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MODEL SELECTION IN PERIODIC AUTOREGRESSIONS†

Philip Hans Franses and Richard Paap

I. INTRODUCTION

Periodic autoregressive time series models [PAR] are models which allow the AR parameters to vary with the seasons, see e.g., Gladyshev (1961), Pagano (1978) and Troutman (1979). Periodic autoregressions, which allow for the presence of stochastic trends, can be useful in economics since such models can describe time series in which trends, cycles and seasons may be related, see Franses (1992a) and Ghysels (1993). Possible economic motivations for time-varying parameters models like PARs are that economic agents may have seasonally varying utility functions (Osborn, 1988), seasonally varying expectations (Franses, 1992b), and/or periodic adjustment costs. Further, Ghysels (1992) documents that the probability of getting out a recession seems to be unequally distributed over the seasons. Moreover, institutional causes may establish that observations on a variable in some season have more impact on the dynamic pattern of this variable than observations in other seasons.

In this paper we focus on the issue of PAR model selection in practice. One aspect of model selection is the choice for the appropriate PAR order. This can be of interest for the evaluation of economic models, see e.g., Osborn (1988), where economic theory prescribes that consumption follows a PAR(1) process. Further, the appropriate PAR order is important for an adequate empirical application of tests for unit roots since too many parameters affect the performance of such tests. In fact, another aspect of PAR model selection is the decision on the number of unit roots. Finally, in case of unit roots, model choice involves a decision on the most suitable differencing filter to ensure (periodic) stationarity of the transformed series.

The outline of the paper is as follows. In section II, we discuss the notation and model representation to be used throughout the paper. In section III, we propose an empirical model selection strategy for PAR processes. In section

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IV we evaluate each of its steps using Monte Carlo replications. In section V, we apply our strategy to several quarterly UK macroeconomic series. In section VI, we conclude with some remarks.

II. NOTATION AND REPRESENTATION

The simplest PAR model which is most suitable to outline the main concepts in this paper is the PAR(2) process for a quarterly observed time series y_t,

$$y_t = \phi_{1s} y_{t-1} + \phi_{2s} y_{t-2} + \varepsilon_t, \tag{1}$$

where ε_t is a standard white noise process, and where the ϕ_{is} are parameters with values that vary with the seasons, $i=1, 2$, $s=1, 2, 3, 4$. A convenient way to represent (1), see Gladyshev (1961), is by using the vector notation, i.e. the quarterly y_t observations are stacked in the annually observed vector $Y_T = (Y_{1T}, Y_{2T}, Y_{3T}, Y_{4T})$, where Y_{sT} is the observation in season s in year T. The model in (1) can then be written as

$$A_0 Y_T = A_1 Y_{T-1} + \omega_T, \tag{2}$$

with

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_{12} & 1 & 0 & 0 \\ -\phi_{23} & -\phi_{13} & 1 & 0 \\ 0 & -\phi_{24} & -\phi_{14} & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 0 & \phi_{21} & \phi_{11} \\ 0 & 0 & 0 & \phi_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

where ω_T is a (4×1) white noise error process containing the stacked ε_t variables.

The parameters in (1) can be estimated by applying ordinary least squares to

$$y_t = \sum_{s=1}^{4} D_{st}(\phi_{1s} y_{t-1} + \phi_{2s} y_{t-2}) + \varepsilon_t, \tag{4}$$

where D_{st} are seasonal dummy variables. The model in (1) can be enlarged by including seasonally varying trends and constants. Note, however, that the underlying constant or trend does not have to be seasonally varying since, e.g., in the PAR(1) case,

$$y_t - \mu = \phi_{1s}(y_{t-1} - \mu) + \varepsilon_t,$$

where μ is a constant, can be written as

$$y_t = \mu_s + \phi_{1s} y_{t-1} + \varepsilon_t,$$

where $\mu_s = (1 - \phi_{1s})\mu$. This implies that estimating PAR models with a constant and a trend anyhow involves estimating a model like (4) with the inclusion of $\Sigma_{s=1}^{4} D_{st}\mu_s$ and $\Sigma_{s=1}^{4} D_{st}\tau_s t$, where t denotes a deterministic trend.

Below we will say that the regression then contains four constants and four trends.

The vector process Y_T in (2) is stationary if the roots of the characteristic equation

$$|A_0 z - A_1| = z^2(z^2 - [\phi_{22}\phi_{13}\phi_{14} + \phi_{22}\phi_{24} + \phi_{21}\phi_{12}\phi_{13} + \phi_{21}\phi_{23} + \phi_{11}\phi_{12}\phi_{13}\phi_{14}$$
$$\phi_{11}\phi_{12}\phi_{24} + \phi_{11}\phi_{14}\phi_{23}]z + \phi_{21}\phi_{22}\phi_{23}\phi_{24}) = 0 \qquad (5)$$

are inside the unit circle, see also Lütkepohl (1991). When a PAR model for y_t implies a multivariate model for Y_T of order 1 as in (2), the solutions to (5) for z are the eigenvalues of the matrix $A_0^{-1}A_1$. When one of the solutions for z equals one, the following nonlinear parameter restriction is imposed on the PAR(2) model,

$$1 - [\phi_{22}\phi_{13}\phi_{14} + \phi_{22}\phi_{24} + \phi_{21}\phi_{12}\phi_{13} + \phi_{21}\phi_{23} + \phi_{11}\phi_{12}\phi_{13}\phi_{14} + \phi_{11}\phi_{12}\phi_{24}$$
$$+ \phi_{11}\phi_{14}\phi_{23}] + \phi_{21}\phi_{22}\phi_{23}\phi_{24} = 0 \qquad (6)$$

holds true. Note that (6) reduces to $\phi_{11}\phi_{12}\phi_{13}\phi_{14} = 1$ in case of a PAR(1) process.

Using the expressions in (1), (2) and (5), we can characterize how any unit roots in the y_t process are related to those in the Y_T system. For the PAR(2) process in (1) there are three possibilities, i.e. zero, one or two unit roots. For higher order processes, there may be more unit roots. To save space we only consider the PAR(2) process, also since an analysis of PAR(p) models, $p > 2$, proceeds along similar lines. When y_t does not contain a unit root, the vector process Y_T is stationary, and the corresponding y_t process is (periodically) stationary. Note that, strictly speaking, the y_t process is not stationary in case a PAR model can be fitted, since the (autoco-) variances are not constant over time. A process y_t is therefore said to be periodically stationary when the related vector process Y_T does not contain a unit root. When y_t contains a unit root at the zero frequency, i.e. the $(1 - B)$ filter, with $B^k z_t = z_{t-k}$, is required to obtain periodic stationarity, it is clear that the restriction $\phi_{1s} + \phi_{2s} = 1$ holds for (1). In that case the characteristic equation (5) becomes

$$|A_0 z - A_1| = z^2(z - \phi_{21}\phi_{22}\phi_{23}\phi_{24})(z - 1) = 0. \qquad (7)$$

Hence, one zero frequency unit root in y_t carries over to a single unit root in the Y_T process. Note that the latter unit root is also at the zero frequency. The same applies in case y_t has a seasonal unit root -1, which assumes that the $(1 + B)$ filter is appropriate to make the y_t series periodically stationary, see Hylleberg *et al.* (1990) for a discussion of seasonal and nonseasonal unit roots in quarterly time series. The root -1 for the y_t series means that $\phi_{2s} - \phi_{1s}$ is equal to 1 and that the relevant characteristic equation is again (7). Thus, a seasonal unit root in y_t results in a nonseasonal unit root in the Y_T process.

However, in case there is a single unit root in the Y_T process, this root is not necessarily related to the $(1 - B)$ or $(1 + B)$ filter for the y_t series. In fact, in case of one unit root in Y_T there are three linear combinations of the Y_{sT} series which are stationary. These three cointegration relations ensure that the Y_{sT} series do not diverge too much in case of exogenous shocks. Generally, the three cointegration relationships can be written as $Y_{1T}-\alpha_1 Y_{4T}$, $Y_{2T}-\alpha_2 Y_{4T}$, and $Y_{3T}-\alpha_3 Y_{4T}$. In turn, this implies that the appropriate differencing filter for y_t is equal to $(1-\gamma_s B)$ under the restriction that $\gamma_1\gamma_2\gamma_3\gamma_4=1$. When all γ_s are equal to 1, the $(1-B)$ filter emerges, and when they are all equal to -1, the $(1+B)$ filter is needed. Further, when some or all γ_s values are unequal to 1 or -1, one should apply the $(1-\gamma_s B)$ filter to ensure the periodic stationarity of the γ_t process. In the latter case, it is said that γ_t is periodically integrated of order 1, to be denoted as PI(1), see Osborn *et al.* (1988) and Boswijk and Franses (1992).

The PAR(2) process for y_t in (1) can also have two unit roots. When the y_t series contains the seasonal unit roots $\pm i$, it can be written as $y_t=-y_{t-2}+\varepsilon_t$, and hence $\phi_{1s}=0$ and $\phi_{2s}=-1$. Note that, in fact, the process is not periodic anymore. This also applied to the cases where $+1$ and -1 are the unit roots, i.e. when $\phi_{1s}=0$ and $\phi_{2s}=1$. In these two cases, the corresponding vector Y_T has two zero frequency unit roots, i.e. $|A_0 z-A_1|=z^2(z^2-2z+1)=0$, see (7). Of course, similar to the single unit root case, the Y_T process can also have two unit roots at the zero frequency which do not correspond these pairs of (non-)seasonal unit roots. For example, when $\phi_{1s}=2$ and $\phi_{2s}=-1$, y_t is an I(2) process. Similarly, one can define a PI(2) process. In the present paper we abstract from I(2) type processes, and consider the analysis of such processes to be a topic for further research.

It is clear that higher order PAR processes can possess more (non-) seasonal unit roots. For example, under specific parameter restrictions, a PAR(5) process can be written as $(1-B^4)y_t=\alpha_s(1-B^4)y_{t-1}+\varepsilon_t$, i.e. it may contain one nonseasonal unit root and three seasonal unit roots. The corresponding vector Y_T has then 4 unit roots, i.e. there is no cointegration relationship between the Y_{sT} elements.

In the next section we will discuss an empirical model selection strategy for PAR(p) processes, which includes order selection, testing for unit roots, and the selection of the appropriate differencing filter for the y_t series.

III. A MODEL SELECTION STRATEGY

The first step in a model selection strategy for periodic autoregressive time series models is a decision on the order p of the autoregression. Generally, there are two types of methods often applied in practice. The first is to use model selection criteria as Akaike's information criterion (AIC),

$$\mathrm{AIC}(p)=n\log\hat{\sigma}^2+8p, \tag{8}$$

where $\hat{\sigma}^2$ is RSS/n, with RSS is the residual sum of squares, and where n is the effective sample size, or the Schwarz criterion (SC),

$$\mathrm{SC}(p)=n\log\hat{\sigma}^2+4p\log n. \tag{9}$$

Note that in (8) and (9) account has been taken of the fact that one estimates parameters for each of the four seasons. An alternative approach is given by using F type tests for the deletion of the parameters ϕ_{ls}, for some value of l. The order of a PAR is set equal to p when some or all $\phi_{ps} \neq 0$, while the $\phi_{p+1,s} = 0$ for all s. These F tests are applied to PARs with decreasing orders, where the initial order may be set at 4 or 8. The partial autocorrelation function can be useful to decide on this initial model order. Alternatively, one may estimate PARs of order 8, 7, etc, and test whether the residuals display periodic autocorrelation. The order p is chosen when the LM test statistic calculated from the auxiliary regression

$$\hat{\varepsilon}_t = \sum_{i=1}^{p} \sum_{s=1}^{4} \kappa_{is} D_{st} y_{t-i} + \sum_{s=1}^{4} \gamma_{1s} D_{st} \hat{\varepsilon}_{t-1}, \tag{10}$$

for the significance of the four lagged $\hat{\varepsilon}_t$ variables, indicates that the hypothesis of no periodic autocorrelation cannot be rejected, see Franses (1992a).

When the order of the PAR is chosen, the second step is to test for the presence of periodicity. The null hypothesis is that $\phi_{ls} = \phi_l$ for all s, and where $l = 1,\ldots,p$. This is an important step since tests for any unit roots can be affected by the inclusion of too many parameters. The null hypothesis of no periodicity can be tested using an F type test. Given that this hypothesis is not related to the number of unit roots in the model, it can be expected that the F test follows a standard F distribution under the null hypothesis.

Assuming that the autoregressive parameters display periodicity indeed, i.e. that the F test in the previous step rejects the null hypothesis, the third step is the calculation of the eigenvalues of $A_0^{-1}A_1$. When the order p is such that the model in (2) should be enlarged by including $A_2 Y_{T-2}$, one should calculate the characteristic roots of $|A_0 z^2 - A_1 z - A_2| = 0$. It is our experience, see also section V below, that most quarterly time series can be described by models like (2), i.e. models of order 1 for Y_T. A test for a single unit root in the Y_T series, which is based on checking whether the largest eigenvalue of $A_0^{-1}A_1$ equals 1, is given in Fountis and Dickey (1989). A test for the same hypothesis, which is based on testing nonlinear parameter restrictions like in (6) in (1), is proposed in Boswijk and Franses (1992). This test is calculated as

$$\text{BF} = (\text{sign}(g(\hat{\phi}) - 1))[\log(RSS_0/RSS_1)]^{1/2}, \tag{11}$$

where RSS_0 corresponds to the restricted nonlinear regression model, and RSS_1 corresponds to the linear unrestricted model, and where $g(\hat{\phi})$ is a function of the periodic autoregressive parameters. This $g(\hat{\phi})$ can be obtained as follows. The characteristic equation $|A_0 z - A_1| = 0$ can be written as $z^4 - \beta_1 z^3 - \ldots - \beta_4 = 0$, where not all β_i have to be unequal to zero, and $g(\phi)$ equals $\beta_1 + \ldots + \beta_4$. Under the null hypothesis of a single unit root, the BF follows a standard Dickey-Fuller distribution, see Boswijk and Franses

(1992). In the latter paper it is also shown that the empirical performance of the BF test is superior to the Fountis and Dickey (1989) test.

The Fountis and Dickey (1989) and Boswijk and Franses (1992) approaches may not be easily extended to test for more than a single unit root in models for Y_T. A simple method to deal with this hypothesis is to rewrite (2) as

$$\Delta Y_T = (A_0^{-1} A_1 - I) Y_{T-1} + A_0^{-1} \omega_T, \tag{12}$$

where Δ is the first order differencing filter for annual time series, and to apply the Johansen (1991) cointegration method to test for the number of unit roots, see Franses (1994). Note that the Δ filter for Y_T corresponds to the Δ_4 filter for the quarterly observed y_t series. However, when the order of the PAR is small, say 1 or 2, it is easily understood that the empirical performance of the such tests may not be optimal, see Franses (1994) for simulation results. Hence, in case of small PAR models, we propose, first, to calculate the eigenvalues of $A_0^{-1}A_1$, second, to decide informally on whether there may be one or more unit roots, and, third, in case of the hypothesis of a single unit root, to apply the BF test given in (11).

Although the parameters in PAR models vary with the seasons, it may be the case that the autoregressive polynomial $\phi_{ps}(B)$ can be written as a product of $\lambda(B)$ and $\eta_{ls}(B)$, where $\lambda(B)$ is, e.g., $(1-B)$, $(1+B)$ or $(1+B^2)$. Hence, a PAR process can contain (non-)seasonal unit roots. As discussed in the previous section, the presence of (non-)seasonal unit roots implies parameter restrictions in the PAR model. These can simply be tested using standard F tests. When the number of unit roots in the PAR under the null hypothesis and under the alternative hypothesis is equal, the tests asymptotically follow standard distributions. For example, when it is found that there is a single unit root in Y_T, i.e. that in principle the transformation $(1-\gamma_s B)$ with $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$ is required, the F type test for the hypothesis $\gamma_s = 1$ follows a standard $F(3,\ldots)$ distribution under the null hypothesis given the imposed nonlinear restriction $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$. Note that testing for the restriction $(1-B)$ in a periodic model assumes that the order of the PAR is at least equal to 2, since, implicitly, this restriction has already been tested in the first step in case a PAR(1) process is adequate. Further, in case one hypothesizes that there is more than one unit root in the multivariate model for Y_T, one may start with testing for restrictions implied by (non-)seasonal unit roots. If the null hypothesis is not rejected, one can proceed with estimating a PAR model for the $(1-B)y_t$ series, and test whether this model contains a single unit root. For example, in a PAR(2) process the latter unit root implies the validity of the restriction $\phi_{21}\phi_{22}\phi_{23}\phi_{24} = 1$, as can be observed from (7).

In the next section we will evaluate the proposed steps in the selection strategy using Monte Carlo replications. The emphasis will be on the choice of the PAR order and on the performance of the tests for parameter restrictions.

IV. MONTE CARLO EVIDENCE

This section reports on the results of several Monte Carlo simulations. The effective sample size in all simulations is set equal to 120. The number of replications equals 5000. The nominal signficance level of the tests is 5 percent.

First we investigate the empirical performance of the autoregressive order selection methods. The expressions of the AIC and SC criteria are given in (8) and (9). The LM_{par} is the test for periodicity in the residuals, see (10). The F test is the F type test for the significance of the four $\phi_{l+1,s}$ parameters where l is the current model order. We start with generating PAR processes of order 2 and estimate PAR processes of orders 4 through 1. In Table 1, we

TABLE 1

Autoregressive Order Selection in Periodic Autoregressions with no Unit Roots in the Vector Process. Based on 5000 Monte Carlo Replications. Sample Size is 120 Observations. The Cells Report the Frequencies that a Certain Model is Selected. The Data Generating Process is a Periodic Autoregression of Order 2

			Order of periodic autoregression			
DGP[1]	*Roots of* $A_0^{-1}A_1$	*Criterion*[2]	*1*	*2*	*3*	*4*
I	$0.06 \pm 0.11i$	SC	33.4	66.3	0.3	0.0
		AIC	2.1	79.8	12.4	5.7
		LM_{par}	3.9	82.1	5.2	4.6
		F test	4.2	85.7	5.4	4.7
II	0.38, 0.42	SC	0.0	99.7	0.3	0.0
		AIC	0.0	82.1	12.2	5.8
		LM_{par}	0.0	85.2	4.6	5.0
		F test	0.0	89.7	4.9	5.4
III	$0.59 \pm 0.08i$	SC	0.0	99.7	0.3	0.0
		AIC	0.0	82.1	11.7	6.2
		LM_{par}	0.0	84.8	5.0	4.9
		F test	0.0	89.6	5.2	5.2
IV	0.78, 0.83	SC	0.0	99.7	0.3	0.0
		AIC	0.0	82.1	12.0	5.9
		LM_{par}	0.0	84.9	4.7	5.4
		F test	0.0	89.4	4.8	5.8

[1]The data generating processes are all PAR(2) processes, see (1). The parameters ϕ_{11}, ϕ_{21}, $\phi_{12}, \ldots, \phi_{24}$ are 0.70, 0.40, 0.90, 0.30, 0.80, −0.30, 0.60 and −0.40 for I, 2.00, −0.70, 1.40, −0.50, 1.70, −0.60, 1.40 and −0.76 for II, 1.99, −0.90, 1.40, −0.70, 1.70, −0.95, 2.25 and −0.60 for III, and 2.00, −1.10, 1.80, −0.90, 1.90, −0.80, 1.89 and −0.81 for IV.

[2]The expressions of the AIC and SC criteria are given in (8) and (9). The LM_{par} is the test for periodicity in the residuals, see (10). The F test is the F type test for the significance of the four $\phi_{l+1,s}$ parameters.

report on the selection frequencies of each of these PAR models in case the PAR(2) models have roots ranging from close to zero to close to unity. It can be seen that the SC criteria performs best in case the roots are not too close to zero. With the AIC one will be inclined to opt for too high an order in about 18 percent of the cases. The F type tests for the significance of the $\phi_{l+1,s}$ and for the periodicity in the autocorrelation function of the residuals from a PAR show selection frequencies which are reasonably stable over the DGPs. Similar results are reported in Table 2, where the DGP is again a PAR(2) process, though now with one unity solution to the characteristic equation for Y_T. Comparing the selection frequencies with those in Table 1, it can be observed that the number of unit roots does not seem to affect order selection, and also that the F test performs well. In Table 3 we display the order selection results in case a subset PAR(2) is the data generating process, i.e. we set one or more parameters ϕ_{2s} equal to zero. The cases I through IV in Table 3 are the cases in which 0 through 3 of these ϕ_{2s} are set equal to zero. The performance of the SC is the most affected. The AIC as well as the F tests display similar results as in Tables 1 and 2.

Until now we only generated and estimated periodic processes. It is also interesting to see whether one also finds low order periodic PAR models to be appropriate in case conventional seasonal time series processes, as e.g., seasonal ARIMA processes, are the DGP. In Table 4 we report the results of

TABLE 2

Autoregressive Order Selection in Periodic Autoregressions with one Unit Root in the Vector Process. Based on 5000 Monte Carlo Replications. Sample Size is 120 Observations. The Cells Report the Frequencies that a Certain Model is Selected. The Data Generating Process is a Periodic Autoregression of Order 2

			Order of periodic autoregression			
DGP[1]	*Roots of* $A_0^{-1}A_1$	*Criterion*[2]	*1*	*2*	*3*	*4*
I	1.0, 0.6	SC	0.0	99.8	0.2	0.0
		AIC	0.0	82.8	11.4	5.8
		LM_{par}	0.0	85.4	4.4	4.6
		F test	0.0	90.4	4.7	4.9
II	1.0, 0.8	SC	0.0	99.6	0.4	0.0
		AIC	0.0	81.4	12.5	6.2
		LM_{par}	0.0	84.9	4.4	5.1
		F test	0.0	89.7	4.8	5.5

[1]The data generating processes are all PAR(2) processes, see (1). The parameters $\phi_{11}, \phi_{21}, \phi_{12}, \ldots, \phi_{24}$ are 1.96, −0.96, 1.93, −0.93, 1.85, −0.85, 1.79 and −0.79 for I, 1.70, −0.70, 1.95, −0.95, 2.20, −1.20, 2.00 and −1.00 for II.

[2]The expressions of the AIC and SC criteria are given in (8) and (9). The LM_{par} is the test for periodicity in the residuals, see (10). The F test is the F type test for the significance of the four $\phi_{l+1,s}$ parameters.

selecting PAR(p) processes in case $\Delta_4 y_t = \alpha \Delta_4 y_{t-1} + \varepsilon_t$ is the DGP for some values of α. The correct model order is 5, but given the number of redundant parameters to be estimated when considering a PAR model here, one may expect less favourable results as in Tables 1 and 2. Setting the maximum value of p at 8, we observe that when α is small, say 0.2, most criteria indicate too small a model order. When this α value increases, the performance of the order selection methods largely improves, and a similar performance as in the Tables 1 and 2 can be observed. Comparable results emerge when the DGP is a low order nonperiodic process with seasonal unit roots. In Table 5 we report on the frequencies of order selection for five different cases, and the general impression is that, roughly speaking, in at least 75 percent of the cases one would detect the correct model order.

TABLE 3

Autoregressive Order Selection in Periodic Subset Autoregressions. The Data Generating Process is a Periodic Autoregression of Order 2. Based on 5000 Monte Carlo Replications. Sample Size is 120 Observations. The Cells Report the Frequencies that a Certain Model is Selected

			Order of periodic autoregression			
DGP[1]	*Roots of* $A_0^{-1}A_1$	*Criterion*[2]	*1*	*2*	*3*	*4*
I	$0.22 \pm 0.03i$	SC	3.0	96.5	0.5	0.0
		AIC	0.1	81.2	12.4	6.3
		LM_{par}	0.2	84.9	5.1	4.8
		F test	0.2	89.3	5.5	5.0
II	0.45	SC	43.5	56.5	0.0	0.0
		AIC	4.2	78.5	11.7	5.6
		LM_{par}	6.9	78.3	4.5	5.2
		F test	7.4	82.5	4.7	5.4
III	0.89	SC	34.5	65.4	0.1	0.0
		AIC	2.8	79.3	12.2	5.7
		LM_{par}	5.0	81.1	4.3	4.7
		F test	5.3	85.3	4.5	5.0
IV	0.70	SC	62.2	37.7	0.1	0.0
		AIC	10.7	72.8	11.0	5.4
		LM_{par}	15.5	70.6	4.1	4.9
		F test	16.2	74.2	4.5	5.1

[1]The data generating processes are all PAR(2) processes, see (1). The parameters $\phi_{11}, \phi_{21}, \phi_{12}, \ldots, \phi_{24}$ are 0.70, 0.55, 0.90, 0.40, 0.80, −0.45, 0.60 and −0.50 for I, 0.70, 0.55, 0.90, 0.40, 0.80, −0.45, 0.60 and 0.00 for II, 0.70, 0.55, 0.90, 0.40, 0.80, 0.00, 0.60 and 0.00 for III, and 0.70, 0.55, 0.90, 0.00, 0.80, 0.00, 0.60, and 0.00 for IV.

[2]The expressions of the AIC and SC criteria are given in (8) and (9). The LM_{par} is the test for periodicity in the residuals, see (10). The F test is the F type test for the significance of the four $\phi_{i+1,s}$ parameters.

TABLE 4

Periodic Autoregressive Order Selection when the DGP is the Nonperiodic ARIMA(0,1,0)$_4$ × (1,0,0) Process: $\Delta_4 y_t = \alpha \Delta_4 y_{t-1} + \varepsilon_t$. Based on 5000 Monte Carlo Replications. Sample Size is 120 Observations. The Cells Report the Frequencies that a Certain Model is Selected. The Correct Model Order is 5

		Order of periodic autoregression							
α in DGP	*Criterion*[1]	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
0.2	SC	0.0	0.0	0.0	95.7	4.2	0.1	0.0	0.0
	AIC	0.0	0.0	0.0	47.9	27.9	10.6	7.1	6.5
	LM_{par}	0.0	0.0	0.0	57.1	18.3	5.2	6.0	6.1
	F test	0.0	0.0	0.0	61.9	19.5	5.8	6.5	6.4
0.4	SC	0.0	0.0	0.0	64.5	35.0	0.5	0.0	0.0
	AIC	0.0	0.0	0.0	11.8	56.4	14.0	9.0	8.7
	LM_{par}	0.0	0.0	0.0	22.5	53.5	5.1	5.9	5.7
	F test	0.0	0.0	0.0	24.2	57.7	5.5	6.4	6.3
0.6	SC	0.0	0.0	0.1	12.0	86.9	0.8	0.1	0.0
	AIC	0.0	0.0	0.0	0.5	65.4	15.9	8.9	9.2
	LM_{par}	0.0	0.0	0.0	1.5	73.3	6.0	5.9	6.7
	F test	0.0	0.0	0.0	1.5	78.6	6.4	6.4	7.1
0.8	SC	0.2	0.0	0.3	0.5	98.2	0.7	0.1	0.0
	AIC	0.0	0.0	0.0	0.0	65.8	14.9	9.8	9.5
	LM_{par}	0.0	0.0	0.0	0.0	74.5	5.4	6.4	6.8
	F test	0.0	0.0	0.0	0.0	80.1	5.9	6.9	7.1
1.0	SC	0.1	0.0	0.0	0.0	98.8	1.0	0.1	0.0
	AIC	0.0	0.0	0.0	0.0	63.8	14.9	10.6	10.7
	LM_{par}	0.0	0.0	0.0	0.0	72.5	6.0	7.2	7.5
	F test	0.0	0.0	0.0	0.0	77.8	6.5	7.6	8.1

[1]The expressions of the AIC and SC criteria are given in (8) and (9). The LM_{par} is the test for periodicity in the residuals, see (10). The *F* test is the *F* type test for the significance of the four $\phi_{l+1,s}$ parameters.

Overall, we conclude that order selection does not seem to be affected by the number of unit roots, that the SC often detects the correct model order, and that the *F* test for the significance of $\phi_{l+1,s}$ shows satisfactory empirical performance since its success rate appears to be reasonably stable over the various DGPs. Hence, as an empirical strategy we recommend to use the SC to select the model order *p*, provided that the *F* test for $\phi_{p+1,s}=0$ does not reject the null hypothesis. This approach will also be followed in the next section where we analyze a set of empirical time series.

The second set of simulations concerns the *F* type test for nonperiodicity in the AR parameters in an estimated PAR model. In the Tables 6 and 7 we show the fractiles of this *F* test in case the DGP is a nonperiodic AR(1) or AR(2) process, possibly with one or two unit roots. Comparing the fractiles

TABLE 5

Periodic Autoregressive Order Selection when the DGP is the Nonperiodic Process: $\lambda(B)y_t = 0.5\lambda(B)y_{t-1} + \varepsilon_t$, where $\lambda(B)$ Corresponds to Components of the $(1-B^4)$ Filter. Based on 5000 Monte Carlo Replications. Sample size is 120 Observations. The Cells Report the Frequencies that a Certain Model is Selected

		Order of periodic autoregression[1]							
$\lambda(B)$	*Criterion*[2]	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
$(1+B)$	SC	9.6	90.3	0.1	0.0	0.0	0.0	0.0	0.0
	AIC	0.2	78.8	10.5	4.4	2.2	1.5	1.1	1.2
	LM_{par}	0.5	68.0	3.9	4.1	3.8	3.6	5.2	5.8
	F test	0.5	71.6	4.2	4.4	3.8	3.9	5.4	6.0
$(1-B^2)$	SC	0.0	9.9	89.7	0.3	0.0	0.0	0.0	0.0
	AIC	0.0	0.3	74.9	11.4	5.2	3.1	2.5	2.6
	LM_{par}	0.0	0.7	70.9	3.8	4.4	4.2	4.4	5.6
	F test	0.0	0.8	75.4	4.0	4.7	4.5	4.6	6.0
$(1+B^2)$	SC	0.0	16.5	83.1	0.4	0.0	0.0	0.0	0.0
	AIC	0.0	0.6	75.3	11.0	5.5	3.1	2.1	2.4
	LM_{par}	0.0	1.1	70.4	3.8	4.5	4.3	4.3	5.5
	F test	0.0	1.2	75.0	4.1	4.9	4.5	4.6	5.8
$(1-B)(1+B^2)$	SC	0.0	0.0	16.5	83.1	0.3	0.0	0.0	0.0
	AIC	0.0	0.0	0.7	69.2	13.8	7.0	4.3	4.9
	LM_{par}	0.0	0.0	1.8	70.3	5.1	5.7	4.8	5.3
	F test	0.0	0.0	1.9	75.5	5.7	6.1	5.1	5.7
$(1+B)(1+B^2)$	SC	0.0	0.0	34.1	65.5	0.4	0.0	0.0	0.0
	AIC	0.0	0.0	3.0	68.9	13.0	6.2	4.3	4.6
	LM_{par}	0.0	0.0	5.6	67.2	5.0	4.2	5.1	6.5
	F test	0.0	0.0	6.1	71.4	5.4	4.5	5.5	7.0

[1]The correct order for the DGPs is 2, 3, 3, 4, 4, respectively.
[2]The expressions of the AIC and SC criteria are given in (8) and (9). The LM_{par} is the test for periodicity in the residuals, see (10). The F test is the F type test for the significance of the four $\phi_{i+1,s}$ parameters.

of the corresponding theoretical F distribution, it can be seen from both tables that the empirical fractiles closely match the theoretical ones. This applies to the stationary AR processes as well as to the nonstationary models.

Finally, we investigate whether the F test for restrictions related to seasonal and nonseasonal unit roots follows the standard F distribution, as conjectured in the previous section. In Table 8, we report the empirical fractiles of the F test for the parameter restrictions corresponding to the $(1-B)$ and the $(1+B)$ filter in a PAR(2). It can be observed from a comparison of the first column of Table 8 with the next six columns that again the empirical and theoretical fractiles closely match. Hence, the conjecture that F type tests for parameter restrictions follow standard F distributions in case

TABLE 6

Fractiles of F-Statistic for the Null Hypothesis $\phi_1=\phi_2=\phi_3=\phi_4=\phi$ in the PAR(1) Model $y_t=\phi_s y_{t-1}+\varepsilon_t$ when the DGP is the Nonperiodic AR(1) Process $y=\phi y_{t-1}+\varepsilon_t$. Based on 5000 Monte Carlo Replications. Sample Size is 120 Observations

	$F(3,115)$	$\phi=0.6$	$\phi=0.8$	$\phi=1.0$
5%	0.12	0.12	0.11	0.10
10%	0.19	0.20	0.19	0.18
20%	0.34	0.34	0.33	0.32
50%	0.79	0.81	0.78	0.78
80%	1.57	1.61	1.58	1.58
90%	2.13	2.16	2.09	2.11
95%	2.68	2.66	2.64	2.65
Mean	1.02	1.03	1.01	1.00
Var.	0.72	0.74	0.72	0.73

TABLE 7

Fractiles of F-Statistic for the Null Hypothesis $\phi_{11}=\phi_{12}=\phi_{13}=\phi_{14}=\phi_1$ and $\phi_{21}=\phi_{22}=\phi_{23}=\phi_{24}=\phi_2$ in a PAR(2) Process $y_t=\phi_{1s}y_{t-1}+\phi_{2s}y_{t-1}+u_t$ when the DGP is the Nonperiodic AR(2) Process $y_t=\phi_1 y_{t-1}+\phi_2 y_{t-2}+\varepsilon_t$. Based on 5000 Monte Carlo Replications. Sample Size is 120 Observations

	The roots of the AR(2) process					
	$F(6, 110)$	(0.6, 0.8)	(0.6, 1.0)	(0.8, 1.0)	(1.0, 1.0)	(0.6, −1.0)
5%	0.27	0.26	0.26	0.27	0.26	0.26
10%	0.36	0.35	0.35	0.36	0.37	0.36
20%	0.51	0.50	0.50	0.51	0.51	0.50
50%	0.90	0.90	0.90	0.89	0.88	0.89
80%	1.46	1.46	1.47	1.43	1.43	1.43
90%	1.83	1.84	1.84	1.81	1.83	1.82
95%	2.18	2.18	2.21	2.17	2.21	2.18
Mean	1.02	1.02	1.02	1.01	1.01	1.01
Var.	0.37	0.37	0.38	0.37	0.36	0.38

the number of unit roots is the same under the null and alternative hypotheses seems to be verified by our simulation outcomes in the Tables 6–8.

V. SEVERAL UK MACROECONOMIC TIME SERIES

To illustrate our model selection strategy with empirical non-simulated time series, we analyze 19 quarterly UK macroeconomic time series. All series have been log transformed, except for the interest rate and stockbuilding.

TABLE 8

Testing for (Non-)seasonal Unit Roots in Periodic Autoregressions of Order 2. Fractiles of F-statistic for the Null Hypothesis that $\phi_{1s}+\phi_{2s}=1$ or that $\phi_{1s}-\phi_{2s}=1$, Given the Restriction that there is One Unit Root in the PAR Process. Based on 5000 Monte Carlo Replications. Sample Size is 120 Observations

		Data generating process[1]					
		Nonseasonal unit root			Seasonal unit root		
	$F(3, 111)$	I	II	III	IV	V	VI
5%	0.12	0.13	0.11	0.11	0.10	0.12	0.11
10%	0.19	0.20	0.18	0.19	0.18	0.20	0.19
20%	0.34	0.34	0.33	0.33	0.32	0.32	0.33
50%	0.79	0.80	0.78	0.80	0.77	0.79	0.76
80%	1.57	1.58	1.55	1.54	1.54	1.55	1.52
90%	2.13	2.11	2.11	2.08	2.12	2.11	2.11
95%	2.69	2.62	2.64	2.67	2.69	2.69	2.65
Mean	1.02	1.02	1.00	1.00	0.99	1.00	1.00
Var.	0.72	0.71	0.70	0.70	0.72	0.70	0.72

[1]The data generating process for I, II and III is $y_t-y_{t-1}=\phi_s(y_{t-1}-y_{t-2})+\varepsilon_t$, with ϕ_s is 0.96, 0.93, 0.85, 0.79 for I, 0.70, 0.95, 1.20, 1.00 for II and 1.25, 0.80, 2.00, 0.50 for III. The roots of $A_0^{-1}A_1$ are 1.0, 0.6 for I, 1.0, 0.8 for II and 1.0, 1.0 for III, respectively. The data generating process for IV, V and VI is $y_t+y_{t-1}=\alpha_s(y_{t-1}+y_{t-2})+\varepsilon_t$, with α_s equal to $-\phi_s$. Hence, the processes I, II and III correspond to IV, V and VI with respect to $\alpha_s=-\phi_s$, and the roots of $A_0^{-1}A_1$.

These series have also been studied in Osborn (1990) and Franses and Romijn (1992). In the first study it has been found using nonperiodic models that many of these quarterly series only have a nonseasonal unit root. An application of the Johansen (1991) method to vector autoregressions of order one for the annual Y_T series, see (12), yields that several UK series are periodically integrated, see Franses and Romijn (1992). Given that periodic models may not be accurate in all cases, and also given that the performance of the Johansen method for the Y_T series may not be adequate in case the PAR order is only 1 or 2, we expect that our model selection strategy will result in a more thorough univariate analysis of the UK macroeconomic time series.

In the first column of Table 9 we list the UK variables we investigate. We consider PAR(p) models with and without trends. All models contain four seasonal constants and/or seasonal trends. The orders of the PAR in each of the models is reported in the columns 2 and 4. These orders are chosen using the SC provided that the F test for $\phi_{l+1,s}$ does not reject the null hypothesis. The orders considered for initial model selection are 1 to 4. It can be seen that for a few variables this order equals 3, while for the majority of the time series an adequate value for p turns out to be equal to 2. In some regression models we include dummy variables to capture outlying observations. These observations are detected by considering the residual plot in case the residuals do not display normality in the initial model. All empirical models are tested for first and fourth order residual autocorrelation, ARCH effects and normality of the residuals. Sometimes we find ARCH patterns which we cannot remove by including additional lags. In the columns 3 and 5 of Table 9 we report the F test results for the hypothesis of no periodicity in the autoregressive parameters. This null hypothesis is rejected for 14 of the 19 time series in case the model does not include trends, and for 11 of the 19 time series in case the auxiliary regression includes four trends. Since we focus on periodic models in this paper, we decide not to pursue with an analysis of the variables stockbuilding, vacancies, stock prices, interest and exchange rate, and we refer to the results in Osborn (1990). We also do not further consider Consumption Durables since the four trends are highly significant in a non-periodic AR model, with a joint F value of 11.227.

The remaining 13 variables which can be modeled using PAR models, with or without four trends, are displayed in the first column of Table 10. In the next two columns we display the results of the eigenvalues of the $A_0^{-1}A_1$ matrix and the BF test in case the auxiliary regression contains no trend variables, while in columns four and five we report similar results in case this regression does include trend variables. The inclusion of trend terms ensures that the distribution of the BF test statistic is invariant to the value of the drift parameters, see Banerjee *et al.* (1993) for a discussion. From the columns two and four we observe that there may be only a single unit root in all these time series, and the BF test results emphasize this conjecture since the null hypothesis cannot be reject at a reasonable significance level for all variables. The only exception is imports where the null hypothesis of a single unit root is rejected at a 10 percent level in case the auxillary regression contains four trends.

The sixth column reports on the estimated F test statistic values for the joint significance of the four trend variables in the auxiliary regressions. In case the models contain a unit root in the Y_T process, the distribution of the latter F test is not the standard F distribution, cf. Nankervis and Savin (1987). In that case the distribution is shifted to the right. Using Monte Carlo simulations we calculated the critical values of this F statistic in case the process $y_t = \alpha_s y_{t-1} + \varepsilon_t$ with $\prod_{s=1}^{4} \alpha_s = 1$ is the data generating process and the regression model includes $D_{st} y_{t-1}$ and four constants and four trends. The 5 percent significance level of the F test for the joint hypothesis that the trends can be deleted from the model is 3.43, while the 10 percent significance level is 2.88. These figures are robust to variations in the values of α_s in the data generating process as long as $\prod_{s=1}^{4} \alpha_s = 1$ holds. Comparing the F test values in the sixth column of Table 10 with these critical values, we may conclude that only GDP, public investment, imports and productivity have significant trend variables.

TABLE 9
Periodic Autoregressions of Order p Fitted to Quarterly UK Macroeconomic Variables[1]

	Auxiliary regression[6]			
	Without trend		*With trend*	
Variable[2]	p^3	$F_{per}{}^4$	p^3	$F_{per}{}^4$
GDP	2	11.715**	2	5.564**
Total consumption	1	35.574**	1	0.465
Durables	3	8.380**	1^5	1.503
Non-durables	1	31.129**	1	1.664
Govt. consumption	3	8.235**	3	4.234**
Investment	2	7.087**	2	2.733**
Private	1	4.430**	1	1.086
Public	2	25.887**	2	3.483**
Stockbuilding	2^5	2.081	2^5	2.603**
Trade balance	2^5	3.713**	2^5	3.010**
Exports	2^5	5.629**	2^5	3.357**
Imports	1^5	4.731**	1	3.500**
Vacancies	3	1.297	2^5	2.211
Workforce	1	10.444**	2	3.474**
Productivity	2^5	12.901**	2	6.833**
Prices	3^5	5.399**	3^5	4.157**
Stock prices	2	1.383	2	1.775
Interest rate	2^5	0.762	3	1.096
Exchange rate	2	1.102	2	1.283

**Significant at a 5% level.
[1]The sample size for all variables is 55.1–88.4, except Private and Public investment which start in 62.1, Stock prices and interest rate which start in 63.1 and Exchange rate which starts in 73.1.
[2]The definitions of the variables can be found in Osborn (1990).
[3]The estimated models are checked for first and fourth order residual autocorrelation, first order ARCH effects, and normality. The model order p is selected using SC, under the condition that an F test does not reject this order versus $p+1$, and that the diagnostic checks for residual autocorrelation do not reject the empirical specification.
[4]The F test for the null hypothesis of no periodicity in the autoregressive parameters, which follows a standard F distribution under the null.
[5]The estimated model displays ARCH effects at a 5% level, but the model cannot be improved by adding extra lags.
[6]Dummy variables to capture outlying observations in the PAR models are included for Interest rate (73.3, 81.3, 85.1), Productivity (72.1), Workforce (59.2), Total consumption (79.3, 80.2), Public investment (63.1), GDP (71.1), Prices (79.3), and Exports (67.4, 68.1).

A closer, though unreported, look at the autoregressive parameters in the estimated PAR processes suggests that only the $(1-B)$ filter may be relevant, and that the $(1+B)$ filter seems far from adequate. In the final column of Table 10 we report the F type test for the null hypothesis that the $(1-B)$ filter is appropriate to remove the stochastic trend from the time series. This

TABLE 10

Testing for Unit Roots in Periodic Autoregressions for Several Quarterly UK Macroeconomic Variables

	Auxiliary regression					
	Without trend		*With trend*			
Variable	*Roots*	BF^1	*Roots*	BF^1	$F_{trend}{}^2$	$F_{(1-B)}{}^3$
GDP	0.982, −0.002	−0.908	0.777, −0.029	−2.215	8.142**	7.755**
Total Cons.	1.020	1.108	0.783	−1.658	1.999	26.768**
Nondurables	1.010	0.586	0.852	−1.304	3.146	31.235**
Govt. Cons.	0.997, 0.149, 0.0004	−0.389	0.796, 0.117, 0.010	−1.883	2.275	4.506**
Investment	0.948, −0.002	−1.414	1.072, 0.001	1.596	2.653	10.155**
Private	0.991	−0.146	0.589	−2.338	1.905	4.467**
Public	1.055, 0.002	0.662	0.834, 0.005	−1.577	13.991**	1.166
Trade balance	0.675, 0.004	−2.217	0.532, 0.001	−2.000	1.288	6.217**
Exports	0.980, −0.002	−1.074	0.769, −0.002	−1.936	2.117	6.292**
Imports	1.003	0.105	0.485	−3.282	5.713**	5.582**
Workforce	1.015	0.584	0.932, 0.002	−0.917	1.309	9.499**
Productivity	0.972, −0.013	−1.484	0.778, −0.049	−1.992	9.508**	8.661**
Prices	1.011, 0.482, 0.010	0.490	0.810, 0.539, 0.013	−2.528	3.196	2.607

**Significant at a 5% level.

[1]The test statistic for a single unit root in a periodic autoregression, see (11).

[2]An F test for the significance of the four trend variables. This test is performed in the unrestricted model, i.e. the unit root is not imposed on the periodic autoregressive parameters. A simulated 5% critical value for this F test is 3.43.

[3]The F test for the null hypothesis that the periodic autoregressive polynomial can be decomposed as $(1-B)\eta_{is}(B)$, given the restriction that there is a single unit root in the models for Y_T. For all variables this F test is calculated for the model which does not include four trend variables, except for GDP, public investment imports and prices.

F test is calculated for the model which does not include four trend variables, except for GDP, public investment, imports and productivity. We observe that the $(1-B)$ filter may be useful for public investment and for prices, while for the other variables the parameter restriction corresponding to the $(1-B)$ filter can be rejected.

It can be concluded from the results in Tables 9 and 10 that several quarterly UK variables display periodic variation in their dynamic patterns, and that, except for imports, these variables have a single unit root. The evidence in Table 10 further indicates that several quarterly UK variables are periodically integrated of order 1. The results for the two consumption variables are in line with those obtained in Osborn (1988) and Osborn and Smith (1989).

VI. CONCLUDING REMARKS

In this paper we propose a model selection strategy for quarterly time series when they can be modeled using periodic autoregressions. Model selection involves decisions on the order of the autoregressions, on the number of unit roots in the annual series containing the quarterly observations, and on the possible decomposition of the autoregressive polynomial in terms which correspond to (non-)seasonal unit roots in the quarterly series. Our strategy mainly relies on tests for parameter restrictions in the periodic models. We evaluate it in a Monte Carlo study. The results indicate that its empirical performance is broadly satisfactory. We apply our method to 19 quarterly UK macroeconomic time series, and we find that several series can be described by a low order periodic autoregression with a single unit root.

Our empirical analysis considers only univariate time series. Of course, it is most likely that these variables are generated from multivariate processes. An obvious extension of our strategy is therefore to consider its application to multivariate periodic models.

One of the essential characteristics of periodically integrated time series models is that seasons, cycles and trends can not be separated in the traditional sense. For example, the zero-frequency unit root in the annual vector series can not be extracted from the series without affecting the seasonal pattern. Hence, one of the basic assumptions underlying seasonal adjustment methods is violated. Some empirical results for the UK series in the present paper may therefore be interpreted as evidence that the application of seasonal correction methods may not be useful for every seasonal time series.

Finally, one of the empirical time series, i.e. public investment, can be described using a periodic AR(1) process for the first order differenced time series, where four periodic trends are included. At first sight this seems to be a strange model, and in fact it might reflect the presence of a seasonal unit root. However, the presence of such a root in each of the seasons can be rejected. One possibility is now that for public investment the number of seasonal unit roots varies with the seasons. This suggests an interesting topic for further research, i.e. to extend the Hylleberg *et al.* (1990) method to cope with such a variation.

Econometric Institute, Erasmus University, Rotterdam

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