TECHNOLOGICAL FORECASTING AND SOCIAL CHANGE 45, 287-297 (1994)

Gompertz Curves with Seasonality

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ABSTRACT

This paper considers an extension of the usual Gompertz curve by allowing the parameters to vary over the seasons. This means that, for example, saturation levels can be different over the year. An estimation and testing method is proposed and illustrated with an example.

Introduction

A Gompertz trend curve is typically used for forecasting market development. This curve can be particularly useful for many empirical cases as it assumes an asymmetric growth pattern, that is, the period of the introduction stage and the takeoff and growth stage is shorter than the period of the maturity stage, in which there is a decreasing growth rate. This in contrast to, for example, the logistic trend curve, which assumes a symmetric growth pattern. See Meade [1] for a survey of the Gompertz and other trend curves.

The standard Gompertz curve is characterized by three parameters, which obviously is one of the reasons why it is often applied in practice. However, sometimes there can be some doubt whether three parameters is enough, especially when one is interested in forecasting the quarterly or monthly market development, for example. This is because there can be products for which the demand in the fourth quarter is higher than in other quarters because of Christmas, that is, for which the saturation level varies over the seasons. Alternatively, there may be consumer goods that were originally used as gifts in some seasons, but for which after some years the sales are equally distributed throughout the year. The latter goods may show seasonally varying inflexion points, and hence varying growth parameters. If a Gompertz curve with nonseasonal parameters is fitted to sales series with such seasonal patterns, one can expect that these parameters will be inappropriately estimated. This suggests that for some cases it may be worthwhile to consider a Gompertz trend curve with parameters that are allowed to vary over the seasons. The present paper focuses on such a model.

In the next section the general model is discussed, and the effects of allowing the parameters to vary over the seasons are illustrated with several figures. Section 3 deals with the estimation of the model, and also with a strategy to test whether the parameters

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 0040-1625/94/$7.00

are varying indeed. Section 4 illustrates the model with an example. The final section contains a discussion of the results and of possible extensions.

Gompertz Curves with Seasonality

The mathematical representation of a process X_t that can be characterized by a Gompertz curve is

$$X_t = \alpha.\exp(-\beta.\exp(-\gamma.t)) \tag{1}$$

where α, β, and γ are positive valued unknown parameters. The α denotes the saturation level, and t is a deterministic trend, $t = 0,1,2,3. \ldots$ The starting value of X_t is $X_0 = \alpha.\exp(-\beta)$.

When X_t is a seasonally observed time series, one has s observations per year. Denote the series containing the observations in season s when they are viewed as separate annual series by X_{sT}, where T refers to the annual observation interval. From (1) it can be seen that the Gompertz curves for the annual series X_{sT} are similar to the model for the seasonally observed series X_t. This fact is illustrated in Figure 1 for the case where α is 100, β is 4, and γ is 0.02. The first part of the figure displays the pattern of (1), and the second part the patterns of X_{sT} when s is set equal to 4.

An extension of (1), in which all parameters are allowed to vary over the seasons, can be represented by

$$X_t = \alpha_s.\exp(-\beta_s.\exp(-\gamma_s.t)) \tag{2}$$

where the index s reflects that the parameters can obtain different values in different seasons. The model in (2) has $3s$ unknown parameters to be estimated. To have an impression of the effects of seasonally varying parameters, let us first consider Figure 2 for the case of seasonally varying saturation levels. It can be seen that a series with increasing seasonal variation is obtained. Secondly, consider Figure 3 in which the graphs of X_t and X_{sT} are given when (2) is the data generating process with $\alpha_s = \alpha$ and $\gamma_s = \gamma$. From these graphs it can be observed that a seasonally varying β parameter causes, in some seasons, that the point of inflexion is reached in an earlier stage than in other seasons, that in the first few years the process can look like a process with increasing seasonal variation, and that the observations converge to a common saturation level. The effects of a varying β are similar to the effects of a varying γ parameter. The latter can be seen from Figure 4, in which (2) is depicted with $\alpha_s = \alpha$ and $\beta_s = \beta$. The graphs in Figures 1 through 4 share the characteristic that the annual series with the observations per season do not intersect. When the model in (1) is considered with all parameters seasonally varying, one can infer from Figure 5 that it can then occur that the ranking of the annual series can change, and hence that, for example, "summer can become winter."

Summarizing, a Gompertz trend curve with parameters that are allowed to vary over the seasons as in (2) can describe a rich class of possible patterns of seasonally observed market development. Of course, in practice one may want to test whether the parameters are indeed varying.

Estimation and Testing

A well-known estimation method of the parameters in (1), which may be extended to estimating the time-varying parameters in (2), is the three-point method, see, for example, Granger [2]. This method constructs three data points by taking some average of the first, middle, and the last observations. Given the corresponding values of trend t, one can insert these three data points in (1) to obtain three equations with three unknown

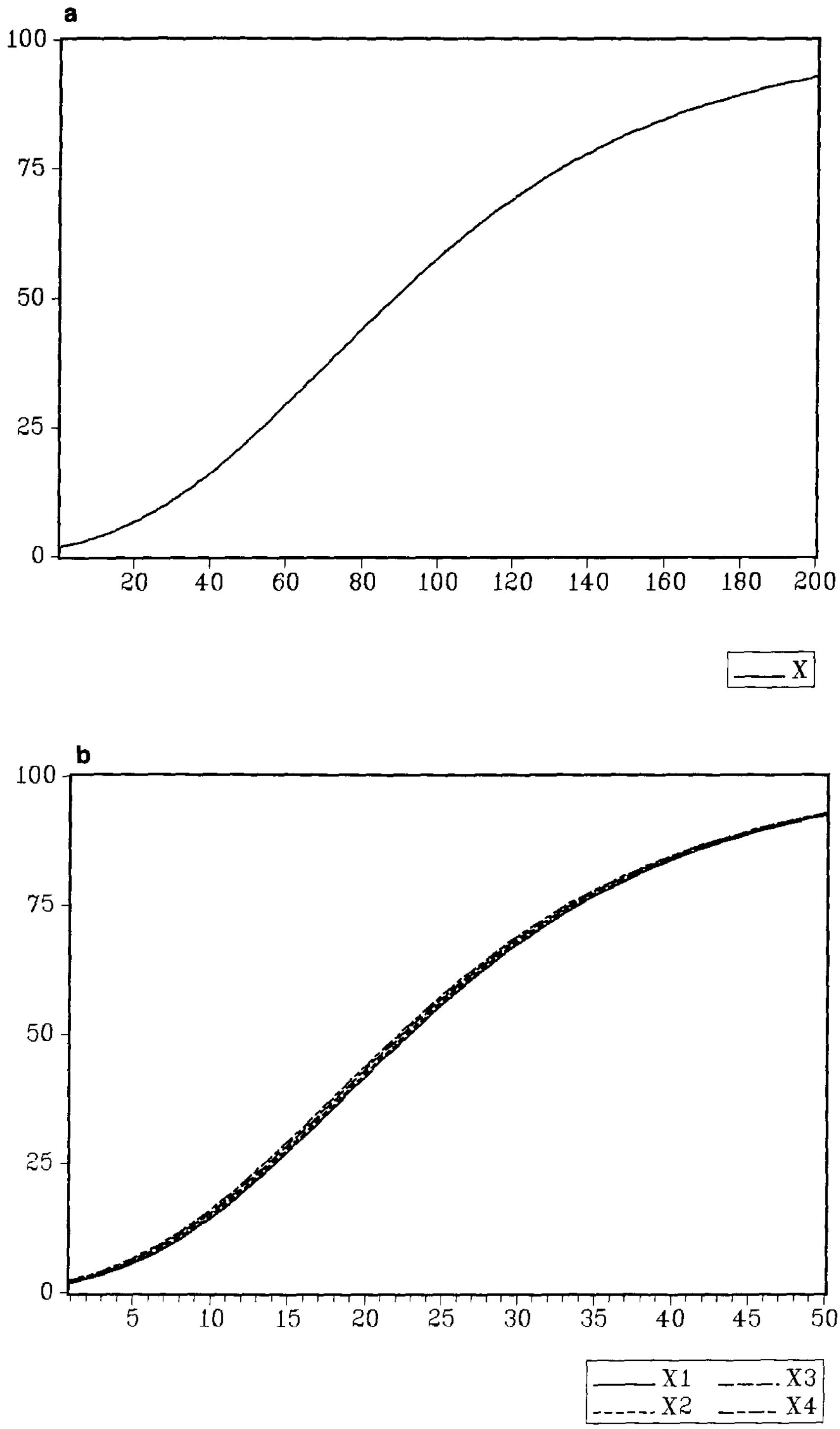


Fig. 1. (A) A Gompertz curve with constant parameters: $X_t = 100\exp(-4\exp(-0.02t))$. The quarterly observations X_t. (B) A Gompertz curve with constant parameters: $X_t = 100\exp(-4\exp(-0.02t))$. The annual observations per quarter, X_{sT}, where $s = 1,2,3,4$.

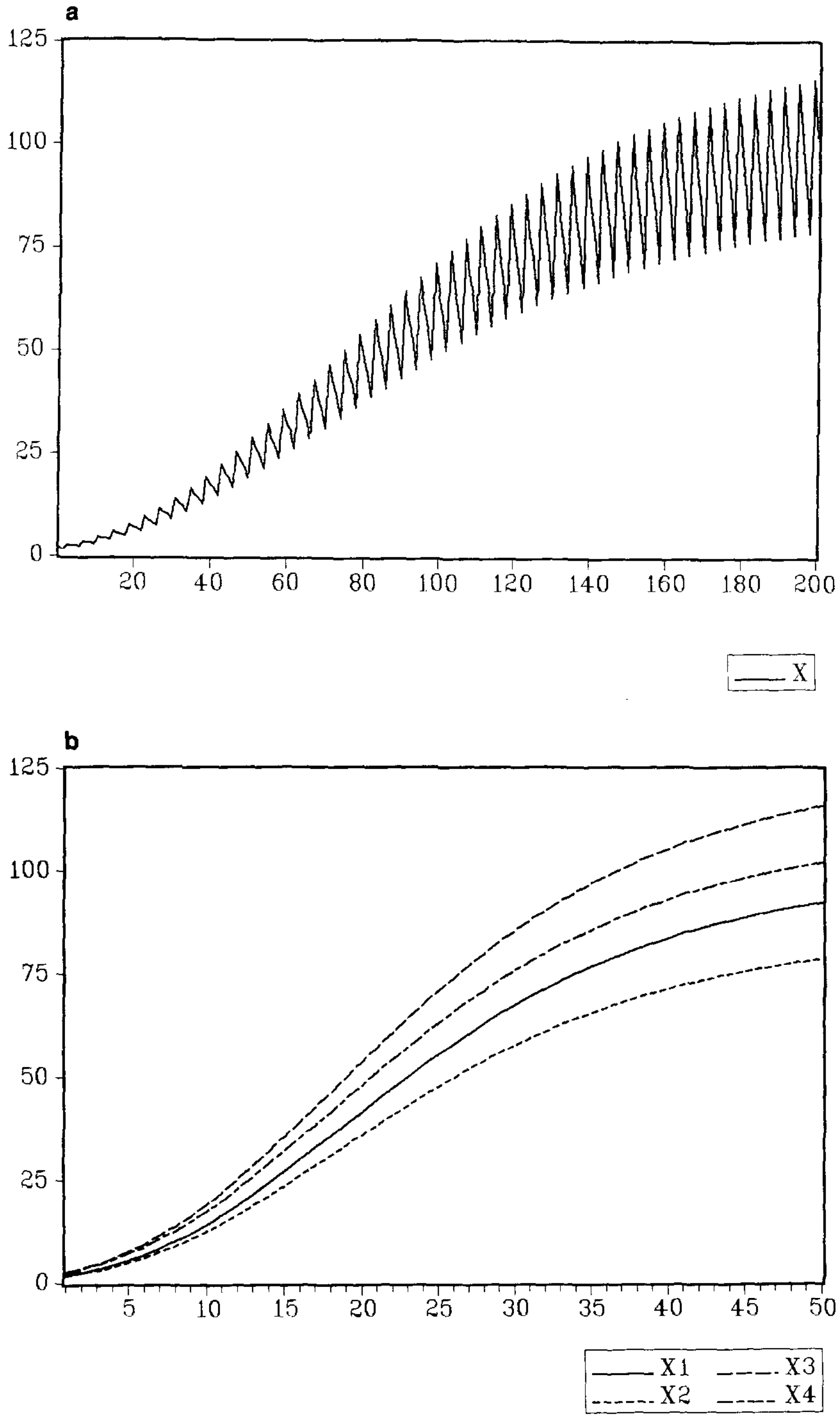


Fig. 2. (A) A Gompertz curve with a time-varying saturation level $X_t = \alpha_s(-4\exp(-0.02t))$, with $\alpha_1 = 100$, $\alpha_2 = 85$, $\alpha_3 = 125$ and $\alpha_4 = 110$. The quarterly observations, X_t. (B) A Gompertz curve with a time-varying saturation level $X_t = \alpha_s(-4\exp(-0.02t))$, with $\alpha_1 = 100$, $\alpha_2 = 85$, $\alpha_3 = 125$, and $\alpha_4 = 110$. The annual observations per quarter, X_{sT}, where $s = 1,2,3,4$.

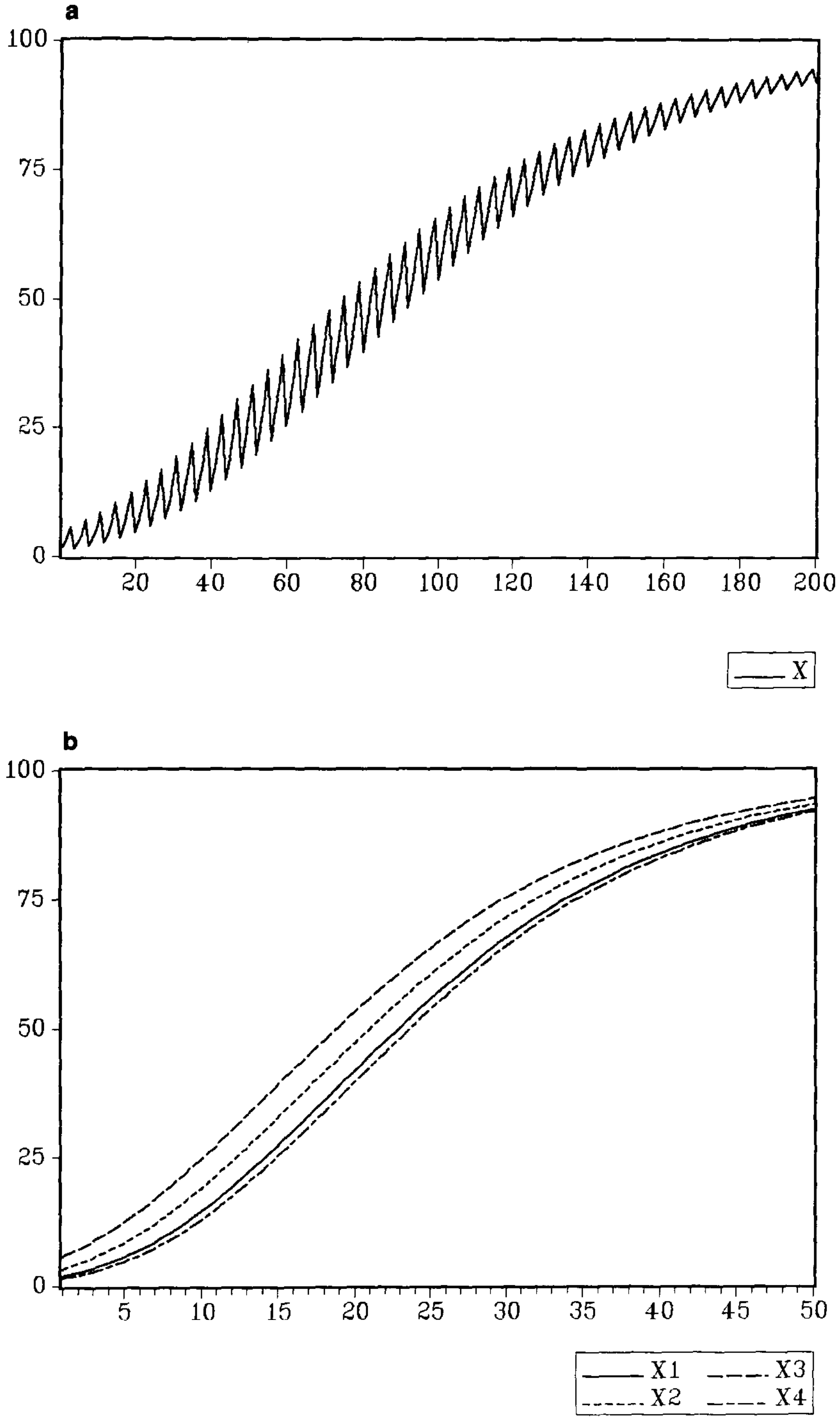


Fig. 3. **(A) A Gompertz curve with a time-varying β parameter: $X_t = 100[-\beta_s \exp(-0.02t)]$, with $\beta_1 = 4$, $\beta_2 = 3.5$, $\beta_3 = 3$, and $\beta_4 = 4.5$. The quarterly observations, X_t. (B) A Gompertz curve with a time-varying β parameter: $X_t = 100(-\beta_s \exp(-0.02t))$, with $\beta_1 = 4$, $\beta_2 = 3.5$, $\beta_3 = 3$, and $\beta_4 = 4.5$. The annual observations per quarter, X_{sT}, where $s = 1,2,3,4$.**

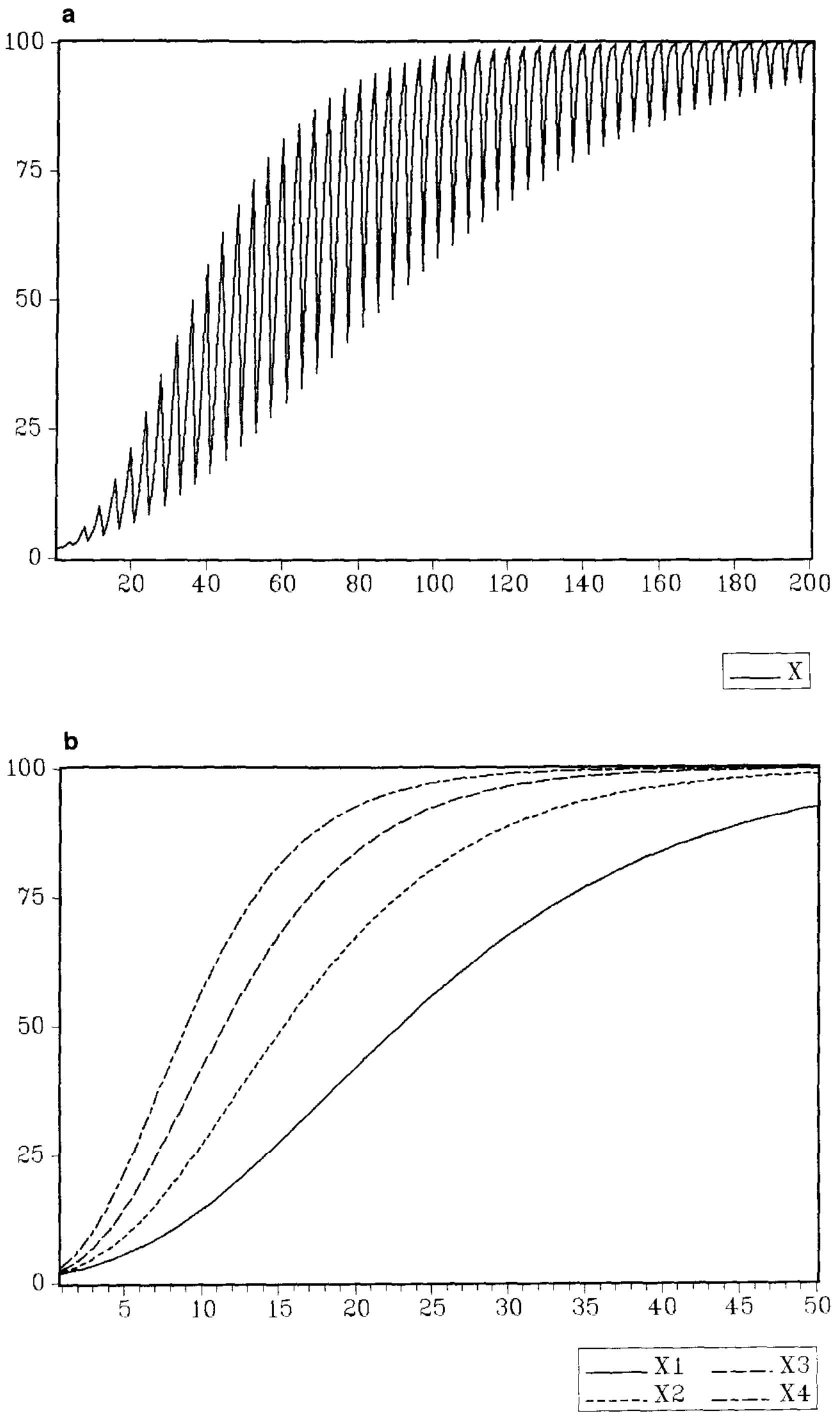


Fig. 4. (A) A Gompertz curve with a time-varying γ parameter: $X_t = 100(-4\exp(-\gamma_s t))$, with $\gamma_1 = 0.02$, $\gamma_2 = 0.03$, $\gamma_3 = 0.04$, and $\gamma_4 = 0.05$. The quarterly observations, X_t. (B) A Gompertz curve with a time-varying γ parameter: $X_t = 100(-4\exp(-\gamma_s t))$, with $\gamma_1 = 0.02$, $\gamma_2 = 0.03$, $\gamma_3 = 0.04$, and $\gamma_4 = 0.05$. The annual observations per quarter, X_{sT}, where $s = 1,2,3,4$.

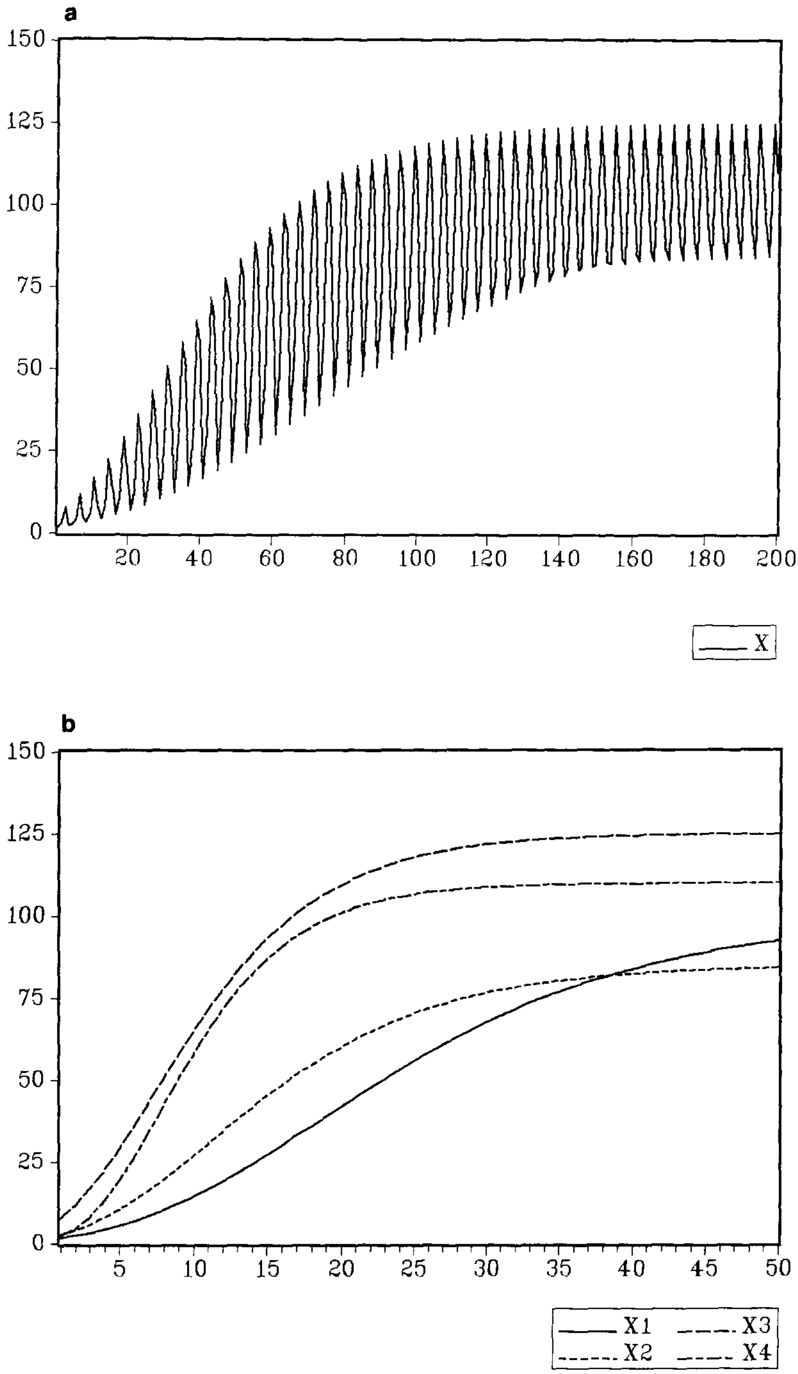


Fig. 5. **(A) A Gompertz curve in which all parameters are time-varying: $X_t = \alpha_s[-\beta_s\exp(-\gamma_s t)]$, with $\alpha_1 = 100$, $\alpha_2 = 85$, $\alpha_3 = 125$, $\alpha_4 = 110$, $\beta_1 = 4$, $\beta_2 = 3.5$, $\beta_3 = 3$, $\beta_4 = 4.5$, and $\gamma_1 = 0.02$, $\gamma_2 = 0.03$, $\gamma_3 = 0.04$, $\gamma_4 = 0.05$. The quarterly observations, X_t. (B) A Gompertz curve in which all parameters are time-varying: $X_t = \alpha_s[-\beta_s\exp(-\gamma_s t)]$, with $\alpha_1 = 100$, $\alpha_2 = 85$, $\alpha_3 = 125$, $\alpha_4 = 110$, $\beta_1 = 4$, $\beta_2 = 3.5$, $\beta_3 = 3$, $\beta_4 = 4.5$, and $\gamma_1 = 0.02$, $\gamma_2 = 0.03$, $\gamma_3 = 0.04$, $\gamma_4 = 0.05$. The annual observations per quarter, X_{sT}, where $s = 1,2,3,4$.**

parameters. For a series X_t that is presumed to be generated by a model as (2), the use of this method amounts to constructing three data points for each of the annual series X_{sT}. A drawback of this method, however, is that it will be difficult to test whether the parameters are indeed seasonally varying. Therefore, we have to rely on an alternative estimation method.

Recently, Franses [3] proposed a simple method to estimate the three parameters in a standard Gompertz process. This method is based on rewriting (1) as

$$\log X_t = \log\alpha - \beta.\exp(-\gamma.t) \qquad (3)$$

where log denotes the natural logarithm. Taking the first order difference of $\log X_t$ gives

$$\log X_t - \log X_{t-1} = \beta\exp(-\gamma t)[1 - \exp(\gamma)] \qquad (4)$$

Rewriting (4) and taking a natural logarithm again yields

$$\log(\log X_t - \log X_{t-1}) = \gamma t + \log(-\beta + \beta\exp(\gamma)) \qquad (5)$$

The parameters β and γ in this linear model, when an error term is added, can now be estimated by applying nonlinear least squares. A range of α values can be estimated using

$$\hat{\alpha}_t = \exp(\log X_t + \hat{\beta}\exp(-\hat{\gamma}.t)) \qquad (6)$$

One can then opt for the mean of $\hat{\alpha}_t$ as an estimate for α. Note from (5) that forecasts for X_t can be made without knowing the value of α.

This method can be extended to model (2) in a straightforward way. First, rewrite (2) as

$$\log X_t = \log\alpha_s - \beta_s.\exp(-\gamma_s.t) \qquad (7)$$

Taking the sth order differences of this $\log X_t$ gives

$$\log X_t - \log X_{t-s} = -\beta_s\exp(-\gamma_s.t)[1 - \exp(\gamma_s.S)] \qquad (8)$$

Rewriting (8) and again taking a natural logarithm yields

$$\log(\log X_t - \log X_{t-s}) = -\gamma_s t + \log(-\beta_s + \beta_s\exp(\gamma_s.s)) \qquad (9)$$

The parameters β_s and γ_s in this equation (9) can now be estimated by applying nonlinear least squares to the regression model

$$\log(\log X_t - \log X_{t-s}) = -\sum_{j=1}^{s} D_{jt}\gamma_j t + \sum_{j=1}^{s} D_{jt}\mu_j + \varepsilon_t \qquad (10)$$

where the D_{st} are s seasonal dummy variables, and where

$$\mu_j = \log(-\beta_j + \beta_j\exp(\gamma_j S)) \qquad (11)$$

A sequence of α_s values is found from

$$\hat{\alpha}_t = \exp(\log X_t + \hat{\beta}_s\exp(-\hat{\gamma}_s.t)) \qquad (12)$$

and the mean of $\hat{\alpha}_t$ in each of the season can be considered to be an estimate of α_s.

In practical occasions it may occur that $\log X_t - \log X_{t-s}$ is negative or zero, and hence that the natural logarithm of $\log X_t - \log X_{t-s}$ cannot be calculated. In that case the corresponding observations can be treated as missing observations, and (10) can be estimated for the remaining sample.

The above estimation method for the parameters in (2) is particularly useful as it facilitates testing for the significance of the differences between the parameters over the seasons. A Likelihood Ratio (LR) test for the equality of the γ_s, to be denoted as LR($\gamma_s = \gamma$), can be constructed by comparing the log likelihood of (10) with that of the regression

$$\log(\log X_t - \log X_{t-s}) = -\gamma t + \sum_{j=1}^{s} D_{jt}\mu_j + \varepsilon_t \tag{13}$$

Under the null hypothesis (13), this LR($\gamma_s = \gamma$) is asymptotically $\chi^2(s-1)$ distributed. Similarly, a LR($\beta_s = \beta$) can be constructed. Further, a LF($\gamma_s = \gamma, \beta_s = \beta$) can be constructed by comparing (10) with the regression

$$\log(\log X_t - \log X_{t-s}) = -\gamma t + \mu + \varepsilon_t \tag{14}$$

Finally, a test for the equality of the α_s is constructed from the regression

$$\varphi(B)\hat{\alpha}_t = \kappa + \sum_{j=1}^{s-1} D_{jt}\kappa_j + \psi_t \tag{15}$$

where $\varphi(B)$ is an autoregressive polynomial, which is used to whiten the error process. There is seasonality in the α values if the relevant F test statistic for the hypothesis that the three κ_j are equal to zero exceeds some critical value. The $\hat{\alpha}_{st}$ can then be found by regressing $\hat{\alpha}_t$ on s seasonal dummies.

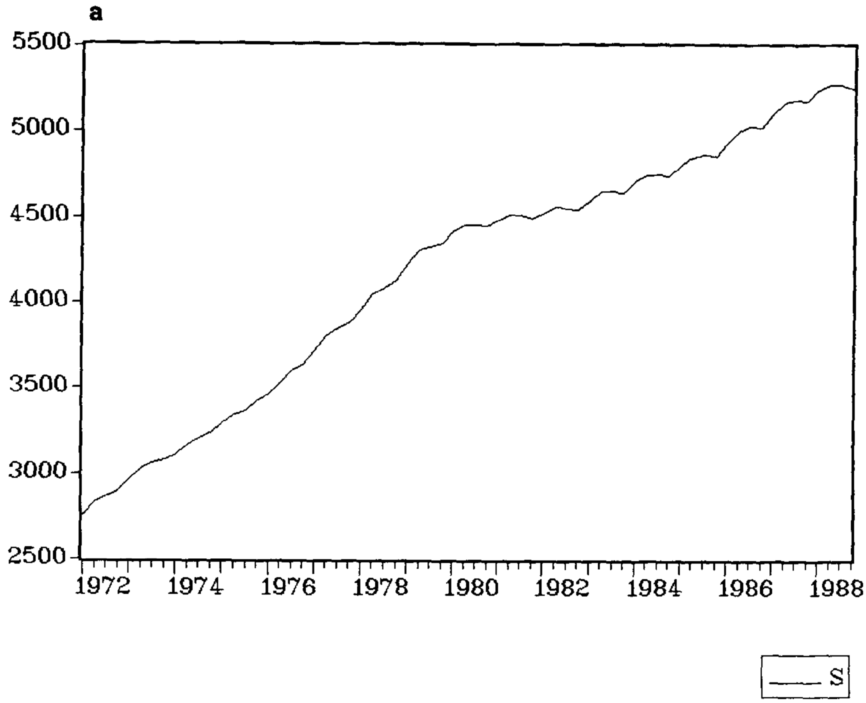
An Illustration

To illustrate the practical relevance of the Gompertz trend curve with varying parameters as in (2), and of the estimation method in the previous section, I consider the Dutch quarterly stock of cars series, 1972.1–1988.4, as it is depicted in Figure 6. In Franses [3] it has already been found that the annual stock of cars series can adequately be described with a Gompertz process. To have an impression whether there is some seasonality in the series, it can be instructive to have a look at the first differenced time series $X_t - X_{t-1}$, plotted for each of the seasons. In Figure 6b, the graphs of the $X_{sT} - X_{s,T-1}$ are depicted for s equal to 2, 3, and 4. It is clear that the differences between the seasons is small until about 1977, but that after that period the differences converge to some constant that varies with the seasons. This suggests that there may be seasonality in the Gompertz curve parameters for these quarterly data.

The model in (10) is estimated and the tests for parameter restrictions LR($\gamma_s = \gamma$), LR($\beta_s = \beta$) and LR($\gamma_s = \gamma, \beta_s = \gamma$) obtain values of 0.013, 0.009, and 0.016, respectively. The final estimation results are $\hat{\gamma}_s = \hat{\gamma} = 0.021$ and $\hat{\beta}_s = \hat{\beta} = 0.786$. The s saturation levels, however, do seem to be seasonally varying. To whiten the errors, an autoregression of order 2 with seasonal dummies should be fitted to the $\hat{\alpha}_t$ series. The $F(3,62)$ test statistic for the hypothesis that the κ_j in (15) are equal to zero has a value of 32.461. Hence, there seems to be evidence for the presence of seasonally varying saturation levels. The estimated seasonal saturation levels ($\times 1000$) are 6258, 6292, 6272, and 6238, for the quarters 1 through 4, respectively.

Discussion

In this paper an extension of the well-known Gompertz trend curve is proposed and applied. This extension allows the parameters to vary over the seasons. Because the conventional three-point estimation method has the drawback that it is difficult to verify whether the parameters are indeed seasonally varying, an alternative estimation and testing approach has been proposed. Application of this method to the Dutch stock of cars series yields that this series shows seasonally varying saturation levels.



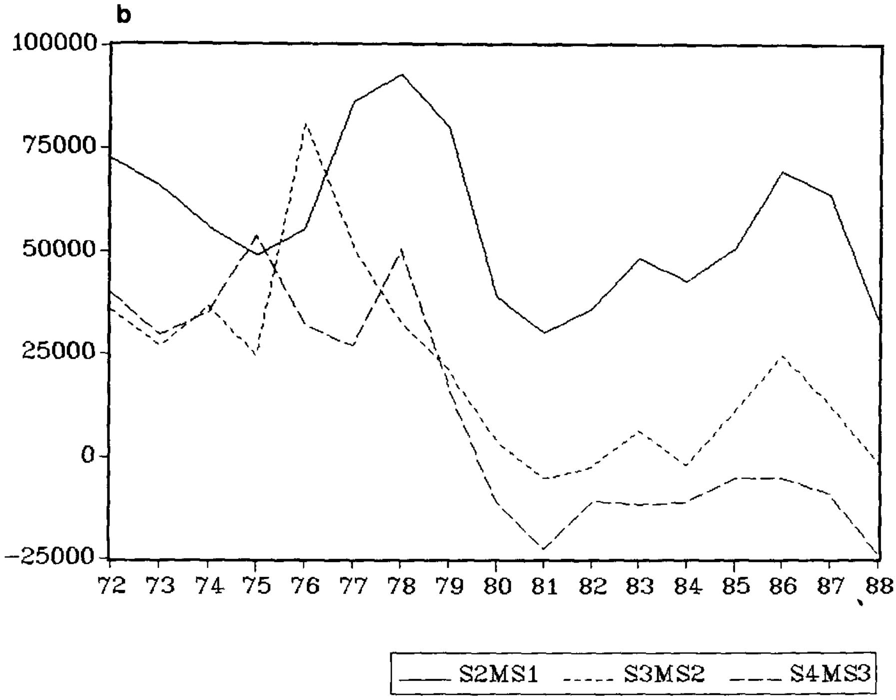


Fig. 6. (A) The quarterly stock of cars in The Netherlands, 1972.1–1988.4 (× 1000). The quarterly observations, X_t. (B) The quarterly stock of cars in The Netherlands, 1972.1–1988.4. The annual observations per quarter, $X_{sT} - X_{s-1T}$, where $s = 2,3,4$.

The example in the present paper extends the Gompertz curve in only one direction, that is, seasonally varying saturation levels. Future applications in, for example, the area of tourism and the flower industry, may indicate that other parameters may also be nonconstant over the seasons. A straightforward theoretical extension of the material in this paper is found in the investigation and application of estimating techniques in cases of other trend curves, such as the logistic curve, which have periodically varying parameters.

A previous version of this paper was presented at the International Symposium on Forecasting 1992 in Wellington, New Zealand, and at Curtin University in Perth, Australia. Helpful comments from participants, from Nigel Meade, and from an anonymous referee are gratefully acknowledged. Thanks are due to the Royal Netherlands Academy of Arts and Sciences for their financial support.

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Received December 8, 1992