A vector of quarters representation for bivariate time series

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A VECTOR OF QUARTERS REPRESENTATION  
FOR BIVARIATE TIME SERIES  

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ABSTRACT  
In this paper it is shown that several models for a bivariate nonstationary quarterly time series are nested in a vector autoregression with cointegration restrictions for the eight annual series of quarterly observations. Or, the Granger Representation Theorem is extended to incorporate, e.g., seasonal and periodic cointegration.  

1. INTRODUCTION  

The use of periodic models for univariate nonstationary seasonal time series has been advocated in recent studies as Osborn (1988) and Franses (1991). The basic idea is to treat the annual series containing the seasonal observations as separate series, and to consider multiple time series models for the vector of the annual variables. The present paper considers an extension of this idea.
to the bivariate case. Within this framework, it is argued that several non-nested bivariate models for nonstationary seasonal time series are implied by a vector autoregression with cointegration restrictions and parameter restrictions. Hence, the Granger Representation Theorem given in Engle and Granger (1987), is extended to incorporate, e.g., periodic cointegration as well as seasonal cointegration.

Section 2 deals with the representation issue. For notational convenience I only consider bivariate quarterly time series, though extensions are easily constructed. Section 3 covers some of the cointegration restrictions that imply often applied bivariate models. The final section discusses the possible practical relevance of considering vector autoregressions for empirical data, and focuses on the issue of model selection.

2. REPRESENTATION

Consider two time series $x_t$ and $y_t$ that are observed quarterly, i.e. 4 times per year, with $t = 1, \ldots, n$. Further, consider the annual series $X_T$ and $Y_T$ that contain the 4 observations per year when stacked in a vector, with $T = 1, \ldots, N$, i.e., $X_T = (X_{1T}, X_{2T}, X_{3T}, X_{4T})$. It is assumed that $n = 4N$ and that $Z_T = (X_T, Y_T)'$ can be described by the multivariate model

$$A_0 Z_T = \mu^* + A_1 Z_{T-1} + \ldots + A_p Z_{T-p} + \varepsilon_T$$

where the $A_i$, $i = 0, \ldots, p$, are $(8 \times 8)$ parameter matrices, where $\mu^*$ is an $(8 \times 1)$ vector of constants, and where $\varepsilon_T$ is an $(8 \times 1)$ vector white noise process with covariance matrix $\sigma^2 I_8$. The model in (1) is a general periodic model in which the means, variances and autoregressive dynamics are allowed to vary over the seasons.

For our purposes of checking cointegration restrictions between the elements of $Z_T$, it is most convenient to rewrite (1) by pre-multiplying with $A_0^{-1}$, and by writing the emerging VAR as

$$\Delta Z_T = \mu + \sum_{i=1}^{p-1} \Gamma_i \Delta Z_{T-i} + \Pi Z_{T-p} + \omega_T,$$

(2)
(cf. Johansen, 1991), where the $A_i$ and $II$ are $(8 \times 8)$ parameter matrices, which are functions of the $A_i$, $i = 0, \ldots, p$, and where the $\mu$ and $\omega_T$ are $(8 \times 1)$ vectors of constants and errors processes, respectively. The $\Delta$ is the usual first order differencing operator, defined by $\Delta Z_T = (1 - L)Z_T = Z_T - Z_{T-1}$. Note that $\Delta$ for the annual series corresponds to the $A_4$ filter for quarterly series, with $\Delta_4 x_t = x_t - x_{t-4}$. The $\omega_T$ are independent variables with mean zero and variance matrix $\Lambda$, where this $\Lambda$ is not a diagonal matrix.

With respect to the general vector autoregressive model in (2), the Granger Representation Theorem roughly amounts to the following. When the matrix $II$ is of full rank, the system is stationary. When $II$ has rank zero, all variables in the vector $Z_T$ are integrated of order one. And, when $II$ has a rank $r$ such that $0 < r < 8$, one can write $II = \alpha \beta^\prime$, where $\beta$ is an $(8 \times r)$ matrix of cointegration vectors and $\alpha$ is an $(8 \times r)$ matrix of adjustment parameters. For details of this theorem, see Engle and Granger (1987) and Johansen (1991). Linear restrictions on the cointegration space can be formulated as

$$\beta = H \varphi$$

where $H$ and $\varphi$ are $(8 \times s)$ and $(s \times r)$ matrices, respectively, where $r \leq s \leq 8$. In Johansen (1991) a maximum likelihood method is developed to test for the value of $r$ and for the validity of the restrictions $H$.

3. Cointegration Restrictions and Implied Models

This section considers particular restrictions on the cointegration space and the models that are implied by these restrictions. Of course, there are many examples one can think of, but in order to save space, I only review some of the models which are currently the most often used in practice.

The value of the rank $r$ of $II$, as well as the adequacy of the restrictions $H$, can imply the adequacy of models for the quarterly $z_t = (x_t, y_t)'$ series. To save space, I assume that a model for $x_t$ is the equation of interest. For example, the annual series $X_{sT}$ and $Y_{sT}$, $s = 1, \ldots, 4$, are integrated processes when $r = 0$, and one may want to consider

$$\phi_s(B) \Delta_4 x_t = \mu_s + \gamma_s(B) \Delta_4 y_t + \varepsilon_{st}$$

(4)
as the first row of a bivariate model. The $\phi_s(B)$ and $\gamma_s(B)$ are seasonally varying polynomials in the backward shift operator $B$, which operates on quarterly observations. The $\mu_s$ are the seasonal means, and the $\varepsilon_{st}$ is an error process with a periodic variance. Of course, in practice one can check whether any assumptions on $\varepsilon_{st}$ and on periodicity of the polynomials are valid.

When the matrix $\beta$ contains seven columns, i.e., when $r$ equals 7, and the restrictions

\[
H = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-\alpha & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -\alpha & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & -\alpha & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -\alpha & 0 & 0 & 1
\end{bmatrix},
\]

(5)

apply, there are $(1,-1)$ relations between the $X_{st}$, $(1,-1)$ relations between the $Y_{st}$, and the variables $X_{st} - \alpha Y_{st}$ are stationary. A corresponding bivariate model is now

\[
\phi_s(B)\Delta_1 x_t = \mu_s + \gamma_s(B)\Delta_1 y_t + \psi_s(x_\alpha y_t)_{t-k} + \varepsilon_{st}.
\]

(6)

Note that the parameters which reflect adjustment to disequilibrium errors are seasonally varying. The $\Delta_1$ filter is the first order differencing filter for quarterly time series, and it is implied by the $(1,-1)$ cointegration relations between the annual series. The value of $k$ corresponds to the order of the VAR in (2), i.e., $k$ equals $4p$. In case there are error correcting variables that represent equilibrium relations which are valid for all quarters, such as in fact the $(x_t-\alpha y_t)$ in (6), the $k$ can take other values.

The model used in Davidson et al. (1978) is a restricted, i.e. nonperiodic version of

\[
\phi_s(B)\Delta_4 x_t = \mu_s + \gamma_s(B)\Delta_4 y_t + \psi_s(x-y)_{t-k} + \varepsilon_{st},
\]

(7)

and it is implied by the restrictions
When the $-1$ in (8) are replaced by $-\phi_s$, where $\phi_s$ is a seasonally varying parameter, (7) becomes

$$\phi_s(B)\Delta_4 x_t = \mu_s + \gamma_s(B)\Delta_4 y_t + \psi_s(x - \phi_s y)_{t-k} + \varepsilon_{st}. \quad (9)$$

This model is called a periodic cointegration model, which is found useful in some practical occasions, see, e.g., Franses and Kloek (1991). In Birchenhall et al. (1989) a similar model is proposed, in which it is assumed that $y_t$ may be described by a periodically integrated model, such as $y_t = \lambda_s y_{t-1} + \varepsilon_t$ with $\lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1$, albeit that not all $\lambda_s$ are equal to one, see also Osborn (1988), see also Boswijk and Franses (1992).

Under certain restrictions, one cointegration relationship between the elements of $X_{sT}$ and $Y_{sT}$ can imply the model

$$\phi_s(B)\Delta_4 x_t = \mu_s + \gamma_s(B)\Delta_4 y_t + \psi_s[(1+B+B^2+B^3)x - \alpha_{12}(1+B+B^2+B^3)y]_{t-k} + \varepsilon_{st}. \quad (10)$$

This model reflects that the $x_t$ and $y_t$ series are seasonally integrated, and that there is cointegration at frequency 0, see Engle et al. (1993). The full seasonal cointegration model, without lagged $\Delta_4 x_t$ and $\Delta_4 y_t$, is

$$\Delta_4 x_t = \gamma_{11}[(1+B+B^2+B^3)x_{t-1} - \alpha_{12}(1+B+B^2+B^3)y_{t-1}] \quad (11)$$

$$+ \gamma_{12}(-1+B-B^2+B^3)x_{t-1} - \alpha_{22}(-1+B-B^2+B^3)y_{t-1}]$$

$$- (\gamma_{13}+\gamma_{14}B)(-1+B^2)x_{t-2} - \alpha_{32}(-1+B^2)y_{t-2} - \alpha_{41}(-1+B^2)x_{t-3}$$

$$- \alpha_{42}(-1+B^2)y_{t-3},$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (8)$$
see equation (5) in Engle et al. (1993). It can now easily be derived that
cointegration at frequency $\nu_2$ is implied by the restrictions

$$H' = [1, -1, 1, -1, -\alpha_{22}, \alpha_{22}, -\alpha_{22}, \alpha_{22}],$$

(12)

and that cointegration at frequency $\nu_4$ is related to

$$H' = \begin{bmatrix} -\alpha_{41} & 1 & \alpha_{41} & 1 & -\alpha_{42} & -\alpha_{32} & \alpha_{42} & \alpha_{32} \\ 1 & \alpha_{41} & -1 & -\alpha_{41} & -\alpha_{32} & \alpha_{42} & \alpha_{32} & -\alpha_{42} \end{bmatrix},$$

(13)

In summary, it can be seen from the expressions in (4) through (13) that
restrictions on the parameters of (2) can imply several nonnested models for a
bivariate quarterly time series.

4. DISCUSSION

In this paper it is shown that a vector autoregression with cointegration res-
trictions for the annual series of quarterly observations nests several models
for bivariate nonstationary quarterly time series. An implication is that the
Granger Representation Theorem is extended to cover the seasonal, as well as
the periodic, cointegration model.

The main conclusion to be drawn from the exercise in the previous section
is that when one assumes the adequacy of a certain bivariate time series model
for quarterly observations in practice, one implicitly makes an assumption on
the number of unit roots in the multivariate system and on the validity of
particular restrictions on the cointegration vectors. Naturally, this calls
for a model selection strategy, with which one can verify whether one's
empirical model is compatible with the assumed number of unit roots and the
parameter restrictions.

Given the vector autoregressive model framework, one may now be inclined
to use the Johansen (1991) maximum likelihood method for model selection in
empirical occasions. However, there are several practical problems involved.
First, the number of annual observations may not be very large, while the
number of parameters to be estimated can be. For example, in a VAR(1) for $Z_T$,
the number of parameters in $H$ equals 64. Second, the empirical performance of
the method deteriorates in case the system gets large. Finally, the small sample behavior of the tests for the linear restrictions on the cointegration vectors may be different from the asymptotic behavior.

There are several alternative methods. An informal approach is given in Franses (1992). This method amounts to checking whether the annual series $X_{sT}$ and $Y_{sT}$ are cointegrated using the standard Engle and Granger (1987) tests. When the null hypothesis of no cointegration is rejected for each of the four seasons, there are at least 4 cointegration relations between the elements of $Z_T$. Only when there is cointegration at all the frequencies, as in (11), there are four such relations. Hence, when there is cointegration at only some of the frequencies, this informal method may be useful.

A more formal approach consists of two steps. The first is to check the number of unit roots in the individual systems of $X_T$ and $Y_T$. The sum of these numbers is then the maximum number of unit roots in the $Z_T$ system. Conversely, the sum of the cointegration relations in models for $X_T$ and $Y_T$ is the minimum value of the rank of $\Pi$ in (2). The methods in Boswijk and Franses (1992) and in Franses (1992b) can be used for detecting the numbers of unit roots and cointegration relations in the individual series, respectively. The sum of these numbers may already indicate the exclusion of several bivariate models. The second step consists of estimating a bivariate time series model with periodic parameters for $x_t$ and $y_t$. The emerging parameter estimates are then used for $A_i$ in (1), with which one can calculate the elements of the $\Pi$ matrix in (2). In many empirical occasions, this $\Pi$ matrix will not contain only nonzero elements. Hence, it may be possible to relate the rank of $\Pi$ to particular parameter restrictions in the bivariate model for $(x_t,y_t)$. The latter restrictions can then be tested using nonlinear estimation techniques. When one has decided on the number of cointegration relations between the elements of $Z_T$, $F$ type tests can be used for further model selection steps. For example, when the rank of $\Pi$ equals 4, and one considers selection between (9) and (7), one can estimate (9) in the unrestricted form

$$\phi_s(B)\Delta_4 x_t = \mu_s + \gamma_s(B)\Delta_4 y_t + \psi_s x_{t-k} + \kappa_s y_{t-k} + \epsilon_{st} \quad (14)$$

and test whether $\kappa_s$ equals $-\psi_s$ for all $s$. Given that the number of unit roots is equal under the null and alternative hypothesis, the $F$ test asymptotically follows a standard $F$ distribution.
An extensive comparison of the various methods in empirical examples and in Monte Carlo experiments is an interesting topic for further research.

NOTES

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REFERENCES


Franses, P.H. (1992b), A multivariate approach to modeling univariate seasonal time series, conditionally accepted for *Journal of Econometrics*.

