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A Method to Select Between Gompertz and Logistic Trend Curves

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ABSTRACT

In this paper a simple method is proposed to select between two often applied trend curves; the Gompertz and the logistic curve. The method is based on one auxiliary regression. Two applications illustrate its merits.

Introduction

The Gompertz and logistic trend curves are often applied in forecasting market development, see Gregg et al. [1] and Meade [2] for extensive overviews. Although these curves can describe similar behavior in some phases of this development, one of the most important differences is that the Gompertz process is asymmetric, whereas the logistic curve is a symmetric process. Therefore, using an inappropriate growth curve can have a substantial impact on forecasting.

Though the selection of an appropriate curve appears to be important, the choice between the two models is usually made using criteria based on forecasting errors, on the plausibility of the estimated saturation levels, or on visual evidence obtained from depicting the data points in a particular way, see for example, Gregg et al. [1], Young and Ord [3] *inter alia*. In this paper, I propose an alternative selection method, which is based on one auxiliary regression, and on a significance test for one parameter.

In the next section, some aspects of the Gompertz and logistic curve are discussed, and the simple selection method is explained. The differences of this method with other selection methods are highlighted as well. In section 2, the new approach is applied to two empirical series to illustrate its merits. The third section concludes this paper.

1. A Selection Method

The Gompertz trend curve for a time series X_t is given by

$$X_t = a_1 \cdot \exp(-b_1 \cdot \exp(-c_1 t)), \qquad (1)$$

where t represents time, and where a_1 is the saturation level and $a_1, b_1, c_1 > 0$, or by

$$\log(-\log(X_t/a_1)) = \log b_1 - c_1 t \qquad (2)$$

where log denotes the natural logarithm. The logistic curve for X_t can be written as

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$$X_t = a_2 \cdot (1 + b_2 \cdot \exp(-c_2 t))^{-1}, \tag{3}$$

where $a_2, b_2, c_2 > 0$, or

$$\log((a_2 - X_t)/X_t) = \log b_2 - c_2 t \tag{4}$$

Recently, in Franses [4], a simple estimation method for the Gompertz curve has been developed and applied. This method uses the fact that the model in (1) can be rewritten as

$$\log(\Delta \log X_t) = d_1 - c_1 t, \tag{5}$$

where Δ is the differencing filter defined by $\Delta z_t = z_t - z_{t-1}$, and where d_1 is a nonlinear function of b_1 and c_1. This transformation can be made irrespective of the value of a_1. Note that in practice values of $\Delta \log X_t$ can be negative, and hence that the additional log transformation is then not appropriate. One way of dealing with such observations is to replace them by interpolated observations. To treat the corresponding observations as missing is an alternative strategy. Of course, when many observations have negative $\Delta \log X_t$ values, one may question the adequacy of a Gompertz or logistic curve in the first place. Interpolation or smoothing may also be useful when successive observations are almost equal, see, for example, Franses [4].

The model for $\log(\Delta \log X_t)$ in (5) is linear, and it seems worthwhile to investigate whether the logistic model in (3) can be rewritten analogously. A first step is to take logs of both sides of (3), and to apply the differencing filter Δ, which results in

$$\begin{aligned} \Delta \log X_t &= \log[(1 + b_2 \exp(-c_2 t + c_2))/(1 + b_2 \exp(-c_2 t))] \\ &\approx b_2 \exp(-c_2 t) \cdot (\exp c_2 - 1)/(1 + b_2 \exp(-c_2 t)) \end{aligned} \tag{6}$$

since the latter expression only obtains values between 0 and 1. Taking logs of both sides of (6) and some rewriting yields

$$\log(\Delta \log X_t) \approx d_2 - c_2 t + (\log X_t - \log a_2) \tag{7}$$

where d_2 is a nonlinear function of b_2 and c_2. An expression related to (7) can be found in Harvey [5, eq. (7)].

Typical graphs of the $\log(\Delta \log X_t)$ series for the Gompertz and logistic curves are depicted in Figure 1.

Obviously, one may find values of a_1, a_2, b_1, b_2, c_1 and c_2 for which these graphs look similar. Hence, one may not want to rely on visual evidence only. From (7) it is clear that for the logistic curve, the $(\log X_t - \log a)$ element in (7) ensures that $\log(\Delta \log X_t)$ is a nonlinear function of time. A simple parametric selection method between (1) and (3) may therefore be given by the auxiliary regression

$$\log(\Delta \log X_t) = \delta + \gamma t + \tau t^2, \tag{8}$$

and a test for the significance of the τ parameter based on its t ratio. Of course, one may also want to consider variables like t^{-1} or $t^{-1/2}$ instead of t^2. Even when having seemingly similar patterns in graphs like those in Figure 1, one can expect that the test based on τ in (8) can be a powerful selection tool. Finally, the models as they are expressed in Harvey [5] can be written in a similar way for model selection purposes.

The selection method between a Gompertz and a logistic curve based on (8) uses all the in-sample observations. Hence, no observations are lost because of out-of-sample forecasting performance evaluation. This may be important in several practical occasions since we usually have to rely on small samples. A graphical device as Figure 1 can also

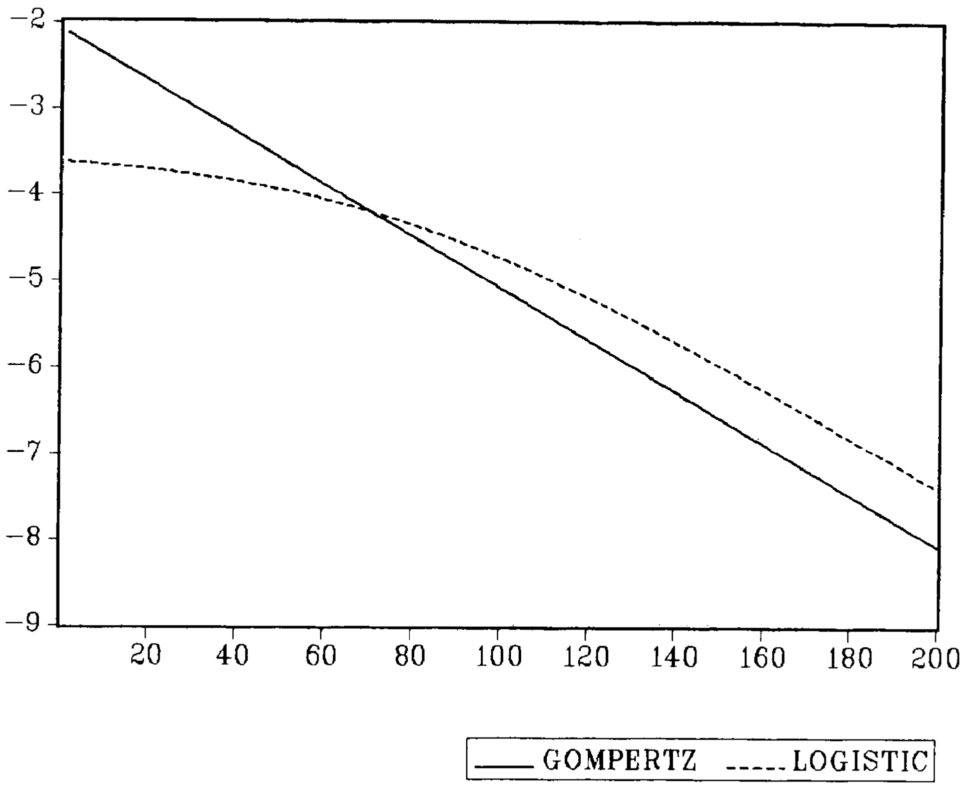

Fig. 1. Log($\log X_t - \log X_{t-1}$) for a Gompertz and a logistic process.

be helpful. Gregg et al. [1] and Young and Ord [3] propose to draw graphs for the $\log(-\log(X_t/a_1))$ transformed time series or for the $\log((a_2-X_t)/X_t)$ transformed series to see whether for the X_t observations the Gompertz or the logistic curve may be adequate, respectively. One drawback of this method is that we have to rely on visual evidence only, although one may consider regressions like (8) to gain some insights. A second and more important drawback is that one has to know the values of a_1 and a_2 in advance. This may be the case for some applications, but for others it is precisely these a_1 and a_2 one wants to estimate.

Lee and Lu [6] propose to consider data-based transformed models, which generalize each of the growth curves. A Box-Cox type of approach can then be used to select between (1) and (3). Again, the value of a_1 or a_2 is assumed to be known. Further, this approach introduces an additional Box-Cox parameter to be estimated from the same set of observations.

Finally, one may argue that a coefficient of determination can be useful for model selection. The models in (1) and (3) and in (2) and (4) are, however, nonnested, and this may complicate a straightforward comparison of these R^2 coefficients. In a small Monte Carlo experiment the method in (8) is evaluated when either (2) or (4) with the inclusion of a random error term is the data generating process. The (unreported) results indicate that the empirical size can be somewhat low, that is, below the nominal level, whereas the empirical power of the test is close to 100%.

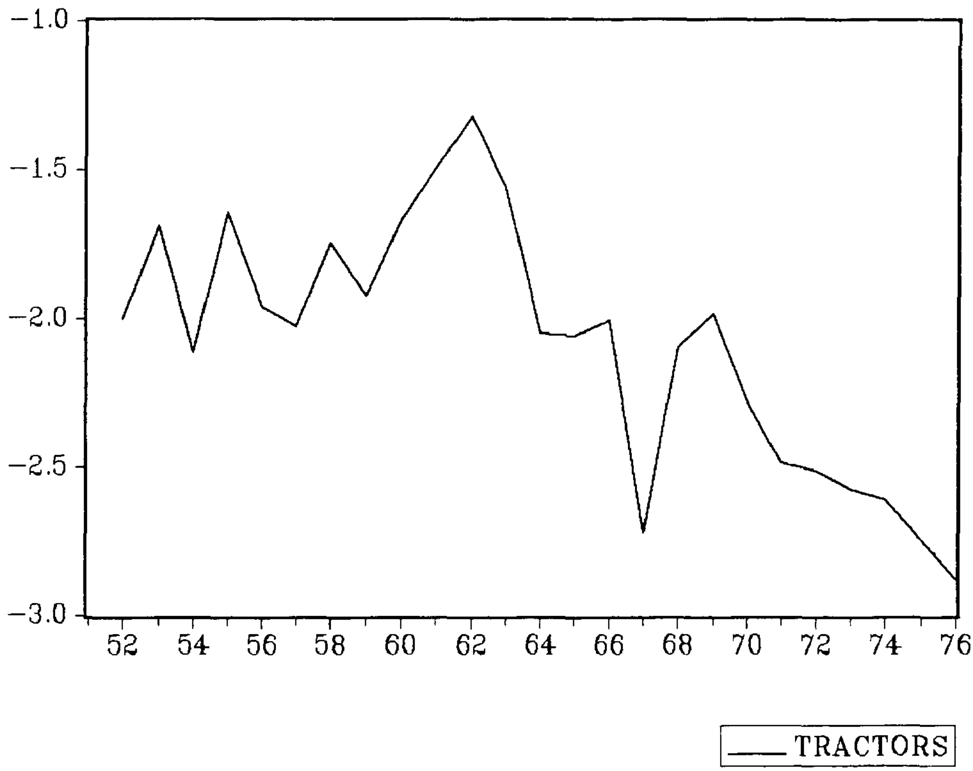

Fig. 2. Log($\log X_t - \log X_{t-1}$) for the tractors in Spain.

2. Applications

To illustrate the merits of the proposed model selection method, I consider two examples. The first is taken from Mar-Molinero [7], and it concerns the tractors in Spain data series. In that paper and in Meade [2], it has been argued that a logistic curve fits these data best, see also Oliver [8] and Harvey [5]. This conjecture can be verified by looking at the graph of the $\log(\Delta \log X_t)$ series in Figure 2, although we have to interpret this figure with caution.

Anyhow, this figure seems similar to that for the logistic curve in Figure 1. The t ratio of the τ parameter in the regression as (8) obtains a value of -3.740, which is significant at a 5% level.

The second example is given by the Dutch annual (smoothed) stock of cars series, as it is analysed in Franses [4], where a Gompertz curve has been fitted to this series. The graph of the $\log(\Delta \log X_t)$ series is depicted in Figure 3, and there is some visual evidence that a model like (5) may indeed be appropriate.

The t value of the τ parameter is 1.031, which is not significant at a 10% level, and hence the Gompertz curve seems indeed appropriate for the stock of cars series.

3. Conclusion

In this paper a simple parametric method is proposed to choose between a Gompertz and a logistic trend curve. Two examples indicate its practice use. Moreover, it seems possible to extend this method to other types of trend curves.

Fig. 3. **Log(logX_t − logX_{t-1}) for the (smoothed) stock of cars in the Netherlands.**

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