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Statistics & Probability Letters 30 (1996) 235–240

STATISTICS & PROBABILITY LETTERS

Temporal aggregation in a periodically integrated autoregressive process

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Received July 1995; revised October 1995

Abstract

A periodically integrated autoregressive process for a time series which is observed S times per year assumes the presence of $S-1$ cointegration relations between the annual series containing the seasonal observations, with the additional feature that these relations are different across the seasons. This means that there is a single unit root in the vector autoregression for these annual series. In this paper it is shown that temporally aggregating such a process does not affect the presence of this unit root, i.e. the aggregated series is also periodically integrated.

1. Introduction

Temporal aggregation in flow-type univariate time series processes is studied, e.g. in Amemiya and Wu (1972) and Weiss (1986) with respect to the order of the ARMA process for the aggregated time series. Recently, the focus is also on the effects of temporal aggregation on unit roots. For example, Hylleberg et al. (1993) investigate to what extent seasonal unit roots in monthly series are preserved in quarterly time series. In this paper we consider the effects of temporally aggregating a so-called periodically integrated process.

Section 2 of the present paper is dedicated to a brief review of several concepts for periodically integrated autoregressions. Section 3 presents the main results.

2. Periodic integration

Consider a time series y_t which is observed S times a year during N years. The index t runs from 1 to $n = SN$. Assume that y_t can be described by a periodic autoregressive process of order p [PAR(p)],

$$y_t = \mu_s + \phi_{1s} y_{t-1} + \cdots + \phi_{ps} y_{t-p} + \varepsilon_t \quad (s = 1, 2, \ldots, S), \tag{1}$$

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SSDI 0167-7152(95)00225-1

were ε_t denotes a standard white noise process with variance σ^2. The index s indicates that the parameters are assumed to be time-varying, cf. Jones and Brelsford (1967) and Troutman (1979). Note that the mean of y_t may be δ, i.e. nonperiodic, since (1) can originate from a PAR(p) process for $y_t - \delta$. The AR order p corresponds to the maximum order, i.e. $p = \max(p_s)$, where p_s is the AR order per season.

The process in (1) can be rewritten in vector notation by stacking the observations y_t in the $(S \times 1)$ vector of annual observations per season, to be denoted as Y_T with $T = 1, \ldots, N$, and representing (1) by

$$A_0 \begin{bmatrix} Y_{1,T} \\ Y_{2,T} \\ Y_{3,T} \\ \vdots \\ Y_{S,T} \end{bmatrix} = \mu + A_1 \begin{bmatrix} Y_{1,T-1} \\ Y_{2,T-1} \\ Y_{3,T-1} \\ \vdots \\ Y_{S,T-1} \end{bmatrix} + A_2 \begin{bmatrix} Y_{1,T-2} \\ Y_{2,T-2} \\ Y_{3,T-2} \\ \vdots \\ Y_{S,T-2} \end{bmatrix} + \cdots + A_P \begin{bmatrix} Y_{1,T-P} \\ Y_{2,T-P} \\ Y_{3,T-P} \\ \vdots \\ Y_{S,T-P} \end{bmatrix} + \omega_T, \tag{2}$$

see e.g. Osborn (1991). The μ and ω_T are partitioned conformily with $Y_T = (Y_{1,T}, Y_{2,T}, \ldots, Y_{S,T})'$. Model (2) corresponds with a VAR(P) process for Y_T, with $P = 1 + [(p-1)/4)]$, where $[\cdot]$ denoting the entire-function. ω_T is a vector white noise process with covariance matrix $\sigma^2 I_S$. The matrices $A_0, \ldots, A_P$ are $(S \times S)$ parameter matrices.

The vector process for Y_T, which can be described by (2), is stationary if the roots of the characteristic equation

$$|A_0 - A_1 z - \cdots - A_P z^P| = 0 \tag{3}$$

are outside the unit circle. In that case we say that the univariate series y_t is periodically stationary. If a single solution to (3) equals unity, while the other solutions are outside the unit circle, there are $S-1$ so-called cointegrating relationships between the S elements of Y_T, see Engle and Granger (1987) and Johansen (1991). Cointegration relations correspond to stationary linear combinations of the $Y_{s,T}$ ($s = 1, 2, \ldots, S$) variables. A common representation of a cointegration model for (2) is given by rewriting it as

$$\Delta Y_T = \mu^* + \Gamma_1 \Delta Y_{T-1} + \cdots + \Gamma_{P-1} \Delta Y_{T-(P-1)} + \Pi Y_{T-P} + \omega_T^*, \tag{4}$$

where Δ is the first differencing filter for annual time series, where μ^*, Γ_1 to Γ_{P-1} and ω_T^* are functions of μ, A_0^{-1} and the A_1 to A_{P-1} matrices, and where

$$\Pi = A_0^{-1}\left(\sum_{i=1}^{P} A_i - I_S\right). \tag{5}$$

When Π has reduced rank $S-1$, it can be decomposed as $\alpha\beta'$, where α and β are $S \times (S-1)$ matrices. The matrix β contains the cointegration parameters. It can be shown that in case of this reduced rank $S-1$, the $S-1$ variables $\beta' Y_T$ are stationary, i.e. do not contain a unit root. Hence, model (4) contains time series variables which are all stationary. The α parameters can be interpreted as dynamic adjustment parameters. There are several methods to test for cointegration between the $Y_{s,T}$ variables ($s = 1, 2, \ldots, S$). An elegant method is developed in Johansen (1991), which involves formal tests of the rank of Π. In Franses (1994) this method is applied to the case of periodic time series as (1).

When there are $S-1$ cointegrating relations between the S variables, these relations are given by $Y_{j,T} - \delta_j Y_{j-1,T}$ for $j = 2, \ldots, S$. In case the δ_j values are not equal across the seasons, the y_t process is called periodically integrated. Boswijk and Franses (1995) propose tests for periodic integration.

An illustrative example of the expressions in (1)–(5) is given by the periodic first-order autoregressive model for a quarterly observed series y_t with zero mean, or

$$y_t = \phi_s y_{t-1} + \varepsilon_t, \quad s = 1,2,3,4, \tag{6}$$

where ϕ_s is unequal to zero, with $\phi_s \neq \phi$ for all $s = 1, 2, 3, 4$, and where some of the ϕ_s can be larger than 1. Stacking y_t into $Y_T = (Y_{1,T}, Y_{2,T}, Y_{3,T}, Y_{4,T})'$ ensures that (6) can be written as

$$A_0 Y_T = A_1 Y_{T-1} + \varepsilon_T, \tag{7}$$

with

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_2 & 1 & 0 & 0 \\ 0 & -\phi_3 & 1 & 0 \\ 0 & 0 & -\phi_4 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & \phi_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic equation for (7) reads

$$|A_0 - A_1 z| = (1 - (\phi_1\phi_2\phi_3\phi_4)z) = 0. \tag{8}$$

The process Y_T is periodically stationary if $|\phi_1\phi_2\phi_3\phi_4| < 1$, and periodically integrated if $\phi_1\phi_2\phi_3\phi_4 = 1$. To investigate the presence of cointegration relations between the $Y_{s,T}$ in (7), it is convenient to rewrite (7) as

$$\Delta_1 Y_T = \Pi Y_{T-1} + \omega_T, \tag{9}$$

where $\omega_T = A_0^{-1}\varepsilon_T$, with

$$\Pi = \begin{bmatrix} -1 & 0 & 0 & \phi_1 \\ 0 & -1 & 0 & \phi_1\phi_2 \\ 0 & 0 & -1 & \phi_1\phi_2\phi_3 \\ 0 & 0 & 0 & \phi_1\phi_2\phi_3\phi_4 - 1 \end{bmatrix}. \tag{10}$$

Under the restriction $\phi_1\phi_2\phi_3\phi_4 = 1$, this matrix has rank 3. Hence, for this quarterly time series, there are three cointegrating relations between the elements of Y_T, and these are $Y_{2,T} - \phi_2 Y_{1,T}$, $Y_{3,T} - \phi_3 Y_{2,T}$ and $Y_{4,T} - \phi_4 Y_{3,T}$.

3. Temporal aggregation

Consider the temporal aggregation of a y_t series, which is observed S times per year and which can be described by a PAR process as in (1), to a series ya_t, that is observed S^* times, with $kS^* = S$ and k usually is 6, 4, 3 or 2. In general, the ya_t series can be described by a periodic ARMA process [PARMA], see also Weiss (1984) for nonperiodic processes.

For example, consider the aggregation of the quarterly series y_t, which is generated from (6) into a bi-annual series z_τ, where τ runs from 1 through $n/2$. The observations z_τ are generated as $z_1 = y_1 + y_2$, $z_2 = y_3 + y_4$, and so on. Substituting the appropriate expressions for y_i, $i = 1, \ldots, n$, one easily obtains that

z_τ follows a periodic ARMA process of order (1, 1),

$$z_\tau = \alpha_s z_{\tau-1} + v_\tau + \beta_s v_{\tau-1} \tag{11}$$

with

$$\alpha_1 = ((\phi_2 + 1)\phi_1\phi_4)/(\phi_4 + 1), \tag{12}$$

$$\alpha_2 = ((\phi_4 + 1)\phi_2\phi_3)/(\phi_2 + 1), \tag{13}$$

where the $v_\tau + \beta_s v_{\tau-1}$ is a moving average process of order 1 with parameter β_s, which also varies with the season. Note that this assumes that ϕ_2 and ϕ_4 are not equal to -1. If this is the case, z_τ becomes a white noise process. From (12) and (13) it is clear that $\alpha_1 = \alpha_2$ only when a specific nonlinear parameter restriction holds. This suggests that temporal aggregation of a PAR generically yields a PARMA process.

Aggregating the y_t series in (6) to an annually observed time series x_T yields

$$x_T = \phi_1\phi_2\phi_3\phi_4 x_{T-1} + v_T - \theta v_{T-1}, \tag{14}$$

which is a nonperiodic ARMA(1, 1) process.

Proposition. *Let $\{Y_T\}$ be an $(S \times 1)$ cointegrated process with cointegrating rank $S-1$ and $S \times (S-1)$ matrix of cointegrating vectors β. Define $\beta_\perp \neq 0$ to be an S-vector than spans the null space of β, i.e. $\beta'\beta_\perp = 0$. Let A be an arbitrary $(S \times S^*)$ matrix, $S^* < S$, and let $\{X_T = A'Y_T\}$ be an S^*-vector process. Then X_T is a stationary process iff $M(A) \subset M(B)$, where $M(\cdot)$ denotes the column space, or equivalently iff $A'\beta_\perp = 0$; otherwise, $\{X_T\}$ is a cointegrated process with cointegrating rank $S^* - 1$.*

Proof. Johansen's version of the Granger representation theorem (Theorem 4.1) states that

$$Y_T = \beta_\perp(\alpha_\perp'\Psi\beta_\perp)^{-1}\alpha_\perp' S_T + U_T, \tag{15}$$

where $\alpha_\perp$ and Ψ are as defined in Johansen (1991), where S_T is an S-vector random walk, and where U_T is an $(S \times 1)$ stationary process. Because $\beta_\perp$ and $\alpha_\perp$ are $(S \times 1)$ vectors, there is one common trend in the system: $\alpha_\perp' S_T$ is a scalar random walk. The same single common trend appears in the system

$$X_T = A'\beta_\perp(\alpha_\perp'\Psi\beta_\perp)^{-1}\alpha_\perp' S_T + A'U_T, \tag{16}$$

unless $A'\beta_\perp = 0$, so that the linear combinations $A'Y_T$ annihilates the common trend and thus is stationary. Finally, Johansen's Theorem 4.1 implies that S^*-vector process with a single common trend is cointegrated with cointegrating rank $S^* - 1$. □

For the periodic AR process y_t in (1) with a single unit root in the multivariate system (2), it applies that β' is the $(S-1) \times S$ matrix

$$\beta' = \begin{bmatrix} -\delta_2 & 1 & 0 & . & . & . & 0 \\ 0 & -\delta_2 & 1 & & & & 0 \\ 0 & 0 & -\delta_3 & 1 & . & . & 0 \\ \vdots & \vdots & 0 & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & & & -\delta_S & 1 \end{bmatrix}, \tag{17}$$

and that $\beta_\perp$ is given by

$$\beta_\perp' = [1, \delta_2, \delta_2\delta_3, \ldots, \delta_2\delta_3..\delta_{S-1}, \delta_2\delta_3..\delta_S]. \tag{18}$$

For temporal aggregation of the process Y_T in (2), the matrix A' in the proposition is given by the $(S^* \times S)$ matrix

$$A' = \begin{bmatrix} 1..1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1..1 & 0 & 0 & . & 0 \\ . & . & & & & & . \\ . & . & & & & & . \\ 0 & 0 & & & & & 1..1 \end{bmatrix} \tag{19}$$

where the (1 . . 1) parts are $(1 \times k)$ vectors. The event that all S^* elements of $A'\beta_\perp$ are equal to zero, where A and $\beta_\perp$ are given by (18) and (19), has measure zero. Hence, if a periodically integrated process for an S times observed y_t series, i.e. a process with $S - 1$ cointegration relations, is aggregated to an S^* times observed series ya_t, this ya_t series generically has $S^* - 1$ cointegrating relations between its annually observed components.

An additional intuitive argument is that the only way to get rid of the unit root $z = 1$ is to use some differencing filter. Temporal aggregation of y_t does not involve such a filter, and hence the unit root is not removed.

For the periodically integrated AR(1) process in (6) the cointegration vectors are $(-\phi_2, 1, 0, 0)'$, $(0, -\phi_3, 1, 0)'$ and $(0, 0, -\phi_4, 1)'$, and the (4×1) vector $\beta_\perp$ is $(\phi_1, \phi_1\phi_2, \phi_1\phi_2\phi_3, 1)$. Temporal aggregation to the z_τ in (11) implies that

$$A' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{20}$$

and hence z_τ is periodically integrated unless $A'\beta_\perp = 0$, or $\phi_2 = \phi_4 = -1$, see below (12) and (13). The process in (11) can be represented in vector notation as $B_0 Z_T = B_1 Z_{T-1} + \psi_T$, where ψ_T is a moving average type error process. The characteristic equation is $|B_0 - B_1 z| = (1 - (\phi_1\phi_2\phi_3\phi_4)z) = 0$, which is the same as that in (8). An expression for the cointegration vector can be found from

$$B_0^{-1}B_1 - I = \begin{bmatrix} -1 & ((\phi_2 + 1)\phi_1\phi_4)/(\phi_4 + 1) \\ 0 & \phi_1\phi_2\phi_3\phi_4 - 1 \end{bmatrix}. \tag{21}$$

The rank of the matrix in (21) is at least one, and the cointegration relation between $Z_{1,T}$ and $Z_{2,T}$ is easily found.

To obtain the annual series X_T in (14), one uses $A' = [1, 1, 1, 1]$. Hence, the X_T is an integrated series provided that $\phi_1 + \phi_1\phi_2 + \phi_1\phi_2\phi_3 \neq -1$.

In summary, in this paper it is shown that temporally aggregating a univariate periodically integrated autoregressive time series [PIAR], i.e. a process of the multivariate representation contains a single unit root, yields a PIAR again.

Acknowledgements

Both the authors thank the Royal Netherlands Academy of Arts and Sciences for its financial support and a referee for useful comments.

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