Determining The Optimal Order Picking Batch Size In Single Aisle Warehouses

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BIBLIOGRAPHIC DATA AND CLASSIFICATIONS		
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DETERMINING THE OPTIMAL ORDER PICKING BATCH SIZE IN SINGLE AISLE WAREHOUSES

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Abstract

This work aims at investigating the influence of picking batch size to average time in system of orders in a one-aisle warehouse under the assumption that order arrivals follow a Poisson process and items are uniformly distributed over the aisle's length. We model this problem as an $M/G^k/1$ queue in which orders are served in batches of exactly k orders. The average time in system of the $M/G^k/1$ queue is difficult to obtain for general service times. To circumvent this obstacle, we perform an extensive numerical experiment on the average time in system of the model when the service time is deterministic ($M/D^k/1$) or exponentially distributed ($M/M^k/1$). These results are then compared with the corresponding times in system of the actual model taken from simulation runs. A variance analysis is carried out and its result elicits that the $M/D^k/1$ queue is a very good approximation for the average time in system of orders. Correspondingly, the optimal picking batch size of the real system can be approximated by the optimal batch size when service time is deterministic.

Keywords: order picking, warehousing, batch picking, batch size, batch service queue

1 Problem description, assumptions and notations

1.1 Problem description

Order picking, the process of retrieving items from their storage locations to fill customers' orders, is known as the most time-consuming and laborious component of the warehousing activities. Nowadays, with the introduction of e-commerce, customers can order things they need electronically with their computers or mobile phones via the Internet. Because the process of ordering is fast, the customer expects a comparable fast delivery (Roodbergen [11]). This fact makes order picking operations become a strong candidate for productivity improvement studies. There are four essential factors that greatly influence the performance and efficiency of the order picking operations. They are: (a) layout of the warehouse, (b) the routing and sorting policy, (c) the storage strategy and (d) the batching method. While the first three topics have received much attention from researchers, the last one is largely unexplored.

In this paper we consider the problem of determining the 'optimal' batch size for the order picking operation in a single-aisle warehouse (Figure 1). We use the following assumptions: every order has a quantity of one, arrivals of orders follow a Poisson process and items are uniformly distributed over the aisle's length. Single-line, single-item orders that arrive in a random pattern can commonly be observed in mail order and online retailing companies that focus on specialized product types, such as books, computers or CD's. For small batches, the service time will mainly be determined by the set-up time, that is, the time needed to acquire an empty pick device (pallet, cart, bin), to retrieve a pick list and, at the end of the pick tour, to set down the pick device and confirm the picks. For large batches, the service time will be mainly determined by picking and traveling activities. In this respect, there is a similarity with the optimization of manufacturing batch sizes (see, for example, Tielemans [13]). The major difference is that the processing time is not linear in the batch size (as is the case in manufacturing models), since the travel time gives an additional non-linear service time component.

Figure 1 A single-aisle rectangular warehouse

Clearly, we can increase the efficiency of the order picking process in such environments by serving a group of orders instead of individual orders. If the order picker starts a tour for every order, the capacity may even be insufficient to serve all orders. If the order picker waits to have a sufficient large number of orders, the average time in system of the orders may be longer than desired. The critical issue is, therefore, to determine how many orders an order picker should serve in a tour to minimize the average time in system of orders.

1.2 Assumptions and notations

In modeling, this system, we use the following assumptions:

- Single item orders (meaning that every order consists of only one item), and
- Inter-arrival times of orders are independent identically and exponentially distributed with mean $\frac{1}{\lambda}$.
- Service times of batches (with the same size) are also independent identically distributed random variables.
- Picking is performed per batch of exactly q orders.
- There is infinite queue space.
- Setup is performed per batch and setup time is independent of the batch size.
- Products (items to be picked) are uniformly stored over the aisle.

We use the following notations:

λ	the arrival rate of orders
$s = 1/\mu$	the average service time of an order (note that this is also the service time of the batch)
τ	the set up time of a batch (supposed to be insensitive to the batch size)
r	picking rate (items/ minute)
L_0	the length of the aisle (in minutes travel)
q	the batch size (number of orders in a picking batch)
p_j	the probability that there are j orders in the system at a random epoch
p_j^+	the probability that there are j orders in the system at departure epochs
$P(z)$	the probability generating function of $\{p_j\}$
$P^+(z)$	the probability generating function of $\{p_j^+\}$
W	the average time in system of orders (or average throughput time of orders)
L	the average system size (number of orders in system at a random epoch)

2 Literature review on $M/G^q/1$

The problem described above can be modeled as an $M/G^q/1$ queueing model where M indicates the exponentially distributed inter-arrival time, G^q indicates that the service time has a general distribution form and services are performed in batch of exactly q orders. Obviously, this queueing model has many applications in both distribution logistics and production management but, surprisingly, not many results, which systematically deal with computational aspects of system measurements such as the average system size and the average queue time, are available. Although many approximation methods have been developed for many kinds of none-Markovian queues (see Tijms [14] and Van Hoorn [8]), no approximation scheme on $M/G^q/1$ has been found.

Foster and Nyunt [4] derive the equilibrium distribution of the system-size at instants just after departures as follows:

$$P^+(z) = \frac{(q-\rho)(z-1)\prod_{j=1}^{q-1}\frac{(z-\delta_j)}{(1-\delta_j)}}{\frac{z^q}{K(z)}-1} \qquad (1)$$

Where $P^+(z) = \sum_{j=0}^{\infty} p_j^+ z^j$ is the probability generating function of the system-size at departure epochs, $K(z)$ is the Laplace-Stieltjes transform of the cumulative service time distribution function. $\rho = \frac{\lambda}{\mu}$. And, δ_j with $j = 1,..,(q-1)$ are $(q-1)$ roots inside the unit circle of the characteristic equation $z^q = K(z)$. It follows from Rouche's theorem that this equation has exactly $(q-1)$ roots inside the unit circle. A detailed explanation can be found in Saaty [12] (p. 175).

Foster and Perera [5] show that the relation between the probability generating function of the system size at random epochs, $P(z)$, and at departure epochs, $P^+(z)$, can be expressed by the following formula:

$$P(z) = \frac{1-z^q}{q(1-z)} P^+(z) \qquad (2)$$

Substituting (1) into the right-hand size of (2), we obtain:

$$P(z) = \frac{(1-z^q)(1-\frac{\rho}{q})\prod_{j=1}^{q-1}\frac{(z-\delta_j)}{(1-\delta_j)}}{1-\frac{z^q}{K(z)}} \qquad (3)$$

If we know the form of service time distribution then the steady state probability $\{p_n\}$ can be obtained theoretically by successive differentiation of $P(z)$. Nevertheless this work is cumbersome when q becomes large.

3 Time in system analysis

Again, we can theoretically find the average time in system of orders of an $M/G^q/1$ queue by taking limitation of $P'(z)$ when z reaches to 1. We have:

$$W = \frac{1}{\lambda}\frac{d}{dz}P(z)\Big|_{z=1}$$

We note that, for $z=1$, $P(z)$ in equation (3) is indeterminate of the $0/0$ form. Therefore, we proceed as follows. Let $N(z)$ and $D(z)$ denote the numerator and denominator of the right-hand side of equation (3) respectively. Then we use following well-known result in queueing theory (see Madan [20]):

$$W = \frac{1}{\lambda}\frac{d}{dz}P(z)\Big|_{z=1} = \frac{1}{\lambda}P'(1) = \frac{1}{\lambda}\lim_{z\to 1}\frac{N'(z)D''(z)-D'(z)N''(z)}{2\left(N'(z)\right)^2}$$

$$= \frac{1}{\lambda}\frac{N'(1)D''(1)-D'(1)N''(1)}{2\left(N'(1)\right)^2} \tag{4}$$

When the service time has a general form, it is cumbersome to get numerical results for W. Chaudhry [1] mentions a closed-form expression in terms of the roots of certain character equations for computing the queueing time but he only considers the queueing time of the last customer in the service batch. This time is, of course, different from the waiting time of a random customer. When the service time has particular forms, we may come up with a more concrete formulation for computing values of W. Especially, when the service time is an exponentially distributed random variable, we can derive a closed-form formula for W. In the remaining part of this paper, average times in system under deterministic and exponential service time are considered.

3.1 Average time in system of orders in an $M/M^q/1$ queue

This is a very special case of the $M/G^q/1$ queues because the $M/M^q/1$ queue can be modeled as a non-birth-death Markovian queue. As shown in Gross and Harris (p. 125) [6]:

$$p_n = \begin{cases} \dfrac{p_0(1-r_0^{n+1})}{1-r_0} & (1 \le n \le q) \\ \dfrac{p_0 \lambda r_0^{n-q}}{\mu} & (n > q) \end{cases} \tag{5}$$

$$p_0 = \frac{1-r_0}{q}$$

Where r_0 is the unique root (real and less than 1) of the characteristic equation:

$$\mu r_0^{q+1} - (\lambda+\mu) r_0 + \lambda = 0 \tag{6}$$

By definition, $L = \sum_{n=0}^{\infty} n p_n$. Replacing p_n with the corresponding formula given in (5) and after some algebraic operations (see appendix A) we have:

$$L = \frac{q-1}{2} + \frac{\lambda}{\mu q}\frac{r_0^{1-q}}{1-r_0} + r_0\left(\frac{\lambda(r_0^{-q} - r_0^{1-q}) - \mu r_0}{\mu q}\right)\left(\frac{1-r_0^q - q r_0^{q-1}(1-r_0)}{(1-r_0)^2}\right)$$

Applying Little's formula we obtain:

$$W = \frac{1}{\lambda}\left[\frac{q-1}{2} + \frac{\lambda}{\mu q}\frac{r_0^{1-q}}{1-r_0} + r_0\left(\frac{\lambda(r_0^{-q} - r_0^{1-q}) - \mu r_0}{\mu q}\right)\left(\frac{1-r_0^q - q r_0^{q-1}(1-r_0)}{(1-r_0)^2}\right)\right] \tag{7}$$

In order to get the value of W, we first solve the characteristic equation (6) and then substitute the root into (7). Numerical computation is easy in this case.

3.2 Average time in system of orders in an $M/D^q/1$ queue

If the service time is deterministic we have:

$$\begin{aligned} K(z) &= \sum_{j=0}^{\infty} \frac{e^{-\lambda q s}(\lambda q s)^j}{j!} z^j \\ &= e^{-q\rho} \sum_{j=0}^{\infty} \frac{(q\rho z)^j}{j!} \\ &= e^{-q\rho} e^{q\rho z} \\ &= e^{-q\rho(1-z)} \end{aligned} \tag{8}$$

Substituting (8) into (3), we obtain:

$$P(z) = \frac{(1-z^q)(q-\rho)\prod_{j=1}^{q-1}\frac{(z-\delta_j)}{(1-\delta_j)}}{q\left(1-\frac{z^q}{e^{-q\rho(1-z)}}\right)} \tag{9}$$

Where δ_j, $j = 1,..,(q-1)$, now become the $(q-1)$ roots inside unit circle of the equation:

$$z^q = e^{-q\rho(1-z)} \tag{10}$$

In the literature, several solution methods have been proposed for finding the roots of this equation. The common technique used is transforming (10) into $\left\lceil \frac{q-1}{2} \right\rceil$ independent equations each having only one root inside the unit circle. These roots and their conjugate roots form $(q-1)$ roots that we need (see Appendix D). The literature on this topic can be found in Chaudhry [1] or Chaudhry et al. [2].

When the $(q-1)$ roots of the equation (10) are known, we can apply the formula given in (4) to find W. As mentioned early, successive differentiations are cumbersome when the batch size is large, but in this case we only need to take the first and second order derivation of the numerator and denominator function of the generating function. Added to this, the derivative operator is available in many common mathematical software packages, such as Maple or Matlab. This favorableness makes it possible to perform numerical analysis on values of W even for large values of the batch size.

4 Experiment design

We start from a problem called 'seed' problem. In the seed problem, the aisle length equals 40 seconds of walking, the picking time per item is 20 seconds, the setup time of a batch is 90 seconds and the mean inter-arrival time of orders is 60 seconds. This configuration is typical for a shelf-type warehouse. From this seed problem, we expand one parameter at a time and fix the others. The feasible range of the batch size is between its lower bound and infinity. For the ease of computation, we choose the maximum batch size equal to 30. In practice, this upper limit is determined by the order picker's capacity (or the capacity of a group of order pickers who work together on the same batch). Each time when we vary the value of one parameter, the mean service time of a batch will change (see the average service time formula given in Appendix B). These changes, of course, will lead to fluctuation in the value of the traffic density. Based on the general condition for a queueing system to reach equilibrium, a lower bound for the batch size is determined as shown in Appendix C.

To be able to make comparisons, we define 3 ranges for values of the batch lower bound: low, medium and high. The low range is between 1 and 5, the medium range is between 6 and 10, and the high range covers the remaining possibilities. For each set of parameters, 6 values are randomly chosen in such a way that their corresponding batch size lower bounds constitute 3 groups of two values and each group will fall into one of the different ranges defined above. Table 1 below shows 25 sets of parameters that we used for experimenting in which set 1 is the seed set. Corresponding lower bounds as well as traffic density ranges are listed in the table. It is noted that for a given set of parameters, the traffic density is highest when the batch size equals its lower bound and it decreases when the batch size increases.

Table 1 Parameters of test problems

Set up time (τ)	Picking rate (r)	Aisle's length (L_0)	Arrival rate (λ)	Lower bound (q_{LB})	Traffic density range		Set index
					Minimum	Maximum	
1.5	3	0.667	1	4	0.426366	0.975133	1
0	3	0.667	1	2	0.376366	0.778	2
0.2	3	0.667	1	2	0.383032	0.878	3
2	3	0.667	1	5	0.443032	0.955667	4
4	3	0.667	1	8	0.509699	0.981556	5
7	3	0.667	1	13	0.609699	0.967081	6
8	3	0.667	1	14	0.643032	0.993695	7
1.5	10	0.667	1	3	0.193032	0.9335	8
1.5	8	0.667	1	3	0.218032	0.9585	9
1.5	2	0.667	1	6	0.593032	0.940571	10
1.5	1.5	0.667	1	9	0.759699	0.966733	11
1.5	1.4	0.667	1	10	0.807318	0.985558	12
1.5	1.35	0.667	1	11	0.833773	0.988271	13
1.5	3	0	1	3	0.383333	0.833333	14
1.5	3	0.4	1	4	0.40914	0.868333	15
1.5	3	0.8	1	5	0.434946	0.9	16
1.5	3	1.5	1	7	0.480108	0.922619	17
1.5	3	3	1	11	0.576882	0.969697	18
1.5	3	4	1	14	0.641398	0.97381	19
1.5	3	0.667	0.5	2	0.213183	0.764	20
1.5	3	0.667	1.1	5	0.469002	0.941233	21
1.5	3	0.667	1.2	6	0.511639	0.928686	22
1.5	3	0.667	1.5	9	0.639548	0.9501	23
1.5	3	0.667	1.7	11	0.724822	0.987468	24
1.5	3	0.667	1.9	15	0.810095	0.981746	25

From the table we can see that we have covered a very broad traffic density range, between 0.193 and 0.994. So we believe that our experiment will present most real life situations.

As we will see later, the average system time of $M/G^q/1$ is a convex function of batch size and in every case the optimal batch size (the value at which the average time in system reaches the minimum) is very close to the lower bound. Therefore, considering batch size values in a reasonably narrow range, not too far from the lower bound, is sufficient. This means that the upper batch size value is not really a restriction.

To investigate influences of picking batch size on the average time in system of orders (or average throughput time of orders) we built the following models.

4.1 Exponential service time model

Numerical results for the system under the exponential service case were obtained from a $M/M^q/1$ model built in Microsoft Excel using Visual Basic for Applications ([16]). The core of the model is the formula given in (7). Outputs of the model (average time in system of orders) under many cases were compared with corresponding results obtained from QTS software of Gross and Harris (see Gross and Harris [7]). In all cases, no difference was found. By building this model, we can compute values of average time in system over a range of batch sizes (Gross and Harris's package only allows to evaluate the system at a single batch size value). Running times of the model in all cases were negligible.

4.2 Deterministic service model

In order to find numerical results of $M/D^q/1$ queue, we need to take the first and second order derivates of the numerator and denominators function of the steady state probability generating function (see the formula given in (9)). We used derivative operators built-in Maple 6.01 ([15]) to perform this task.

As we mentioned earlier, finding robust roots for the characteristic equations (10) is not an easy task. Sensitive factors, which may influence the quality of roots and CPU running time, are starting values of variables (see e.g. Chaudhry et al. [2] and Powell [10]). In this model, we used the starting scheme given by Gross & Harris (see QTS software – $M/D/c$ model, Gross and Harris [7]). As a consequence of derivative operations, the model consumed considerable time especially when the maximum value of the batch size was large. It might take up to 30 seconds (on a Pentium III processor PC), when the upper limit of batch size equaled 30.

4.3 Simulation model

To get roughly numerical results for the system under actual service time distribution, we built a simulation model using AutoMod 9.1 ([18]). Then we used AutoStat 3.1, a companion module of AutoMod 9.1, as an efficient tool for making warming-up analysis and performing batch running. The model did need warming-up time specially when the batch size was close to its lower bound. This fact can be explained as follows. When the batch size is close to the lower bound, the corresponding traffic density of the system gets very close to 1 and as a consequence the system is not stable and thus takes some times to reach equilibrium. We experienced that different test sets required different warming-up time periods. We made several pilot runs and decided to take 4 hours as the common warming-up period for all test sets.

Various works had been done for validating the model. The model was considered in both deterministic and exponential service case and its outcomes were compared with results obtained from the analytical models respectively. Even in the worst case only a small difference was found. The simulation run length was set to 8 hours. The number of replications for each test set was chosen in such a way that it was sufficient to provide a 95% confidence interval with a half-width of less than 2.5% of the sample mean. Again, we experienced that different test sets required different numbers of replications. For example, set 2,4,8,10 needed about 20 replications while set 11,12,13 asked for more than 100 replications.

5 Some results and discussions

For each set of parameters listed in Table 1, we used the 3 models mentioned above to compute the average time in system of orders, for all batch size values between the corresponding lower bound and the upper limit. For the deterministic and exponential model we took the mean service time of a batch equal to the average service time defined by the formula given in Appendix C. We found the following results.

5.1 Average time in system of orders is a convex function of batch size

We first consider the behavior of the average time in system of orders when the service time has a general form. The average time in system of orders appears to be a convex function of the batch size between its lower and upper bounds and this result holds for all test sets. When we increase the batch size from its lower bound, the average time in system of orders decreases, rapidly reaches minimum and then from the minimum it monotonously increases. For large batch size values, the average time in system behaves as a linear function of batch size. The reason for this finding is as follows. When the batch size value is large the average time in system is mainly determined by the service time component (as there are many arrivals during a batch service period). Added to this, for a large value of the batch size the order picker has to entirely traverse the aisle to pick all requested orders in a batch. As a consequence, the travel is insensitive to batch size for large batches. The set up time of a batch is also constant. Therefore, the picking time, which is a linear increasing function of the batch size, stipulates the linearly behavior of the average time in system. Figure 2 depicts the average time in system as a function of the batch size for some test sets.

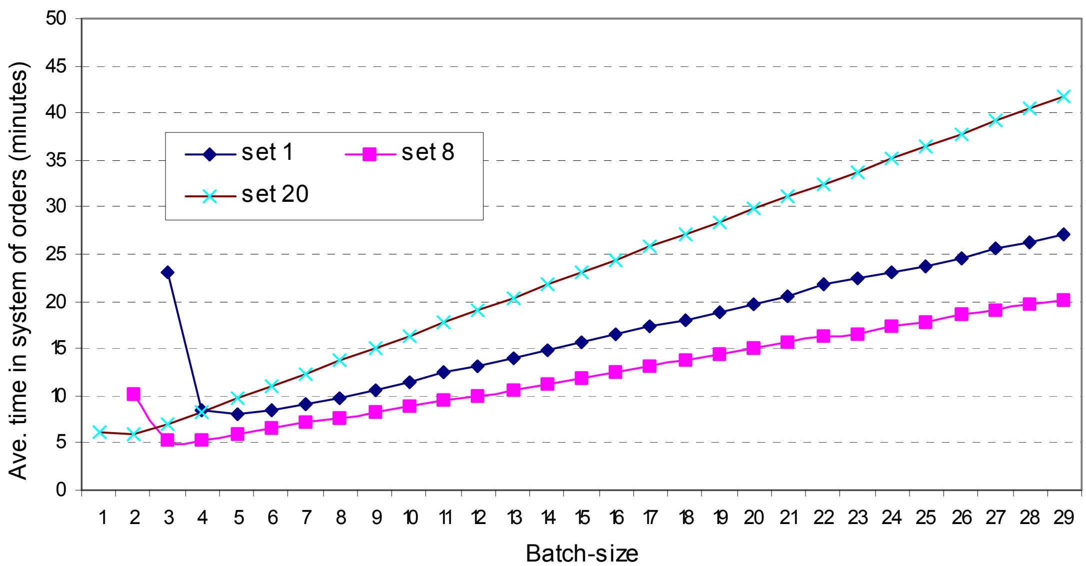

Figure 2 Average time in system of $M/G^q/1$, based on simulation runs, as a function of q.

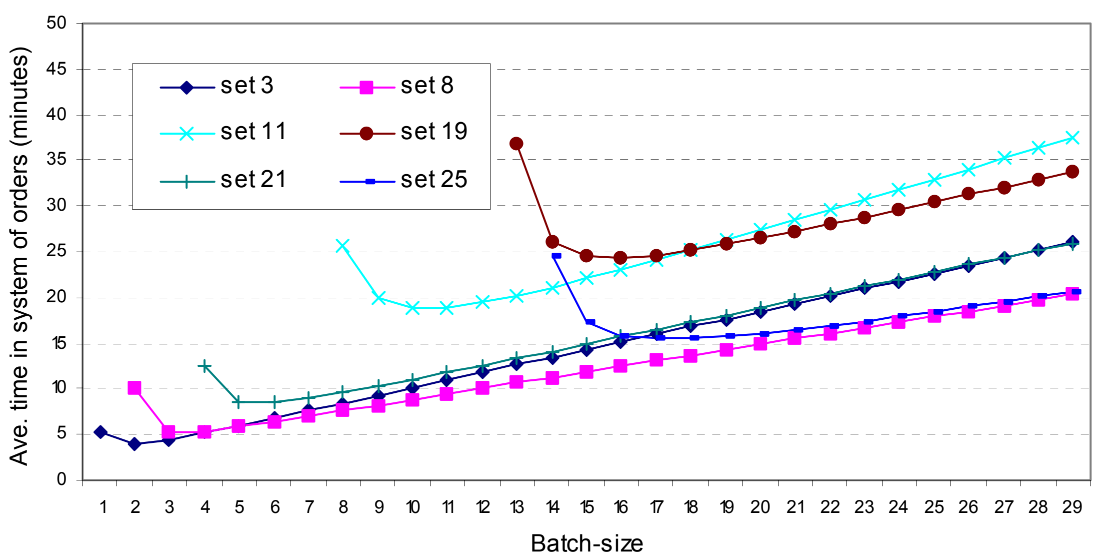

Figure 3 Average time in system of $M/D^q/1$ as a function of q

As an illustration of the above result, we also sketched out shapes of average time in system when service time was deterministic and exponential.
Figure 3 shows the convex property of average time in system for the deterministic case. The average time in system of order behaves as in the previous case; all curves are convex and smooth. Figure 4 shows the average time in system when service time is exponential.

We tested the convex property of average time in system of order not only for the sets of parameters listed on Table 1 but also for many other sets and we found that in all cases the result remains unchanged.

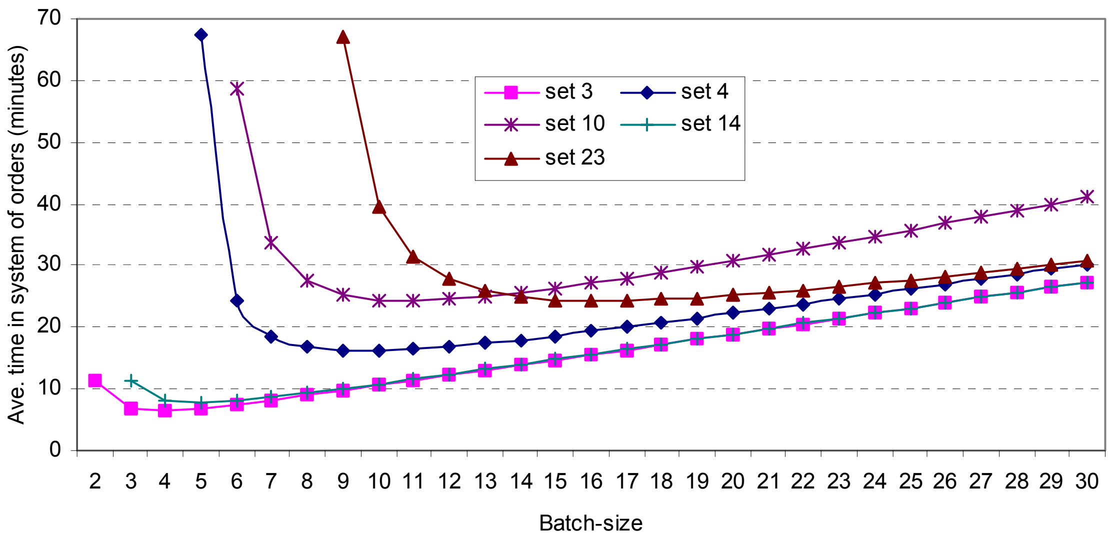

Figure 4 Average time in system of $M / M^q / 1$ as a function of q

5.2 Average time in system of orders is always highest when service time is exponential

For every set of parameters, we compared the throughput times of $M / M^q / 1$, $M / D^q / 1$ and $M / G^q / 1$ at different values of batch size and found that the average time in system of orders was always the highest for exponential service time. Figure 5 gives an illustration for the case of test set 3. This result seems quite obvious but rather difficult to explain in an analytical way because in the deterministic and general service time case we do not have closed-forms for computing the average time in system of orders.

Another interesting point is that the optimal batch size for every test set is always highest when the service time distribution is exponential (see in the first half of Table 2). This result is very useful because it defines an upper limit for finding the optimal batch size of the actual system. This upper bound is rather easy to compute as in the case of exponential service time we have a closed form for computing the average time in system of orders.

Next to above findings, a remark should be made is that the optimal batch size in the case of deterministic or general distribution service time is very close to its lower bound. Therefore, if we trace the optimum (the batch size that gives the minimum average time in system) from the lower bound and each time we increase the batch size value by 1 (order), it is likely that we will reach the optimum rapidly.

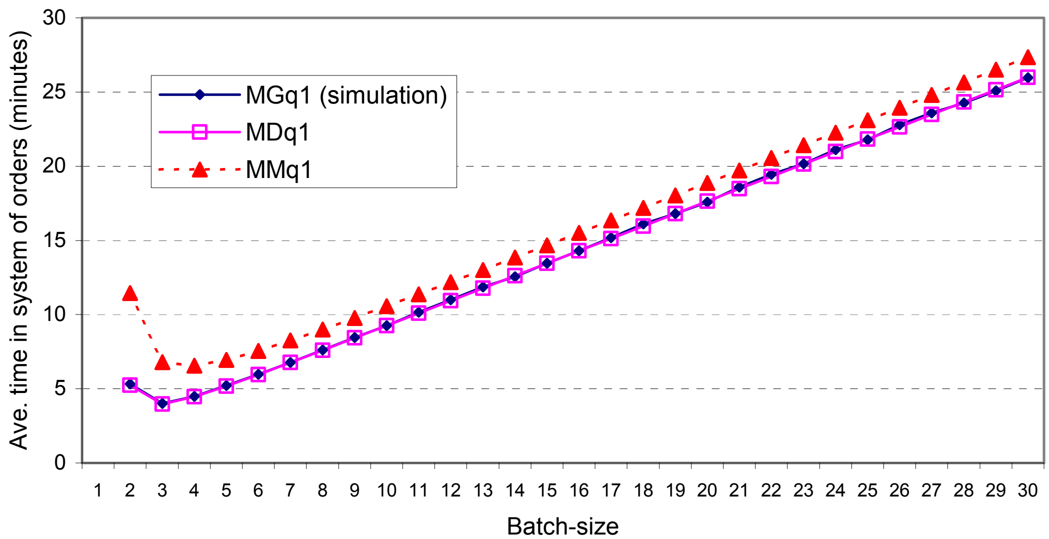

Figure 5 Average time in system of orders for test set 3

For every set, we observed that the difference between the average time in system obtained from the deterministic and general service time model is very small and this holds for all feasible batch size values (note that the average time in system in the case of general service time is the mean value obtained from simulation runs with running time and number of replications mentioned above). Therefore, we hypothesize that, with respect to the average time in system, these two models are statistically indifferent. We used SPSS ([17]) to test this hypothesis. Table 3 shows results of this test. The results point out that the null hypothesis was rejected for only 3 sets (of in total 25 sets). In other words, for 22 sets no statistical difference between the results of the deterministic model and the outcomes of the simulation runs was detected.

The optimal picking batch size for a system with general service time is extremely difficult to find, if not impossible. The above findings suggest that we can use the $M/D^q/1$ queue as an approximation for determining the optimal picking batch size of the actual system. We propose the following heuristic procedure:

- Calculate the batch size lower bound q_{LB} (see Appendix C).
- Find the optimal batch size for the exponential service time model $q_{M/M^q/1}^{opt}$.
- Determine the upper bound
  $q_{UB} = \min\left\{q_{M/M^q/1}^{opt},\ \textit{capacity of an order picker (or a group of order pickers)}\right\}$.
- $q_{M/G^q/1}^{opt} = q_{M/D^q/1}^{opt}, \quad q_{M/D^q/1}^{opt} \in [q_{LB}, q_{UB}]$.

Table 2 Differences analysis results[1]

Test set	q_{LB}	$q_{M/D^q/1}^{opt}$	$q_{M/G^q/1}^{opt}$	$q_{M/M^q/1}^{opt}$	Optimal average time in system (W^{opt})		
					$M/D^q/1$	$M/G^q/1$	% difference[2]
1	4	6	6	8	7.99	8	0.12
2	2	2	2	3	3.27	3.23	1.24
3	2	3	3	4	3.98	4.02	1.00
4	5	7	6	10	9.39	9.32	0.75
5	8	10	10	16	14.72	14.74	0.14
6	13	16	16	24	22.40	22.31	0.40
7	14	17	17	28	24.85	24.65	0.81
8	3	4	4	6	5.26	5.21	0.96
9	3	4	4	6	5.50	5.48	0.36
10	6	8	8	11	11.62	11.67	0.43
11	9	11	12	16	18.88	18.96	0.42
12	10	13	14	19	22.52	22.47	0.22
13	11	15	15	21	25.17	24.56	2.48
14	3	4	4	5	4.90	4.93	0.61
15	4	5	5	7	6.68	6.83	2.20
16	5	6	6	9	8.60	8.68	0.92
17	7	8	8	13	12.11	12.1	0.08
18	11	13	13	21	19.48	19.53	0.26
19	14	17	16	27	24.33	24.34	0.04
20	2	3	3	3	6.00	6.04	0.66
21	5	7	6	9	8.54	8.61	0.81
22	6	7	8	11	9.09	9.06	0.33
23	9	11	10	16	11.07	11.16	0.81
24	11	14	14	21	12.92	12.91	0.08
25	15	18	19	29	15.46	15.34	0.78

[1] Average time in system of orders of the general distribution service time system is taken from simulation runs
[2] Absolute value

Table 3 Significance test of differences between average times in system of orders obtained with deterministic and general service time model

Paired-Samples T test with 95% confidence interval of the difference				
Test set	Degree of freedom	*t*-value	Critical value	Insignificant?
1	26	0.756	2.056	yes
2	28	2.372	2.048	no
3	28	3.051	2.048	no
4	25	0.964	2.06	yes
5	22	0.907	2.074	yes
6	17	0.974	2.11	yes
7	16	0.993	2.12	yes
8	27	2.47	2.052	yes
9	27	1.021	2.052	yes
10	24	0.732	2.064	yes
11	21	0.882	2.08	yes
12	20	0.993	2.179	yes
13	19	1.211	2.093	yes
14	27	0.073	2.052	yes
15	26	0.087	2.056	yes
16	25	0.592	2.06	yes
17	23	0.23	2.069	yes
18	19	0.627	2.101	yes
19	16	1.205	2.12	yes
20	28	3.503	2.048	no
21	25	0.749	2.06	yes
22	24	1.174	2.064	yes
23	21	0.754	2.08	yes
24	19	0.927	2.093	yes
25	15	0.067	2.131	yes

$q^{opt}_{M/D^q/1}$ can be chased by the following procedure:

$$W_{M/D^q/1}^{q^{opt}} = \infty;$$

$$\textbf{\textit{for}}\ q = q_{LB} + 1\ \textbf{\textit{to}}\ q_{LB}\ \textbf{\textit{do}}\ q^{opt}_{M/D^q/1} = q$$

$$\textbf{\textit{if}}\ W_{M/D^q/1}^{q} < W_{M/D^q/1}^{q^{opt}}\ \textbf{\textit{then}}\ \left(W_{M/D^q/1}^{q^{opt}} = W_{M/D^q/1}^{q};\ q^{opt} = q\right);$$

6 Conclusions

Following conclusions could be drawn from this study:

- ✳ The average time in system is a convex function of the picking batch size.
- ✳ $M/D^q/1$ is a good approximation for the system.
- ✳ The optimal order picking batch size can be found by just considering the average time in system of orders of the deterministic service time system over a tight range of batch sizes. The upper bound of the range is the minimum of the optimal picking batch size given by the corresponding $M/M^q/1$ model and the maximum number of orders that an order picker (or a group of order pickers) can serve in a picking tour. The lower bound is derived from the equilibrium condition.

The problem we discussed above is the simplest case of the on-line order batching problem, the problem of determining the optimal picking batch size for order pickers when orders arrive online. We intend to relax some assumptions (mentioned in section 1) to cope with more complex and realistic situations. One straightforward direction is to consider multi-aisle warehouses. With this extension we have to take into account the effect of routing methods. Another possibility is to investigate the influence of class-based storage strategies on the batching decision.

Appendix A *Average queue size of $M/M^q/1$*

As shown in Gross and Harris [6] and in many other queueing theory materials:

$$p_n = \begin{cases} \dfrac{p_0(1-r_0^{n+1})}{1-r_0} & (1 \le n \le q) \\[2ex] \dfrac{p_0 \lambda r_0^{n-q}}{\mu} & (n > q) \end{cases}$$

$$p_0 = \frac{1-r_0}{q}$$

Substitution into the definition formula of the average system size:

$$\begin{aligned} L &= \sum_{n=0}^{\infty} np_n \\ &= \sum_{n=0}^{q-1} np_n + \sum_{n=q}^{\infty} np_n \end{aligned}$$

$$= \sum_{n=1}^{q-1} n \frac{1-r_0}{q} \frac{1-r_0^{n+1}}{1-r_0} + \sum_{n=q}^{\infty} n \frac{1-r_0}{q} \frac{\lambda r_0^{n-q}}{\mu}$$

$$= \left(\frac{1}{q} \sum_{n=1}^{q-1} n - \frac{1}{q} \sum_{n=1}^{q-1} n r_0^{n+1} \right) + \frac{1-r_0}{q} \frac{r_0^{-q} \lambda}{\mu} \sum_{n=q}^{\infty} n r_0^n$$

$$= \frac{1}{q} \frac{(q-1)q}{2} - \frac{r_0}{q} \sum_{n=1}^{q-1} n r_0^n + \frac{\lambda}{\mu q} (r_0^{-q} - r_0^{1-q}) \left(\sum_{n=1}^{\infty} n r_0^n - \sum_{n=1}^{q-1} n r_0^n \right)$$

$$= \frac{q-1}{2} + \frac{\lambda}{\mu q} (r_0^{-q} - r_0^{1-q}) \sum_{n=1}^{\infty} n r_0^n + \left(\frac{\lambda}{\mu q} (r_0^{-q} - r_0^{1-q}) - \frac{r_0}{q} \right) \sum_{n=1}^{q-1} n r_0^n$$

Since $\sum_{n=1}^{\infty} n r_0^n = r_0 \frac{1}{(1-r_0)^2}$ *(where* $r_0 < 1$)

and $\sum_{n=1}^{q-1} n r_0^n = r_0 \left(\frac{1 - r_0^q - q r_0^{q-1}(1-r_0)}{(1-r_0)^2} \right)$

We finally obtain:

$$L = \frac{q-1}{2} + \frac{\lambda}{\mu q} \frac{r_0^{1-q}}{1-r_0} + r_0 \left(\frac{\lambda(r_0^{-q} - r_0^{1-q}) - \mu r_0}{\mu q} \right) \left(\frac{1 - r_0^q - q r_0^{q-1}(1-r_0)}{(1-r_0)^2} \right) \quad \blacklozenge$$

Appendix B *Expected batch service time*

The expected travel time to pick q items, where items are uniformly distributed over the aisle, can be computed as follow:

$$E(x) = 2L_0 \int_{x=0}^{1} x f(x).dx$$

$$= 2L_0 \int_{x=0}^{1} x \left(q x^{q-1} \right) dx$$

$$= 2L_0 \frac{q}{q+1} x^{q+1} \Big|_0^1$$

$$= 2L_0 \frac{q}{q+1}$$

Where L_0 denotes the length of the aisle (in minutes). The service time consists of 3 components: setup, picking and travel time. We suppose that the setup time is insensitive to the size of batch and the picking time per item is constant. Only the travel time has a stochastic nature, so the average service time of a batch size q can be computed as follows.

$$S(q) = \; setuptime \; + \; E(picking\ time) \; + \; E(travel\ time)$$

$$= \; \tau + \frac{q}{r} + E(x)$$

$$= \tau + \frac{q}{r} + 2L_0 \frac{q}{q+1} \qquad \blacklozenge$$

Appendix C *Compute lower bound of batch size q*

As a common condition for every queuing system to reach equilibrium the traffic density must be less than 1. Hence:

$$\frac{\lambda s}{q} < 1$$

$$\Leftrightarrow \; \lambda\left(\tau + \frac{q}{r} + \; 2\frac{q}{q+1}L_0\right) < q$$

$$\Leftrightarrow \; \frac{\tau}{q} + \frac{2L_0 q}{q+1} < \frac{1}{\lambda} - \frac{1}{r}$$

$$\Leftrightarrow \; \begin{cases} q^2 + q\left(1 - \dfrac{\tau + 2L_0}{\alpha}\right) - \dfrac{\tau}{\alpha} > 0 & where \quad \alpha = \dfrac{1}{\lambda} - \dfrac{1}{r} \\ \alpha > 0 & \end{cases} \qquad (11)$$

It is clear that equation $q^2 + q\left(1 - \frac{\tau + 2L_0}{\alpha}\right) - \frac{\tau}{\alpha} = 0$ has two real roots ($q_1 > q_2$):

$$q_1 = \frac{-\left(1 - \dfrac{\tau + 2L_0}{\alpha}\right) + \sqrt{\left(1 - \dfrac{\tau + 2L_0}{\alpha}\right)^2 + 4\dfrac{\tau}{\alpha}}}{2}$$

and,

$$q_2 = \frac{-\left(1-\frac{\tau+2L_0}{\alpha}\right)-\sqrt{\left(1-\frac{\tau+2L_0}{\alpha}\right)^2+4\frac{\tau}{\alpha}}}{2}$$

It is easy to see that q_1 is positive while q_2 is negative, therefore:

$$(11) \Leftrightarrow \begin{cases} q > q_1 \\ r > \lambda \end{cases}$$

$$\Leftrightarrow \begin{cases} q_{LB} = \text{int}(q_1)+1 \\ r > \lambda \end{cases}$$

For a given set of λ, r, τ and L_0, we can determine a lower bound for the batch size by simply solving the equation above and taking the smallest integer that is greater than the positive root we have found. It is noted that, the picking rate must be greater than the arrival rate of orders otherwise the equilibrium condition will be broken. ♦

Appendix D *Finding roots for characteristic equation (10) (Muller's method)*

Our characteristic equation (10) can be rewritten as follows (for more details see Chaudhry [1]).

$$(10) \Leftrightarrow z^q = e^{-q\rho(1-z)+2\pi ni} \qquad \textit{for } n=1,2,\ldots,q \textit{ and } i=\sqrt{-1}$$

$$\Leftrightarrow z = e^{-\rho(1-z)+2\pi ni/q} \quad (*)$$

It is clear that for each value of n, *(*)* has a unique root. Solving these equations for $n = 1, 2, \ldots, \left\lceil \frac{q-1}{2} \right\rceil$ and then taking conjugates of these roots we obtain $(q-1)$ roots inside the unit circle. ♦

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