certainly not the case here. Furthermore, it can be seen that the forecasts can be too high or too low about equally often. These simulation experiments strongly suggest that considering the incorrect MSBJ model can yield biased forecasts. Furthermore, it emerges that the usual specification checks are often not discriminative enough to reject this incorrect model.

The results of biased forecasts are however not uniquely established by considering an MSBJ model while an FDSD model is adequate. When similar experiments as above are carried out for the case where β_1, β_2 and β_3 in (4.49) are set equal to -0.5, -0.4 and 0.2, respectively, similar results as in figure 4.1 emerge. These unreported simulation results show that 13 out of 100 times the MSBJ model is rejected at a 10% level by autocorrelation checks, and that the figure for m closely resembles figure 4.1. Hence, the MSBJ model may not be useful for forecasting, even when it is the DGP.

(a)



(b)

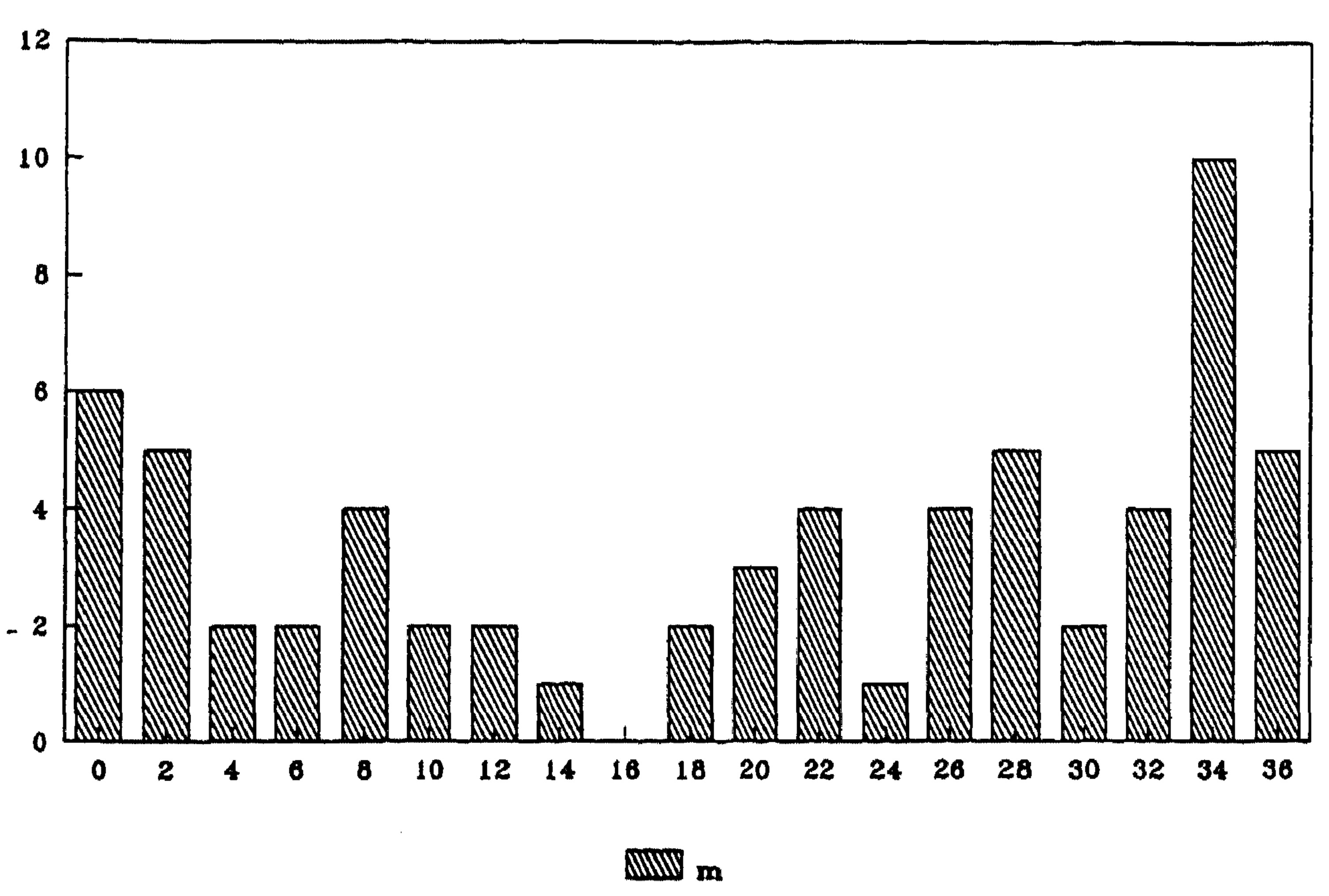


Figure 4.1 Forecast performance evaluation of an MSBJ model when an FDSD model is the DGP, in 69 and 64 replications m is the number of times the true value exceeds the forecasted value

Table 4.9

Estimation results of models for $\Delta_1 lnp$, $\Delta_1 lnip$ and $\Delta_1 lnqc$

	$\Delta_1 lnp$		$\Delta_1 lnip$		$\Delta_1 lnqc$	
Model variables[(1)]						
C	0.097**	(0.017)	0.018**	(0.007)	-0.851**	(0.077)
D_1	-0.038	(0.032)	-0.056**	(0.012)	2.859**	(0.184)
D_2	-0.092**	(0.023)	0.001	(0.010)	-0.250	(0.473)
D_3	0.051**	(0.021)	-0.023**	(0.010)	1.378**	(0.077)
D_4	-0.088**	(0.032)	-0.022**	(0.010)	0.607**	(0.163)
D_5	-0.109**	(0.019)	-0.051**	(0.011)	0.846**	(0.072)
D_6	0.032	(0.021)	-0.026**	(0.010)	0.847**	(0.080)
D_7	0.044	(0.031)	-0.137**	(0.017)	0.547**	(0.084)
D_8	-0.697**	(0.029)	0.025**	(0.010)	0.823**	(0.063)
D_9	-0.211**	(0.021)	0.057**	(0.012)	1.021**	(0.065)
D_{10}	-0.260**	(0.017)	0.023**	(0.010)	1.029**	(0.108)
D_{11}	-0.263**	(0.017)	0.009	(0.010)	0.510**	(0.132)
AR_1	-0.273**	(0.099)			0.396	(0.248)
AR_{12}			0.388**	(0.064)		
MA_1			-0.401**	(0.078)	-0.815**	(0.274)
MA_4			-0.216**	(0.079)		
Evaluation criteria[(2)]						
BP(12)	9.293		7.849		9.925	
BP(24)	22.049		30.363		23.377	
$\hat{\sigma}$	0.035		0.027		0.122	
ARCH(1)	1.180		4.090**		1.512	
T_{norm}	1.194		0.892		3.165	

** Significant at a 5% level. Standard deviations in brackets.

(1) The model contains a constant C, 11 seasonal dummies, D_1,..,D_{11}, where D_1 corresponds to January, autoregressive terms at lag p, AR_p, and moving average terms at lag q, MA_q.

(2) The evaluation criteria are the Box-Pierce test statistics for k lags. Under the null this BP(k) follows a $\chi^2(k-r)$ distribution with $k-r$ degrees of freedom, where r is the number of ARMA parameters (Pierce 1971). The $\hat{\sigma}$ is the standard deviation of the residuals. ARCH(1) is a test statistic for ARCH errors, see Engle (1982). T_{norm} is a $\chi^2(2)$ statistic for normality.

The effects of inappropriately considering the MSBJ model while the FDSD model may be suitable can also be illustrated with some empirical examples. Consider again the series *lnp*, *lnip* and *lnqc* for which, according the test results in table 4.6, an FDSD is the preferred type of model. These models, in which the orders p and q of the ARMA specification have been found after a brief specification search, are given in table 4.9, together with their estimation results and some evaluation criteria. These criteria include the familiar Box-Pierce test for residual autocorrelation, for which it is shown in Pierce (1971) that the inclusion of seasonal dummy variables does not effect its asymptotic distribution. The estimated standard deviation of the residuals is given to facilitate a comparison of models (4.49) and (4.50). Test results for ARCH errors and normality are also reported. Unreported linearity tests did not yield significant outcomes. The estimation method in case of MA terms is iterative least squares. From this table it is obvious that the FDSD type of model gives a fairly good representation of the data for all three variables. Most parameters for the seasonal dummies are highly significant, and the results of the evaluation criteria do not provide strong arguments to reject the specifications.

Table 4.10

Estimation results of models for $\Delta_1\Delta_{12}lnp$, $\Delta_1\Delta_{12}lnip$ and $\Delta_1\Delta_{12}lnqc$

	$\Delta_1\Delta_{12}lnp$		$\Delta_1\Delta_{12}lnip$		$\Delta_1\Delta_{12}lnqc$	
Model variables(1)						
MA_1	-0.338**	(0.104)	-0.436**	(0.076)	-0.337**	(0.113)
MA_{12}	-0.715**	(0.104)	-0.571**	(0.078)	-0.733**	(0.103)
MA_{13}	0.322**	(0.104)	0.363**	(0.078)	0.359**	(0.103)
Evaluation criteria(2)						
BP(12)	6.606		8.813		9.661	
BP(24)	15.325		20.185		15.848	
$\hat{\sigma}$	0.036		0.028		0.125	
ARCH(1)	0.434		6.125**		1.010	
T_{norm}	1.035		0.910		1.852	

** Significant at a 5% level. Standard deviations in brackets.

(1)(2) See notes table 4.9.

Table 4.11

Evaluation of the 36 months out-of-sample forecasting performance

Criterion[1]	*lnp* MSBJ	*lnp* FDSD	*lnip* MSBJ	*lnip* FDSD	*lnqc* MSBJ	*lnqc* FDSD
ME	-0.074	-0.064	0.042	0.022	-0.200	0.039
MAE	0.074	0.067	0.044	0.033	0.221	0.124
*max*AE	0.179	0.196	0.109	0.112	0.607	0.346
MAPE	1.229	1.116	0.942	0.691	2.117	1.171
MSE	0.007	0.006	0.003	0.002	0.079	0.022
RMSE	0.081	0.079	0.051	0.044	0.280	0.148
m	0	4	33	24	5	22
U (×100)	1.339	1.303	1.102	0.943	2.641	1.392
PIMSE of FDSD	5.402		24.17		72.22	
SIGNSE	25*		24*		22	
Signed Rank	2.781*		-4.305*		5.232*	
Rank Sum SE	0.248		2.106*		3.221*	
PE	-0.158		2.117*		1.994*	

* Significant at a 5% level

(1) The forecast error is defined as the true value y minus the forecasted value f. Forecast evaluation criteria are the mean error, ME, mean absolute error, MAE, maximum value of absolute error, *max*AE, mean average percentage of the absolute errors, MAPE, and (root) mean squared error, (R)MSE. m denotes the number of times $y-f > 0$. U is Theil's (1966) inequality coefficient. PIMSE denotes the percentage improvement of forecasts from the FDSD model with respect to mean squared error. SIGNSE refers to the sign test which reports the number of times the squared error of FDSD is smaller than that of MSBJ in pairwise comparison. The Wilcoxon *signed rank* test statistic refers to the ranks of positive differences between the forecasts. *Rank Sum* refers to the Wilcoxon test for differences in forecast performance with respect to squared error SE or to percentage error PE. Positive values for this test indicate that the FDSD model is better. Definitions and asymptotic results for the Wilcoxon tests can be found in Lehmann (1975).

The estimation and evaluation results of models of type (4.49) ,which will be the competitors in the forthcoming forecasting exercise, are displayed in table 4.10. These models show significant estimated parameters and no significant residual autocorrelation. Hence the choice for an MSBJ model might also be defended.

To evaluate the FDSD and MSBJ models in tables 4.9 and 4.10 with respect to their forecasting performance, forecasts for the levels of the series for 36 months out-of-sample are generated. The values of several forecast evaluation criteria are given in table 4.11. A test to investigate whether there are significant differences between the forecasts is the Wilcoxon signed-rank test, see, e.g., Flores (1989). The results for this test indicate that there are statistically significant differences indeed. The general result with respect to the criteria ME through Theil's (1966) U statistic seems to be that the FDSD model outperforms the MSBJ model. It is also clear that for *lnip* and *lnqc* the numbers of positive forecast errors m from using an FDSD model are close to what might have been expected, while those when using a MSBJ model are out of any reasonable range. These empirical results for m seem to confirm the simulation evidence. From the results of the Wilcoxon rank sum test for squared errors and percentage errors, and of a sign test, it appears that most differences between the models are significant and are in favor of the FDSD model. However, for the airline series the differences in forecasting performance between the MSBJ and the FDSD model are not that striking, although some forecasting improvement can be witnessed. Summarizing, appropriately taking account of the type of seasonality and nonstationarity in monthly data clearly can improve forecasting performance.

Multivariate models

The selection of seasonal models for monthly univariate time series can also have an effect on the construction of multivariate models. An example may be given by the study of Heuts and Bronckers (1988), where it is found that industrial production does not help to explain and to forecast the monthly Dutch truck sales. Seasonality in industrial production is modeled with a model like (4.49), while there is no seasonal model for the two truck sales variables. Hence, it might be interesting to see whether these models could have had an impact on the final model. Note that this resembles studies on effects of seasonal adjustment upon relationships between variables, see, e.g., Sims (1974) and Wallis (1974). I will consider four monthly Dutch time series for the period 1978–1988 and I will use the observations for 1989 and some of 1990 to evaluate the forecasting performance. The series are new truck sales, *lnqt*, new car sales, *lnqc*, the industrial production index, *lnip*, and the index of the production of building materials, *lnbm*. The untransformed observations for three of these series can be found in the Appendix A.2 through A.4. The graphs of the series are displayed in figures 4.2 and 4.3.

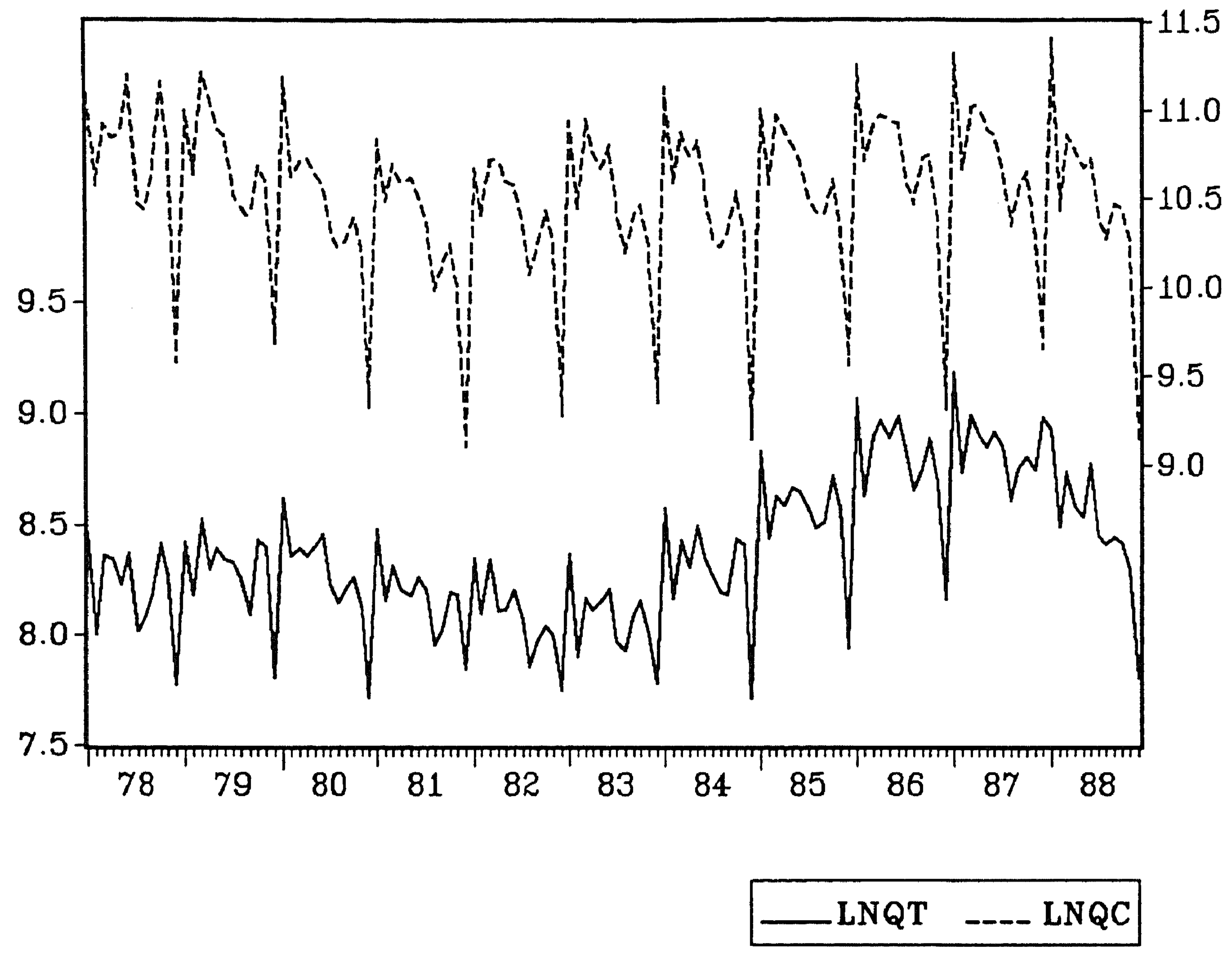


Figure 4.2. Monthly truck sales, *lnqt*, and car sales, *lnqc*, in the Netherlands

The differences between this and Heuts and Bronckers' sample are, first, the use of aggregated truck sales, and, second, that the sample period has been shifted to recent years to avoid outlier problems. This implies that the results below may not be exactly comparable to those in their study.

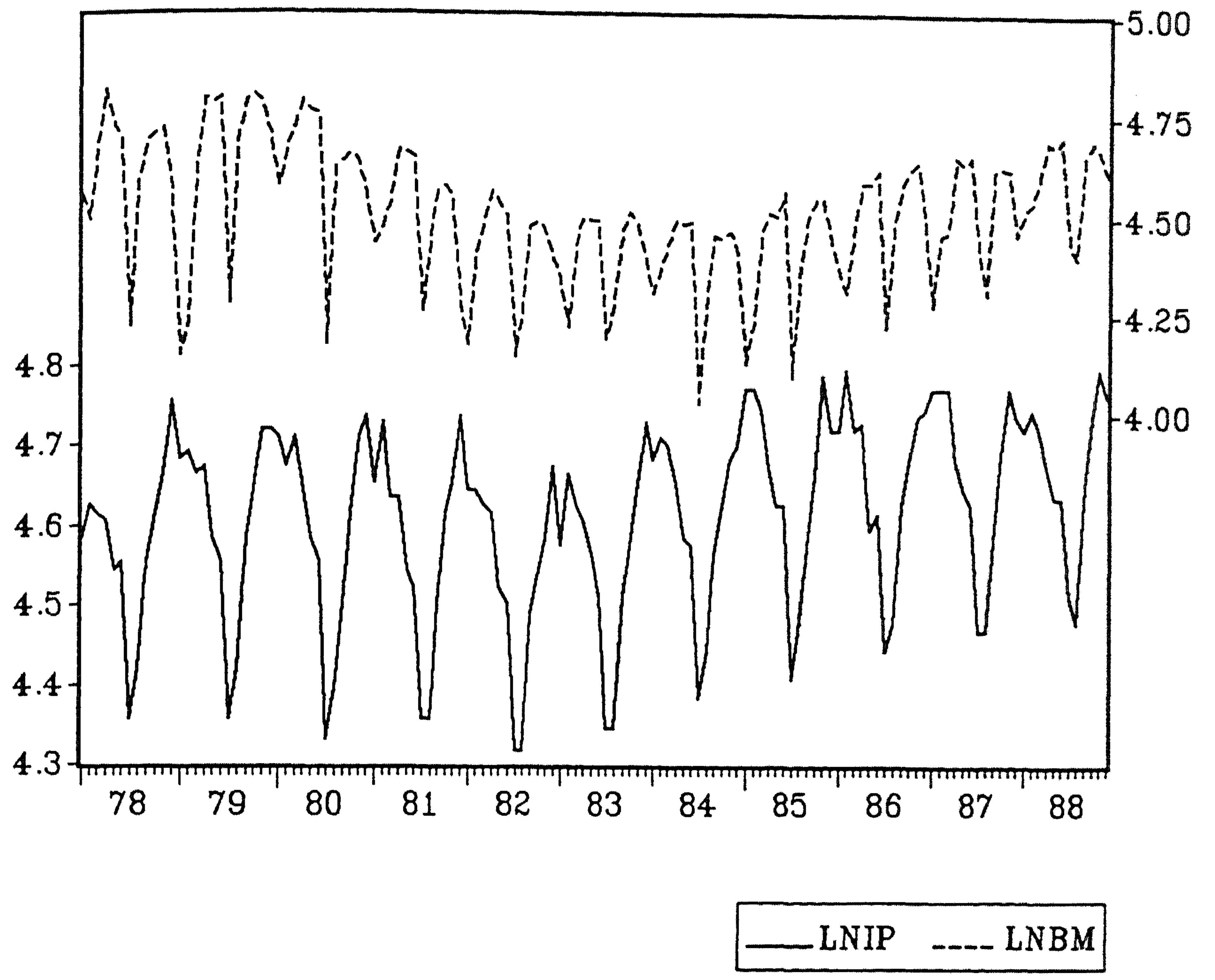


Figure 4.3 Monthly indices of industrial production, *lnip*, and of the production of building materials, *lnbm*, in the Netherlands

Table 4.12 summarizes the results of the application of the procedure for testing for (seasonal) unit roots to the four monthly series. The figures 4.2 and 4.3, as well as the significance of the corresponding parameter estimates, indicate that the auxiliary regressions should include seasonal dummies. The general outcome is that for each of the series an FDSD model may be suitable, although the hypothesis that $\pi_2=0$ for *lnbm* can only be rejected at a 15% level. From the plots in figure 4.2 it seems that there are outlying observations for *lnqt* as well as for *lnqc* in December 1987, January 1988 and February 1988. The cause for this can be found by the termination of the government subsidies for industrial investments in January 1988. Its anouncement in November 1987 induced many firms to buy, e.g., new trucks and small business vans, the latter falling into the cars category. This hoarding behavior caused an increase of the registrations in December 1987 and a decrease of these in subsequent months. The inclusion of three dummy variables, where the sum of the values is set equal to one to prevent from assuming permanent effects in this and forthcoming test equations, does not very much change the results in table 4.12.

Table 4.12

Testing for (seasonal) unit roots in some empirical monthly time series, II

	Variables (1978–1988) [1]			
t–statistics	*lnqt*	*lnqc*	*lnip*	*lnbm*
π_1	-1.507	-2.205	-0.310	-1.455
π_2	-3.336**	-3.207**	-2.783**	-2.341
π_3	-1.119	-3.483**	-1.697	-1.054
π_4	-4.384**	-3.610**	-2.553	-4.002**
π_5	-5.238**	-4.223**	-3.445**	-4.101**
π_6	-5.259**	-4.507**	-3.836**	-4.209**
π_7	2.431	2.911	1.096	2.343
π_8	-4.244**	-4.577**	-2.616	-4.026**
π_9	-1.657	-1.377	-0.686	-2.340
π_{10}	-3.898**	-3.538**	-4.201**	-3.638**
π_{11}	-1.108*	0.122	0.594	0.179
π_{12}	-4.987**	-4.187**	-3.576**	-3.468**
F–statistics				
π_3,π_4	10.519**	13.888**	4.985*	8.731**
π_5,π_6	15.066**	10.419**	7.396**	9.360**
π_7,π_8	16.486**	15.496**	7.604**	13.600**
π_9,π_{10}	7.673**	6.384**	10.901**	6.863**
π_{11},π_{12}	14.359**	12.602**	7.705**	8.392**
$\pi_3,..,\pi_{12}$	28.109**	19.425**	13.461**	15.965**

** Significant at a 5% level.

* Significant at a 10% level.

(1) All auxiliary regressions include constant and seasonal dummies, and it also applies that $\varphi^*(B)=1$ and that the number of observations is 120.

Since the four monthly series appear to have a nonseasonal unit root, these variables can be first order differenced to obtain stationarity. A multivariate time series model can now be constructed for these monthly growth rates. Such a model is, however, misspecified when the levels of the series are cointegrated, see, e.g., Engle and Yoo (1987). In the present case it is unclear which variables are exogenous and which are not, and it may therefore be suitable to apply the Johansen (1988) cointegration approach, see also section 2.2 for an exposition. An adequate order of the VAR is found to be equal to 4 which is estimated with 128 observations. The estimated eigenvalues of interest are λ_1=0.149, λ_2=0.069, λ_3=0.055, λ_4=0.012. The corresponding test statistics have the values $L(r\leq 0)$=38.523, $L(r\leq 1)$=17.871, $L(r\leq 2)$=8.706 and $L(r\leq 3)$=1.519. Comparing these with the critical values in table 2.2, i.e. those tabulated under the assumption that there is only a constant in the DGP, shows that none of these is significant. It can thus be concluded that there is no cointegration between the four variables.

Table 4.13
Estimation results of a simplified VAR(3)
containing truck and car sales and two production indices

$$\Gamma_0' = \begin{bmatrix} -0.536 & -1.172 & 0.085 & -0.047 \\ 1.047 & 2.321 & -0.083 & -0.024 \\ 0.511 & 0.932 & -0.073 & -0.274 \\ 0.727 & 1.597 & -0.103 & 0.237 \\ 0.489 & 1.510 & -0.126 & 0.172 \\ 0.544 & 1.057 & -0.180 & 0.151 \\ 0.586 & 1.247 & -0.151 & 0.109 \\ 0.425 & 0.902 & -0.292 & -0.341 \\ 0.371 & 0.924 & -0.177 & 0.058 \\ 0.500 & 1.313 & 0.031 & 0.230 \\ 0.742 & 1.385 & 0.092 & 0.130 \\ 0.523 & 0.823 & 0.067 & 0.110 \\ 0.710 & 0.280 & & \\ -0.482 & & & \\ -0.410 & -0.331 & & \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} -0.570 & & & \\ & -0.417 & & \\ & & -0.643 & \\ & 0.155 & & -0.606 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} -0.245 & & & \\ & -0.312 & 0.877 & \\ & & -0.230 & \\ & & & -0.310 \end{bmatrix}$$

$$\Gamma_3 = \begin{bmatrix} & & 0.333 & \\ & -0.255 & 0.785 & \\ & & & \\ & & & -0.186 \end{bmatrix}$$

This means that the matrix $\boldsymbol{\Pi}$ in (2.36) is the null matrix, and hence that a VAR(3) for the first order differenced variables can be constructed, or

$$\Delta_1 \boldsymbol{y}_t = \boldsymbol{\Gamma}_0 \boldsymbol{\mu} + \boldsymbol{\Gamma}_1 \Delta_1 \boldsymbol{y}_{t-1} + \boldsymbol{\Gamma}_2 \Delta_1 \boldsymbol{y}_{t-2} + \boldsymbol{\Gamma}_3 \Delta_1 \boldsymbol{y}_{t-3} + \boldsymbol{\varepsilon}_t, \tag{4.52}$$

with $\boldsymbol{y}_t = (lnqt,\ lnqc,\ lnip,\ lnbm)_t'$. The $\boldsymbol{\mu}$ is a (15x1) vector containing a constant, 11 seasonal dummies where the first reflects January and three dummies, and $\boldsymbol{\Gamma}_0$ is a (4x15) matrix of parameters. The three additional dummies are now given in the conventional way, i.e. with ones and zeros, to ease interpretation. The general model appears to be overparameterized. A simplification search using several model selection criteria, and where diagnostic checks as in section 3.4 are used to detect inadequate models, resulted in a smaller model. Its parameter estimates are displayed in table 4.13.

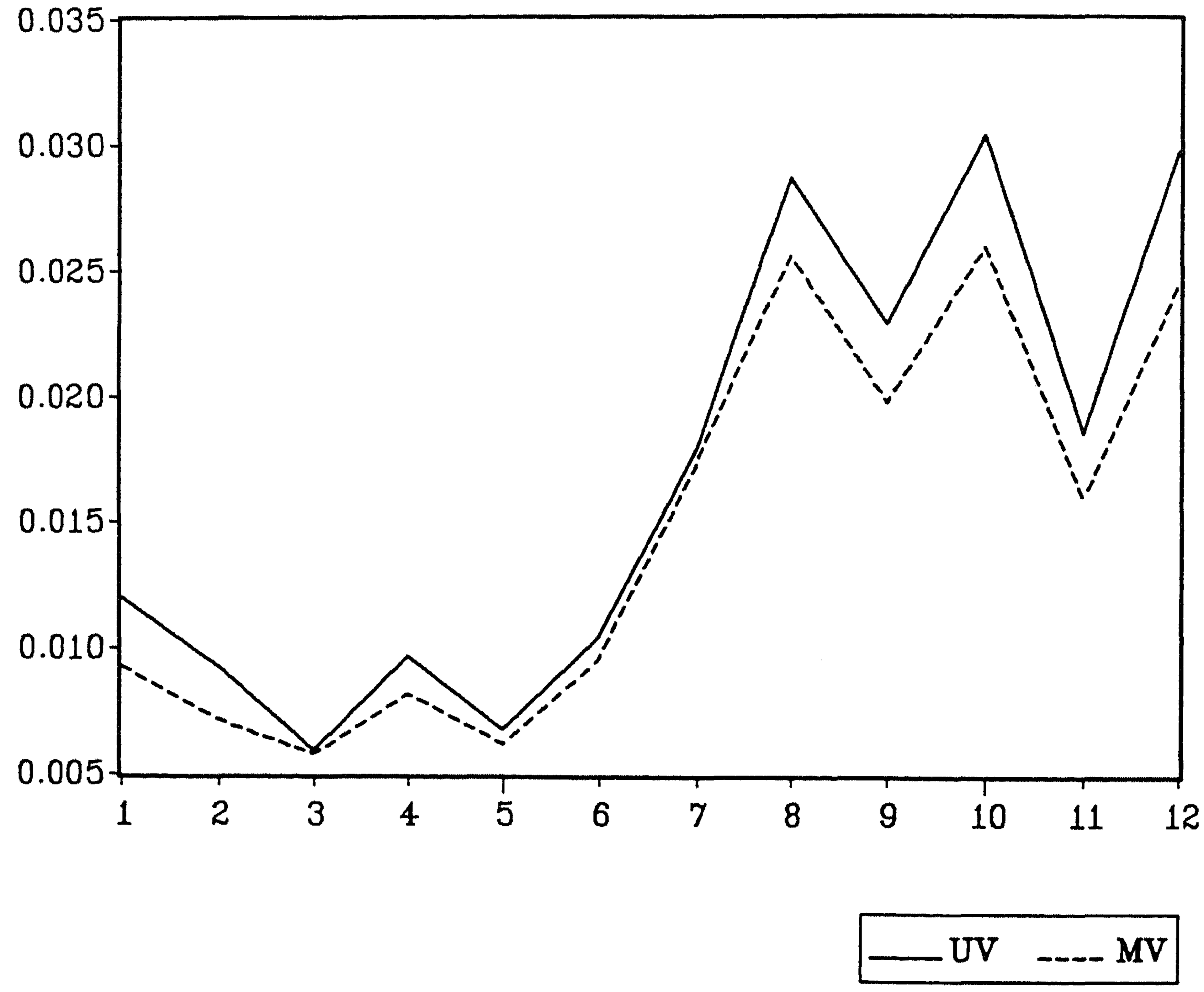


Figure 4.4 Average mean squared error of prediction of a univariate model, *UV*, and a multivariate model, *MV*

From these estimation results it can be seen that the four variables are largely explained by their own past and by deterministic seasonal dummy variables, which might have been expected from the graphs in figures 4.2 and 4.3. The industrial production index only effects the truck sales with a lag of three months. To see whether this has an impact on forecasting, the following forecast exercise, which is comparable to the one in Heuts and Bronckers (1988), is carried out. The models to be considered are the VAR(3) in table 4.13, and a univariate FDSD model where the order of the AR part is set equal to 2. Forecasts for 12 periods ahead are generated from a shifting origin, i.e. from December 1988 through May 1989. For each horizon, I calculate the mean squared error of prediction (MSEP). The averages of these 6 MSEPs are depicted in figure 4.4. From this figure it is obvious that the multivariate model yields a better forecasting performance than the univariate model for all 12 horizons. So, not only does the variable *lnip* help to explain *lnqt* within sample, it also helps to predict future values. This example shows that appropriately taking account of the seasonal behavior of each of the univariate series may be useful for the construction and merits of a multivariate model.

Outliers and linearity

There is already a vast literature on seasonality, outliers and (non-) linearity in economic time series. Examples are methods for outlier detection in a linear model context, see for recent studies, e.g., Bruce and Martin (1989) and Peña (1990), and tests for linearity assuming the absence of outlying observations such as in Chan and Tong (1986). In case such methods, or nonlinear models, are applied to time series consisting of seasonal observations, one usually initially transforms the original series to get rid of seasonal influences, see, e.g., Maravall (1983). So, in general, these issues are studied separately while assuming that the others have been appropriately handled.

At this point I wish to draw attention to the way seasonality is treated and its possible effect on outliers and linearity. More precisely, the incorrect transformation of a seasonal time series containing some additive outliers can result in a time series which shows large sequences of outliers, and hence for which a nonlinear model seems to be more appropriate. A formal treatment of this issue is likely to involve complicated formulas, and hence, at present, I will only use some empirical and simu-

To provide an illustration, consider the following simple experiment. Similar to the experiments for the forecasting issue, monthly data are generated from model (4.51) with parameters for the seasonal dummies as in case (a) for 1950.01 to 1970.12, although only those starting from 1958.01 will be used. Furthermore, construct the series add_t which is given by

$$add_t = \begin{cases} -4,\ 12,\ -8,\ 6 & \text{in } 1958.12,\ 1959.01,\ 1959.02,\ 1959.03 \\ 9,\ -7,\ 8,\ -6 & \text{in } 1964.11,\ 1964.12,\ 1965.01,\ 1965.02 \\ 0 & \text{otherwise} \end{cases} \tag{4.53}$$

When the series $ys_t=y_t+add_t$ is constructed, it is obvious that this ys_t contains some additive outliers as can be seen from figure 4.5. Note that such additive outliers are by no means uncommon in economic time series, where, e.g., a strike at a registration office can induce that part of the registrations have to be made up in the next period.

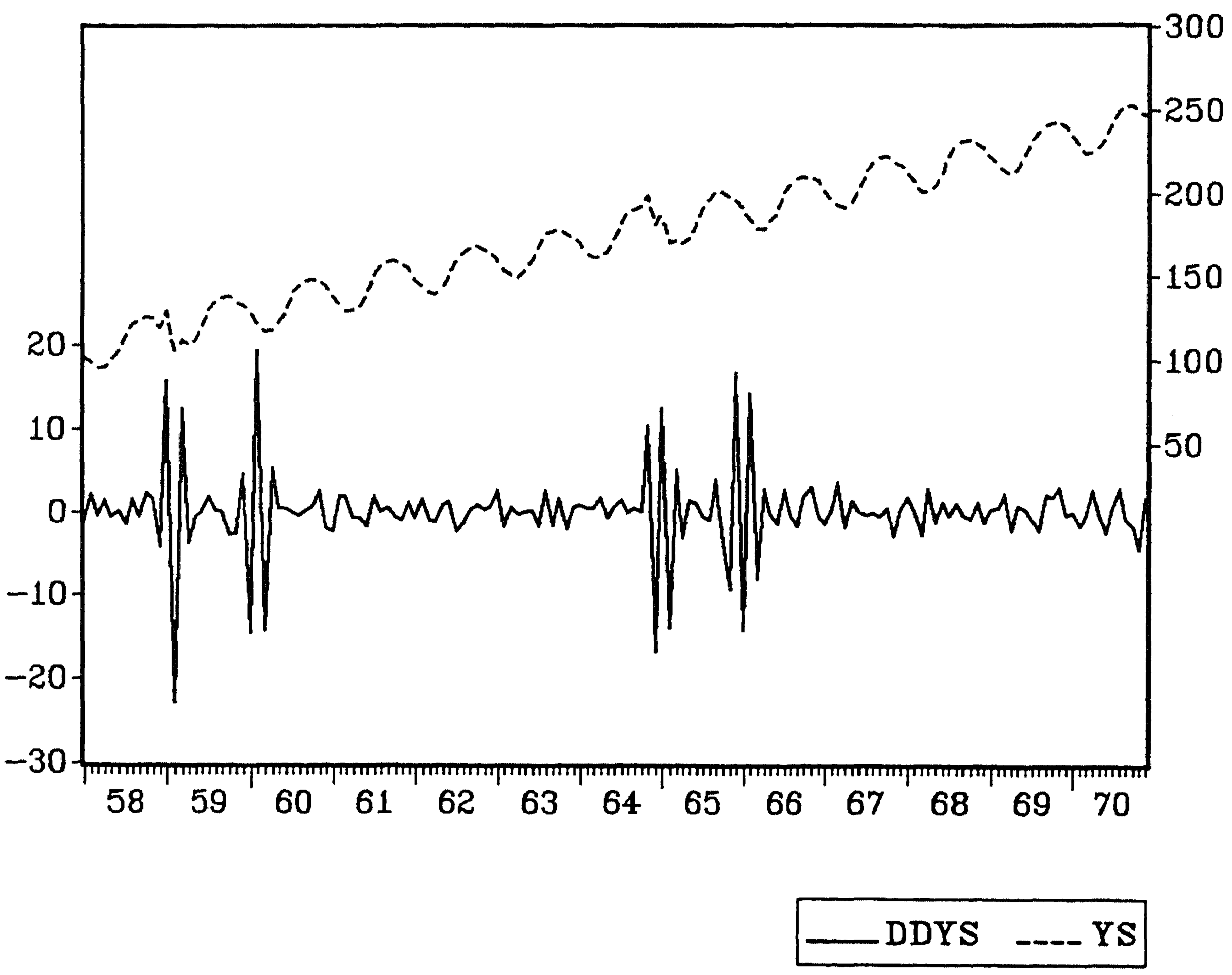


Figure 4.5 Artificial monthly time series ys and $\Delta_1\Delta_{12}ys$

An often applied step in modeling monthly time series is the transformation of the series into $\Delta_1\Delta_{12}ys_t$. The plot of this series is also given in figure 4.5. Suppose now that one encounters the series $z_t = \Delta_1\Delta_{12}ys_t$ without any prior knowledge of the transformation. Fitting a univariate ARMA to z_t would result in patches of outliers. Alternatively, anyone familiar with the literature on bilinear models, and especially with the graphs of typical series, see, e.g., Granger and Andersen (1978) and Subba Rao and Gabr (1984), may presumably start fitting a bilinear model to z_t to account for its 'temporary bubbles', or at least start testing for the presence of nonlinearity. However, assuming that the underlying seasonal process is (4.51), a simple FDSD model can be fitted, and applying the Abraham and Yatawara (1988) test statistic given in (3.45) reveals that there are some additive outliers indeed. Note however that in case the y_t is generated by an MSBJ model as in (4.49) and, analogously, the series ys_t is constructed, then similar graphical evidence emerges.

To empirically illustrate the possible phenomenon that additive outliers can establish that an incorrect transformation introduces nonlinearity, I consider the monthly *lnqt* and *lnu* series already used in table 4.6. From these results it emerges that an FDSD model for these variables would be appropriate. The *lnu* series is used in Subba Rao and Gabr (1984) where a rather complicated bilinear model is fitted to the series. The test for linearity here is the one developed in Keenan (1985), an expression of which was already given in (3.47). It is a general linearity test in the sense that it is not necessary to construct a specific nonlinear alternative model. This can be a drawback, especially compared with tests that test against a certain alternative, and that may therefore possess higher power in some occasions. From studies as Chan and Tong (1986), Tsay (1986), Luukkonen *et al.* (1988) and Lee *et al.* (1989), it emerges that in large samples Keenan's test often has reasonably high power in case bilinear models are the data generating processess. Given the experiment above, it is just this type of models that deserves our special attention here. Furthermore, in case of monthly data it may be difficult to determine the alternative hypothesis for the construction of Lagrange Multiplier tests since one can select between an enormous amount of candidates.

The first regression of the Keenan test is to fit an autoregressive model of order M to the, possibly transformed, time series, see (3.47). This choice of M is important. For example, when M is too small, incorrect sizes may occur because the test will then also respond to remaining autocorrelation in the residuals. Hence, in the applications below the M will be set equal to 12, 24 or 36, the choice of which will be based on the usual checks of the first 48 residual autocorrelations.

One might now argue that the order of M is likely to be misspecified in the current cases of interest. When a series is inadequately doubly differenced, a moving average polynomial is introduced with a root near or equal to one. In principle, the value of M is never large enough. This implies that even when the double filter does not introduce nonlinearity, the test will indicate rejection of the null hypothesis in case M is set at a certain value. To gain some insight in the behavior of the Keenan test in such boundary cases, some simulation experiments have been carried out when the data generating process is

$$y_t = \phi_1 y_{t-1} + \beta_1 y_{t-1}\varepsilon_{t-1} + \varepsilon_t + \theta_1\varepsilon_{t-1}, \tag{4.54}$$

and the parameters can take different values. The number of replications is 350, the number of observations is 100, and M is set equal to 4 and 8. A selection of the results is displayed in table 4.14. They seem to indicate that the Keenan test is sensitive to the specification of M, but that a unit root in the MA(1) model does not have a distorting effect on the empirical size of the test.

Table 4.14

Empirical rejection frequencies of the Keenan (1985) test for linearity

DGP is given by $y_t = \phi_1 y_{t-1} + \beta_1 y_{t-1}\varepsilon_{t-1} + \varepsilon_t + \theta_1\varepsilon_{t-1}$

ϕ_1	β_1	θ_1	$M = 4$	$M = 8$
1.00	0	0	3.4	2.8
0.70	-0.30	-0.30	72.0	62.6
0.20	-0.80	-0.30	45.4	39.4
0	-0.40	-0.50	79.1	61.4
0	0	-0.50	5.1	5.1
0.40	0.20	-0.95	52.0	42.8
0	0	-1.00	6.0	5.1

One might also argue that the Tsay (1986) test should preferably be used because it is an extension of the Keenan test, and, as expected, often obtains higher power. However, the Tsay test involves a regression equation containing $M(M+1)/2$ variables, which implies, e.g., 300(!) regressors in case $M=24$. Even in our large sample case, this does not seem sensible.

The variables $\Delta_1 lnqt_t$ and $\Delta_1 lnu_t$ are regressed on a constant and 11 seasonal dummies. The residuals of these regressions are used for testing for linearity, the results of which are displayed in table 4.15. In the same table test results are given for the $\Delta_1\Delta_{12}$ transformed variables. From these outcomes it can be seen that linearity is accepted for the variables where seasonality is treated according to the FDSD model, and that linearity is rejected for the inappropriately transformed variables.

The result for lnu_t confirms the apparent success of detecting the nonlinearity of, and fitting a bilinear model to, $\Delta_1\Delta_{12} lnu_t$, as in Subba Rao and Gabr (1984). One may now argue that the inclusion of eleven dummy variables could have weakened the influence of any additive outliers. The unreported estimation and testing results for an FDSD model for lnu_t show that such a model may indeed be adequate. And, an application of Abraham and Yatawara's (1988) procedure reveals the presence of several outlying observations with patterns as in (4.53).

Table 4.15

Testing linearity in two empirical monthly time series

Variable	M	Keenan[(1)]
$\Delta_1 lnqt$	12	0.093
$\Delta_1 lnu$	24	0.479
$\Delta_1\Delta_{12} lnqt$	24	7.566**
$\Delta_1\Delta_{12} lnu$	36	9.721**

** Significant at a 5% level.

(1) This test statistic approximately follows an $F(1,n-2M-2)$ distribution under the null hypothesis of linearity.

It can be concluded that an appropriate transformation of a monthly time series, necessary to account for the eventual presence of seasonality and nonstationarity, can lead to the acceptance of linearity. Furthermore, incorrect filtering of the data can result in rejection of linearity, which can cause subsequent modeling to involve unnecessary, and computationally cumbersome, steps, such as e.g. the fitting of bilinear models. It would now also be interesting to investigate whether the success of fitting a bilinear model to the residuals of the linear model for the approximately $\Delta_1\Delta_{12}$ transformed variable in Maravall (1983) is also due to this transformation. In case one however decides to fit a linear model, one would probably encounter a large amount of outliers. This corresponds, e.g., to the recent findings of Bruce and Martin (1989), where patches of outliers emerged for the $\Delta_1\Delta_{12}$ transformed variable in their example 6.

Some additional comments are in order. The first is that it is of course not true that an appropriate filter for monthly data automatically implies linearity. Some experience with several other unemployment series mostly supports the above findings, but also reveals that if a correctly transformed variable shows some nonlinearity, then the $\Delta_1\Delta_{12}$ transformed does so. Secondly, in case the Keenan test statistic indicates the presence of nonlinearity it is unclear which nonlinear model is the most appropriate, although the available power studies suggest that a bilinear model can be. Furthermore, it obviously seems worthwhile to carry out some extensive Monte Carlo simulations to emphasize the empirical outcomes. On the whole, however, I feel that some arguments are provided for the simultaneous treatment of seasonal processes, outliers and linearity.

4.4 A MULTIVARIATE APPROACH

It is clear from the previous sections, and also from the surveys in e.g., Nerlove *et al.* (1979), Hylleberg (1986), and Ghysels (1990b), that there are several types of models for seasonality in economic time series. Examples are models in which a series is assumed to be seasonally integrated, models in which seasonality can be represented by deterministic dummies, and the periodic autoregression model (Osborn 1988). Procedures designed to discriminate between some of these models are e.g., the method to test for the presence of seasonal unit roots, as developed in Hylleberg *et al.* (1990), and the method to check for the noninvertibility of the airline model as given in section 4.1. Selection between most common seasonal models is however not straightforward. One difficulty is that some of these models are not nested. An additional problem is that there are economic time series for which a seasonally integrated model is inappropriate, but where the parameters for seasonal dummies in a model with deterministic seasonality are not constant over time, see Osborn (1990a).

A more general class of models may therefore be needed, in which most common models are nested, and which also contains intermediate cases between some of the current models. In this section, I will argue that a vector autoregressive model for the, possibly nonstationary, vector of annual series of observations per season can provide such a class. As it has already been pointed out in Osborn (1990b), it can be shown that several of the univariate models imply the presence or absence of a number of cointegration relationships between these annual series. Hence, a model selection test procedure may be given by the maximum likelihood procedure for testing for cointegration as developed in Johansen (1988). For notational convenience and illustrative purposes, I only discuss quarterly time series, although the multivariate approach may theoretically be extended to e.g., bimonthly or monthly time series. In the first part of this section, I apply this approach to several empirical univariate series. To save space I will consider only a limited number of examples, and the reader is referred to Franses (1991a) for a detailed account. Furthermore, the empirical modeling strategy discussed in chapter 3 is followed, but not all testing outcomes will be reviewed. An important issue in these applications is the mutual comparison of the graphs of the annual series for they may give additional insights. The second part of this section considers an extension of this new approach to bivariate time series.

Univariate time series

Consider $4n$ observations for a univariate quarterly time series y_t, and the seasonal dummies D_{it}. Two simple models for y_t, t=1,..,4n, are given by

$$\Delta_4 y_t = \eta_t, \tag{4.55}$$

$$\Delta_1 y_t = \sum_{i=1}^{4} \delta_i D_{it} + \eta_t, \tag{4.56}$$

where Δ_k is again defined by $\Delta_k y_t = y_t - y_{t-k}$. The η_t is assumed to follow an ARMA process, and hence the white noise process ε_t is a special case.

An alternative model for y_t may be given by an autoregressive model of order p for the vector $\boldsymbol{q}_t' = (q_{1t}, q_{2t}, q_{3t}, q_{4t})'$ consisting of the n annual observations per quarter i, i=1,2,3,4, or

$$\boldsymbol{q}_t = \boldsymbol{\Pi}_1 \boldsymbol{q}_{t-1} + .. + \boldsymbol{\Pi}_p \boldsymbol{q}_{t-p} + \boldsymbol{\mu} + \boldsymbol{\omega}_t, \tag{4.57}$$

where now t=1,..,n, and the $\boldsymbol{\mu}$ and $\boldsymbol{\omega}_t$ are (4×1)-vectors with $\boldsymbol{\omega}_t \sim N_4(0,\Lambda)$. The order p in (4.57) may usually be low, because it can be shown that the value of p implies that each q_{it} follows an ARMA($4p$,$3p$) process, which in turn implies that y_t follows an ARMA($16p$,$12p$) process with periodically varying parameters and error variances. In fact, in the applications below it will be seen that p=1 may often be appropriate. Model (4.57) can be called a vector of quarters (VQ) representation in case of quarterly data.

The relationships between the quarters within one year are reflected by the matrix $\boldsymbol{\Lambda}$. For example, model (4.55) implies that Λ is a diagonal matrix. To further illustrate the flexibility of the vector autoregressive model, consider the model $y_t = \alpha y_{t-1} + \varepsilon_t$. Then, one can write

$$\begin{aligned} q_{1t} &= \alpha q_{4,t-1} + \varepsilon_{1t} \\ q_{2t} &= \alpha^2 q_{4,t-1} + \varepsilon_{2t} + \alpha\varepsilon_{1t} \\ q_{3t} &= \alpha^3 q_{4,t-1} + \varepsilon_{3t} + \alpha\varepsilon_{2t} + \alpha^2\varepsilon_{1t} \\ q_{4t} &= \alpha^4 q_{4,t-1} + \varepsilon_{4t} + \alpha\varepsilon_{3t} + \alpha^2\varepsilon_{2t} + \alpha^3\varepsilon_{1t}, \end{aligned} \tag{4.58}$$

and one obtains

$$\boldsymbol{\Pi}_1 = \begin{bmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha^2 \\ 0 & 0 & 0 & \alpha^3 \\ 0 & 0 & 0 & \alpha^4 \end{bmatrix},$$

and

$$\Lambda = \sigma_\varepsilon^2 \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ \alpha & 1+\alpha^2 & \alpha+\alpha^3 & \alpha^2+\alpha^4 \\ \alpha^2 & \alpha+\alpha^3 & 1+\alpha^2+\alpha^4 & \alpha+\alpha^3+\alpha^5 \\ \alpha^3 & \alpha^2+\alpha^4 & \alpha+\alpha^3+\alpha^5 & 1+\alpha^2+\alpha^4+\alpha^6 \end{bmatrix}.$$

This model $y_t = \alpha y_{t-1} + \varepsilon_t$ is a very restricted model, since given (4.57) it assumes that $\alpha_s = \alpha$ and that $\varepsilon_{st} = \varepsilon_t$. Note that the relations between the quarters within one year can also be understood from multiplying both sides of (4.57) with a lower triangular matrix $\boldsymbol{\Pi}_0$.

In case $\boldsymbol{q}_t$ consists of $I(1)$ components, there may be linear combinations of the q_{it} which are $I(0)$. Given that the 4 variables q_{it} constitute a complete system with respect to process y_t, it seems natural to apply the Johansen (1988) maximum likelihood procedure for testing for the number of cointegration vectors in (4.57). For that purpose it is convenient to rewrite (4.57) in case p=1, into

$$\Delta_1 \boldsymbol{q}_t = \boldsymbol{\Pi} \boldsymbol{q}_{t-1} + \boldsymbol{\mu} + \boldsymbol{\omega}_t, \tag{4.59}$$

where $\boldsymbol{\Pi} = -\boldsymbol{I}+\boldsymbol{\Pi}_1$. The process generating $\boldsymbol{q}_t$ is asymptotic stationary in case r=rank($\boldsymbol{\Pi}$) is 4. There is no cointegration in case r equals 0. In case $0 < r < 4$, one can write $\boldsymbol{\Pi}=\boldsymbol{\alpha\beta}'$, where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $(4\times r)$ matrices. The matrix $\boldsymbol{\beta}$ contains the cointegration vectors, which have the property that $\boldsymbol{\beta}'\boldsymbol{q}_t$ is stationary. Johansen (1988) develops procedures to test for the value of r, and also to investigate linear hypotheses in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. It boils down to the choice of the r linear combinations of elements of $\boldsymbol{q}_t$ which have the largest partial correlation with $\Delta \boldsymbol{q}_t$ after correcting for $\boldsymbol{\mu}$. The eigenvectors of the relevant canonical correlation matrix, say **C**, are the columns of $\boldsymbol{\beta}$. The corresponding eigenvalues λ_i, where $\lambda_i \geq \lambda_{i+1}$, are used to test for the number of cointegration vectors using the likelihood ratio test statistic

$$Tr(r) = -n \sum_{i=r+1}^{4} \log(1-\lambda_i). \tag{4.60}$$

Several fractiles for this statistic are given in table 2.2.

To test for linear restrictions on the parameters $\boldsymbol{\beta}$, it is suitable to define a $(4\times s)$-matrix **H**, where $r \leq s \leq 4$, which reduces $\boldsymbol{\beta}$ to the parameters

$\boldsymbol{\varphi}$, or $\boldsymbol{\beta}$=H$\boldsymbol{\varphi}$. For brevity, I shall denote these restrictions by their matrix **H**. Furthermore, note that in practice we often take $s = r$. The eigenvalues ξ_i of the canonical correlation matrix under the assumption that the restrictions **H** are valid are now compared with the λ_i to test for significant differences. The test statistic is

$$Q = n \sum_{i=1}^{r} \log(1-\xi_i)/(1-\lambda_i)\}, \tag{4.61}$$

and it follows a $\chi^2(r(4-s))$ distribution under the null hypothesis, see Johansen and Juselius (1990).

This multivariate approach to the, possibly nonstationary, annual series of the quarterly observations of y_t gives an opportunity to test which model, i.e. (4.55), (4.56) or others, may be appropriate for y_t. The idea is thus to check whether the series q_{it} have common nonstationary components. First, consider the case where r is equal to 4 and where $\boldsymbol{\Pi}$ in (4.59) is of full rank. This implies that an observation in some quarter can be explained by all other quarters in the previous year. A special case is $\boldsymbol{\mu}$=0, $\boldsymbol{\Pi}_1=\psi \boldsymbol{I}$ with $|\psi|<1$, for which it is easy to see that (4.59) is equivalent to the stationary model

$$y_t = \psi y_{t-4} + \varepsilon_{st}. \tag{4.62}$$

From model (4.56) it can be seen that pairs of successive q_{it} are cointegrated with parameters (1,-1), see also Osborn (1990b). Or, while dropping the time index, this model assumes the presence of the cointegration relationships (q_2-q_1), (q_3-q_2), and (q_4-q_3), which imply the presence of the fourth, i.e. (q_1-q_4). In terms of model (4.59) this means that r equals 3 and that the restrictions on the columns of $\boldsymbol{\beta}$, which are given by

$$\mathbf{H}_{31}= \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.63}$$

are accepted. This can be tested via the procedure given in (4.61). It is also possible to test for the presence of seasonal unit roots. Recall that Δ_4 can be written as $(1-B)(1+B)(1+B^2)$. The $(1-B)$ part corresponds to the nonseasonal unit root, and the $(1+B)$ and $(1+B^2)$ correspond to the seasonal unit roots -1, $\pm i$. For example, the presence of the roots 1, $\pm i$ can be accepted in case the filter $(1\text{-}B)(1+B^2)$ is appropriate to make the series y_t stationary. So, in case $r = 3$, one can check for the presence of root -1 by testing the restrictions

$$\mathbf{H}_{32}= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

When the restictions $\mathbf{H}_{31}$ and $\mathbf{H}_{32}$ are rejected, one has encountered three general cointegration relationships, which may be represented by

$$q_i = \sum_{j \neq i}^{4} \gamma_{ij} q_j, \tag{4.64}$$

for i=1,2,3, where the γ_{ij} are functions of the elements of $\boldsymbol{\beta}$. This can be solved as

$$q_i = \phi_{i\gamma} q_4, \tag{4.65}$$

where $\phi_{i\gamma}$ is a function of the γ_{ij}. Restricting the $\phi_{i\gamma}$ to ϕ_1, $\phi_1\phi_2$ and $\phi_1\phi_2\phi_3$, respectively, compares to a nonstationary periodic first order autoregressive model (Osborn, 1988), which is given by

$$y_t = \phi_i y_{t-1} + \varepsilon_t, \tag{4.66}$$

with $\phi_1\phi_2\phi_3\phi_4$=1 but not all ϕ_i=1. From (4.66) it is now also easy to see that indeed successive quarters are cointegrated, albeit not with the vector (1,−1).

In case there are no cointegration relationships between the elements of $\boldsymbol{q}_t$, then each q_{it} series follows a separate random walk. It is clear that r=0 implies that a model of type (4.55) is appropriate. The intermediate cases in which r is equal to 1 or 2 provide intermediate cases between (4.55) and (4.56). Again one may want to test for the presence of seasonal unit roots. When r=2, and the restrictions

$$\mathbf{H}_{21}= \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{H}_{22}= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

are not rejected, the roots 1, -1 and $\pm i$ can not be rejected, respectively.

Similarly, the rejection of the hypotheses

$$H_{11} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \qquad H_{12} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

implies that the roots 1, $\pm i$, and -1, $\pm i$ are not present in case $r=1$. When all restrictions H_{rj}, $j=1,2$, are rejected, one ends up with a periodic model with an error correction term $\beta' q_t$. The "error" of overdifferencing is now corrected by one or two variables which contain linear relationships between the annual series.

Table 4.16

Estimation and testing results of VQ models of order 1

Var.[1]	Π_1				$\Lambda_{ii}^{1/2}$ [2]	Correlations[3]			BP
A	0.094	0.260	-0.218	0.837	2.648	0.106	0.509	0.568	2.560
	0.161	0.610	-0.205	0.427	1.480		0.619	0.522	2.222
	0.093	-0.170	0.249	0.806	2.086			0.753	5.850
	0.355	-0.246	0.039	0.804	2.955				2.287
B	-0.155	0.371	0.200	0.592	1.603	0.847	0.714	0.732	3.389
	-0.533	0.613	0.104	0.823	1.673		0.814	0.813	2.858
	-0.596	0.221	0.425	0.973	1.619			0.875	8.421*
	-0.850	0.915	0.064	0.832	2.114				7.185
C	0.091	-0.422	0.621	0.701	6.075	0.884	0.811	0.648	9.296*
	-0.103	-0.170	0.707	0.601	7.601		0.917	0.798	2.976
	-0.035	-0.448	1.090	0.390	7.657			0.830	5.860
	-0.114	-0.544	0.593	1.043	8.556				0.499
D	0.425	0.194	0.714	-0.331	12.67	0.367	0.272	0.151	7.521
	0.073	0.284	0.550	-0.045	10.10		0.613	0.465	3.552
	0.400	-0.586	0.597	0.354	14.02			0.551	3.087
	-0.253	0.180	0.545	0.349	19.86				2.324

* Significant at a 10% level. Box-Pierce test statistic $\sim \chi^2(4)$.

(1) The series are A: disposable income in Japan; B: total consumption in Japan; C: demand for money in Finland; D: new car sales in the Netherlands.

(2) The values of the standard deviations have been multiplied by 100.

(3) Only the off-diagonal elements are reported.

To illustrate this multivariate procedure, I consider four series. The reader is referred to Franses (1991a) for additional examples. The order of a reasonably adequate vector autoregressive model in all the applications appears to be equal to 1. I have determined the value of p by taking the smallest model for which the residual autocorrelations do not indicate that the model is seriously misspecified. Experience with cointegration testing in practice has shown that too small a model often results in an incorrectly large size of the test procedures, while too large a model induces a reduction of their power, see also Boswijk and Franses (1991). The estimation and testing results of the unrestricted VARs are displayed in table 4.16. These include the estimates of the matrix Π_1, of the square roots of the diagonal elements of Λ, and of the correlations between the ω_{it}. Furthermore, the values of a residual autocorrelation test statistic are given, and the general impression is that p=1 may often be adequate indeed. It should also be noted that especially in small samples the significance of a residual autocorrelation may well be caused by one or two observations. This has also motivated the decisions for p=1 rather than automatically extending the models to a higher order. Moreover, it turns out that often similar outcomes emerge in case p is set equal to 2.

The first two series are given in Engle *et al.* (1991). There it is suggested that disposable income in Japan may follow a process like (4.55), and that total consumption does too, approximately. Graphs of the four series for the separate quarters of income are given in figure 4.6. This figure may show about a "typical" pattern of a process like model (4.56), i.e. the patterns of pairs of quarters seem to evolve similarly over time, and also the distances between the individual lines seem to be rather constant. Furthermore, there is only one brief period where one of the inequalities $q_{4t} > q_{3t} > q_{2t} > q_{1t}$ is violated. The results of the cointegration test procedure for this series are displayed in the first column, i.e. under A, of table 4.17. The values of the eigenvalues are such that the hypothesis that r is equal to 3 can not be rejected. Moreover, the null hypothesis of cointegration of the sequential quarters with parameters (1,-1) is accepted. This implies that for this series a model of type (4.56) may be appropriate. Additional evidence for this result may be provided by the estimation results of an FDSD model, cf. (4.50), for this income series. These are a coefficient of 0.71 for $\Delta_1 z_{t-4}$, with standard deviation of 0.06, and a coefficient of -0.60 for the first order moving average term, with a standard deviation of 0.10. The estimates of the parameters for the constant and three dummies are highly significant. These estimation results also indicate that the detection of any restricted cointegration relationships between the quarters may not be effected by nonwhite and possibly periodic error processes in models like (4.56).

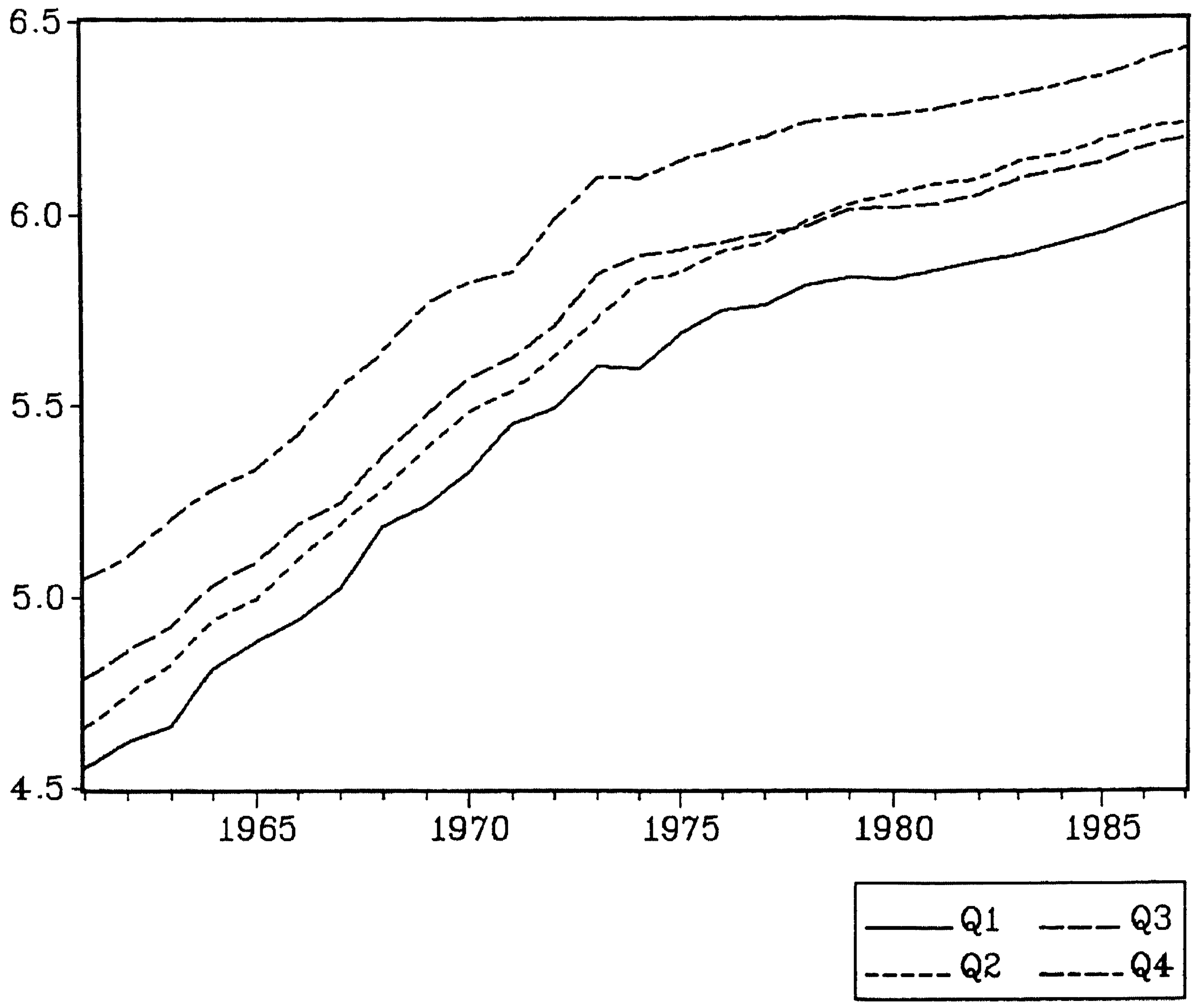


Figure 4.6 Disposable income in Japan, 1961-1987

The graphs corresponding to the total consumption series are given in figure 4.7. Again, there seems to be graphical evidence for the presence of several cointegration relationships. This suggestion is confirmed by the results in table 4.17, second column. The hypothesis that $r=3$ can not be rejected, but the restrictions given in H_{31} and H_{32} are not appropriate. These results appear also to be valid in case p is set equal to 2, a value which may be reasonable given the results in Engle *et al.* (1991). This suggests that the consumption in Japan, as in the United Kingdom

(Osborn, 1988), may be modeled using a periodic model. Estimation for such a model might now be done by regressing q_{it} on q_{it-1}, see also (4.66). Tentative estimation results, where in some equations first order moving average terms are included, are $\hat{\phi}_1$=1.083, $\hat{\phi}_2$=0.977, $\hat{\phi}_3$=1.013 and $\hat{\phi}_4$=0.931, giving that $\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4$ equals 0.998, which is remarkably close to 1. It can be seen that the adopted model selection test procedure may well be able to discriminate between a model like (4.56) and a possibly seasonally varying parameter model, even though the ϕ_i's are close to unity.

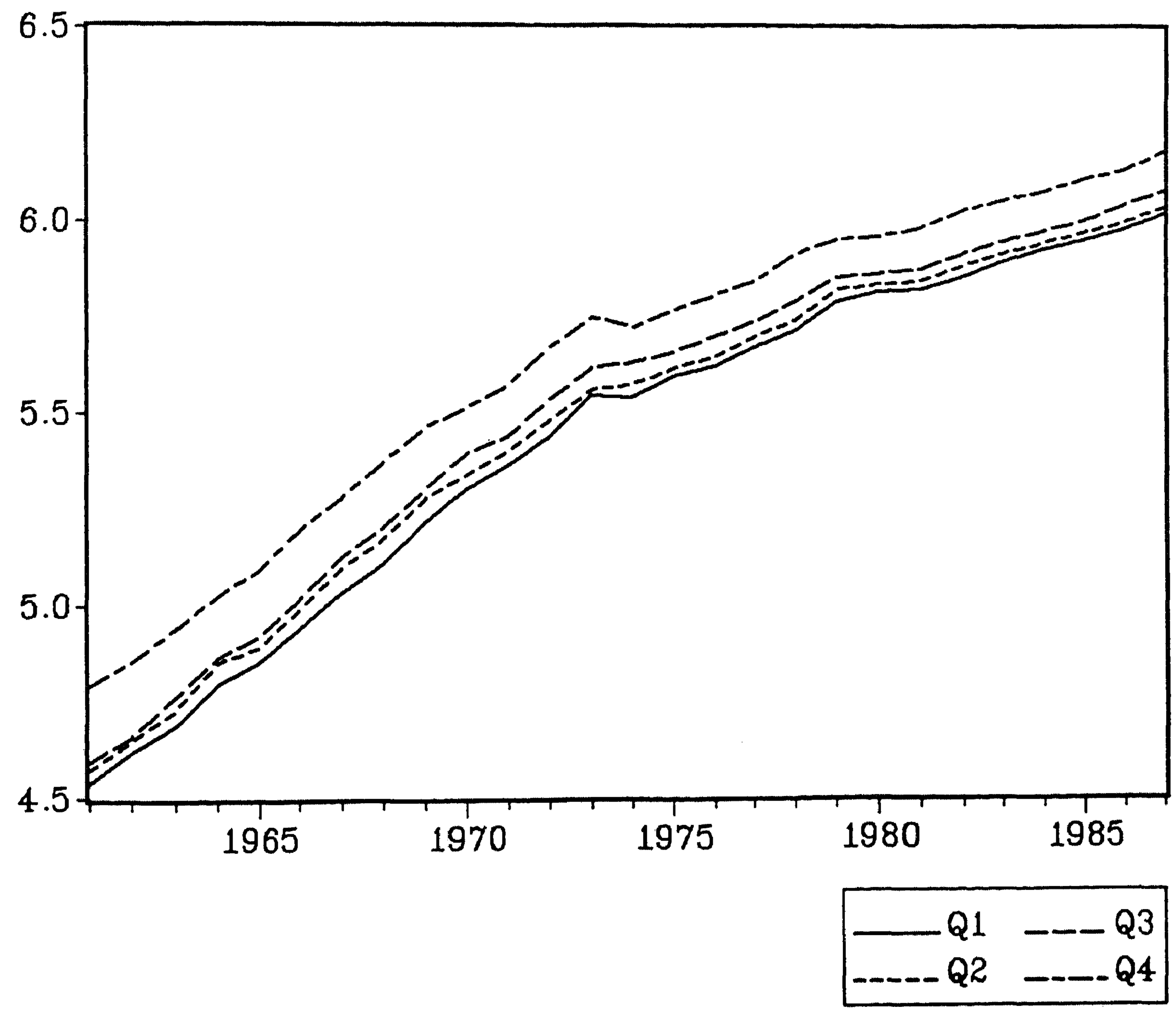


Figure 4.7 Total consumption in Japan, 1961-1987

Table 4.17

Maximum likelihood cointegration results for VQ models

	A	B	C	D
n-1	26	26	24	28
λ_1	0.854	0.837	0.659	0.744
λ_2	0.609	0.554	0.359	0.500
λ_3	0.445	0.328	0.217	0.226
λ_4	0.119	0.235	0.014	0.158
$Tr(3)^{(1)}$	3.297	6.951	0.341	4.832
$Tr(2)$	18.587**	17.286*	6.201	12.016
$Tr(1)$	42.989**	38.268**	16.872	31.446**
$Tr(0)$	93.052**	85.416**	42.672	69.598**
r	3	3	0	2
Hypothesis		$Q^{(2)}$		
H_{r1}	0.753	12.721**		14.323**
H_{r2}	23.167**	9.585**		40.447**

** Significant at a 5% level.

* Significant at about a 5% level.

(1) The $Tr(r)$ is given in (4.60). Critical values can be found in table 2.2. For an exposition of the Johansen procedure, see section 2.2.

(2) Test statistic is given in (4.61). It follows a $\chi^2(r(4-s))$ distribution under the null hypothesis.

The plots of the annual series for the quarters of the demand for money in Finland, the observations of which are given in Johansen (1989), are depicted in figure 4.8. The patterns of the q_{it} deviate from those in the previous figures. The lines show many intersections, for some pairs this occurs more than once, and also the distances between the series change rather often. So, these figures suggest that there are no cointegration relationships between the series. One may tentatively suggest that the patterns in figure 4.8 are typical for a seasonally integrated series because there are several years in which "summer becomes winter". These graphs should however interpreted with care since nonseasonal, but quarterly measured, series may show similar patterns. The results in column

3 under C of table 4.17 indicate that r is equal to 0 indeed.

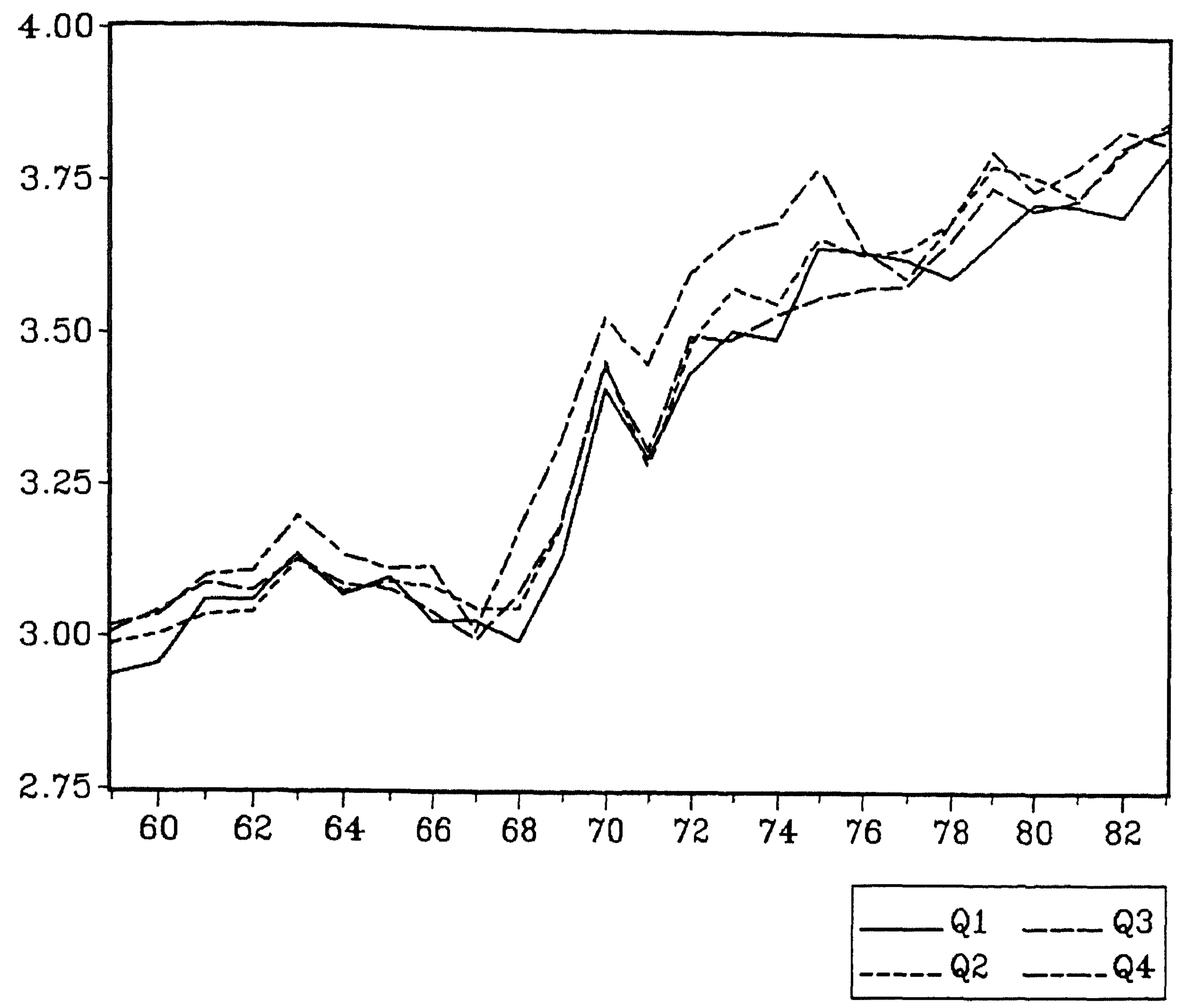


Figure 4.8 Demand for money in Finland, 1959-1983

The final series concerns car sales in the Netherlands. From figure 4.9 it appears that it might be an example of a series which partly shows the pattern of a seasonally integrated series (about up to 1978), and partly exhibits the characteristics of a series generated by a model like (4.56) (after 1978). The latter may be confirmed by the results in table 4.12 where the monthly car sales series between 1978 and 1988 is shown to be conveniently modeled by a first order integrated process with deterministic seasonality. Hence, it might be expected that a model between (4.55) and (4.56) may be appropriate for this series for the entire sample period.

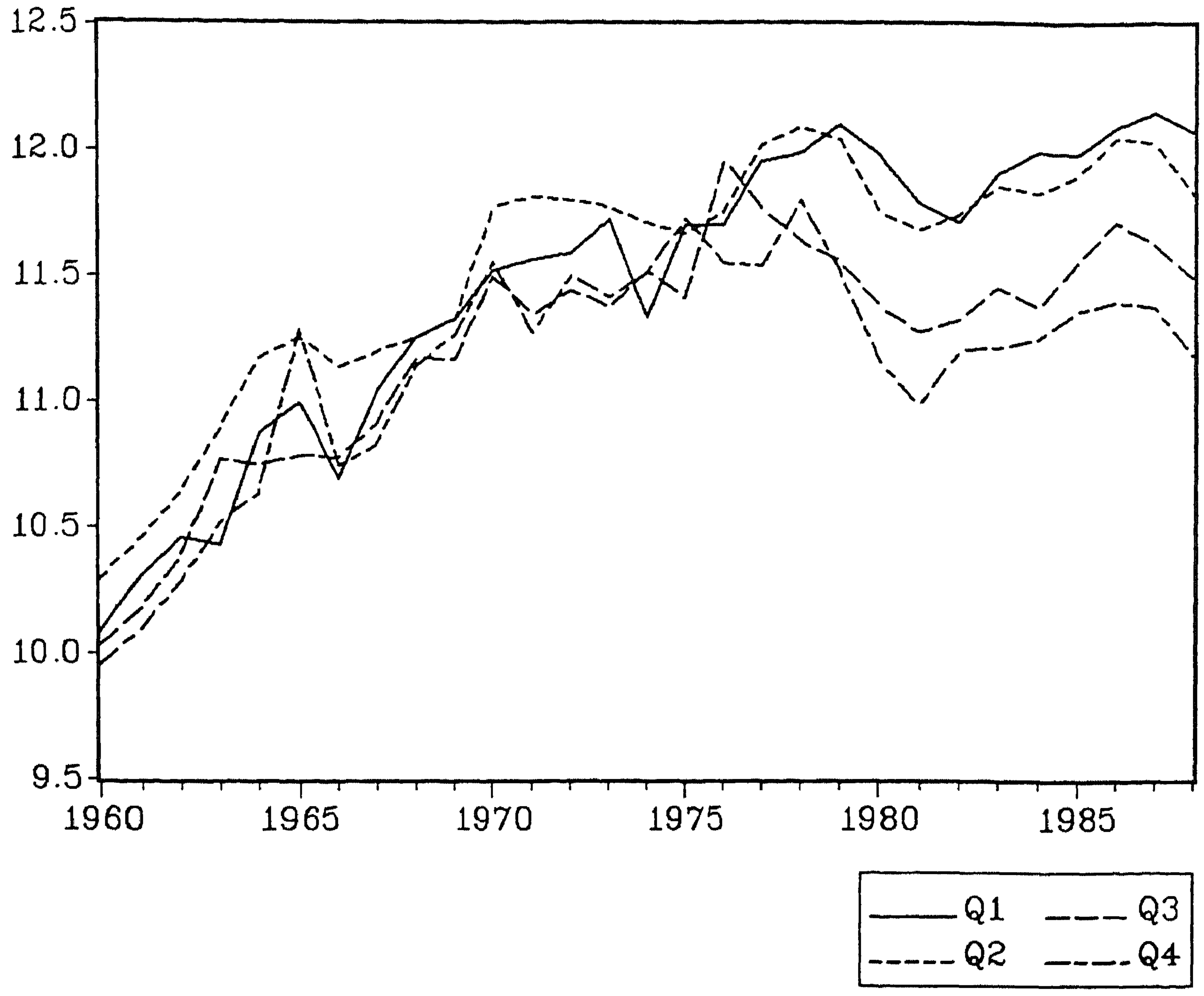


Figure 4.9 New car sales in The Netherlands, 1960–1988

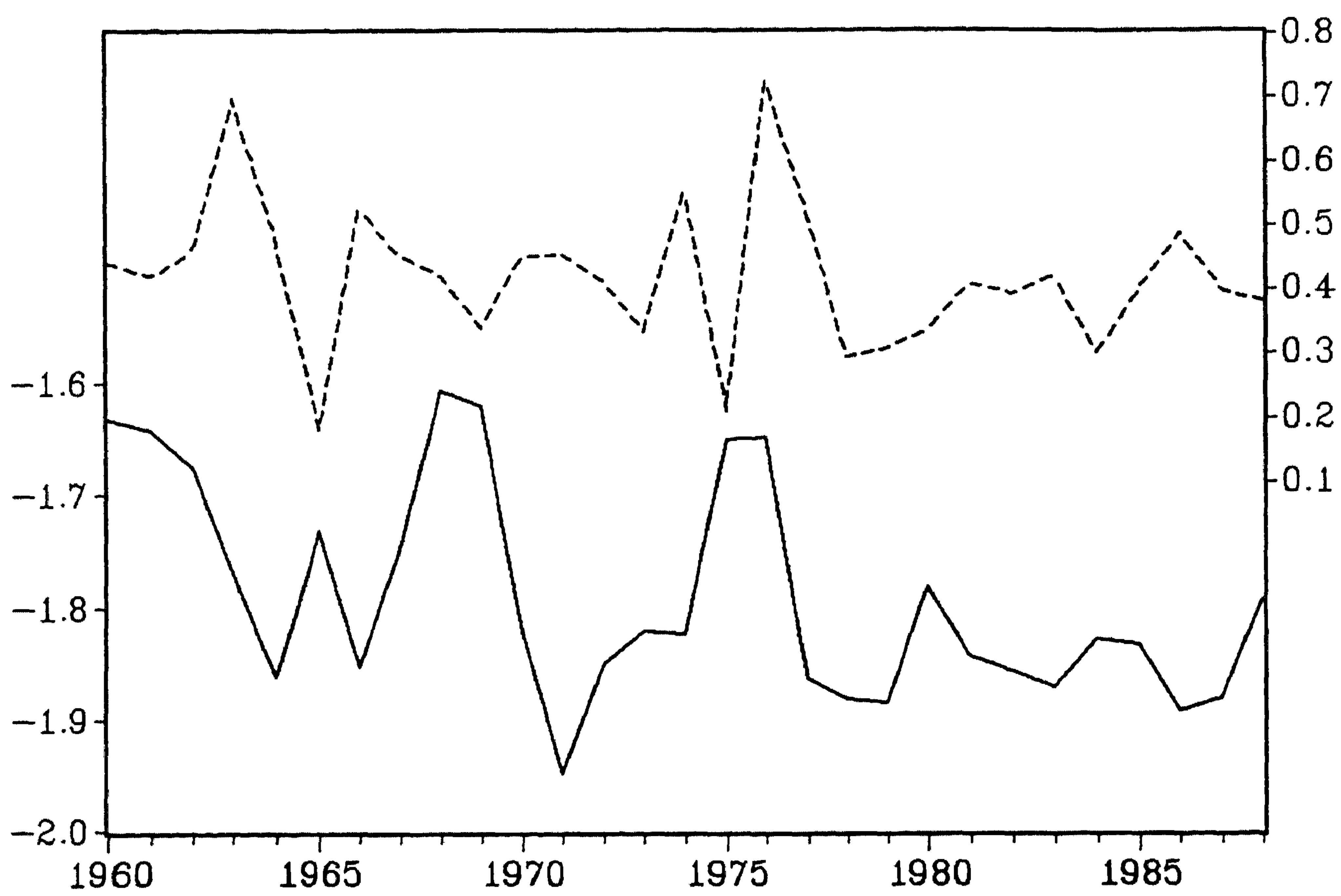


Figure 4.10 Two cointegration relations between the quarterly observations of new car sales

The results in column D of table 4.17 show that there are two cointegration relationships, but that there are no seasonal unit roots. The error correction variables are found by premultiplying q_t by (-0.432, 0.167, 0.759, -0.456) and (0.299, -0.897, 0.218, 0.241). In figure 4.10 these linear combinations have been plotted, and they seem to be stationary indeed.

From these applications it appears that model selection for univariate quarterly time series via a cointegration method applied to a vector of quarters (VQ) model is reasonably straightforward. An additional tool seems to be the inspection of the graphs of the annual series per quarter. They may be useful when looking for periods in which "summer becomes winter" or in which the quarters behave similarly and with constant distances over time. Of course, these graphs should not be treated as providing the utmost evidence since nonwhite error processes may blur the patterns.

The proposed multivariate approach may naturally be applied to bimonthly and monthly series, although practical experience is not yet available. It is however well-known that the power of the applied cointegration test can be low in case the system gets large, and hence it may turn out that such higher order extensions are not reliable in practice. Recent research on $I(2)$ processes, see Johansen (1990), might make it probable to incorporate an $I(1,1)$ model for a univariate seasonal time series in the set of models. This model requires the use of the doubly, i.e. a seasonal as well as a first order, differencing filter. It is easy to recognize that this implies that the annual series per season are $I(2)$ and that there are s-1 cointegration relations between their first differences. Note from figure 4.8 that Finnish money demand may then provide an example.

Bivariate time series

The issue of seasonality in the quarterly time series of United Kingdom log consumption c_t and log income y_t has been studied in, e.g., Davidson *et al.* (1978), Hylleberg *et al.* (1990), Birchenhall *et al.* (1989), and Osborn *et al.* (1988). A recurring aspect in most of these studies is the decision on the type of seasonality in both series, and how it effects their interrelationship. Typical choices are that the individual series may be modeled as quarterly growth rates, while the model includes seasonal dummy variables, or as annual growth rates. These models assume the presence of one or four unit roots, respectively. It may however occur, and it seems to be so for the consumption series (cf. Osborn *et al.* 1988), that neither model is appropriate. The assumption of four unit roots can lead to overdifferencing, while on the other hand the parameters for the seasonal dummies in a model for the variable in first differences do not show constant patterns (cf. Osborn 1990a).

It is now convenient to consider the bivariate series in a vector of quarters (VQ) representation as in (4.57), or

$$\begin{pmatrix} q_c \\ q_y \end{pmatrix}_t = \begin{pmatrix} \mu_c \\ \mu_y \end{pmatrix} + \Pi_1 \begin{pmatrix} q_c \\ q_y \end{pmatrix}_{t-1} + \ldots + \Pi_p \begin{pmatrix} q_c \\ q_y \end{pmatrix}_{t-p} + \begin{pmatrix} \varepsilon_c \\ \varepsilon_y \end{pmatrix}_t, \tag{4.67}$$

and to test for cointegration relationships between the four elements of q_c and of q_y, or across elements of q_c and q_y. The number of these relations r^* and certain restrictions can lead to simple bivariate models relating c_t to y_t. The Johansen (1988) procedure again seems to be most convenient for model selection. Strictly speaking, the bivariate models that are derived from (4.67) are periodic models with seasonallly varying error variances. For notational convenience, I will assume in the sequel that the parameters and error variances are constant over time. In practice one may, of course, want to check this.

There are several possible outcomes of the cointegration exercise. Suppose that for the univariate consumption series the number of cointegration relations is r_c, and analogously for the income series is r_y. In that case there are $4-r_c$ and $4-r_y$ unit roots left in the VQ models for the univariate series. One result of considering a VQ model including both series can be that c_t and y_t have common seasonal components. Obviously, another result may be that $r_c+r_y=r^*$. It can also occur that although the Δ_1 filter is not appropriate for a single series, it may turn out to be valid in a bivariate VQ model. Or, some cointegration relations in the total system establish new relations in one of the single series.

The value of r^* can be 0,1,2,..,8. In case r^* equals 0, all annual series follow independent random walks, and one may want to consider a restricted bivariate model like

$$\Delta_4 c_t = \alpha + \sum_{i=1} \beta_i \Delta_4 c_{t-i} + \sum_{j=1} \gamma_j \Delta_4 y_{t-j} + \eta_t. \tag{4.68}$$

The η_t process in (4.68) can be represented by a general ARMA model. The other extreme case is when r^* is 8, which implies that the system is stationary, and hence a model like

$$c_t = \alpha + \sum_{d=1}^{3} \delta_d D_{dt} + \sum_{i=1} \beta_i c_{t-i} + \sum_{j=0} \gamma_j y_{t-j} + \eta_t, \tag{4.69}$$

can be considered.

Specific bivariate models may be appropriate when $0 < r^* < 8$, and when restrictions on the cointegration vectors are valid. These restrictions are given in an $(8\times s)$ matrix $\mathbf{H}$, where $r^* \leq s \leq 8$. Suppose there are 7 cointegration relationships, and the restrictions

$$\mathbf{H}_7 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}, \tag{4.70}$$

hold true, then it easy to see that there are cointegration relationships with parameters (1,-1) between the quarters of income, between consumption and income in each quarter, and hence there are also such relationships between the quarters of consumption. Or, the variables $q_{sc}-q_{s-1c}$, $q_{sc}-q_{sy}$ and also $q_{sy}-q_{s-1y}$ for $s = 2,3,4$ are all $I(0)$. This case is obviously rather special, which is reflected by the large value of r^* and by the specific restrictions. A corresponding simplified bivariate model is

$$\Delta_1 c_t = \alpha + \sum_{d=1}^{3} \delta_d D_{dt} + \sum_{i=1} \beta_i \Delta_1 c_{t-i} + \sum_{j=0} \gamma_j \Delta_1 y_{t-j} \tag{4.71}$$

$$+ \sum_{k=1} \psi_k (c-y)_{t-k} + \eta_t,$$

in which the c_t and y_t are cointegrated at the nonseasonal level. A restricted version of (4.71) is implied by (4.67) in case r^* equals 6, with certain assumptions for the parameters, and the restrictions

$$\mathbf{H}_6 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.72}$$

can not be rejected, or

$$\Delta_1 c_t = \alpha + \sum_{d=1}^{3} \delta_d D_{dt} + \sum_{i=1} \beta_i \Delta_1 c_{t-i} + \sum_{j=0} \gamma_j \Delta_1 y_{t-j} + \eta_t \tag{4.73}$$

An interesting case, at least for many practical occasions, emerges when $r^*=4$. Now several types of restrictions can be tested, each of which yielding a distinct bivariate model. The model which is used in studies like Davidson *et al.* (1978), or,

$$\Delta_4 c_t = \alpha + \sum_{i=1} \beta_i \Delta_4 c_{t-i} + \sum_{j=0} \gamma_j \Delta_4 y_{t-j} + \sum_{k=1} \psi_k (c-y)_{t-k} + \eta_t, \tag{4.74}$$

is, apart from some parameter assumptions, implied by the acceptance of

$$\mathbf{H}_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{4.75}$$

Rejection of this hypothesis $\mathbf{H}_{41}$ may suggest that a model like

$$\Delta_4 c_t = \alpha + \sum_{i=1} \beta_i \Delta_4 c_{t-i} + \sum_{j=0} \gamma_j \Delta_4 y_{t-j} + \sum_{k=1} \psi_k (c-\phi_s y)_{t-k} + \eta_t, \tag{4.76}$$

is more adequate. This model might be called a periodic cointegration specification and it formalizes the idea that a relationship between c_t and y_t can have a periodic structure (cf. Osborn *et al.* 1988). In Birchenhall *et al.* (1989) a seasonal model of consumption is proposed in which this type of structure is modeled explicitly, although their specification differs from (4.76). Given that the elements of $\boldsymbol{q}_c$ as well as those of $\boldsymbol{q}_y$ are now assumed not to be related, it seems possible to check whether model (4.76) is appropriate via 4 separate tests for cointegration between q_{sc} and q_{sy} for s = 1,2,3,4.

A model which can also be rationalized by economic theory (cf. Osborn 1990b) is

$$\Delta_1 c_t = \alpha + \sum_{d=1}^{3} \delta_d D_{dt} + \sum_{i=1} \beta_i \Delta_1 c_{t-i} + \sum_{j=1} \gamma_j \Delta_4 y_{t-j} \tag{4.77}$$
$$+ \sum_{k=1} \psi_k (c-\xi(y+y_{-1}+y_{-2}+y_{-3}))_{t-k} + \eta_t,$$

for which the restrictions

$$\mathbf{H}_{42} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\xi \\ 0 & 0 & 0 & -\xi \\ 0 & 0 & 0 & -\xi \\ 0 & 0 & 0 & -\xi \end{bmatrix}, \tag{4.78}$$

on the cointegration relationships should be valid. One reasonable choice for ξ can be 0.25 for it implies cointegration of c_t with the annual mean of the y_t observations. A model related to (4.77) is the model in which there is only one cointegration relationship, or r^*=1, which may imply a model like

$$\Delta_4 c_t = \alpha + \sum_{i=1} \beta_i \Delta_4 c_{t-i} + \sum_{j=0} \gamma_j \Delta_4 y_{t-j} \tag{4.79}$$

$$+ \sum_{k=1} \psi_k((c+c_{-1}+c_{-2}+c_{-3})-\phi(y+y_{-1}+y_{-2}+y_{-3}))_{t-k} + \eta_t,$$

reflecting that consumption and income show integrated seasonal behavior, and that their annual sums cointegrate with vector $(1,-\phi)$. Note that this model is nested in the general seasonal cointegration model which is proposed in Engle *et al.* (1991).

Another possible type of models is that in which y_t may be modeled in first differences, while the c_t can be considered in levels, or

$$c_t = \alpha + \sum_{d=1}^{3} \delta_d D_{dt} + \delta T_t + \sum_{i=1} \beta_i c_{t-i} + \sum_{j=0} \gamma_j \Delta_1 y_{t-j} + \eta_t, \tag{4.80}$$

in which the parameters may vary across the seasons, or vice versa. This model indicates that it may occur that c_t, which appears to be (seasonally) integrated in a univariate analysis, turns out to be stationary around a deterministic trend T_t when it is combined with $\Delta_1 y_t$. In a VQ framework this implies that there is only one unit root left, or r^* equals 7, and that the restrictions in the matrix

$$\mathbf{H}_{\Delta y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.81}$$

can not be rejected.

There are of course a host of intermediate cases for which it is not straightforward to write down simple bivariate counterparts. It may then be more convenient to stick to a VQ model like (4.67), although an interpretation of such a model may not be easy.

An application considers total consumption and disposable income in the United Kingdom from the first quarter in 1955 to the fourth quarter in 1988. Graphs of the series are given in figure 4.11.

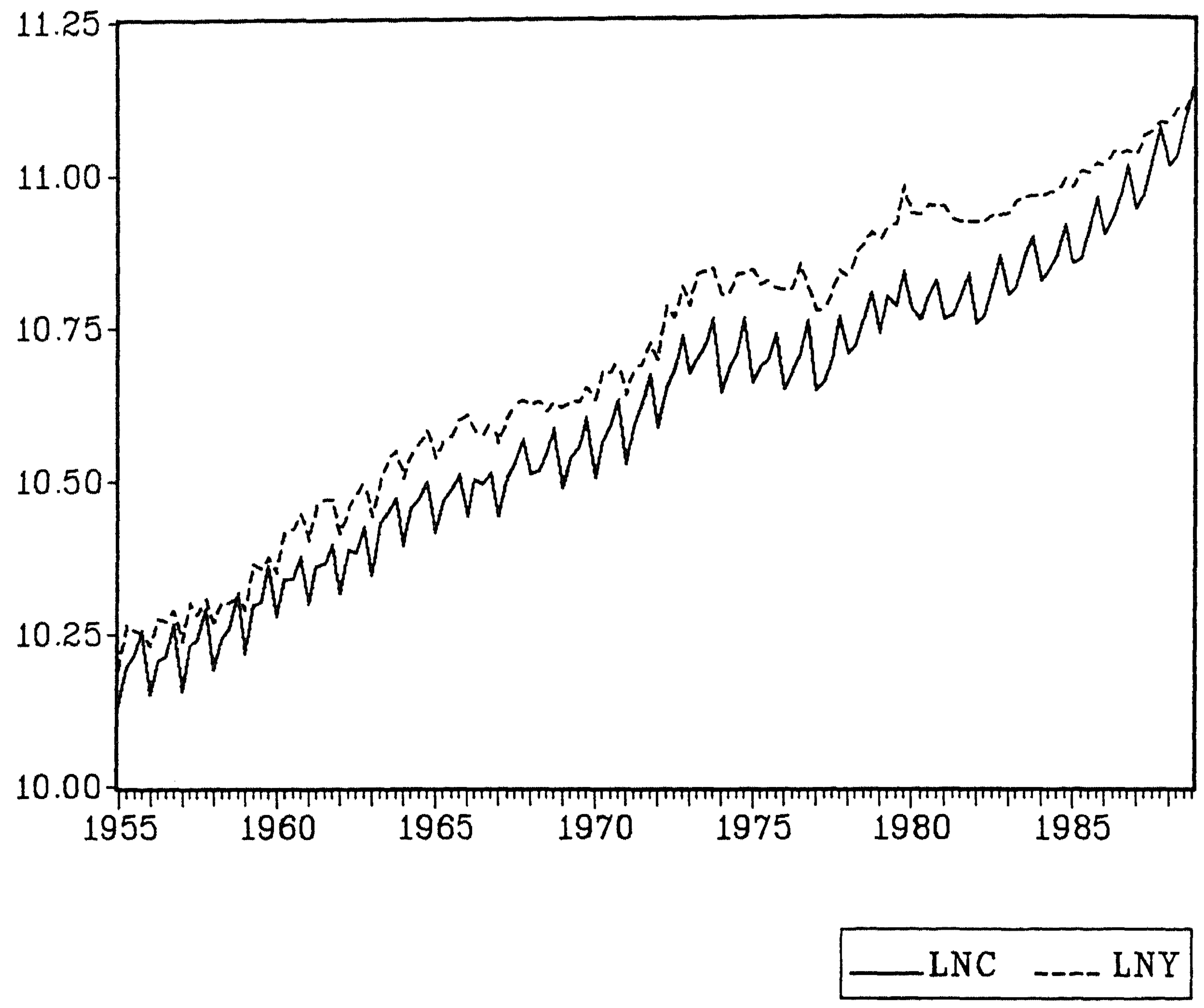


Figure 4.11 Total consumption and disposable income in the United Kingdom, 1955.1-1988.4

Details of the variable definitions can be found in Osborn (1990a). From the results in Osborn (1990a), Hylleberg *et al.* (1990) and Osborn *et al.* (1988) it emerges that a Δ_4 filter may be appropriate for the c_t

series, while the income series shows characteristics of a variable which requires the use of a Δ_1 filter. The empirical evidence is however often not very strong.

The graphs of the four annual series of the quarterly observations are displayed in figures 4.12 and 4.13.

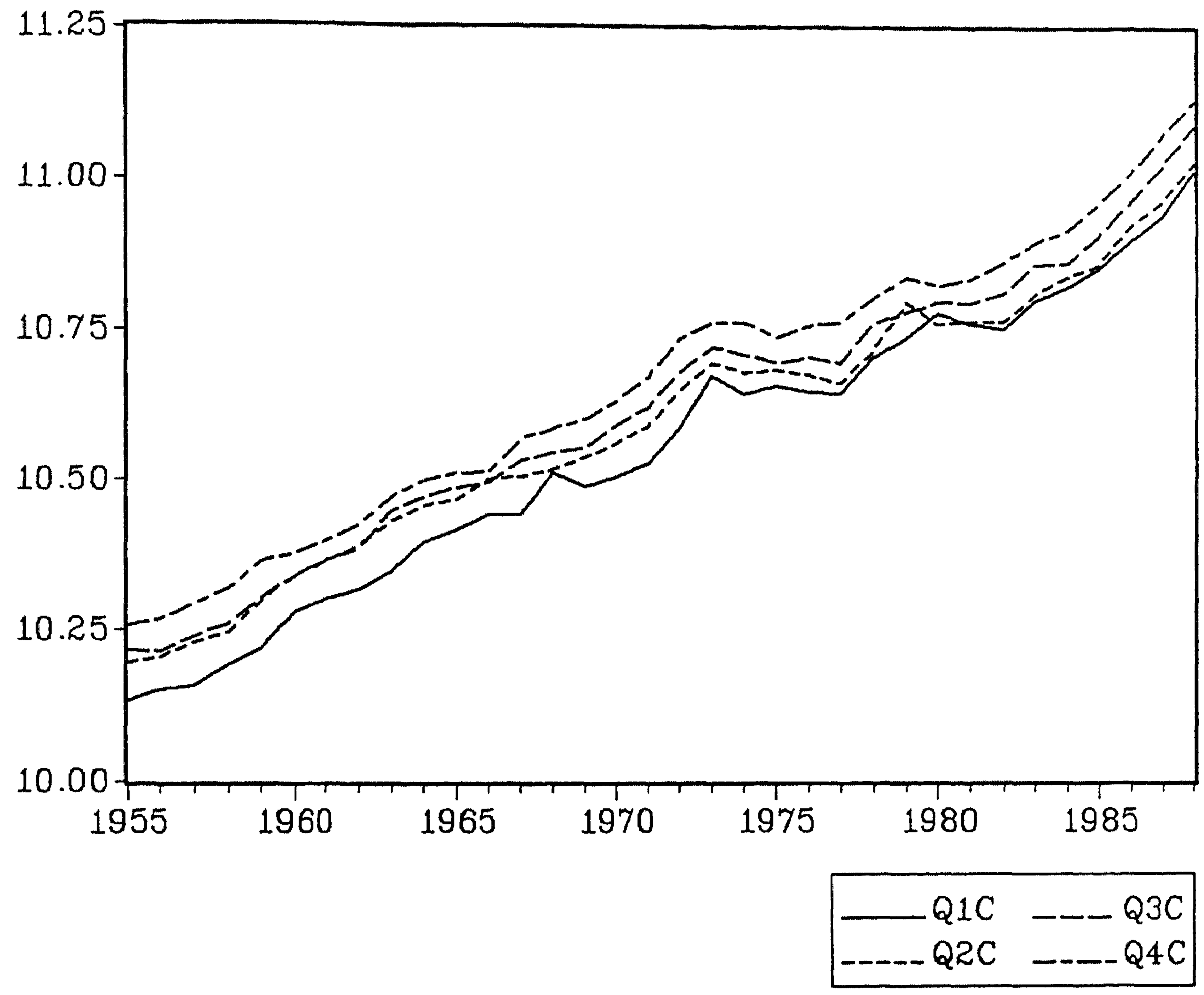


Figure 4.12 Quarterly total U.K. consumption, 1955-1988

From figure 4.12 it can be seen that consumption shows rather stable seasonal patterns because the distances between the quarters 1, 3 and 4 are reasonably constant over time, while the observations for the second quarter shift from the neighbourhood of the third quarter to that of the first quarter. The four series for income do intersect rather often and there are some years in which clearly "summer becomes winter".

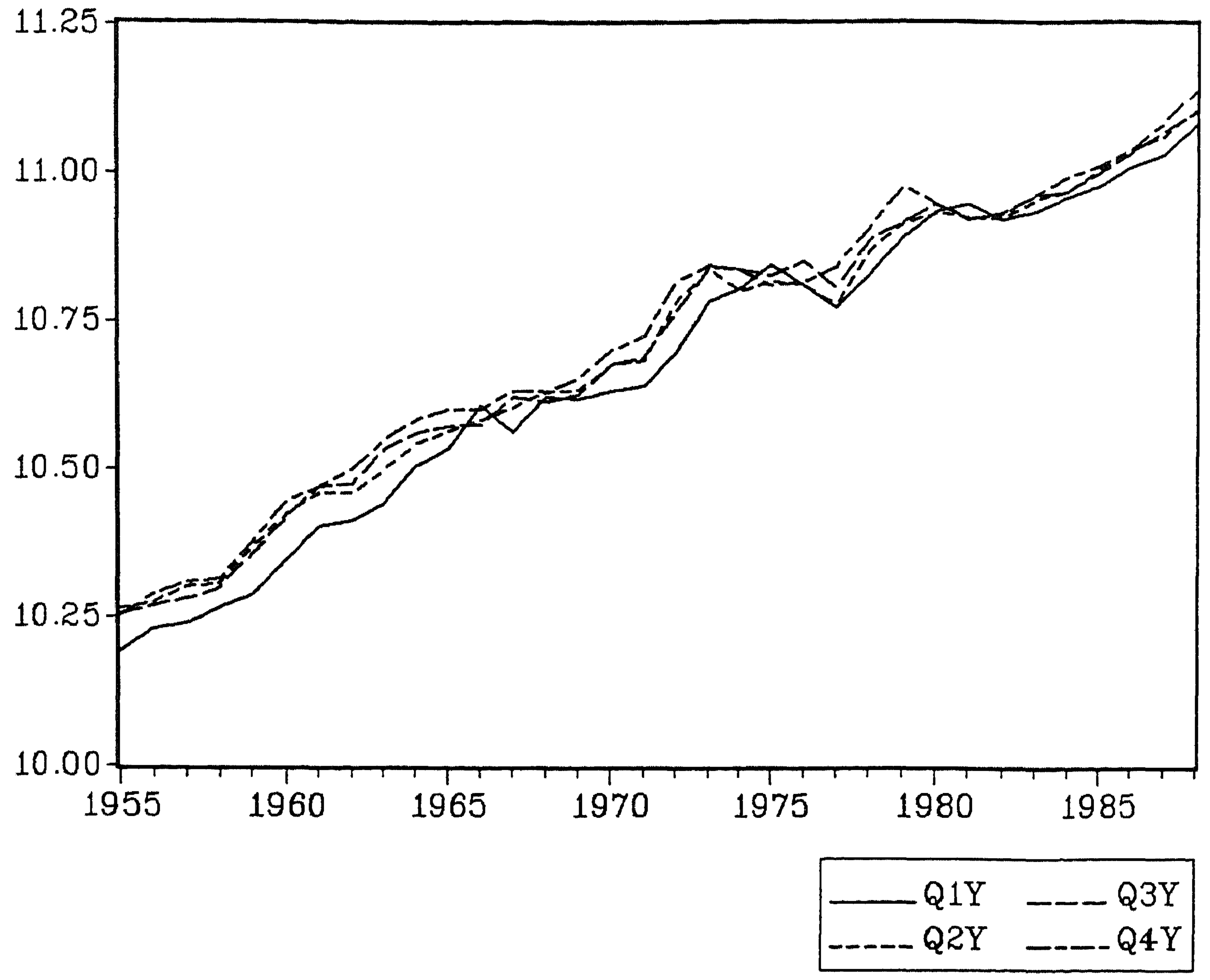


Figure 4.13 Quarterly disposable U.K. income, 1955–1988

An adequate VQ model for the univariate series of consumption as well as of income appears to be of order 1. The unrestricted estimation results are given in the first part of table 4.18. A check on residual autocorrelation indicates that only one of the equations suffers from some autocorrelation. The results of the test for cointegration for the separate series are displayed in the second part of table 4.18. It turns out that for both series the number of cointegration relationships equals 2. Hence it is expected that no simple univariate time series model is adequate. For both series there remain two unit roots in the systems for the quarters.

Table 4.18

VQ models of order 1 for consumption and income:
Estimation and testing for cointegration results (n=33)

	Π_1				$\Lambda_{ii}^{1/2\,(1)}$	Correlations[2]			BP[3]
q_c	−0.176	0.273	0.104	0.856	1.510	0.580	0.519	0.266	6.138
	−0.327	0.095	0.720	0.480	1.970		0.795	0.659	3.166
	−0.430	0.270	0.443	0.754	1.980			0.883	1.981
	−0.239	−0.152	0.311	1.093	1.944				2.037
q_y	0.194	0.063	0.017	0.761	1.565	0.458	0.121	0.051	4.896
	−0.056	0.621	−0.568	0.997	2.278		0.803	0.806	2.833
	−0.001	0.290	−0.222	0.932	2.142			0.780	2.087
	−0.206	0.220	−0.064	1.045	2.744				8.897*

	λ_1	λ_2	λ_3	λ_4	$Tr(0)$	$Tr(1)$	$Tr(2)$	$Tr(3)$
q_c	0.732	0.680	0.237	0.021	90.664**	47.259**	9.637	0.695
q_y	0 815	0.648	0.344	0.018	104.65**	48.943**	14.515	0.583

** Significant at a 5% level.
* Significant at a 10% level.
(1) Multiplied by 100.
(2) Only the off-diagonal elements are reported.
(3) This Box-Pierce test statistic follows a $\chi^2(4)$ distribution.

As might have been expected, the VQ model in (4.67) is also of order 1. The estimation and cointegration test results are displayed in table 4.19. From these outcomes it appears that there are at least 4 cointegration vectors. Note that the critical values for the tests whether $r^* \leq 7$ through $r^* \leq 5$ are not available in the standard tables in Johansen and Juselius (1990). The critical values applied here were kindly provided by Professor Johansen in a personal correspondence.

Table 4.19

A VQ model of order 1 for both consumption and income: Estimation and testing for cointegration results (n=33)

Π_1								$\Lambda_{ii}^{1/2\,(1)}$
0.064	-0.075	0.355	0.702	-0.031	-0.183	-0.322	0.529	1.304
0.205	-0.094	0.979	0.229	-0.102	-0.394	-0.387	0.535	1.709
-0.034	0.237	0.833	0.525	0.110	-0.443	-0.575	0.400	1.634
-0.015	0.029	0.584	0.937	0.131	-0.197	-0.488	0.061	1.643
0.068	-0.724	0.645	0.028	0.314	-0.165	-0.185	1.014	1.308
-0.107	-0.508	0.280	0.695	0.212	-0.017	-0.669	1.090	1.652
-0.123	0.250	-0.449	0.667	0.119	-0.106	-0.100	0.766	1.874
0.010	-0.288	0.366	0.499	0.008	-0.555	-0.129	1.088	2.256

Correlations(2)							BP(3)
0.486	0.520	0.327	0.445	0.555	0.350	0.205	5.510
	0.788	0.714	0.070	0.720	0.675	0.718	3.442
		0.879	-0.200	0.622	0.657	0.608	1.994
			-0.436	0.575	0.606	0.711	7.481
				0.287	0.075	-0.242	3.744
					0.831	0.712	1.922
						0.765	2.063
							6.000

i	1	2	3	4	5	6	7	8
λ_i	0.916	0.806	0.704	0.637	0.441	0.312	0.239	0.156
$Tr(r\leq i\text{-}1)$	255.6**	173.8**	119.7**	79.58**	46.14*	26.96+	14.61+	5.601+

** Significant at a 5% level.

* Significant at a 10% level.

\+ Significant at a 20% level.

(1)(2)(3) see table 4.18.

Assuming that r^* is 4, it is possible to check whether the VQ model with cointegration relationships has a simple bivariate counterpart. The test statistic for the hypothesis H_{41} obtains a value of 108.91, and, compared to a χ^2 distribution with 4(8-4)=16 degrees of freedom, it implies the rejection of this hypothesis. Hence, a Davidson *et al.* (1978) type of model for U.K. total consumption for 1955 to 1988 seems not to be appropriate. It may now be that a so-called periodic cointegration model like (4.76) is adequate. The testing results in table 4.20, where the Johansen procedure is applied to each pair of annual series, shows that there is little evidence of periodic cointegration.

Table 4.20

Testing for periodic cointegration

Variables	λ_1	λ_2	$\beta(\lambda_1)^{(1)}$	$Tr(1)$	$Tr(0)$
q_{1c}, q_{1y}	0.371	0.027	(1,-0.938)	0.897	16.207*
q_{2c}, q_{2y}	0.250	0.008	(1,-0.920)	0.255	9.762
q_{3c}, q_{3y}	0.292	0.125	(1,-0.902)	4.399	15.799*
q_{4c}, q_{4y}	0.239	0.110	(1,-0.970)	3.853	12.849

* Significant at a 10% level.

(1) The parameters in the first hypothesized cointegration relationship.

The test statistic for the hypothesis $\mathbf{H}_{42}$ with ξ=0.25 yields a value of 127.38, which also implies a strong rejection. Similar values are obtained in case the value of ξ is varied from 0.19 to 0.25 with steps of 0.01.

The restrictions in $\mathbf{H}_7$ yield a test statistic value of 25.968 when it is assumed that r^* is 7. Comparison with the $\chi^2(7)$ table indicates that these restrictions are rejected. Similar outcomes emerge for $\mathbf{H}_6$. The $\mathbf{H}_{\Delta y}$ restrictions in (4.81) yield a test outcome of 14.414. The critical value of the $\chi^2(7)$ distribution corresponding to a 2.5% significance level of 16.0, and hence there is some evidence that these restrictions can be accepted. The estimation results for a model like (4.80), which is found after a brief specification search as in chapter 3, are displayed in table 4.21. It seems that each quarter can be modeled differently, i.e. there are distinct relations between c_t and past c_t, $\Delta_1 y_{t-i}$, and trend. Unreported diagnostic checks indicate that there is no serious misspecification. These empirical outcomes compare to those in Osborn (1988), in which a periodically integrated model for (nondurable) consumption is close to being adequate, albeit that $\Delta_1 y_t$ appears to be an omitted variable.

Table 4.21

Estimation results of a periodic model for consumption[1]:

$$\phi_{ps}(B)c_t=\mu_s+\delta_s T_t+\gamma_{us}(B)\Delta_1 y_t+\epsilon_t$$

Parameters[2]	Quarter 1		2		3		4	
μ_s	2.236	(0.851)	0.482	(0.967)	0.936	(0.704)	0.774	(0.546)
δ_s	0.006	(0.002)	-0.001	(0.002)	0.004	(0.002)	0.003	(0.001)
ϕ_{1s}	0.771	(0.083)	0.958	(.095)	0.908	(0.069)	0.540	(0.091)
ϕ_{4s}							0.386	(0.103)
γ_{0s}	0.319	(0.134)	0.281	(0.113)				
γ_{1s}	0.600	(0.158)			0.174	(0.093)		
γ_{2s}	0.376	(0.102)					0.349	(0.078)
γ_{3s}	0.196	(0.083)						

(1) Standard deviations in parentheses.

(2) The parameter indices correspond to a quarter s and to the order of the polynomials in B. For example, γ_{1s} is the parameter for $\Delta_1 y_{t-1}$ in s.

The bivariate extension of the VQ method proposed in this section has some advantages. One of these is that model selection, testing for integration and for cointegration in bivariate quarterly time series can be done all in one run. Secondly, it does not require the construction of models for the seasonal behavior in the separate series before a bivariate analysis is started. In fact, it makes an explicit use of these seasonal and possibly periodic fluctuations. Finally, several common, but nonnested, bivariate models are all nested within the VQ model.

The obvious limitations of this VQ approach are established by the number of elements in the vector for which the analysis attains a reasonable level of accuracy. This level is determined by the number of annual observations, and by the probable reduction in power of the Johansen procedure in case of higher dimensions. Hence, an extension to many more than two variables seems to be unlikely.