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Multi-step Forecast Error Variances for Periodically Integrated Time Series

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ABSTRACT

A periodically integrated (PI) time series process assumes that the stochastic trend can be removed using a seasonally varying differencing filter. In this paper the multi-step forecast error variances are derived for a quarterly PI time series when low-order periodic autoregressions adequately describe the data. The forecast error variances display seasonal variation, indicating that observations in some seasons can be forecast more precise than those in others. Two examples illustrate the empirical relevance of calculating forecast error variances. A by-product of the analysis is an expression for the seasonally varying impact of the stochastic trend.

KEY WORDS seasonality; periodic integration; forecasts

INTRODUCTION

A useful model for seasonal time series that allows seasonal fluctuations to depend on the stochastic trend is a periodically integrated autoregression (PIAR). A PIAR process describes a time series by a periodic autoregression (PAR) and it assumes that a seasonally varying differencing filter is needed to remove the stochastic trend. Hence, periodic integration (PI) assumes that the appropriate differencing filter for a quarterly time series is $(1 - \phi_s B)$, where B is the familiar backward shift operator. The ϕ_s is a seasonally varying parameter, and typical empirical values of ϕ_s are close but unequal to unity. Given that not all ϕ_s are equal to 1, PI allows multiplicativity between the seasonal fluctuations and the stochastic trend. Clearly, PI nests the standard integration case since the $(1 - \phi_s B)$ filter nests the $(1 - B)$ filter. The first (empirical) example of a PI process is given in Osborn (1988). Recently, Boswijk and Franses (1994) discuss several further aspects of PI processes as, for example, model selection and testing. Additional examples of the empirical adequacy of PI processes for macroeconomic time series are given in Franses and Paap (1994).

In the present paper, we provide forecast error variances for the multi-step out-of-sample forecasts for the levels of a PI time series. Since the differencing filter for a PI time series varies with the season, the forecast intervals display seasonal fluctuations. We limit our analysis to periodic autoregressions of low orders to save space and also since Franses and Paap (1994) document that these models often emerge in practice. In principle, extensions to

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higher-order models are straightforward. Furthermore, the focus here is on quarterly time series, although similar results can readily be obtained for monthly or other seasonal variables. Finally, strictly speaking, the $(1 - \phi_s B)$ filter reflects periodic integration of order 1. Our analysis can routinely be extended to periodic integration of higher orders, but we do not pursue this here.

The outline of our paper is as follows. In the next section we first discuss some preliminaries. In the third section we describe the moving average representation which is useful to derive the forecast error variances presented in the fourth section. For a PIAR(1) process we also discuss a by-product of the analysis in this paper, which is an expression for the matrix displaying the seasonally varying impact of stochastic trends. In the fifth section we calculate empirical forecast error variances for PIAR processes for quarterly non-durable consumption in the UK and for real quarterly GNP in Germany. In the final section we present conclusions.

PRELIMINARIES

In this section we discuss some notational issues for PIAR processes. Consider a quarterly observed time series y_t, where t runs from 1 to n, and consider the corresponding skip-sampled vector series Y_T which is the (4×1) vector series $(Y_{1T}, Y_{2T}, Y_{3T}, Y_{4T})'$ where Y_{sT} denotes the observation in season s in year T and where T runs from 1 to $N = n/4$. Furthermore, consider a standard white-noise process ε_t, which is a zero-mean uncorrelated process with constant variance σ^2. Similar to y_t, this ε_t process can be skip-sampled. This results in the (4×1) vector process $\varepsilon_t = (\varepsilon_{1T}, \varepsilon_{2T}, \varepsilon_{3T}, \varepsilon_{4T})'$, where ε_{sT} is a drawing from the white-noise process ε_t in season s and in year T. To calculate empirical forecast intervals, we may assume that ε_t is a Gaussian white-noise process, and hence that $\varepsilon_T \sim N(0, \sigma^2 I_4)$, where I_4 is the (4×4) identity matrix.

Notation and purpose of analysis

In this paper we assume that y_t can, at most, be described by a periodic autoregression of order 2, (PAR(2)), which can be denoted as

$$y_t = \mu_s + \phi_{1s} y_{t-1} + \phi_{2s} y_{t-2} + \varepsilon_t \tag{1}$$

where ϕ_{1s} and ϕ_{2s} are periodically varying parameters. The μ_s is a seasonally varying constant term which does not necessarily reflect that the underlying mean of the series y_t varies with the season. The y_t process displays seasonal heteroscedasticity because of the periodic autoregressive parameters. Sometimes it may be useful to consider additional seasonal heteroscedasticity by allowing the variance of ε_t to vary with the seasons. For notational convenience this modification is not pursued here, but if needed, the expressions below can all be easily modified accordingly.

The PAR(2) process in equation (1) is a non-stationary process in the sense that its autocovariance function varies with the seasons. Therefore, a more convenient representation of equation (1) in order the check for the presence of stochastic trends in y_t is the so-called vector of quarters (VQ) representation

$$A_0 Y_T = \mu + A_1 Y_{T-1} + \varepsilon_T \tag{2}$$

where μ is the (4×1) vector stacking the μ_s (see Tiao and Grupe, 1980, and Osborn, 1991, *inter alia*). For the PAR(2) process in equation (1), the A_0 and A_1 are

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_{12} & 1 & 0 & 0 \\ -\phi_{23} & -\phi_{13} & 1 & 0 \\ 0 & -\phi_{24} & -\phi_{14} & 1 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 0 & 0 & \phi_{21} & \phi_{11} \\ 0 & 0 & 0 & \phi_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3)$$

where A_0 is a full-rank matrix. Note that the expressions for A_0 and A_1 in equation (3) indicate that a VQ model of order 1 as in equation (2) is sufficient to capture the dynamics of a PAR(p) process with $p \leqslant 4$.

In the present paper the focus is on the forecast error variances corresponding to the multi-step ahead out-of-sample forecasts $\hat{Y}_{N+h}$ where $h = 1, 2, \ldots$ This means that a PAR model like equation (1) is estimated for n observations, i.e. year N is the final year within the sample, and that forecasts are generated from year N onwards. We assume that the PAR model is estimated using all observations until quarter 4 in year N, and hence that the first forecast concerns quarter 1 in year $N+1$. If one wants to have a different starting point, one can rearrange the four elements in the Y_T vector accordingly and again make use of the expressions below. In calculating the forecast error variances we further assume that the parameters ϕ_{is} and σ^2 are known, $i = 1, 2$. On practical occasions, we use $\hat{\phi}_{is}$ and $\hat{\sigma}^2$ to calculate the empirical forecast error variances. The inclusion of the uncertainty caused by parameter estimation into the construction of forecast error variances is considered to be outside the scope of this paper. Finally, to construct the forecast intervals for multi-step ahead forecasts, we use the method proposed by Box and Jenkins (1970), i.e. we analyse equation (2) in a vector moving average (VMA) representation.

Periodic integration

Before we turn to the VMA representation in the next section, we first focus on the issue of unit roots in PAR models like equation (1). The presence of such roots can be investigated by checking the solutions of the characteristic equation

$$|A_0 - A_1 z| = 0 \quad (4)$$

The process y_t is said to be periodically integrated when there is only one unity solution to equation (4) while all other solutions are outside the unit circle, and when the differencing filter for y_t to remove the single stochastic trend equals $(1 - \phi_s B)$, where not all $\phi_s = 1$. Boswijk and Franses (1994) propose a nested testing strategy for periodic integration in PAR processes. Using this method, Franses and Paap (1994) document that quarterly observed macroeconomic time series often have only a single unit root and that the hypothesis $\phi_s = \phi$ for all s can be rejected.

Given that periodic integration assumes the presence of three cointegration relations between the four elements of Y_T, it can easily be derived that a periodically integrated autoregression of order 2 (PIAR(2)) can be written as

$$y_t - \phi_s y_{t-1} = \mu_s + \beta_s(y_{t-1} - \phi_{s-1} y_{t-2}) + \varepsilon_t \quad (5)$$

where $\phi_1\phi_2\phi_3\phi_4 = 1$ and where we denote $\phi_{-j} = \phi_{4-j}$ for j is $0, 1, 2 \ldots$ The β_s and ϕ_s parameters are functions of the ϕ_{1s} and ϕ_{2s} parameters in equation (1). The restriction $\phi_1\phi_2\phi_3\phi_4 = 1$ can be understood by representing equation (5) in the VQ notation similar to equation (2), i.e. equation (5) can be written as

$$\Psi(B)[\Phi_0 Y_T - \Phi_1 Y_{T-1}] = \mu + \varepsilon_T \quad (6)$$

with

$$\Psi(B) = \begin{bmatrix} 1 & 0 & 0 & -\beta_1 B \\ -\beta_2 & 1 & 0 & 0 \\ 0 & -\beta_3 & 1 & 0 \\ 0 & 0 & -\beta_4 & 1 \end{bmatrix} \tag{7}$$

and

$$\Phi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_2 & 1 & 0 & 0 \\ 0 & -\phi_3 & 1 & 0 \\ 0 & 0 & -\phi_4 & 1 \end{bmatrix} \text{ and } \Phi_1 = \begin{bmatrix} 0 & 0 & 0 & \phi_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

Note that the backward shift operator B also operates on annual time series. The characteristic equation (4) for the process in equation (6) is equal to

$$(1 - \beta_1\beta_2\beta_3\beta_4 z)(1 - \phi_1\phi_2\phi_3\phi_4 z) = 0 \tag{9}$$

Hence, any periodically integrated time series can be differenced to obtain (periodic) stationarity using the $(1 - \phi_s B)$ filter with the restriction that the product of the ϕ_s is equal to 1. Again, note that in this paper we focus on periodic integration of order 1, i.e. we only consider the case where $\phi_1\phi_2\phi_3\phi_4 = 1$ and we abstain from the case where additionally $\beta_1\beta_2\beta_3\beta_4 = 1$.

A PI process assumes that there is one stochastic trend driving the time series, and that this trend is affecting the intra-year patterns. This can be observed again from the differencing filter $(1 - \phi_s B)$ which allows for a multiplicative relation between stochastic trend and seasons. This also implies that seasonal patterns may change over time and that these changes may correspond to changes in the underlying stochastic trend. In the next section we derive an explicit expression for this relationship between changing seasonal fluctuations and the stochastic trend.

VECTOR MOVING AVERAGE REPRESENTATIONS

In this section we derive the vector moving average representation for the PIAR(2) process in equation (5) using the VQ expression in equation (6). Given that the PIAR(1) model is nested within the PIAR(2) model and emerges when setting $\beta_s = 0$ for all s, the expressions for the PIAR(1) process immediately follow from those for the PIAR(2) process.

PIAR(2)

To derive the forecast error variances, we write equation (6) in a vector moving average representation. First, we obtain that

$$(\Psi(B))^{-1} = (1 - \beta B)^{-1}(\Omega_0 + \Omega_1 B) \tag{10}$$

with $\beta \equiv \beta_1\beta_2\beta_3\beta_4$ and

$$\Omega_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta_2 & 1 & 0 & 0 \\ \beta_2\beta_3 & \beta_3 & 1 & 0 \\ \beta_2\beta_3\beta_4 & \beta_3\beta_4 & \beta_4 & 1 \end{bmatrix} \text{ and } \Omega_1 = \begin{bmatrix} 0 & \beta_1\beta_3\beta_4 & \beta_1\beta_4 & \beta_1 \\ 0 & 0 & \beta_1\beta_2\beta_4 & \beta_1\beta_2 \\ 0 & 0 & 0 & \beta_1\beta_2\beta_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

In case of a PIAR(1) process, $\Omega_0 = I_4$, $\Omega_1 = 0$, $\beta = 0$, and hence $(\Psi(B))^{-1} = I_4$. Using equations (10) and (11), the process in equation (6) can be written as

$$\Phi_0 Y_T = \Phi_1 Y_{T-1} + \mu^* = (1-\beta B)^{-1}(\Omega_0 \varepsilon_T + \Omega_1 \varepsilon_{T-1}) \tag{12}$$

where $\mu^* = (1-\beta)^{-1}(\Omega_0 + \Omega_1)\mu$. Premultiplying equation (12) with Φ_0^{-1} results in

$$Y_T = \Gamma Y_{T-1} + \mu^{**} + \Phi_0^{-1}(1-\beta B)^{-1}(\Omega_0 \varepsilon_T + \Omega_1 \varepsilon_{T-1}) \tag{13}$$

where $\Gamma = \Phi_0^{-1}\Phi_1$, and $\mu^{**} = \Phi_0^{-1}\mu^*$. Given the restriction $\phi_1\phi_2\phi_3\phi_4 = 1$, the expressions for the Γ and Φ_0^{-1} matrices are

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & \phi_1 \\ 0 & 0 & 0 & \phi_1\phi_2 \\ 0 & 0 & 0 & \phi_1\phi_2\phi_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

and

$$\Phi_0^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \phi_2 & 1 & 0 & 0 \\ \phi_2\phi_3 & \phi_3 & 1 & 0 \\ \phi_2\phi_3\phi_4 & \phi_3\phi_4 & \phi_4 & 1 \end{bmatrix} \tag{15}$$

Note from equation (14) that Γ is an idempotent matrix, i.e. $\Gamma^m = \Gamma$ for $m = 1, 2, 3 \ldots$

The process in equation (13) is a vector moving average process for Y_T (VARMA). The MA part of this model is of infinite order as can be observed from rewriting equation (13) as

$$\begin{aligned} &\Phi_0^{-1}(1-\beta B)^{-1}(\Omega_0\varepsilon_T + \Omega_1\varepsilon_{T-1}) \\ &\quad = (1 + \beta^B + \beta^2B^2 + \beta^3B^3 + \cdots)\Phi_0^{-1}(\Omega_0\varepsilon_T + \Omega_1\varepsilon_{T-1}) \\ &\quad = \Phi_0^{-1}\Omega_0\varepsilon_T + \Phi_0^{-1}\Omega_1\varepsilon_{T-1} + \beta(\Phi_0^{-1}\Omega_0\varepsilon_{T-1} + \Phi_0^{-1}\Omega_1\varepsilon_{T-2}) \\ &\qquad + \beta^2(\Phi_0^{-1}\Omega_0\varepsilon_{T-2} + \Phi_0^{-1}\Omega_1\varepsilon_{T-3}) + \beta^3(\Phi_0^{-1}\Omega_0\varepsilon_{T-3} + \Phi_0^{-1}\Omega_1\varepsilon_{T-4}) + \cdots \\ &\quad = \Pi_0\varepsilon_T + \Pi_1\varepsilon_{T-1} + \beta\Pi_1\varepsilon_{T-2} + \cdots + \beta^{i-1}\Pi_1\varepsilon_{T-i} \end{aligned} \tag{16}$$

with

$$\Pi_0 = \Phi_0^{-1}\Omega_0$$
$$\Pi_1 = \Phi_0^{-1}\Omega_1 + \beta\Phi_0^{-1}\Omega_0$$

where Π_0 is a lower triangular matrix because Y_{iT} precedes $Y_{i+1,T}$ for $i = 1, 2, 3$. In the PIAR(1) case it holds that $\Pi_0 = \Phi_0^{-1}$, $\Pi_1 = 0$ and μ^{**} equals $\Phi_0^{-1}\mu$. When we combine equations (13) and (16), the PIAR(2) process can be written as

$$Y_T = \Gamma Y_{T-1} + \mu^{**} + \Pi_0\varepsilon_T + \Pi_1\varepsilon_{T-1} + \beta\Pi_1\varepsilon_{T-2} + \cdots + \beta^{i-1}\Pi_1\varepsilon_{T-i} \tag{17}$$

This expression can be easily used to derive the infinite vector moving average representation for the Y_T vector process. In fact, recursively substituting lagged Y_T in equation (17), while taking account of the fact that $\Gamma^m = \Gamma$, yields

$$Y_T = Y_0^* + \Gamma\mu^{**}T + \sum_{i=0}^{T-1}\Xi_i\varepsilon_{T-i} \tag{18}$$

where

$$
\begin{aligned}
Y_0^* &= \Gamma Y_0 + (I_4 - \Gamma)\mu^{**} \\
\Xi_0 &= \Pi_0 \\
\Xi_1 &= (\Gamma\Pi_0 + \Pi_1) = \Gamma\Xi_0 + \Pi_1 \\
\Xi_2 &= (\Gamma\Pi_0 + \Gamma\Pi_1 + \beta\Pi_1) = \Gamma\Xi_1 + \beta\Pi_1 \\
&\ldots \\
\Xi_i &= \Gamma\Xi_{i-1} + \beta^{i-1}\Pi_1
\end{aligned}
$$

where T is the deterministic trend and Y_0 is the starting value of the Y_T process.

PIAR(1)

For illustrative purposes we briefly focus on the PIAR(1) case, i.e. equation (5) with $\beta_s = 0$ for all s. After some rewriting it is evident that the expression in equation (18) becomes

$$Y_T = Y_0^* + \Lambda\mu T + \Phi_0^{-1}\varepsilon_T + \Lambda \sum_{i=1}^{T-1} \varepsilon_{T-i} \tag{19}$$

where $\Lambda = \Gamma\Phi_0^{-1}$ i.e.

$$\Lambda = \begin{bmatrix} 1 & \phi_1\phi_3\phi_4 & \phi_1\phi_4 & \phi_1 \\ \phi_2 & 1 & \phi_1\phi_2\phi_4 & \phi_1\phi_2 \\ \phi_2\phi_3 & \phi_3 & 1 & \phi_1\phi_2\phi_3 \\ \phi_2\phi_3\phi_4 & \phi_3\phi_4 & \phi_4 & 1 \end{bmatrix} \tag{20}$$

since $\phi_1\phi_2\phi_3\phi_4 = 1$. This matrix Λ conveys the information on the impact of the stochastic trend in the four seasons, i.e. the impact of the accumulation of shocks in quarter s denoted by $\sum_{i=1}^{T-1}\varepsilon_{s,T-i}$. Given the three cointegration relations between the elements of Y_T the rank of Λ is equal to 1 (cf. Engle and Granger, 1987, Johansen, 1991), i.e.

$$\Lambda = \begin{bmatrix} 1 \\ \phi_2 \\ \phi_2\phi_3 \\ \phi_2\phi_3\phi_4 \end{bmatrix} [1 \ \ \phi_1\phi_3\phi_4 \ \ \phi_1\phi_4 \ \ \phi_1] \tag{21}$$

Furthermore, it can be derived that $\Lambda^k = 2^k\Lambda$ for $k = 1, 2, 3, \ldots$ Note that when all $\phi_s = 1$, the Λ matrix contains only ones, indicating that the impact of the stochastic trend is equal for all seasons.

From equations (19), (20) and (21) it can be observed, for example, that the impact of the accumulation of shocks in the four seasons on the observations in quarter 1 is

$$\sum_{i=1}^{T-1} \varepsilon_{1,T-i} + \phi_1\phi_3\phi_4 \sum_{i=1}^{T-1} \varepsilon_{2,T-i} + \phi_1\phi_4 \sum_{i=1}^{T-1} \varepsilon_{3,T-i} + \phi_1 \sum_{i=1}^{T-1} \varepsilon_{4,T-i} \tag{22}$$

Similar expressions can be derived for the other three quarters. This indicates that given $\phi_s \neq \phi$ for all s, this impact varies with the seasons. Similarly, this applies to the parameters $\Lambda\mu$ that correspond to the deterministic trend component. The (1×4) vector $[1 \ \ \phi_1\phi_3\phi_4 \ \ \phi_1\phi_4 \ \ \phi_1]$ in equation (21) conveys information on the relative importance of the accumulation of shocks in season s. If one wants to test whether some trend components have the same impact, the relevant hypotheses can be formulated in terms of the ϕ_s parameters. Under the restriction that

$\phi_1\phi_2\phi_3\phi_4 = 1$, one can apply χ^2 type test statistics for hypotheses like $\phi_1\phi_4 = 1$ or $\phi_1 = 1$ (see Boswijk and Franses, 1994, for formal proofs).

To conclude this section we note that the presence of three cointegration relations further implies that taking first-order differences of Y_T, which is equivalent to taking annual differences of Y_T, results in a non-invertible vector MA(1) model. In fact, from equation (19) we can derive that

$$Y_T - Y_{T-1} = \Lambda\mu + \Phi_0^{-1}\varepsilon_T + (\Lambda - \Phi_0^{-1})\varepsilon_{T-1} \tag{23}$$

with the characteristic equation for the moving average part

$$\xi(z) = |\Phi_0^{-1} + (\Lambda - \Phi_0^{-1})z| = (1 - \phi_1\phi_2\phi_3\phi_4 z)^3 = (1 - z)^3 = 0 \tag{24}$$

so that the vector model MA(1) has three unit roots.

FORECAST ERROR VARIANCES

In this section we derive explicit expressions for the multi-step-ahead forecast error variances in case the parameters are assumed known. These expressions simply follow from equations (18) and (19) for the PIAR(2) and PIAR(1) cases, respectively.

PIAR(2)
For the PIAR(2) process, we obtain

$$\begin{aligned}
\hat{Y}_{N+1} - Y_{N+1} &= \Xi_0\varepsilon_{N+1}\\
\hat{Y}_{N+2} - Y_{N+2} &= \Xi_0\varepsilon_{N+2} + \Xi_1\varepsilon_{N+1}\\
&\cdots\\
\hat{Y}_{N+h} - Y_{N+h} &= \Xi_0\varepsilon_{N+h} + \Xi_1\varepsilon_{N+h-1} + \cdots + \Xi_{h-1}\varepsilon_{N+1}
\end{aligned} \tag{25}$$

Under the assumption that $\varepsilon_T \sim N(0, \sigma^2 I_4)$, it is obvious that for equation (25)

$$E(\hat{Y}_{N+h} - Y_{N+h}) = 0 \text{ for all } h \tag{26}$$

Furthermore, since ε_T is independent of ε_{T+k} for any $k \neq 0$, one easily obtains

$$\begin{aligned}
E(\hat{Y}_{N+1} - Y_{N+1})^2 &= \sigma^2\Xi_0\Xi_0'\\
E(\hat{Y}_{N+2} - Y_{N+2})^2 &= \sigma^2(\Xi_0\Xi_0' + \Xi_1\Xi_1')\\
&\cdots\\
E(\hat{Y}_{N+h} - Y_{N+h})^2 &= \sigma^2\sum_{k=0}^{h-1}\Xi_k\Xi_k'
\end{aligned} \tag{27}$$

The diagonal elements of the matrices in equation (27) can now be used to calculate the forecast error variances of y_t. The assumption of Gaussianity of ε_t allows us to construct the conventional 90% or 95% forecast intervals.

PIAR(1)
To highlight some specific properties of the forecast error variances from PIAR processes, consider again the PIAR(1) process, for which we obtain that

$$\hat{Y}_{N+1} - Y_{N+1} = \Phi_0^{-1}\varepsilon_{N+1} \tag{28}$$

which, using equation (15), can be decomposed as

$$\hat{Y}_{1,N+1} - Y_{1,N+1} = \varepsilon_{1,N+1}$$
$$\hat{Y}_{2,N+1} - Y_{2,N+1} = \phi_2\varepsilon_{1,N+1} + \varepsilon_{2,N+1}$$
$$\hat{Y}_{3,N+1} - Y_{3,N+1} = \phi_2\phi_3\varepsilon_{1,N+1} + \phi_3\varepsilon_{2,N+1} + \varepsilon_{3,N+1}$$
$$\hat{Y}_{4,N+1} - Y_{4,N+1} = \phi_2\phi_3\phi_4\varepsilon_{1,N+1} + \phi_3\phi_4\varepsilon_{2,N+1} + \phi_4\varepsilon_{3,N+1} + \varepsilon_{4,N+1}$$

Obviously, given equation (26) it holds that

$$E(\hat{Y}_{s,N+1} - Y_{s,N+1}) = 0, \text{ for } s = 1, 2, 3, 4$$

Furthermore, given equation (27), the squared prediction errors for the forecasts for the observations in season s in year $N+1$ ($\text{SPES}_{s,N+1}$) are the diagonal elements of

$$E(\hat{Y}_{N+1} - Y_{N+1})^2 = \sigma^2\Phi_0^{-1}(\Phi_0^{-1})' \tag{29}$$

i.e. these are

$$\text{SPE}_{1,N+1} = \sigma^2$$
$$\text{SPE}_{2,N+1} = [\phi_2^2 + 1]\sigma^2$$
$$\text{SPE}_{3,N+1} = [\phi_2^2\phi_3^2 + \phi_3^2 + 1]\sigma^2$$
$$\text{SPE}_{4,N+1} = [\phi_2^2\phi_3^2\phi_4^2 + \phi_3^2\phi_4^2 + \phi_4^2 + 1]\sigma^2$$

Comparing these expressions for the SPEs with those of the non-periodic integrated process $y_t = y_{t-1} + \varepsilon_t$, which are σ^2, $2\sigma^2$, $3\sigma^2$ and $4\sigma^2$, it is clear that a PI process allows the forecast intervals to vary with the seasons. This property reflects the seasonal heteroscedasticity in the PIAR process that is present within sample. Note that the SPE is smallest in the first quarter since all forecasts are generated from quarter 4 in year N onwards. In fact, if one generates forecasts from quarter i in year N, the SPE in quarter $i+1$ will be smallest, $i = 1, 2, 3$.

Using expressions (25) and (19) it can be derived that the PIAR(1) forecast errors for h years ahead are

$$\hat{Y}_{N+h} - Y_{N+h} = \Phi_0^{-1}\varepsilon_{N+h} + \Lambda\sum_{j=1}^{h-1}\varepsilon_{N+j}$$

Given equation (27), one can easily derive that

$$E(\hat{Y}_{N+h} - y_{N+h})^2 = \sigma^2[\Phi_0^{-1}(\Phi_0^{-1})' + (h-1)\Lambda\Lambda'], \text{ for } h = 2, 3, \ldots \tag{30}$$

and hence that the squared prediction errors $\text{SPE}_{s,N+h}$, based on quarter 4 in year N, are

$$\text{SPE}_{1,N+h} = \sigma^2 + (h-1)[1 + \phi_1^2\phi_3^2\phi_4^2 + \phi_1^2\phi_4^2 + \phi_1^2]\sigma^2$$
$$\text{SPE}_{2,N+h} = [\phi_2^2 + 1]\sigma^2 + (h-1)[\phi_2^2 + 1 + \phi_1^2\phi_2^2\phi_4^2 + \phi_1^2\phi_2^2]\sigma^2$$
$$\text{SPE}_{3,N+h} = [\phi_2^2\phi_3^2 + \phi_3^2 + 1]\sigma^2 + (h-1)[\phi_2^2\phi_3^2 + \phi_3^2 + 1 + \phi_1^2\phi_2^2\phi_3^2]\sigma^2$$
$$\text{SPE}_{4,N+h} = [\phi_2^2\phi_3^2\phi_4^2 + \phi_3^2\phi_4^2 + \phi_4^2 + 1]\sigma^2 + (h-1)[\phi_2^2\phi_3^2\phi_4^2 + \phi_3^2\phi_4^2 + \phi_4^2 + 1]\sigma^2$$

To conclude this section, we state that for PIAR processes of orders 3 and higher, one can derive the expressions for the forecast error variances along similar lines as in this section. A useful strategy is then to write such models as equation (6) since this facilitates the construction of the expressions for the error variances.

TWO APPLICATIONS

In this section the empirical multi-step-ahead forecast error variances are calculated for a PIAR(1) and a PIAR(2) process. The first model was found useful to describe the log of quarterly non-durables consumption in the UK for the sample 1955.1–1988.4 (see Boswijk and Franses, 1994). The PIAR(2) process was found adequate in describing the log of real GNP in Germany for the sample period 1960.1–1990.4 (see Franses, 1995). In this section, we re-estimate these models using the same observations except the last 7 years. These 28 quarterly observations will be used to evaluate the out-of-sample forecasting performance of the models.

PIAR(1): UK consumption non-durables

For the UK non-durables consumption series the parameter estimates for the PIAR(1) process

$$y_t = \mu_s + \phi_s y_{t-1} + \varepsilon_t, \text{ with } \phi_1\phi_2\phi_3\phi_4 = 1 \tag{31}$$

for the sample 1955.1–1981.4 are

$$\hat{\mu}_1 = -0.104 \quad \hat{\mu}_2 = 0.752 \quad \hat{\mu}_3 = -0.359 \quad \hat{\mu}_4 = -0.309$$
$$\hat{\phi}_1 = 1.001 \quad \hat{\phi}_2 = 0.933 \quad \hat{\phi}_3 = 1.036 \quad \hat{\phi}_4 = 1.034$$

and σ equals 0.01158. Formal tests on the equality of the ϕ_s estimates and on the restriction that all ϕ_s are equal to unity, result in rejections of the respective hypotheses. Hence, equation (31) is found to be more appropriate than a non-periodic I(1) model. Moreover, a range of diagnostic tests indicates that equation (31) does not seem to be misspecified (see Boswijk and Franses, 1994, for additional details). The estimation results for equation (31) imply that the relevant elements Λ and $\Lambda\mu$ in equation (19) are estimated as

$$\hat{\Lambda} = \begin{bmatrix} 1 & 1.072 & 1.035 & 1.001 \\ 0.933 & 1 & 0.965 & 0.934 \\ 0.967 & 1.036 & 1 & 0.967 \\ 1.000 & 1.071 & 1.034 & 1 \end{bmatrix} \text{ and } \hat{\Lambda}\hat{\mu} = \begin{bmatrix} 0.021 \\ 0.020 \\ 0.021 \\ 0.021 \end{bmatrix}$$

The elements of the $\hat{\Lambda}$, matrix suggest that the accumulation of the second-quarter shocks is most important since the elements in the second column take the highest values across all four columns. Furthermore, the total accumulation of shocks has the highest overall impact on the observations in the first and fourth quarters since these values are highest across rows.

In Figure 1 we display the estimated forecast error variances of the PIAR(1) process for UK non-durables consumption, measured relative to the first quarter SPE. It can be seen from this figure that the SPEs of this PIAR(1) process show seasonal patterns since, for example, the SPE_4-SPE_1 is constant and about $3\sigma^2$. It appears that, relative to quarters 1 and 4, the time series can be forecast more precise in quarters 2 and 3. In Figure 2 we display the multi-step-ahead forecasts yf_t for 1982.1–1988.4 generated from the PIAR(1) model (31), the time series y_t itself and the 95% forecast confidence intervals. It is clear that all 28 forecasts lie within this region, although the forecast for 1988.4 is close to the boundary.

PIAR(2): GNP in Germany

The second example is provided by a PIAR(2) process for the log of real GNP in Germany, which is estimated for sample period 1960.1–1983.4. The estimation results for this process as

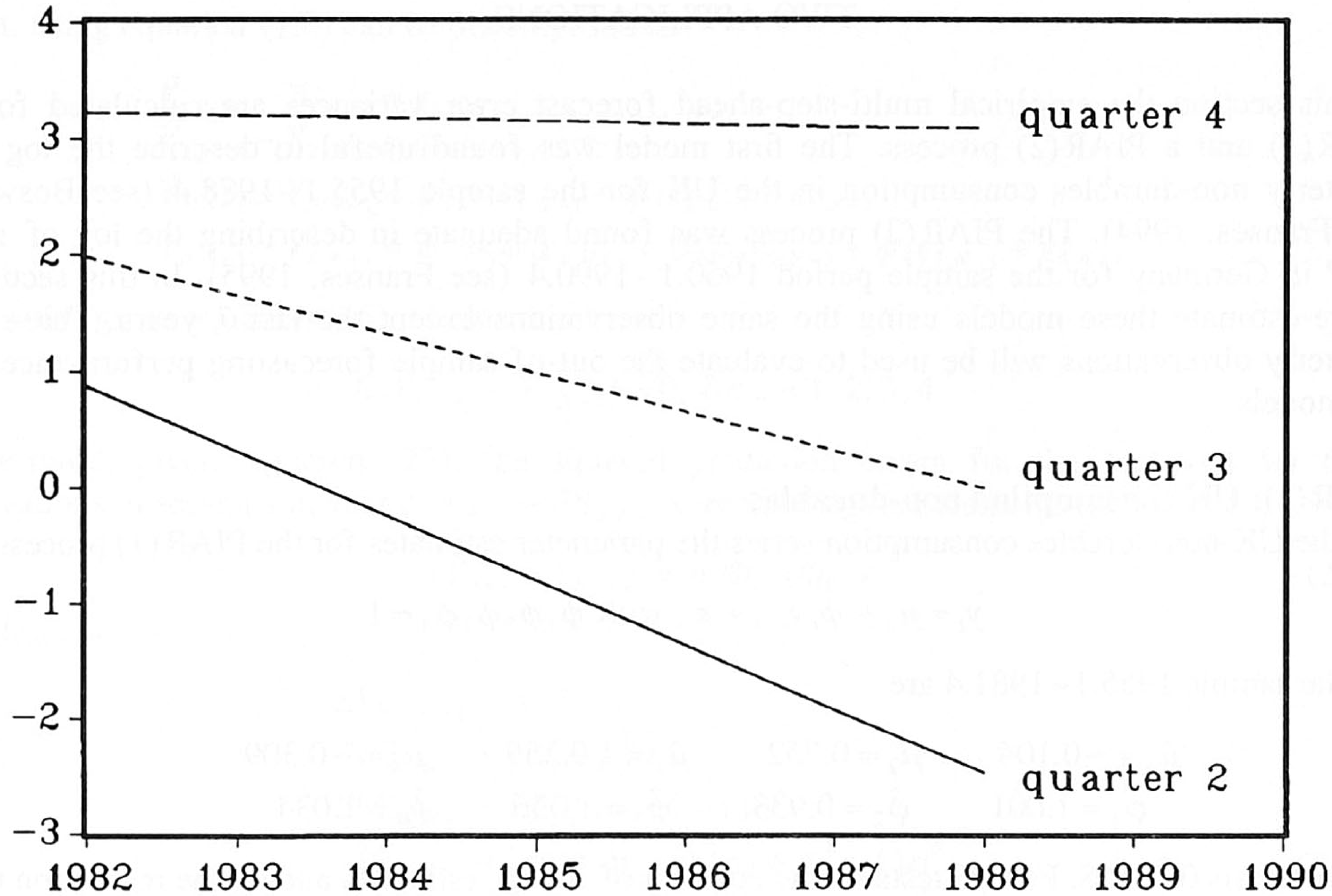

Figure 1. Forecast error variances relative to first quarter. Quarter s denotes SPE_s-SPE_1, for $s = 2, 3, 4$. PIAR(1) for UK non-durables

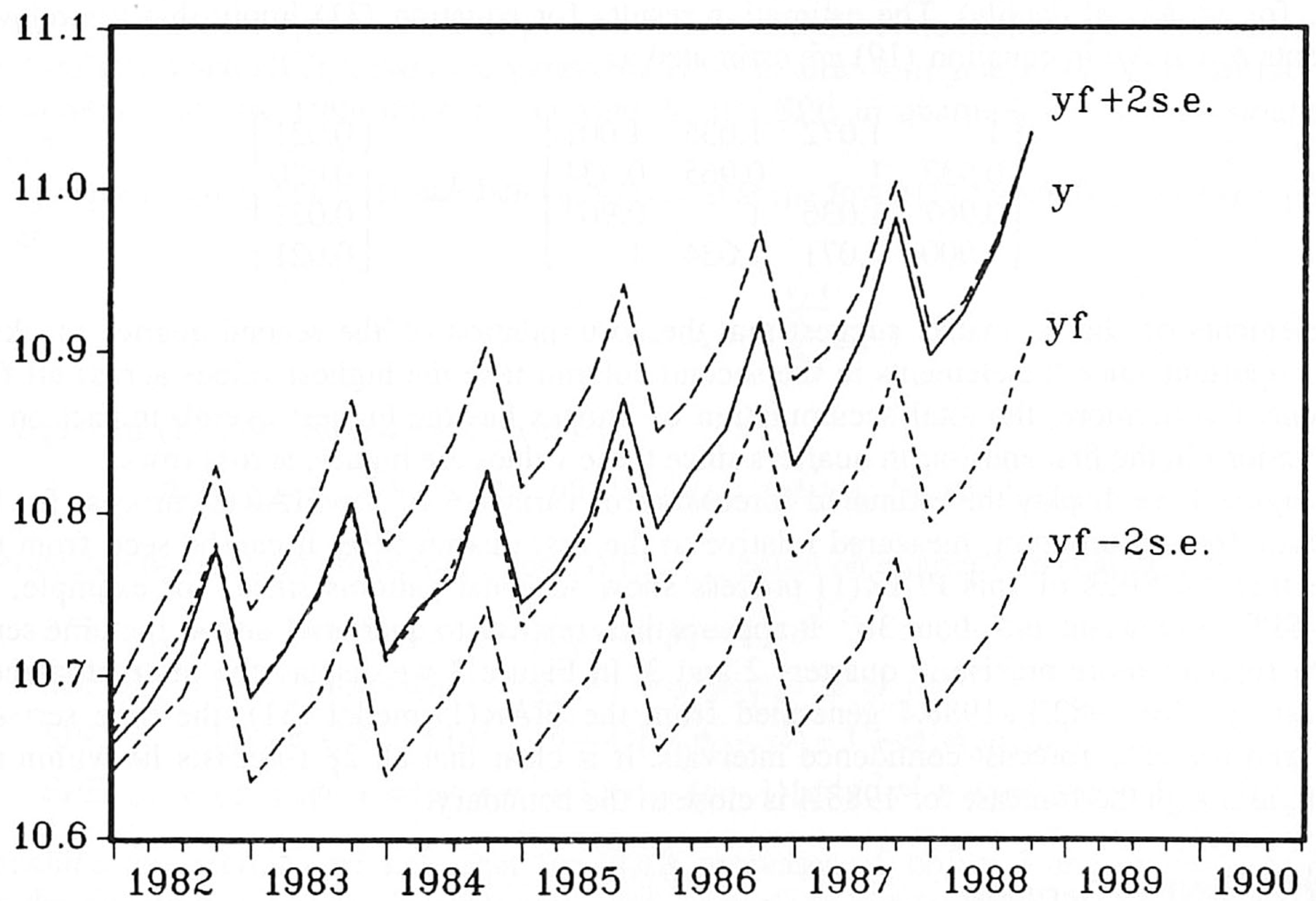

Figure 2. Out-of-sample forecasts for a PIAR(1) process. y is the time series, yf is the forecasted time series and $yf\pm 2se$ indicates the 95% forecast interval.

in equation (5) with $\phi_1\phi_2\phi_3\phi_4 = 1$ are

$$\hat{\mu}_1 = 0.004 \quad \hat{\mu}_2 = 0.137 \quad \hat{\mu}_3 = 0.560 \quad \hat{\mu}_4 = -0.646$$
$$\hat{\phi}_1 = 1.030 \quad \hat{\phi}_1 = 0.954 \quad \hat{\phi}_1 = 0.892 \quad \hat{\phi}_1 = 1.141$$
$$\hat{\beta}_1 = 0.309 \quad \hat{\beta}_2 = -0.665 \quad \hat{\beta}_3 = 0.351 \quad \hat{\beta}_4 = -0.221$$

and $\sigma = 0.0145$. Diagnostic test results reveal that the ϕ_s cannot be set equal to some ϕ, and hence that they are not all equal to 1. With these parameter estimates one can calculate the estimated versions of Γ, Π_0, Π_1 Π_1 and Ξ_i in equation (18).

In Figure 3 the estimated forecast error variances are displayed for this PIAR process of order 2, relative to the first quarter. Again these intervals show a marked seasonal pattern. It can be observed that, relative to the other quarters, the forecasts for the first quarter become increasingly less precise. This reflects the dominance of the stochastic trend and deterministic trend components in this quarter.

In Figure 4, we display the 28 forecasts and the forecast intervals. It is clear that the out-of-sample forecasts are well within the 95% boundaries. In fact, it can be calculated that even for a 75% confidence interval, the true observations do not exceed the boundaries. Only when the confidence interval is based on one standard error does the observation in 1987.1 not lie within this interval.

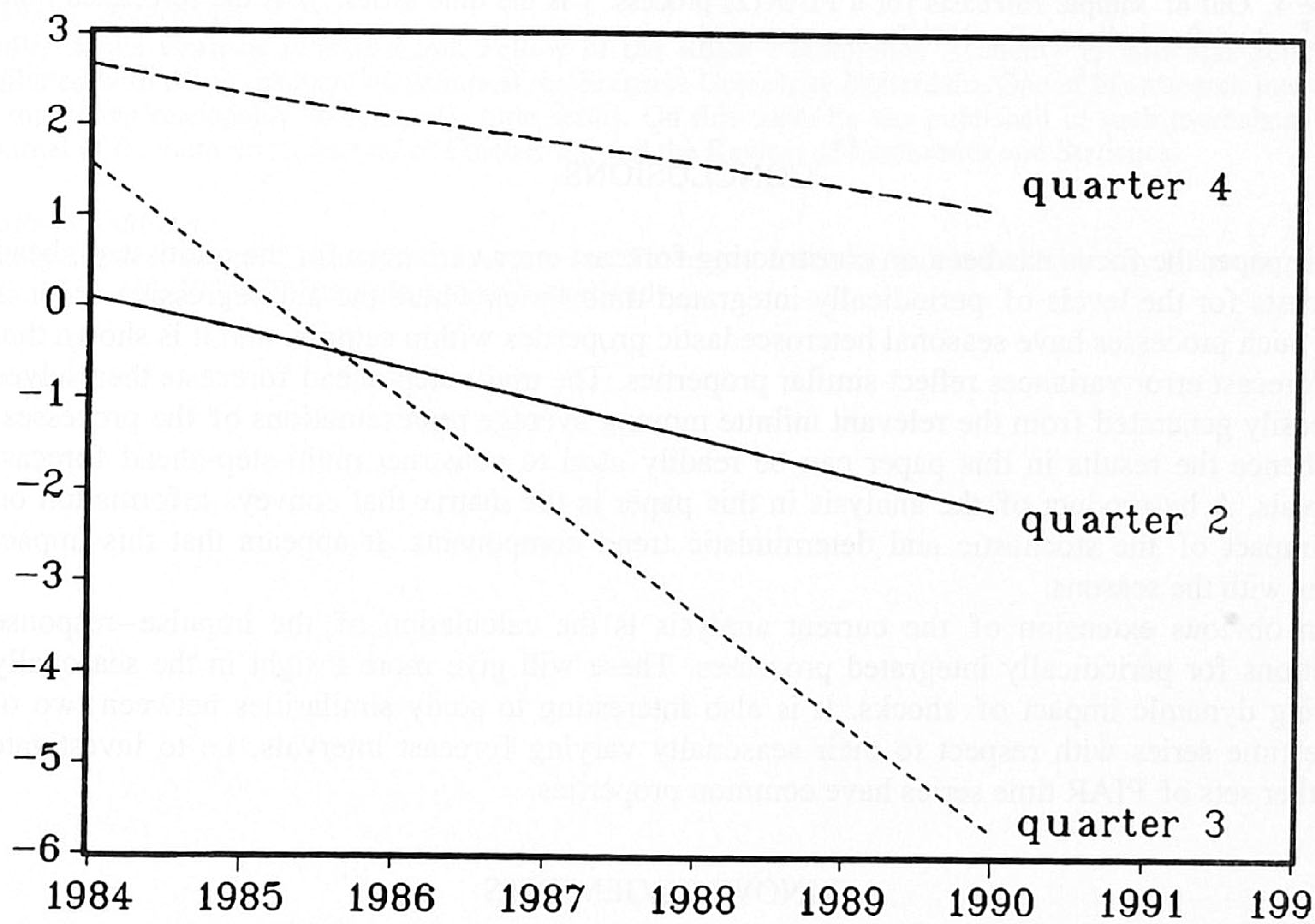

Figure 3. Forecast error variance relative to first quarter. Quarter s denotes SPE$_s$-SPE$_{-1}$, for $s = 2, 3, 4$. PIAR(2) for GNP Germany

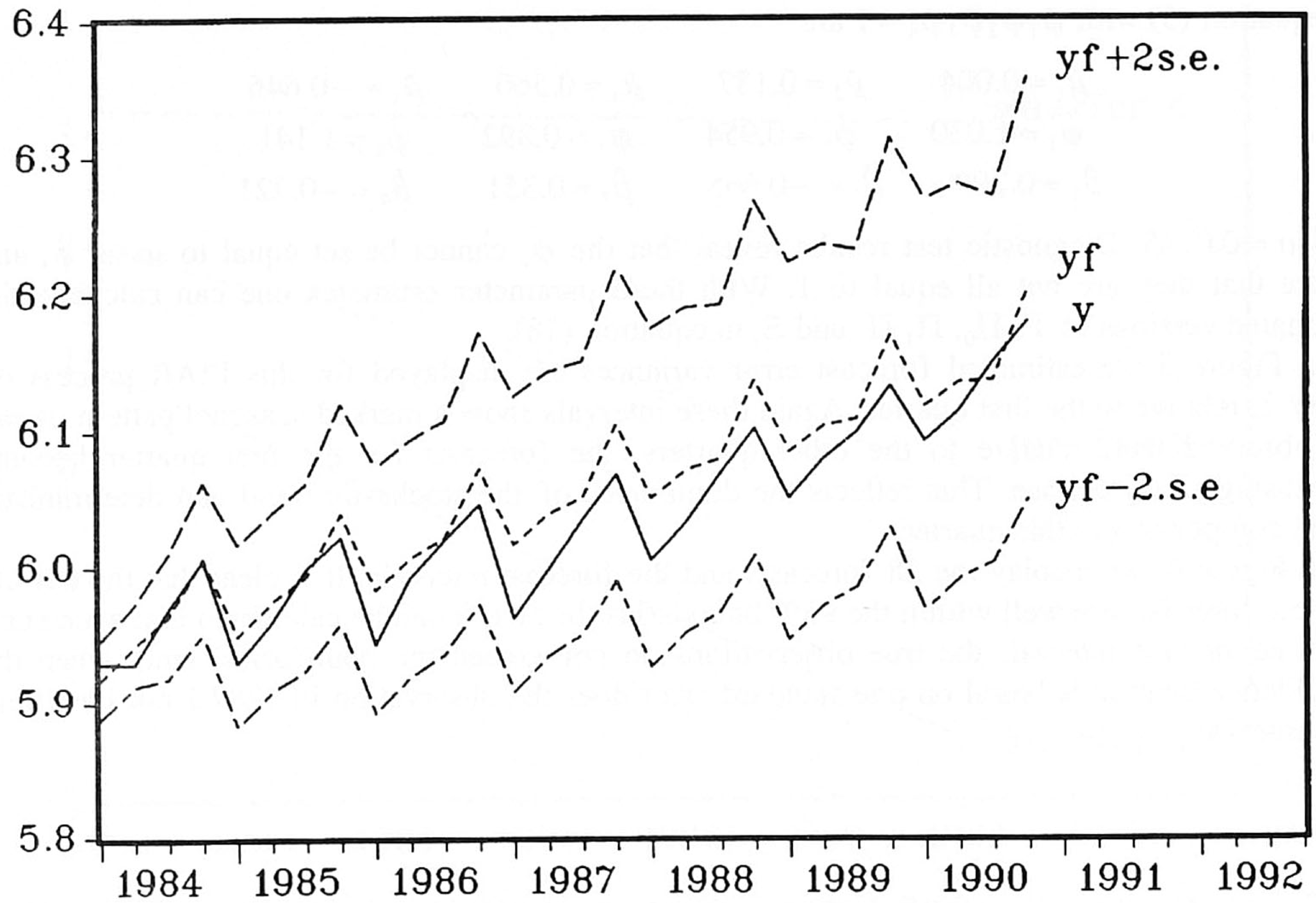

Figure 4. Out of sample forecasts for a PIAR(2) process. *y* is the time series, *yf* is the forecasted time series and *yf*±2*se* indicates the 95% forecast interval.

CONCLUSIONS

In this paper the focus has been on constructing forecast error variances for the multi-step-ahead forecasts for the levels of periodically integrated time series where the autoregressive order is low. Such processes have seasonal heteroscedastic properties within sample, and it is shown that the forecast error variances reflect similar properties. The multi-step-ahead forecasts themselves are easily generated from the relevant infinite moving average representations of the processes, and hence the results in this paper can be readily used to construct multi-step-ahead forecast intervals. A by-product of the analysis in this paper is the matrix that conveys information on the impact of the stochastic and deterministic trend components. It appears that this impact varies with the seasons.

An obvious extension of the current analysis is the calculation of the impulse–response functions for periodically integrated processes. These will give more insight in the seasonally varying dynamic impact of shocks. It is also interesting to study similarities between two or more time series with respect to their seasonally varying forecast intervals, i.e to investigate whether sets of PIAR time series have common properties.

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