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Multiple unit roots in periodic autoregression

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Abstract

In this paper we propose a model selection strategy for a univariate periodic autoregressive time series which involves tests for one or more unit roots and for parameter restrictions corresponding to seasonal unit roots and multiple unit roots at the zero frequency. Examples of models that are considered are variants of the seasonal unit roots model and the periodic integration model. We show that the asymptotic distributions of various test statistics are the same as well-known distributions which are already tabulated. We apply our strategy to three empirical series to illustrate its ease of use. We find that evidence for seasonal unit roots based on nonperiodic models disappears when periodic representations are considered. © 1997 Elsevier Science S.A.

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1. Introduction

Periodic autoregressions (PAR) can yield useful descriptions of seasonally observed time series. Examples of their practical relevance for macroeconomic time series are given in Osborn (1988) and Franses and Paap (1994), among

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many others. The key feature of PAR processes is that the AR parameters are allowed to vary with the season. This implies that the time series is a nonstationary time series since the autocovariance function varies with the season. This fact complicates the analysis of stochastic trends both at the zero frequency and at the so-called seasonal frequencies. To investigate unit root properties of PAR time series, it is therefore useful to write the PAR model in its multivariate form by stacking the seasonal time series into a vector of annual time series. In Franses (1994) it is proposed to apply the Johansen (1988) cointegration testing method to this vector process. This cointegration method can be used to test for nonseasonal and seasonal unit roots in a periodic time series, as well as to investigate the possibility of periodic integration, i.e. the usefulness of a differencing filter that varies with the season. The analysis of the multivariate representation of the PAR process does not impose all restrictions entailed by the univariate model, including seasonal homoskedasticity of the error process. Of course, such flexibility can lead to a reduction of empirical power in case the error process is not seasonally heteroskedastic, and hence may lead to the finding of too many unit roots, see, e.g. Franses and Romijn (1993). In the present paper, we propose a method to investigate multiple and seasonal unit roots imposing all the restrictions in the PAR process. Our approach extends the method in Boswijk and Franses (1996), where the presence of only a single unit root in a PAR process is studied.

We propose a model selection method for PAR processes which involves tests for unit roots at all frequencies of interest. The method can be easily applied in practice, since it only involves tests for the adequacy of certain nonlinear restrictions on the PAR parameters. An important advantage of our procedure is that no new tables with critical values have to be generated since the relevant asymptotic distributions are the same as those derived in Johansen (1988, 1991), for which several critical values are already tabulated in, e.g. Osterwald-Lenum (1992). Examples of models we consider are models where first and seasonal differences are assumed, the seasonal integration model and the periodic integration model. Hence, we also allow for processes where the differencing filter varies with the season. We extend the well-known HEGY procedure (Hylleberg et al., 1990), which concerns seasonal unit root processes in nonperiodic models. We also generalize the approach in Ghysels et al. (1996) by allowing for the possible presence of periodic differencing filters. An important feature of our method is that our generalization in fact amounts to simpler results in the sense that it does not require new asymptotic distributions.

The outline of our paper is as follows. In Section 2, we start off with a discussion of some preliminaries concerning notation and representation. In Section 3, we discuss the impact of multiple unit roots at the zero frequency. In Section 4, we propose our general-to-simple testing procedure for seasonal and nonseasonal unit roots. In Section 5, we evaluate our method through a Monte Carlo experiment. In Section 6, we apply our method to three illustrative quarterly time series. We show that the evidence for seasonal unit roots disappears when we allow for periodic variation in the AR parameters. Furthermore, we find that a periodic differencing filter is most useful to remove the stochastic trend from the data. In Section 7, we conclude this paper with some remarks.

2. Some preliminaries

In this paper we focus on a quarterly time series y_t that can be described by a periodic autoregression of order p (PAR(p)),

$$y_t = \phi_{1s}y_{t-1} + \cdots + \phi_{ps}y_{t-p} + \varepsilon_t, \quad t = 1, 2, \ldots, n, \quad s = 1, 2, 3, 4 \tag{1}$$

or

$$\phi_{p,s}(B)y_t = \varepsilon_t,$$

where ϕ_{is}, $i = 1, \ldots, p$, are periodically varying parameters, B is the backward shift operator, and where ε_t is a standard white noise process. Although some of the ϕ_{ps} parameters can be equal to zero, and (1) allows for seasonally varying autoregressive lag lengths, we assume for the moment that p is equal for all seasons. Furthermore, ε_t is assumed to have nonseasonal variance. Note that, similar to the nonperiodic AR case, the value of p restricts the number of unit roots in (1).

It can be argued that model (1) corresponds to a nonstationary process since the autocovariances of y_t are not constant over time. In order to study unit root properties in y_t, it is therefore most convenient to rewrite (1) in vector notation. For example, for the PAR(2) process, one can write

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_{12} & 1 & 0 & 0 \\ -\phi_{23} & -\phi_{13} & 1 & 0 \\ 0 & -\phi_{24} & -\phi_{14} & 1 \end{bmatrix} \begin{bmatrix} Y_{1,T} \\ Y_{2,T} \\ Y_{3,T} \\ Y_{4,T} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \phi_{21} & \phi_{11} \\ 0 & 0 & 0 & \phi_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_{1,T-1} \\ Y_{2,T-1} \\ Y_{3,T-1} \\ Y_{4,T-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,T} \\ \varepsilon_{2,T} \\ \varepsilon_{3,T} \\ \varepsilon_{4,T} \end{bmatrix}, \tag{2}$$

where $Y_{s,T}$ and $\varepsilon_{s,T}$ are the observations on y_t and ε_t in season s in year $T = 1, \ldots, N = n/4$, see Tiao and Grupe (1980) and Lütkepohl (1991), inter alia. This representation in (2) can be called a vector-of-quarters representation of order 1 (VQ(1)). Denoting $Y_T = (Y_{1,T}, \ldots, Y_{4,T})'$ and $\varepsilon_T = (\varepsilon_{1,T}, \ldots, \varepsilon_{4,T})'$, (2) can be summarized by

$$\Phi_0 Y_T = \Phi_1 Y_{T-1} + \varepsilon_T. \tag{3}$$

Note that PAR processes with orders up to 4 can be written as VQ(1) processes. More generally, a PAR(p) process corresponds to a VQ(P) process with $P = [(p-1)/4] + 1$, where $[x]$ denotes integer part of x. Here we focus on (3) for notational convenience. To investigate the presence of unit roots in y_t it is most convenient to check the solutions of the characteristic equation for (3), i.e.

$$|\Phi_0 - \Phi_1 z| = 0. \tag{4}$$

If one or more solutions to (4) correspond to $z = 1$, then (4) can be expressed in error correction form, i.e.

$$\Delta Y_T = \Pi Y_{T-1} + \nu_T, \tag{5}$$

where $\nu_T = \Phi_0^{-1}\varepsilon_T$, where

$$\Pi = \Phi_0^{-1}\Phi_1 - I, \tag{6}$$

and where $\Delta = (1 - B)$ denotes the first-order differencing filter so that $\Delta Y_T = Y_T - Y_{T-1}$.

Franses (1994) considers testing for (multiple) unit roots using the Johansen (1988) method applied to (5), without imposing the restrictions implied by the original model (1) on Π and Ω, the covariance matrix of ν_t. Suppose for example that $p = 4$. In that case the original model has 16 + 1 parameters ($\{\phi_{is}\}, \sigma^2$), whereas the unrestricted VQ(1) model has 16 + 10 unrestricted parameters in (Π, Ω). The nature of the 9 restrictions implied by the PAR(4) is most easily analyzed by transforming (5) in recursive form:

$$A_0 \Delta Y_T = A_1 Y_{T-1} + \eta_T, \tag{7}$$

where A_0 is a lower-triangular matrix with unit elements on the main diagonal, such that $A_0 \Omega A_0' = \text{diag}(\sigma_1^2, \ldots, \sigma_4^2)$, and hence that $\eta_T = A_0 \nu_T$ has a diagonal covariance matrix, and where $A_1 = A_0 \Pi$. It is easily seen that the PAR(4) model implies $A_0 = \Phi_0$, $A_1 = \Phi_1 - \Phi_0$ and $\sigma_s^2 = \sigma^2$ for $s = 1, 2, 3, 4$. Thus, three of the nine restrictions correspond to periodic homoskedasticity, and the other 6 are restrictions across A_0 and A_1 (or, in the original form, across Π and Ω). Of course, when p is smaller than 4, there will be even more restrictions. The purpose of this paper is to obtain a possible power gain from imposing these restrictions, i.e. by testing for unit roots directly in (1) instead of in the unrestricted VQ model.

3. Periodic autoregression integrated of order 2

If there is one unity solution to (4), Boswijk and Franses (1996) show that any PAR(p) process can be written as

$$\psi_{p-1,s}(B)(1 - \alpha_s B) y_t = \varepsilon_t, \tag{8}$$

where $\psi_{p-1,s}(B)$ is a periodic autoregressive polynomial of order $p-1$ and $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. In VQ notation, this becomes

$$\Psi(B)Y_T^* = \varepsilon_T,$$
$$Y_T^* = (\Xi_0 - \Xi_1 B)Y_T, \tag{9}$$

where

$$\Xi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\alpha_2 & 1 & 0 & 0 \\ 0 & -\alpha_3 & 1 & 0 \\ 0 & 0 & -\alpha_4 & 1 \end{bmatrix} \quad \text{and} \quad \Xi_1 = \begin{bmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{10}$$

and where $\Psi(B) = \Psi_0 - \Psi_1 B - \cdots - \Psi_h B^h$, with $h = [(p-2)/4] + 1$.

It will be useful to start with the case $p = 2$, and then generalize the results to higher-order models. In that case, (8) reduces to

$$(1 - \beta_s B)(1 - \alpha_s B)y_t = \varepsilon_t, \quad s = 1, \ldots, 4,$$

or, since B also operates on α_s, as

$$y_t = \alpha_s y_{t-1} + \beta_s(y_{t-1} - \alpha_{s-1} y_{t-2}) + \varepsilon_t, \tag{11}$$

where $\alpha_0 = \alpha_4$, and the $\Psi(B)$ polynomial in (9) can then be written as

$$\Psi_0 - \Psi_1 B = \begin{bmatrix} 1 & 0 & 0 & -\beta_1 B \\ -\beta_2 & 1 & 0 & 0 \\ 0 & -\beta_3 & 1 & 0 \\ 0 & 0 & -\beta_4 & 1 \end{bmatrix}. \tag{12}$$

Hence, in terms of the α_s and β_s coefficients, the VQ process Y_T reads

$$(\Psi_0 - \Psi_1 B)(\Xi_0 - \Xi_1 B)Y_T = (\Gamma_0 - \Gamma_1 B)Y_T = \varepsilon_T \tag{13}$$

with Γ_0 and Γ_1 defined as

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -(\alpha_2 + \beta_2) & 1 & 0 & 0 \\ \alpha_2\beta_3 & -(\alpha_3 + \beta_3) & 1 & 0 \\ 0 & \alpha_3\beta_4 & -(\alpha_4 + \beta_4) & 1 \end{bmatrix} \quad \text{and}$$

$$\Gamma_1 = \begin{bmatrix} 0 & 0 & -\alpha_4\beta_1 & \alpha_1 + \beta_1 \\ 0 & 0 & 0 & -\alpha_1\beta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

The characteristic equation of the polynomial in (13) can now be written as

$$|\Gamma_0 - \Gamma_1 z| = (1 - \alpha_1\alpha_2\alpha_3\alpha_4 z)(1 - \beta_1\beta_2\beta_3\beta_4 z) = 0. \tag{15}$$

In Boswijk and Franses (1996) it is assumed that $\beta_1\beta_2\beta_3\beta_4 < 1$. In the present paper, we relax this assumption and consider the case where there can be more than a single unit root.

As in (5), the VQ process in (13) can be written in error correction form like

$$\Delta Y_T = (\Gamma_0^{-1}\Gamma_1 - I_4)Y_{T-1} + \Gamma_0^{-1}\varepsilon_T = \Pi Y_{T-1} + \Gamma_0^{-1}\varepsilon_T. \tag{16}$$

Under the assumption of two unity solutions to the characteristic equation (15), i.e. $\alpha_1\alpha_2\alpha_3\alpha_4 = \beta_1\beta_2\beta_3\beta_4 = 1$, the Π matrix in (16) reduces to

$$\Pi = \begin{bmatrix} -1 & 0 & -\alpha_4\beta_1 & \alpha_1 + \beta_1 \\ 0 & -1 & -\alpha_4\beta_1(\alpha_2 + \beta_2) & \alpha_1\alpha_2 + \alpha_2\beta_1 + \beta_1\beta_2 \\ 0 & 0 & -\kappa & \kappa/\alpha_4 \\ 0 & 0 & -\alpha_4\kappa & \kappa \end{bmatrix}, \tag{17}$$

with

$$\kappa = (\alpha_4/\beta_4)(1 + (\alpha_3/\beta_3)(1 + (\alpha_2/\beta_2)(1 + (\alpha_1/\beta_1)))). \tag{18}$$

It can be seen from (17) and (18) that the rank of Π will usually equal 3, and hence that Π can be written as $\Pi = \gamma\lambda'$, where γ and λ are both of dimension (4×3). Consider for instance the case where $\alpha_s = \beta_s = 1$ for all $s = 1, 2, 3, 4$. In this case the four $Y_{s,T}$ series are $I(2)$, see e.g. Haldrup (1994a, b) and Johansen (1992a).

When the restriction $\alpha_1\alpha_2\alpha_3\alpha_4 = \beta_1\beta_2\beta_3\beta_4 = 1$ holds, there is another possibility. When $\kappa = 0$ the Π matrix is of rank 2. This may occur for several possible parameter configurations, but in particular when $\alpha_s = -\beta_s = 1$ for all s. In this case the PAR(2) process reduces to $(1 - B^2)y_t = \varepsilon_t$, and this implies the presence of one nonseasonal unit root and one seasonal unit root at the bi-annual frequency, see, e.g. Hylleberg et al. (1990). We shall return to this situation in the next section. In this section we will elaborate on the $I(2)$ case.

One further insight of a periodically doubly integrated time series can be obtained from considering the vector moving average (VMA) representation. From (11), define the series $y_t^* = (1 - \alpha_s B)y_t$ such that

$$(1 - \beta_s B)y_t^* = \varepsilon_t. \tag{19}$$

It follows from Boswijk and Franses (1996) that the VQ representation of y_t^* can be described as

$$\Delta Y_T^* = (\Theta_0^* + \Theta_1^* B)\varepsilon_T, \tag{20}$$

where

$$\Theta_0^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta_2 & 1 & 0 & 0 \\ \beta_2\beta_3 & \beta_3 & 1 & 0 \\ \beta_2\beta_3\beta_4 & \beta_3\beta_4 & \beta_4 & 1 \end{bmatrix}$$

and

$$\Theta_1^* = \begin{bmatrix} 0 & \beta_3\beta_4\beta_1 & \beta_4\beta_1 & \beta_1 \\ 0 & 0 & \beta_4\beta_1\beta_2 & \beta_1\beta_2 \\ 0 & 0 & 0 & \beta_1\beta_2\beta_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

If we denote $(\Theta_0^* + \Theta_1^* B)\varepsilon_T = u_T$, it follows equivalently that

$$\Delta^2 Y_T = (\Theta_0 + \Theta_1 B)u_T = (\Theta_0 + \Theta_1 B)(\Theta_0^* + \Theta_1^* B)\varepsilon_T, \tag{22}$$

with Θ_0 and Θ_1 defined as

$$\Theta_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha_2 & 1 & 0 & 0 \\ \alpha_2\alpha_3 & \alpha_3 & 1 & 0 \\ \alpha_2\alpha_3\alpha_4 & \alpha_3\alpha_4 & \alpha_4 & 1 \end{bmatrix}$$

and

$$\Theta_1 = \begin{bmatrix} 0 & \alpha_3\alpha_4\alpha_1 & \alpha_4\alpha_1 & \alpha_1 \\ 0 & 0 & \alpha_4\alpha_1\alpha_2 & \alpha_1\alpha_2 \\ 0 & 0 & 0 & \alpha_1\alpha_2\alpha_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

and hence the characteristic equation of the VMA polynomial can be described as

$$\begin{aligned} |\theta(z)| &= |\Theta_0 + \Theta_1 z|\,|\Theta_0^* + \Theta_1^* z| \\ &= (1 - \alpha_1\alpha_2\alpha_3\alpha_4 z)^3(1 - \beta_1\beta_2\beta_3\beta_4 z)^3 = (1 - z)^6 = 0, \end{aligned} \tag{24}$$

when $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ and $\beta_1\beta_2\beta_3\beta_4 = 1$. Since the Δ^2 filter applied to the VQ series Y_T induces 8 unit roots, it can therefore be observed that only 2 of these will be in common, i.e. a single doubly integrated process driving the system.

All of the above representational results are easily generalized to the case where $p > 2$. In that case, write the model as

$$\lambda_{p-2,s}(B)(1 - \beta_s B)(1 - \alpha_s B)y_t = \varepsilon_t,$$

where $\lambda_{p-2,s}(B)$ is a periodic autoregressive polynomial of order $p - 2$. In VQ notation this becomes

$$\Lambda(B)[\Psi_0 - \Psi_1 B][\Xi_0 - \Xi_1 B]Y_T = \varepsilon_T, \tag{25}$$

where $\Lambda(B)$ is a matrix lag polynomial of order $[(p + 1)/4]$. If the characteristic equation $|\Lambda(z)(\Psi_0 - \Psi_1 z)(\Xi_0 - \Xi_1 z)| = 0$ has only two unit roots and all other roots outside the unit circle, then $\Lambda(B)$ is invertible, leading to

$$[\Psi_0 - \Psi_1 B][\Xi_0 - \Xi_1 B]Y_T = [\Lambda(B)]^{-1}\varepsilon_T.$$

Hence, the above results carry through after replacing ε_T by $[\Lambda(B)]^{-1}\varepsilon_T$, a periodically stationary process. For further analysis we need to derive the limiting behavior of the Y_T process.

Lemma 3.1. Let Y_T be generated according to the PAR(p) *process* (25), *or alternatively,*

$$\Delta^2 Y_T = (\Theta_0 + \Theta_1 B)(\Theta_0^* + \Theta_1^* B)\Lambda(B)^{-1}\varepsilon_T, \tag{26}$$

where $\Lambda(B)$ is invertible, $\{\varepsilon_T\}$ i.i.d. $N(0, \sigma^2 I_4)$ *and where $\Theta_0, \Theta_1, \Theta_0^*$ and Θ_1^* are defined in* (21) *and* (23) *with $\alpha_1\alpha_2\alpha_3\alpha_4 = \beta_1\beta_2\beta_3\beta_4 = 1$, and κ in* (18) *is not equal to* 0. *Then we have as $N \to \infty$,*

$$N^{-3/2} Y_{[rN]} \xrightarrow{d} \bar{B}(r) = wa \int_0^r W(u)\,du, \quad r, u \in [0, 1]$$

$$N^{-1/2} Y^*_{[rN]} \xrightarrow{d} B^*(r) = \omega^* a^* W(r), \tag{27}$$

where $[rN]$ denotes the integer part of rN, where $B(r)$ and $B^*(r)$ are (4×1) vector Brownian motion processes with covariance matrices $\omega^2 aa'$ and $\omega^{*2} a^* a^{*\prime}$, $W(r)$ is a standard (scalar) Brownian motion process, and

$$\omega = \sigma(b'a^*)\{b^{*\prime}\Lambda(1)^{-1}\Lambda(1)'^{-1}b^*\}^{1/2},$$
$$\omega^* = \sigma\{b^{*\prime}\Lambda(1)^{-1}\Lambda(1)'^{-1}b^*\}^{1/2}, \tag{28}$$

with

$$a = \begin{bmatrix} 1 \\ \alpha_2 \\ \alpha_2\alpha_3 \\ \alpha_2\alpha_3\alpha_4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ \alpha_1\alpha_3\alpha_4 \\ \alpha_1\alpha_4 \\ \alpha_1 \end{bmatrix}, \quad a^* = \begin{bmatrix} 1 \\ \beta_2 \\ \beta_2\beta_3 \\ \beta_2\beta_3\beta_4 \end{bmatrix} \text{ and } b^* = \begin{bmatrix} 1 \\ \beta_1\beta_3\beta_4 \\ \beta_1\beta_4 \\ \beta_1 \end{bmatrix}. \tag{29}$$

Proof: see Appendix.

3.1. Quasi-differencing an I(2) process

When testing for multiple unit roots in nonperiodic models Pantula (1989) suggests a sequential testing procedure. Assuming that at least a single unit root exists, the differenced time series is tested for the presence of an additional unit root. A similar strategy will be suggested here for periodic AR models. First, one should transform the y_t series such that the resulting time series y_t^* has $Y^*_{s,T}$ components that are at most $I(1)$. In the second step one can check whether y_t^* has multiple unit roots, i.e. nonseasonal and/or seasonal unit roots. New problems arise for periodic models, however, since the way the time series should be transformed in the first step involves quasi-differencing by filters that are unknown and hence have to be estimated. The next theorem suggests how the quasi-differencing filters across the seasons can be obtained.

Theorem 3.1. Let y_t be a PAR(p) *process satisfying the conditions stated in the Lemma* 3.1, *and consider NLS estimation of the* PAR(2) *model*

$$y_t = \sum_{s=1}^{4} \alpha_s D_{s,t} y_{t-1} + \sum_{s=1}^{4} \beta_s D_{s,t}(y_{t-1} - \alpha_{s-1} y_{t-2}) + \eta_t, \tag{30}$$

under the restriction $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$, $t = 1, 2, \ldots, n$, where $D_{s,t}$ are the usual seasonal dummy variables and where $\alpha_{-k} = \alpha_{4-k}$, then it follows that

$$N^2(\hat{\alpha}_s - \alpha_s) = \mathrm{O_p}(1)$$

$$N(\hat{\beta}_s - \beta_s) = \mathrm{O_p}(1)$$

for $s = 1, 2, 3, 4$.

Proof. See Appendix.

The significance of Theorem 3.1 is that in order to exclude the possibility of periodic models integrated of order 2, initial estimates of the periodic coefficients in the periodic differencing filter $(1 - \alpha_s B)$ can be obtained by a nonlinear regression. The auxiliary regression only needs to be of second order, regardless of the actual order of the PAR. Provided that the underlying time series is periodically doubly integrated a 'super–super' ($\mathrm{O_p}(N^2)$) consistent estimate of the periodic coefficients at the first order of integration can be calculated. These estimates form the basis for quasi-differencing the time series. Next, the transformed series can be analyzed in accordance with the guidelines suggested below in Section 4.

4. Model selection

In this section we start with an analysis of y_t time series with $Y_{s,T}$ series that are at most $I(1)$. For such a series we propose test statistics for the number of unit roots in the Y_T process, and tests for parameter restrictions that correspond with seasonal and nonseasonal unit roots. Furthermore, we discuss the impact of trends and constants on the asymptotic distributions of the test statistics and on the time series pattern under the various null hypotheses. Finally, we briefly discuss multiple unit roots in so-called subset PAR models, i.e. the AR order varies with the seasons.

When the $Y_{s,T}$ processes are at most $I(1)$, there are five possible cases: the (4×1) vector process Y_T contains 0, 1, 2, 3, or 4 unit roots, and hence 4, 3, 2, 1, and 0 cointegrating relationships. We consider each case and derive the cointegrating vectors. These vectors imply nonlinear restrictions on the parameters in a PAR(p) process, which can easily be investigated using NLS techniques applied to the PAR(p) process

$$\phi_{p,s}(B)y_t = \varepsilon_t. \tag{31}$$

4.1. A single unit root

The case of a single unit root is covered in Boswijk and Franses (1996). One unit root in Y_T implies three cointegrating relationships, which can be expressed as

$$\begin{aligned} &Y_{4,T} - \alpha_4 Y_{3,T}, \\ &Y_{3,T} - \alpha_3 Y_{2,T}, \\ &Y_{2,T} - \alpha_2 Y_{1,T}. \end{aligned} \tag{32}$$

These three relationships imply the fourth, i.e. $Y_{4,T} - \alpha_4\alpha_3\alpha_2 Y_{1,T}$. Subtracting the stationary variable $\Delta Y_{4,T}$ and dividing by $-\alpha_4\alpha_3\alpha_2$ gives

$$Y_{1,T} - \alpha_1 Y_{4,T-1} \quad \text{with } \alpha_1\alpha_2\alpha_3\alpha_4 = 1. \tag{33}$$

Given (32) and (33), the (periodically differenced) process $y_t - \alpha_s y_{t-1}$ under the restriction $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ is a periodically stationary process. The PAR(p) process can then be written as

$$\phi_{p,s}(B)y_t = \phi_{p-1,s}(B)(1 - \alpha_s B)y_t = \varepsilon_t. \tag{34}$$

Boswijk and Franses (1996) show that the likelihood ratio test

$$\mathrm{LR} = n \log(\mathrm{RSS_r}/\mathrm{RSS_u}) \tag{35}$$

for the hypothesis $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ in (34) follows the 'Johansen (1988) distribution' for rank 3 versus rank 4, where $\mathrm{RSS_r}$ is the residual sum of squares [RSS] of (34) under the nonlinear restriction, and $\mathrm{RSS_u}$ is the RSS of the unrestricted model. For further reference, we denote this test LR_1. In Boswijk and Franses (1996) it is also shown that, conditional on $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$, the F-test for $\alpha_s = 1$, $s = 1, 2, 3$, follows a standard F-distribution. This seems confirmed for small samples by the simulation results in Franses and Paap (1994). Additionally, Boswijk and Franses (1996) show that a joint test for $\alpha_s = 1$ in (34) follows a mixture of a Johansen- and an F- distribution. A drawback of the joint test is that when the null hypothesis $\alpha_s = 1$ is rejected, the y_t series may still have a stochastic trend in case of $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. Therefore, Boswijk and Franses (1996) advocate to use the two step approach. In the sequel of this section, we follow this strategy when investigating seasonal unit roots.

4.2. Two unit roots

In case of two unit roots in the Y_T process, and still under the assumption that Y_T is at most $I(1)$, there are two cointegrating relations between the $Y_{s,T}$ series, like, e.g.

$$Y_{4,T} - \beta_{14}Y_{3,T} - \beta_{24}Y_{2,T}, \tag{36}$$

$$Y_{3,T} - \beta_{13}Y_{2,T} - \beta_{23}Y_{1,T}. \tag{37}$$

These two relations imply two other cointegrating relationships

$$Y_{2,T} - \beta_{12}Y_{1,T} - \beta_{22}Y_{4,T-1}, \tag{38}$$

$$Y_{1,T} - \beta_{11}Y_{4,T-1} - \beta_{21}Y_{3,T-1}. \tag{39}$$

Since there are only two cointegrating relationships between the elements of Y_T, the relations in (38) and (39) should be linear combinations of those in (36) and (37). This implies the following four restrictions on the β parameter values:

$$\begin{aligned}
\beta_{11} &= -\beta_{13}/\beta_{23}\beta_{24}, \\
\beta_{21} &= ((1/\beta_{23}) - \beta_{13}\beta_{14}/(\beta_{23}\beta_{24})), \\
\beta_{12} &= -\beta_{14}\beta_{23}/(\beta_{13}\beta_{14} + \beta_{24}), \\
\beta_{22} &= 1/(\beta_{13}\beta_{14} + \beta_{24}).
\end{aligned} \tag{40}$$

These four parameter restrictions can be tested via imposing the corresponding nonlinear restrictions in the PAR(p) model when it is rewritten as

$$\phi_{p,s}(B)y_t = \phi_{p-2,s}(B)(1 - \beta_{1s}B - \beta_{2s}B^2)y_t = \varepsilon_t. \tag{41}$$

We denote the relevant likelihood ratio test statistic as LR_2. We return with a discussion of its asymptotic distribution in Theorem 4.1 below.

In case the restrictions in (40) cannot be rejected, one may proceed with testing restrictions on the β_{1s} and β_{2s} values, which correspond to, e.g. certain seasonal and nonseasonal unit roots. Conditional on the restrictions in (40), one can construct likelihood ratio statistics LR_2^* for a particular hypothesis. Hence, LR_2^* for $(1 - B)(1 + B)$ in (41) implies that all $\beta_{1s} = 0$ and $\beta_{2s} = 1$. Imposing $\beta_{1s} = 0$ and $\beta_{2s} = -1$ for all s results in a $(1 + B^2)$ filter, i.e. the seasonal unit roots $\pm i$. Note that when $\beta_{1s} = 2$ and $\beta_{2s} = -1$, i.e. the case where the double filter $(1 - B)^2$ is needed, and hence where Y_T is at most $I(2)$, the restrictions in (40) are violated.

4.3. Three unit roots

In case there are three unit roots in the Y_T process, there is only a single cointegrating relation between the $Y_{s,T}$ elements, which can be written as

$$Y_{4,T} - \gamma_{14}Y_{3,T} - \gamma_{24}Y_{2,T} - \gamma_{34}Y_{1,T}. \tag{42}$$

This relation implies three other cointegrating relationships, i.e.

$$\begin{aligned}&Y_{3,T} - \gamma_{13}Y_{2,T} - \gamma_{23}Y_{1,T} - \gamma_{33}Y_{4,T-1},\\ &Y_{2,T} - \gamma_{12}Y_{1,T} - \gamma_{22}Y_{4,T-1} - \gamma_{32}Y_{3,T-1},\\ &Y_{1,T} - \gamma_{11}Y_{4,T-1} - \gamma_{21}Y_{3,T-1} - \gamma_{31}Y_{2,T-1}.\end{aligned} \tag{43}$$

Given (42), there are nine restrictions on the parameters in (43), i.e.

$$\begin{aligned}&\gamma_{11}\gamma_{34} = 1, && \gamma_{21}\gamma_{34} = -\gamma_{14}, && \gamma_{31}\gamma_{34} = -\gamma_{24},\\ &\gamma_{12}\gamma_{24} = -\gamma_{34}, && \gamma_{22}\gamma_{24} = 1, && \gamma_{32}\gamma_{24} = -\gamma_{14},\\ &\gamma_{13}\gamma_{14} = -\gamma_{24}, && \gamma_{23}\gamma_{14} = -\gamma_{34}, && \gamma_{33}\gamma_{14} = 1.\end{aligned} \tag{44}$$

These restrictions can be tested in a rewritten version of (31), i.e.

$$\phi_{p,s}(B)y_t = \phi_{p-3,s}(B)(1 - \gamma_{1s}B - \gamma_{2s}B^2 - \gamma_{3s}B^3)y_t = \varepsilon_t. \tag{45}$$

We denote the likelihood ratio test statistic for the restrictions in (44) in (45) as LR_3. Similar to the case of two unit roots, and conditional on the restrictions (44), one may test for parameter restrictions as $(1 - B)(1 + B^2)$ in (45) using likelihood ratio test statistics LR_3^*.

4.4. Four unit roots

In case of four unit roots, the general PAR(p) model can be written as

$$\phi_{p,s}(B)y_t = \phi_{p-4,s}(B)(1 - B^4)y_t = \varepsilon_t. \tag{46}$$

The test for the hypothesis of four unit roots, which amounts to a linear restriction, will be denoted as LR_4.

Theorem 4.1. Under the hypothesis of q unit roots, we have as $n \to \infty$,

$$\mathrm{LR}_q \xrightarrow{d} \mathrm{trace}\left\{\int_0^1 (\mathrm{d}W(s)\, W(s)' \left(\int_0^1 W(s)W(s)'\,\mathrm{d}s\right)^{-1} \int_0^1 W(s)\,\mathrm{d}W(s)'\right\},$$

where $W(s)$ is a standard q-vector Brownian motion process.

Under the additional hypothesis of a nonperiodic (seasonal) unit root,

$$\mathrm{LR}_q^* \xrightarrow{d} \chi^2(k),$$

where k is the number of additional restrictions tested.

Proof. See Appendix.

Notice that the limiting distribution of LR_q is the same as the one obtained and tabulated in Johansen (1988) for the likelihood ratio tests for p–q cointegrating vectors (and hence q unit roots) in a p-dimensional vector autoregressive process. Thus, the tests proposed here do not require new tables to be generated. This is in contrast to the approach followed in Ghysels et al. (1996), where it is proposed to test for the adequacy of, e.g. the $(1 + B^2)$ filter within the general PAR(p) model. The above theorem indicates that test statistics for such joint hypotheses asymptotically follow distributions that are complicated functions of Johansen- and χ^2-distributions and, hence, that new critical values for those tests have to be generated. An additional disadvantage is that rejection of the null hypothesis leaves open the question how many stochastic trends are driving the time series.

4.5. Summary of our empirical procedure

In practical occasions, the model selection strategy proceeds as follows. The first step is to estimate the order p of the PAR process using some LR based test or one of the familiar information criteria. The simulation results in Franses and Paap (1994) indicate that the number of unit roots in the PAR process does not affect this order selection. In case one suspects $I(2)$ type patterns, one should estimate the α_s in a PAR(2) model, as suggested in Theorem 3.1. In a next step one can analyze the $y_t^* = (1 - \hat{\alpha}_s B)y_t$ series for nonseasonal and seasonal unit roots via imposing nonlinear restrictions in decreasing sequence of unit roots and testing for the number of unit roots using our LR test statistics. Hence, the sequence is first to consider the LR_4 test. Finally, if the number of unit roots is determined, one may check for restrictions like $(1 - B)$ or $(1 + B)$ to investigate specific seasonal and nonseasonal unit roots.

4.6. Constants and trends

In many practical occasions, one may want to enlarge the model in (1) like

$$y_t = \mu_s + \tau_s t + \phi_{1s} y_{t-1} + \cdots + \phi_{ps} y_{t-p} + \varepsilon_t, \quad t = 1, 2, \ldots, n, \tag{47}$$

where μ_s are seasonally varying intercept terms and τ_s are seasonally varying parameters that correspond to the deterministic trend. The inclusion of constants and trends in the regression model has an effect on the asymptotic distribution of the LR_i test statistics, $i = 1, 2, 3, 4$, see also the Appendix. The critical values of the relevant distributions are tabulated in Osterwald–Lenum (1992).

4.7. Subset PAR processes

Until now we have assumed that the PAR process is of order p such that the AR order is equal to p for all seasons s. It may however occur in practice that the AR order in some season s, say p_s, is smaller than p. Furthermore, it may occur that $\phi_{j,s}$ parameters can be set equal to zero for some j or s. In both cases, these models can be called subset PAR processes. Given the expressions for the characteristic equations for the Y_T processes in Section 2, it is clear that the number of possible unit roots in a PAR process is determined by the minimum value of p_s. For example, for a PAR(2) as (11), the characteristic equation becomes $1 - \alpha_1\alpha_2\alpha_3\alpha_4 z = 0$ when only a single β_s value equals zero.

For practical purposes, we recommend that one first tests for unit roots before one checks whether the PAR model is a subset PAR model. This is because the distribution of t-test statistics for the significance of, for example, lagged $(1 - \hat{\alpha}_s B)y_t$ variables depends on the number of (any remaining) stochastic trends.

5. Monte Carlo simulations

We now turn to a small-scale Monte Carlo experiment to assess the finite sample size and power properties of the tests proposed in the previous section, with a specific focus on the validity of the asymptotic results in Theorem 4.1. We consider 7 data generating processes [DGPs], all of which are special cases of the fourth-order periodic autoregression

$$\phi_{4,s}(B)y_t = \mu_s + \varepsilon_t, \tag{48}$$

where in the DGPs we set μ_s at 0 for all s. The DGPs are

(DGP1) $\phi_{4,s}(B) = (1 + 0.8B + 0.6B^2 + 0.4B^3)(1 - \alpha_s B)$

with DGP1nopar: $\alpha_s = 1$ for all s

with DGP1par: $\alpha_s = \{0.8, 1, 1.25, 1\}$

(DGP2) $\phi_{4,s}(B) = (1 + 0.6B^2)(1 - \alpha_s B^2)$

with DGP2nopar: $\alpha_s = 1$ for all s

with DGP2par: $\alpha_s = \{0.8, 1, 1.25, 1\}$

(DGP3) $\phi_{4,s}(B) = (1 + 0.8B)(1 - \alpha_s B + B^2 - \alpha_s B^3)$

with DGP3nopar: $\alpha_s = 1$ for all s

with DGP3par: $\alpha_s = \{0.8, 1.25, 0.8, 1.25\}$

(DGP4) $\phi_{4,s}(B) = (1 - B^4)$.

Table 1
Rejection frequencies of periodic unit root tests at a 5% nominal level number of replications is 2000

		Tests			
DGP	n	LR_1	LR_2	LR_3	LR_4
1nopar	100	0.037	0.274	0.577	0.851
	200	0.049	0.948	1.000	1.000
1par	100	0.035	0.332	0.704	0.944
	200	0.043	0.977	1.000	1.000
2nopar	100		0.028	0.137	0.476
	200		0.038	0.699	0.993
2par	100		0.053	0.189	0.657
	200		0.044	0.791	1.000
3nopar	100			0.032	0.193
	200			0.048	0.594
3par	100			0.079	0.702
	200			0.056	0.998
4	100				0.073
	200				0.063

Note that DGPi involves i unit roots, and that for $i = 1, 2, 3$, DGP$i$nopar is the nonperiodic model, and DGP$i$par is the periodic model. All periodicity in the DGPs is contained in the cointegrating linear combinations, and all short-run dynamics are nonperiodic. The construction of the tests, however, does not involve corresponding parameter restrictions. Furthermore, notice that the DGPs are chosen such that the characteristic roots of the VQ representation are the same for the periodic and nonperiodic DGPs.

Table 1 contains the rejection frequencies of the LR_1 to LR_4 statistics, for a sample size of 100 and 200 observations. All tests are based on the correct order of the PAR(4) model since the simulation results in Franses and Paap (1994) indicate that this order will usually be detected. The entries on the main diagonal of Table 1 represent the empirical size of the test, whereas the off-diagonal cells give the empirical power.

We do not compute rejection frequencies of the LR_i tests for DGPj with $j > i$ for two reasons. First, it is well known that if the DGP contains more unit roots than are tested, then the test will have a higher (asymptotic) rejection frequency than the nominal size. Thus, we should expect values exceeding 0.05 below the diagonal of Table 1, even asymptotically. Secondly, some of the parameters of the model under i unit roots will not be identified when the DGP actually has more than i unit roots. Therefore, convergence of NLS optimization methods will be problematic. The first problem can be solved by employing Johansen's (1992b) sequential testing procedure, based on the work by Pantula (1989). In this procedure, one starts with testing the maximum number of unit roots (in

Table 2
Rejection frequencies of periodicity tests at a 5% nominal level, number of replications is 2000

DGP	Test	n	Nonperiodic DGP (size)	Periodic DGP (power)
1	LR_1^*	100	0.092	0.219
		200	0.071	0.556
2	LR_2^*	100	0.164	0.288
		200	0.095	0.433
3	LR_3^*	100	0.253	0.364
		200	0.130	0.719

this case 4), and only proceeds to testing i unit roots when the hypothesis of $i + 1$ unit roots is rejected. This procedure will have an symptotically controllable size, see Johansen (1992b).

From Table 1, we observe that the empirical sizes are reasonably close to the nominal size of 5%, and that the power of the tests seems to be higher for the periodic DGPs than for their nonperiodic counterparts.

Finally, in Table 2 we report the rejection frequencies of the LR* tests for particular nonperiodic differencing filters, i.e. $(1 - B)$ in DGP1, $(1 - B^2)$ in DGP2 and $(1 - B + B^2 - B^3) = (1 - B)(1 + B^2)$ for DGP3. We observe from Table 2 that the finite sample size of the tests (i.e. the rejection frequencies for DGP$i$nopar) can be quite far from 5%, and seem to converge to the nominal size only slowly. Therefore, it may be worthwhile to investigate the effectiveness of, e.g. bootstrap methods or other small-sample corrections for this testing problem. The power of the tests appear to increase with the sample size, as expected.

6. Applications

In this section we illustrate the empirical usefulness of our method to test for nonseasonal and seasonal unit roots in periodic autoregressions for three quarterly macroeconomic time series, which are selected for no particular reason other than illustrative purposes. These series are Unemployment Rate in Norway for 1966.1–1992.4 (not in logs), (Real) Consumption of Nondurables in the USA for 1947.1–1991.4 (in logs) and Unemployment in Canada for 1960.1–1987.4 (not in logs). We start our empirical analysis with an application of the HEGY test method for nonseasonal and seasonal unit roots in a nonperiodic AR model. The results are summarized in Table 3.

These results indicate that the nonseasonal unit root 1 is present for all three series. Unreported HEGY test results for the first-order differenced time series reveal that these series are at most $I(1)$ at the zero frequency. The seasonal unit

Table 3
Testing for seasonal and nonseasonal unit roots in nonperiodic AR models using the HEGY method, where the auxiliary regression includes four seasonal dummies and a linear trend

Tests[a]	Unemployment Norway	Nondurables Consumption USA	Unemployment Canada
$t(\pi_1)$	− 1.630	− 0.433	− 1.309
$t(\pi_2)$	− 2.626*	− 2.235	− 1.680
$F(\pi_3, \pi_4)$	5.925*	5.061	7.155**
Lags	2	6	5
n^*	102	170	103

[a] The test statistics, the relevant auxiliary regression and the appropriate critical values are given in Hylleberg et al. (1990). Lags denotes the number of lagged $y_t - y_{t-4}$ variables included in the auxiliary regression, and n^* is the number of effective observations. The $t(\pi_1)$-test concerns the nonseasonal unit root 1, the $t(\pi_2)$-test concerns the seasonal unit root -1, and the joint $F(\pi_3, \pi_4)$-test concerns the seasonal unit roots $\pm i$. The unit roots correspond with the $(1 - B)$, $(1 + B)$ and $(1 + B^2)$ differencing filters, respectively.
***Significant at the 1% level.
**Significant at the 5% level.
*Significant at the 10% level.

root -1 is present for all three series when we consider the 5% significance level. When we allow for the 10% significance level, we do not find this seasonal unit root to be present in unemployment in Norway. At the 5% significance level, we obtain evidence for the seasonal unit roots $\pm i$ for two of the three series. In sum, our three example series all have one or more seasonal unit roots in case we consider a nonperiodic model.

To investigate the robustness of the findings in Table 3 to periodicity in the AR parameters, we use the method proposed in Franses (1994). Within this method, the PAR model is allowed to have seasonally heteroskedastic error terms. The main results are presented in Table 4.

The Johansen type test statistics for the rank r of the matrix $\Pi = \gamma\alpha'$ in the VQ(1) model as in (5) indicate that this rank is equal to 1 for the two unemployment series and is equal to 2 or 3 for the consumption variable. These results clearly indicate that the rank is not equal to 0, which corresponds to the adequacy of the $(1 - B^4)$ transformation for y_t (or the Δ transformation for Y_T). The value of 3 of the rank of Π for the consumption series is clearly in contrast to the results in Table 3, where evidence for all four unit roots is reported. Since this cointegration-based method does not impose all restrictions implied by the PAR model, it will be less powerful than our new methods which do impose these restrictions, provided of course that these restrictions, such as seasonal homoskedasticity, are satisfied.

Table 4
Testing for the cointegration rank r of $\Pi = \gamma\alpha'$ in the VQ(1) model (5) using the Johansen method

Tests[a]	Unemployment Norway	Nondurables Consumption USA	Unemployment Canada
λ_1	0.973***	0.808***	0.917***
λ_2	0.411	0.553***	0.415
λ_3	0.248	0.263*	0.260
λ_4	0.088	0.037	0.044
Trace ($r \leqslant 0$)	117.688***	123.023***	91.046***
Trace ($r \leqslant 1$)	23.584	50.457***	23.813
Trace ($r \leqslant 2$)	9.813	15.068	9.346
Trace ($r \leqslant 3$)	2.392	1.641	1.223
N^*	26	44	27
Decision	$r = 1$	$r = 3$	$r = 1$

[a] The tests are the familiar λ_{max} and Trace test statistics, proposed in Johansen (1988, 1991), where λ_i in the table refers to the relevant λ_{max} test. The asymptotic distribution of these tests is given in Osterwald-Lenum (1992). N^* is the effective sample size. Because of the small sample size, we use the critical values displayed in Franses (1994), for N^* is 25 and 50. Several model selection and diagnostic criteria indicate that the VQ(1) model sufficiently describes the annually observed vector time series Y_T.
*** Significant at the 1% level.
** Significant at the 5% level.
* Significant at the 10% level.

Our next step is to apply the LR tests which are proposed in Section 4. The (unreported) HEGY outcomes for the $(1 - B)y_t$ series suggest that $I(2)$ type behavior is not present in our three example series. Before we can apply our tests, we need to decide on the order of the various PAR models. Using diagnostic tests for residual autocorrelation at lags 1 and 1–4 and for periodic patterns in the residual autocorrelation function, we find that the order p can be set at 4 for each time series. The residuals of the estimated PAR(4) model are also checked for the absence of seasonal heteroskedasticity. The F-test values of the auxiliary regression of $\hat{\varepsilon}_t^2$ on a constant and three seasonal dummies obtain the (5%) insignificant values of 1.982, 1.796 and 2.407. We also test whether the AR parameters are indeed periodically varying. In the PAR(4) model, an F-test for the hypothesis of no periodicity follows a standard F-distribution with 12 and $n - 20$ degrees of freedom. See Boswijk and Franses (1996) for the derivation of this distribution. The test results for periodicity are 9.762, 9.797 and 3.102, which indicate a convincing rejection of the null hypothesis of no periodic parameter variation for our three example series.

Table 5
Testing for multiple unit roots in PAR(4) models using LR tests

Tests[a]	Unemployment Norway	Nondurables Consumption USA	Unemployment Canada
LR_4	328.128***	196.432***	232.741***
LR_3	123.700***	133.950***	164.240***
LR_2	70.459***	64.315***	32.004***
LR_1	0.469	2.056	1.040
α_1	1.014	1.053	1.077
α_2	1.057	1.018	0.982
α_3	1.012	1.022	0.971
α_4	0.922	0.913	0.974
$F_{(1-B)}$	7.457***	8.907***	3.895***

[a] The LR_i ($i = 1, 2, 3, 4$) test statistics are discussed in Section 4. The α_s ($s = 1, 2, 3, 4$) estimates concern the parameters in the periodic differencing filter $(1 - \alpha_s B)$ under the restriction $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$, which transforms the y_t series to periodic stationarity. The $F_{(1-B)}$ test concerns an F-test for the restriction $\alpha_s = 1$, conditional on $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. This F-test has a standard F-distribution with 3 and m degrees of freedom, with m equal to $n - 4 - 4p + 1$. The PAR(4) models contain 4 seasonal intercept terms, but no seasonal trends. The results do not change very much when these trends are included.
*** Significant at the 1% level.
** Significant at the 5% level.
* Significant at the 10% level.

The results for the LR test statistics for 1–4 unit roots are given in Table 5. Compared with Table 4, we may expect different findings given that our new method imposes all restrictions implied by the PAR(4) model (including seasonal homoskedasticity) and the diagnostic mentioned above suggest that these restrictions are valid.

The results for the LR_i ($i = 1, 2, 3, 4$) statistics in Table 5 can easily be summarized. Only the LR_1 test value is insignificant, while the null hypotheses corresponding to the LR_2 to LR_4 tests are rejected at the 1% level (or even at the 0.1% level). Hence, there appears to be only a single unit root in each of the three time series considered. Finally, we investigate if this single stochastic trend can be removed using the $(1 - B)$ filter. The values of the $F_{(1-B)}$ test for this hypothesis in the last row of Table 5 indicate a firm rejection. In other words, our three example series all seem periodically integrated of order 1. This means that the appropriate differencing filter for these series is $(1 - \alpha_s B)$ with $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$, where the estimates of α_s are also given in Table 5.

7. Concluding remarks

In this paper we propose and apply a simple testing strategy for periodic autoregressions, which involves an investigation of seasonal unit roots and one or two nonseasonal unit roots. Our method also allows for periodic integration of order 1 and 2. The latter class of methods may yield useful descriptions of seasonal time series since it allows the seasonal fluctuations to depend on the stochastic trend. Our applications show that tests for seasonal unit roots in nonperiodic models may too often detect such roots, while, when allowing for periodically varying AR parameters, the evidence for seasonal unit roots tends to disappear.

One drawback of periodic autoregressions is that the number of parameters increases with the seasonal frequency. Hence, a periodic AR model for monthly time series can involve a huge number of parameters. This would complicate the empirical application of our approach, which seems to work well for quarterly data. To allow the application of our method to monthly data would then imply that one imposes restrictions on the various AR parameters. When such proper restrictions are imposed, the asymptotic results in our paper indicate that our selection strategy can easily be used.

Appendix. Proofs of lemmas and theorems

Proof of Lemma 3.1. The error term $\{\varepsilon_T\}$ is assumed Gaussian and hence satisfies the multivariate invariance principle, see, e.g. Phillips and Durlauf (1986), thus implying that

$$(1/N^{1/2}) \sum_{j=1}^{[rN]} \varepsilon_j \xrightarrow{d} \sigma E(r), \tag{A.1}$$

where $\sigma^2 I_4 = \lim_{N\to\infty} N^{-1}\mathrm{E}((\sum_{j=1}^{N} \varepsilon_j)(\sum_{j=1}^{N} \varepsilon_j)')$ is the covariance matrix of the vector standard Brownian motion $E(r)$ of dimension (4×1). It follows from the analysis of, e.g. Park and Phillips (1989) and Haldrup (1994a, b) that given (26)

$$\begin{aligned}(1/N^{3/2})Y_{[rN]} &= (1/N^{3/2})(\Theta_0 + \Theta_1)(\Theta_0^* + \Theta_1^*)A(1)^{-1} \sum_{k=1}^{[rN]} \sum_{j=1}^{k} \varepsilon_j + o_p(1) \\ &\xrightarrow{d} \sigma(\Theta_0 + \Theta_1)(\Theta_0^* + \Theta_1^*)A(1)^{-1} \int_0^r E(u)\,\mathrm{d}u \equiv \bar{B}(r).\end{aligned} \tag{A.2}$$

It is now a consequence of the results in Boswijk and Franses (1996) that due to the unit roots in the VMA polynomial (24), the matrices $(\Theta_0 + \Theta_1)$ and $(\Theta_0^* + \Theta_1^*)$ both will be of rank 1 and

$$(\Theta_0 + \Theta_1)(\Theta_0^* + \Theta_1^*) = ab'a^*b^{*\prime}, \tag{A.3}$$

where the a, b, a^* and b^* vectors are given in (29). Define now the scalar integrated standard Brownian motion

$$\bar{W}(r) = \sigma\omega^{-1}b'a^*b^{*\prime}A(1)^{-1}\int_0^r E(u)\,\mathrm{d}u, \tag{A.4}$$

with ω given in (28) and the first part of (27) follows straightforwardly. With respect to the second part of (27) it follows from

$$\Delta Y_T^* = (\Theta_0^* + \Theta_1^* B)A(B)^{-1}\varepsilon_T \tag{A.5}$$

that

$$(1/N^{1/2})Y_{[rN]}^* = (1/N^{1/2})(\Theta_0^* + \Theta_1^* B)A(B)^{-1}\sum_{j=1}^{[rN]}\varepsilon_j + \mathrm{o_p}(1)$$

$$\xrightarrow{\mathrm{d}} \sigma(\Theta_0^* + \Theta_1^*)A(1)^{-1}E(r) \equiv B^*(r). \tag{A.6}$$

Similarly to the analysis in Boswijk and Franses (1996) and the procedure above, we now have that

$$B^*(r) = \sigma a^* b^{*\prime}A(1)^{-1}E(r), \tag{A.7}$$

and with the standard Brownian motion

$$W(r) = \sigma\omega^{*-1}b^{*\prime}A(1)^{-1}E(r), \tag{A.8}$$

the required result follows. □

Proof of Theorem 3.1. We shall consider model (30) both under the restriction $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ and the additional restriction $\beta_1\beta_2\beta_3\beta_4 = 1$. Our setup encompasses both situations. For the latter case, we write the model as $y_t = x_t(\gamma) + \varepsilon_t$, where $\gamma = (\alpha_2, \alpha_3, \alpha_4, \beta_2, \beta_3, \beta_4)'$, which we also may write in the condensed form $\gamma = (\alpha', \beta')'$. When the restriction $\beta_1\beta_2\beta_3\beta_4 = 1$ is *not* imposed in the estimation, the γ vector may be redefined such that β is simply $(\beta_1, \beta_2, \beta_3, \beta_4)'$. Define the vector of pseudo regressors

$$\partial x_t/\partial\gamma = (z_t', w_t')',$$

where $z_t = J_1' v_t$, $w_t = J_2' u_t$ with $v_t = (v_{1,t}, v_{2,t}, v_{3,t}, v_{4,t})'$, $u_t = (u_{1,t}, u_{2,t}, u_{3,t}, u_{4,t})'$, $v_{s,t} = D_{s,t}y_{t-1} - \beta_{s+1}D_{s+1,t}y_{t-2}$, $u_{s,t} = D_{s,t}(y_{t-1} - \alpha_{s-1}y_{t-2})$ and with J_1 and J_2 defined as

$$J_1' = \begin{bmatrix} -1/(\alpha_2^2\alpha_3\alpha_4) & 1 & 0 & 0 \\ -1/(\alpha_2\alpha_3^2\alpha_4) & 0 & 1 & 0 \\ -1/(\alpha_2\alpha_3\alpha_4^2) & 0 & 0 & 1 \end{bmatrix}$$

and

$$J_2' = \begin{bmatrix} -1/(\beta_2^2\beta_3\beta_4) & 1 & 0 & 0 \\ -1/(\beta_2\beta_3^2\beta_4) & 0 & 1 & 0 \\ -1/(\beta_2\beta_3\beta_4^2) & 0 & 0 & 1 \end{bmatrix},$$

respectively. When the restriction $\beta_1\beta_2\beta_3\beta_4 = 1$ is not imposed when estimating the parameters, all the derived results will carry through with J_2 substituted by a (4×4) identity matrix. Notice that we also let $\alpha_1 = (\alpha_2\alpha_3\alpha_4)^{-1}$ and $\beta_1 = (\beta_2\beta_3\beta_4)^{-1}$. Furthermore, define $\alpha_0 = \alpha_4$, $\beta_0 = \beta_4$. Hence, we have that the NLS estimates can be found asymptotically as

$$\begin{bmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^n z_t z_t' & \sum_{t=1}^n z_t w_t' \\ \sum_{t=1}^n w_t z_t' & \sum_{t=1}^n w_t w_t' \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^n z_t \eta_t \\ \sum_{t=1}^n w_t \eta_t \end{bmatrix}. \tag{A.9}$$

To derive the order of $\hat{\alpha}$ and $\hat{\beta}$ we require that after appropriate normalization the diagonal submatrices $\sum_{t=1}^n z_t z_t'$ and $\sum_{t=1}^n w_t w_t'$ are nonsingular. To show this we write (A.9) in the following way:

$$\begin{bmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{bmatrix} = \left[\begin{bmatrix} J_1' & 0 \\ 0 & J_2' \end{bmatrix} \begin{bmatrix} \sum_{t=1}^n v_t v_t' & \sum_{t=1}^n v_t u_t' \\ \sum_{t=1}^n u_t v_t' & \sum_{t=1}^n u_t u_t' \end{bmatrix} \begin{bmatrix} J_1' & 0 \\ 0 & J_2' \end{bmatrix} \right]^{-1}$$
$$\times \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \sum_{t=1}^n v_t \eta_t \\ \sum_{t=1}^n u_t \eta_t \end{bmatrix}.$$

Along the lines of Boswijk and Franses (1996) we let $V_{s,T}$ indicate the VQ process of $v_{s,t}$. This is a different process for each s. If the (4×4) matrix $\Psi(B)$ defined in (12) is partitioned such that row number s is ψ_s, it follows that

$$V_{s,T} = \psi_s(B) Y_{s-1,T} = \psi_s Y_{s-1,T} + \psi_s^*(B) \Delta_1 Y_{s-1,T},$$

where $\psi_s^*(B)$ follows from a polynomial decomposition of $\psi_s(B)$. Since the term $\psi_s Y_{s-1,T}$ is doubly integrated whilst $\Delta_1 Y_{s-1,T}$ is integrated of order one, it follows from Park and Phillips (1989) and Haldrup (1994a) that

$$N^{-4} \sum_{t=1}^{n} v_{s,t} v_{q,t} = N^{-4} \sum_{T=1}^{N} V_{s,T}' V_{q,T} = (\psi_s' \psi_q)(N^{-4})$$
$$\times \sum_{T=1}^{N} Y_{s-1,T} Y_{q-1,T} + o_p(1)$$

for $s, q = 1, 2, 3, 4$. In accordance with Lemma 3.1, we therefore have that

$$N^{-4} \sum_{t=1}^{n} v_t v_t' \xrightarrow{d} \omega^2 A \Psi' \Psi A \int_0^1 (\bar{W}(r))^2 \, dr$$

where $A = \text{diag}(a_4, a_1, a_2, a_3)$. In a similar fashion the VQ process corresponding to $u_{s,t}$ reads

$$U_{s,T} = \psi_s^* Y_{s-1,T}^*,$$

where $\Psi^* = I_4$ and ψ_s^* corresponds to column s of this matrix. Because the columns are orthogonal, we have that the expression

$$N^{-2} \sum_{t=1}^{n} u_{s,t} u_{q,t} = N^{-2} \sum_{T=1}^{N} U'_{s,T} U_{q,T}$$

is 0 for $s \neq q$, while for $s = q$ the expression yields

$$N^{-2} \sum_{t=1}^{n} u_{s,t}^2 = N^{-2} \sum_{T=1}^{N} Y_{s-1,T}^* Y_{s-1,T}^*.$$

It follows in accordance with Lemma 3.1 that

$$N^{-2} \sum_{t=1}^{n} u_t u_t' = \omega^{*2} A^* A^* \int_0^1 W(r)^2 \, \mathrm{d}r,$$

where $A^* = \mathrm{diag}(a_4^*, a_1^*, a_2^*, a_3^*)$.

The nonsingularity conditions that have to be met require that the matrices $J_1' A \Psi'$ and $J_2' A^*$ are of full rank equal to 3. Both J_1' and J_2', which are (3×4) matrices, are of full rank by their construction. When the restriction on the β's is not imposed, J_2 is naturally full rank of dimension (4×4). A and A^* are full rank matrices whilst Ψ is of rank 3. It follows that $J_2' A^*$ is full rank and it also follows from the construction of J_1' that the first rank condition will be satisfied. To conclude, the matrices $N^{-4} \sum_{t=1}^{n} z_t z_t'$ and $N^{-4} \sum_{t=1}^{n} w_t w_t'$ are indeed nonsingular.

Now, define $D_N = \mathrm{diag}(N^{3/2} I_3, N^{1/2} I_3)$. It can be deduced from Haldrup (1994a) that in this case

$$N^{1/2} D_N \begin{bmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{bmatrix} = \left[N^{-1} D_N^{-1} \begin{bmatrix} \sum_{t=1}^{n} z_t z_t' & \sum_{t=1}^{n} z_t w_t' \\ \sum_{t=1}^{n} w_t z_t' & \sum_{t=1}^{n} w_t w_t' \end{bmatrix} D_N^{-1} \right]^{-1}$$
$$\times N^{-1/2} D_N^{-1} \begin{bmatrix} \sum_{t=1}^{n} z_t \eta_t \\ \sum_{t=1}^{n} w_t \eta_t \end{bmatrix} = \mathrm{O_p}(1).$$

Therefore,

$$N^2(\hat{\alpha} - \alpha) = \mathrm{O_p}(1), \qquad N(\hat{\beta} - \beta) = \mathrm{O_p}(1).$$

Because $\alpha_1 = (\alpha_2 \alpha_3 \alpha_4)^{-1}$, we also have that

$$N^2(\hat{\alpha}_1 - \alpha_1) = -(\alpha_2 \alpha_3 \alpha_4)^{-1} \sum_{s=2}^{4} (\alpha_s)^{-1} N^2 (\hat{\alpha}_s - \alpha_s) = \mathrm{O_p}(1).$$

A similar argument applies to $N(\hat{\beta}_1 - \beta_1)$. Notice that these results will apply regardless of imposing the restriction $\beta_1 \beta_2 \beta_3 \beta_4 = 1$ in the NLS estimation. □

Proof of Theorem 4.1. Consider the general VQ representation of the PAR(p) model

$$\Phi_0 Y_T = \Phi_1 Y_{T-1} + \cdots + \Phi_p Y_{T-p} + \varepsilon_T, \quad \varepsilon_T \sim \mathrm{IN}(0, \Sigma), \quad T = 1, \ldots, N \tag{A.10}$$

where P is an integer obeying $[(p-1)/4]+1$, where $[\cdot]$ means 'integer of', where $\Sigma = \sigma^2 I_4$ and

$$\Phi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_{12} & 1 & 0 & 0 \\ -\phi_{23} & -\phi_{13} & 1 & 0 \\ -\phi_{34} & -\phi_{24} & -\phi_{14} & 1 \end{bmatrix}, \quad (\Phi_i)_{sj} = \phi_{(4i+s-j),s}, \quad i = 1, \ldots, P \tag{A.11}$$

In error correction form, the model reads as

$$\Delta Y_T = \Pi Y_{T-1} + \Gamma_1 \Delta Y_{T-1} + \cdots + \Gamma_{P-1} \Delta Y_{T-P+1} + \nu_T,$$
$$\nu_T \sim \mathrm{IN}(0, \Omega) \tag{A.12}$$

where $\nu_T = \Phi_0^{-1}\varepsilon_T$, where $\Omega = \Phi_0^{-1}\Sigma\Phi_0^{-1\prime} = \sigma^2(\Phi_0'\Phi_0)^{-1}$, and

$$\Pi = -\Phi_0^{-1}(\Phi_0 - \cdots - \Phi_P), \quad \Gamma_i = -\Phi_0^{-1}(\Phi_{i+1} + \cdots + \Phi_P),$$
$$i = 1, \ldots, P-1. \tag{A.13}$$

The null hypothesis of $(4-r)$ unit roots expressed as

$$\mathrm{H}_0\colon \quad \Pi = \gamma\alpha', \tag{A.14}$$

where the matrix α is defined in Section 4 for $r = 1, 2, 3$; for $r = 0$, take α to be equal to 0. Define the matrix $\Xi = \Pi - \gamma\alpha'$, so that one way to express the model is

$$\Delta Y_T = \gamma\alpha' Y_{T-1} + \Xi Y_{T-1} + \Gamma_1 \Delta Y_{T-1} + \cdots + \Gamma_{P-1} \Delta Y_{T-P+1} + \nu_T,$$
$$\nu_T \sim \mathrm{IN}(0, \Omega), \tag{A.15}$$

and the null hypothesis becomes H_0: $\Xi = 0$.

Suppose that we disregard the restrictions imposed upon (A.15) by the underlying periodic model. Define the full parameter vector $\theta = (\theta_1', \theta_2', \theta_3')'$, where θ_1 contains the free parameters in $(\gamma, \Gamma_1, \ldots, \Gamma_{P-1}, \Omega)$, and where θ_2 and θ_3 contain the identified parameters in α and Ξ, respectively. The dimension of θ_1 is $4r + 16(P-1) + 10$. From Johansen's (1991) results it can be seen that the dimension of θ_2 is $r(4-r)$, and from this it can in turn be deduced that dim θ_3 equals $(4-r)^2$.

The periodic model however implies certain over-identifying restrictions on (A.15). Without the unit root restrictions, the total number of free parameters is $4p+1$. With $(4-r)$ unit roots, this number is $4(p-(4-r)) + 1 + r(4-r)$, so that this hypothesis corresponds to $(4-r)^2$ restrictions, which corresponds exactly with the dimension of θ_3. Likewise, from Section 4 it can be seen that the dimension of the vector of parameters in the cointegrating vectors is $r(4-r)$, which corresponds to the dimension of θ_2 in the unrestricted VQ model. Thus, the underlying periodic structure implies only restrictions on θ_1, i.e. on the 'short-run' parameters (in the sense that these do not characterize unit roots or cointegrating vectors). Let us express these restrictions as $g(\theta_1) = 0$. From the above, it follows that the dimension of the vector function g equals $16P - 4p + 9$, which is exactly equal to the difference in the number of parameters between a four-dimensional VAR(P) and a univariate PAR(p) process.

Let $\Theta = \Theta_1 \times \Theta_2 \times \Theta_3$ denote the unrestricted parameter space, where Θ_i is the parameter space for θ_i, $i = 1, 2, 3$. Next, let $\Theta_1^p = \{\theta_1 \in \Theta_1 : g(\theta_1) = 0\}$, the restricted parameter space corresponding to the periodic structure, and let $\Theta_0^r = \{0\}$, the restricted parameter space corresponding to the hypothesis of $(4 - r)$ unit roots. Similarly, define

$$\Theta^p = \Theta_1^p \times \Theta_2 \times \Theta_3, \quad \Theta^r = \Theta_1 \times \Theta_2 \times \Theta_3^r,$$
$$\Theta^{pr} = \Theta^p \cap \Theta^r = \Theta_1^p \times \Theta_2 \times \Theta_3^r. \tag{A.16}$$

Johansen's likelihood ratio statistic for the hypothesis of $(4 - r)$ unit roots in the unrestricted VAR may be expressed as

$$\mathrm{LR}(\Theta^r \mid \Theta) = -2(\max_{\theta \in \Theta^r} \log L(\theta) - \max_{\theta \in \Theta} \log L(\theta)), \tag{A.17}$$

with $L(\theta)$ the likelihood function. On the other hand, the LR statistic for $(4 - r)$ unit roots in the periodic autoregression is given by

$$\mathrm{LR}(\Theta^{pr} \mid \Theta^p) = \mathrm{LR}(\Theta^r \mid \Theta) + \mathrm{LR}(\Theta^{pr} \mid \Theta^r) - \mathrm{LR}(\Theta^p \mid \Theta). \tag{A.18}$$

Note that the last two terms on the right-hand side are the likelihood ratio statistics for the restriction $g(\theta_1) = 0$, with and without the unit roots imposed. Slightly extending the results of Johansen (1991, Appendix C), it can be shown that $(N^{1/2})(\hat{\theta}_1 - \theta_1)$ is asymptotically independent from $N(\hat{\theta}_2 - \theta_2)$ and $N(\hat{\theta}_3 - \theta_3)$. This implies that

$$\mathrm{LR}(\Theta^{pr} \mid \Theta^p) - \mathrm{LR}(\Theta^r \mid \Theta) = \mathrm{LR}(\Theta^{pr} \mid \Theta^r) - \mathrm{LR}(\Theta^p \mid \Theta) \xrightarrow{\mathrm{P}} 0, \tag{A.19}$$

so that the LR statistic for $(4 - r)$ unit roots in the periodic autoregression is asymptotically equivalent to the LR statistic for $(4 - r)$ unit roots in the unrestricted VAR model. Hence,

$$\mathrm{LR}(\Theta^{pr} \mid \Theta^p) \xrightarrow{\mathrm{d}} \mathrm{trace}\left\{ \int_0^1 (\mathrm{d}W(s) W(s)' \right.$$
$$\left. \times \left(\int_0^1 W(s) W(s)' \, \mathrm{d}s \right)^{-1} \int_0^1 W(s) \, \mathrm{d}W(s)' \right\}, \tag{A.20}$$

where $W(s)$ is a standard$(4 - r)$-dimensional Brownian motion process, see Johansen (1991). Quantiles of this distribution are tabulated in Osterwald–Lenum (1992, Table 0). Extensions to fitted intercepts and linear trends can be proved analogously; see Osterwald–Lenum (1992) for tables with the relevant critical values.

In a similar fashion, we can prove that *with the correct number of unit roots imposed*, the likelihood ratio statistic for restrictions on α has an asymptotic χ^2 distribution under the null hypothesis, whether or not the restriction $g(\theta_1) = 0$ is imposed. Let us denote such restrictions on α by $h(\theta_2) = 0$, and the corresponding restricted parameter space of θ_2 by $\Theta_2^h = \{\theta_2 \in \Theta_2 \colon h(\theta_2) = 0\}$. Likewise, define $\Theta^{phr} = \Theta_1^p \times \Theta_2^h \times \Theta_3^r$, and so on. Then, the likelihood ratio statistic for the restrictions on θ_2 in the periodically integrated AR model satisfies

$$\begin{aligned} \mathrm{LR}(\Theta^{phr} \mid \Theta^{pr}) &= \mathrm{LR}(\Theta^{hr} \mid \Theta^{r}) + \mathrm{LR}(\Theta^{phr} \mid \Theta^{hr}) - \mathrm{LR}(\Theta^{pr} \mid \Theta^{r}) \\ &= \mathrm{LR}(\Theta^{hr} \mid \Theta^{r}) + o_p(1) \xrightarrow{d} \chi^2(m), \end{aligned} \tag{A.21}$$

where m is the dimension of h. The second equality follows from the fact that the LR statistic for the restrictions on θ_1 is independent of whether restrictions have been imposed on α; the limiting χ^2 distribution is proved in Johansen (1991, Appendix C). □

References

Boswijk, H.P., Franses, P.H., 1996. Unit roots in periodic autoregressions. Journal of Time Series Analysis 17, 221–245.

Franses, P.H., 1994. A Multivariate approach to modeling univariate seasonal time series. Journal of Econometrics 63, 133–151.

Franses, P.H., Paap, R., 1994. Model selection in periodic autoregressions. Oxford Bulletin of Economics and Statistics 56, 421–439.

Franses, P.H., Romijn, G., 1993. Periodic integration in quarterly UK macroeconomic variables, International Journal of Forecasting 9, 467–476.

Ghysels, E., Hall, A., Lee, H.S., 1996. On periodic structures and testing for seasonal unit roots. Journal of the American Statistical Association, 91, 1551–1559.

Haldrup, N., 1994a. The asymptotics of single equation cointegration regression models with $I(1)$ and $I(2)$ variables. Journal of Econometrics 63, 153–181.

Haldrup, N., 1994b. Semi-parametric tests for double unit roots. Journal of Business and Economic Statistics 12, 109–122.

Hylleberg, S., Engle, R.F., Granger, C.W.J., Yoo, B.S., 1990. Seasonal integration and cointegration. Journal of Econometrics 44, 215–238.

Johansen, S., 1988. Statistical analysis of cointegration vectors. Journal of Economic Dynamics and Control 12, 113–125.

Johansen, S., 1991. Estimation and hypothesis testing of cointegration vectors in Gaussian vector autoregressive models. Econometrica 59, 1551–1580.

Johansen, S., 1992a. A representation of vector autoregressive processes integrated of order 2. Econometric Theory 8, 188–202.

Johansen, S., 1992b. Determination of cointegration rank in the presence of a linear trend. Oxford Bulletin of Economics and Statistics 54, 383–397.

Lütkepohl, H., 1991. Introduction to Multiple Time Series Analysis, Springer, Berlin.

Osborn, D.R., 1988. Seasonality and habit persistence in a life-cycle model of consumption. Journal of Applied Econometrics 3, 255–266.

Osterwald-Lenum, M., 1992. A note with quantiles of the asymptotic distribution of the maximum likelihood cointegration rank test statistics: four cases. Oxford Bulletin of Economics and Statistics 54, 461–471.

Pantula, S.G., 1989. Testing for unit roots in time series data. Econometric Theory 5, 256–271.

Park, J.Y., Phillips, P.C.B., 1989. Statistical inference in regressions with integrated processes: Part 2. Econometric Theory 5, 95–131.

Phillips, P.C.B., Durlauf, S.N., 1986. Multiple time series regression with integrated processes. Review of Economic Studies 53, 473–495.

Tiao, G.C., Grupe, M.R., 1980. Hidden periodic autoregressive-moving average models in time series data. Biometrika 67, 365–373.