A Simple Approximation to the Renewal Function

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Key Words — **Renewal function, Phase type distribution, Block replacement**

Reader Aids —
Purpose: Present a simple approximation
Special math needed for explanations: Probability
Special math needed to use results: Same
Results useful to: Reliability theoreticians and analysts

Summary & Conclusions — **This article presents a simple, easy to understand approximation to the renewal function; the approximation is easy to implement on a personal computer. The key idea is that, for small values of time, the renewal function is almost equal to the Cdf of the inter-renewal time, whereas for larger values of time an asymptotic expansion — depending upon only the first and second moment of the inter-renewal time — can be used. The relative error is typically smaller than a few percent for Weibull inter-renewal times.**

The simple approximation method works very well with one term if not too much accuracy is required (eg, in the block replacement problem) or if the inter-renewal (failure) distribution is not exactly known (eg, only the first two moments are known). Although the accuracy of the simple approximation can be improved by increasing the number of terms, we do not advocate this strategy, since then speed and simplicity are lost. If high accuracy is required it is better to use another approximating method (eg, power series expansion or cubic splines method).

1. INTRODUCTION

Practical application of renewal theory typically requires knowledge of the renewal function $M(t)$. For most distributions used in reliability theory (eg, Weibull, gamma, and truncated s-normal distributions) no explicit expressions for the renewal function $M(t)$ are available. However, there are many methods to approximate this function ([10] provides an overview), for example the extended cubic splining algorithm [8], the generating function algorithm [6], and power series expansion. The methods 1 & 2 generally apply, but method 1 is not easily implemented, while method 2 suffers from discretization errors and requires many calculations for large values of t. Method 3 is distribution specific. Most of the literature on the power series method is about the Weibull distribution, eg, [13, 14], but, in principle, such a method can be developed for all Cdf's with a power series expansion.

Ref. [7] proposes a procedure for computing the renewal function and related functions of the phase type renewal process [9]. We propose here a similar but much simpler technique, based on the observation that: 1) for small values of t, $M(t)$ is mainly determined by the Cdf $F(t)$, and 2) for large values of t an asymptotic result yields a good approximation [11, pp 6–7]. The approximation applies both for distributions with a coefficient of variation smaller than one (under-dispersed) and for distributions with a coefficient of variation larger than one (dispersed).

Notation

X	r.v. denoting the time between successive renewals of a renewal process
$f(t), F(t)$	pdf, Cdf of X
μ	mean of X
σ	standard deviation of X
c_X	coefficient of variation of X, $c_X \equiv \sigma/\mu$

2. THE APPROXIMATION

The renewal function $M(t)$, indicating the mean number of renewals in [0, t], is defined as:

$$M(t) \equiv \sum_{n=1}^{\infty} F^{(n)}(t) \qquad , t \geq 0 \tag{1}$$

where $F^{(n)}(t)$ denotes the n-fold convolution of $F(\cdot)$ with itself, recursively defined as

$$F^{(n)}(t) \equiv \int_0^t F^{(n-1)}(t-u)dF(u) \qquad , t \geq 0,\ n \geq 2, \tag{2}$$

with $F^{(1)}(t) \equiv F(t)$. Note that $F^{(n)}(t)$ is the probability of n or more renewals in the interval [0, t]. A basic result from renewal theory is the following asymptotic expansion of $M(t)$ (for rigour and proofs we refer to [1, 4]).

$$\lim_{t\to\infty} \left\{ M(t) - \frac{t}{\mu} \right\} = \frac{\sigma^2}{2\mu^2} - \frac{1}{2} \tag{3}$$

We propose the following class of approximations $M_l(t)$ to $M(t)$, $l = 1, 2, ..$, in which $F(\cdot)$ is approximated by another Cdf $\hat{F}(\cdot)$ with the same mean μ and variance σ^2 as $F(\cdot)$ and —

$$M_l(t) = \sum_{n=1}^{l} F^{(n)}(t) + \sum_{n=l+1}^{\infty} \hat{F}^{(n)}(t) \qquad , t \geq 0 \tag{4}$$

$$= \hat{M}(t) + \sum_{n=1}^{l} \{F^{(n)}(t) - \hat{F}^{(n)}(t)\} \qquad , t \geq 0 \tag{5}$$

$\hat{M}(t)$ ≡ renewal function associated with $\hat{F}(t)$.

It is clear that $M_l(0) = M(0) = 0$ and that $M_l(t)$ and $M(t)$ have the same asymptotic expansion (3). Now l and $\hat{F}(t)$ have to be chosen in such a way that (4) or (5) can be evaluated quickly and that $M_l(t)$ is a good approximation to $M(t)$. The choice of $\hat{F}(t)$ depends on the squared coefficient of variation c_X^2 of the Cdf $F(\cdot)$. For $0 < c_X^2 \le 1$ we can always fit an $E_{k-1,k}$ distribution and for $½ \le c_X^2$ we can always fit a K_2 (also called Coxian-2) distribution on the first two moments [10, pp 397–400] (for $½ \le c_X^2 \le 1$, the $E_{k-1,k}$ distribution is a special case of the K_2 distribution). In appendix 1 we define the $E_{k-1,k}$ and the K_2 distribution and show how to choose the parameters to obtain a 2-moment fit.

For the K_2 distribution with parameters λ_1, λ_2 and p we have the following explicit result for the renewal function (see appendix 1)

$$M(t) = \frac{\lambda_1\lambda_2}{\lambda_1(1-p)+\lambda_2}\, t - \frac{\lambda_1(1-p)(\lambda_2-p\lambda_1)}{[\lambda_1(1-p)+\lambda_2]^2} \cdot [1-\exp\{-(\lambda_1(1-p)+\lambda_2)t\}] \quad (6)$$

For the n-fold convolution $F^{(n)}(t)$ of an $E_{k-1,k}$ distribution with parameters λ, p (and k) it follows from conditioning on the total number of (independent) exponential phases (with parameter λ) in [0, t] that

$$F^{(n)}(t) = \sum_{j=0}^{n} \binom{n}{j} (1-p)^j p^{n-j} \cdot \left[1 - \sum_{l=0}^{n(k-1)+j-1} \frac{(\lambda t)^l}{l!} \exp\{-\lambda t\}\right] \quad (7)$$

It is easy to set up a recursive scheme to compute the $F^{(n)}(t)$. (There is no explicit expression for the renewal function of an $E_{k-1,k}$ distribution). From (2) it follows that $F^{(n+1)}(t) \le F(t)F^{(n)}(t)$. Using this we obtain the following (conservative) error bound for truncating the infinite sum in the r.h.s. of (1)

$$\sum_{n=N+1}^{\infty} F^{(n)}(t) \le \frac{F(t)}{1-F(t)} F^{(N)}(t) \quad , t \ge 0. \quad (8)$$

Using (6) in (5), or (7) with the error bound (8) in (4), the approximate terms $\hat{F}^{(n)}(t)$ and $\hat{M}(t)$ can be readily computed. The terms $F^{(n)}(t)$, $n=1,\ldots,l$ can be computed numerically with a multi-dimensional integration routine. In general this is cumbersome, but we show that good approximations can be obtained evaluating only the Cdf $F(\cdot)$ numerically, ie, by choosing $l=1$.

For the choice of l we first observe that the accuracy as well as the computer time required increases with l. For l = 1, 2, 3 we compared the approximation $M_l(t)$ with the tables in [2] for Weibull, gamma, and truncated s-normal distributions. These distributions are widely used to model lifetimes in reliability theory. Results are tabulated in appendixes 2 & 3; also, see [10]. For the three distributions the approximation $M_l(t)$ is best for the gamma distribution followed by the Weibull with $c_X^2 < 1$ and the truncated s-normal distribution. The results are worst for a Weibull distribution with a $c_X^2 >$ 1. In fact the pdf's of the gamma, Weibull and $E_{k-1,k}$ distributions all have the same shape for $0 < c_X^2 \le 1$ [11, pp 396], whereas the pdf of the truncated s-normal distribution has a different shape (eg, $f(0) > 0$). In the following example we test the approximation.

Example: Block Replacement Problem

The renewal function is typically used in the block replacement problem (BRP, see [1, 3]), where a component is both replaced upon failure and preventively after fixed intervals of length t, regardless of the number of failure replacements in between. The cost of failure and cost of preventive replacement are c_f and c_p, respectively. It is easily derived that for an interval of length t the average costs are:

$$g(t) = \frac{c_p + c_f M(t)}{t} \quad , t>0.$$

Sufficient (see [3]), but not necessary conditions for the existence of a unique minimum of $g(t)$ are that $c_p/c_f < ½(1-c_X^2)$ and that $m(t) \equiv dM(t)/dt$ is strictly increasing in t. The latter is true for, eg, Weibull distributions with $c_X^2 < 1$, with t up to about μ. With an algorithm for $M(t)$, the actual minimization of $g(t)$ is quite straightforward. For weif($\cdot$;1.5) and weif($\cdot$;2) failure time distributions and cost ratios c_f/c_p varying from 5 to 20 we calculated the optimal block replacement interval and minimum average costs using the approximation $M_l(t)$ with $l=1$, denoted by $\tilde{t}$, $\tilde{g}$, respectively, and the optimal block replacement interval and minimum average costs obtained using the (exact) tables for the renewal function in [2], denoted by t^* and g^*. We also computed, using these tables, the average costs $g(\tilde{t})$ when using interval $\tilde{t}$ instead of t^*, and the relative difference $\Delta \equiv (g(\tilde{t}) - g(t^*))/g(t^*)$. Table 1 gives the results.

TABLE 1
Effects of the Approximation in the BRP for Weibull Lifetime Distributions.

Distribuiton	c_f/c_p	$\tilde{t}$	$\tilde{g}$	t^*	g^*	$g(\tilde{t})$	Δ(%)
weif(.;1.5)	5	0.75	5.084	0.75	5.179	5.179	—
E{X}=0.9027	10	0.40	8.355	0.40	8.520	8.520	—
c_x^2=0.4599	20	0.24	13.534	0.25	13.740	13.750	0.07
weif(.:2.0)	5	0.53	4.248	0.50	4.308	4.314	0.09
E{X}=0.8862	10	0.34	6.166	0.35	6.220	6.225	0.08
c_x^2=0.2732	20	0.23	8.831	0.25	8.920	8.930	0.11

The fact that the resulting error in the average costs is small compared to the error in the location of the minimum is due to the flatness of the cost curve $g(t)$ near its minimum g^*. So, in the block replacement problem with Weibull failure time distributions, the approximation $M_l(t)$ yields fully satisfactory results for $l=1$. In this case no integrals have to be evaluated since an explicit expression for $F(t)$ exists.

Remark

For small values of t the approximation to the weif$(\cdot;2)$ renewal function using phase type distributions given in [7] can be improved in a very straightforward manner using (5) with $l=1$. Doing this reduces the relative error for small values of t from the order of 10% to 1%. Such an improvement can be important since, for example, in block replacement problems one is typically interested in preventing failures and in many cases the optimal replacement interval is relatively small (see table 1).

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APPENDIX 1

A r.v. X has a K_2 (also called Coxian-2) distribution if —

$$X = \begin{cases} X_1 \text{ with probability } p \\ X_1 + X_2 \text{ with probability } 1-p \end{cases}$$

where X_1 and X_2 are independently, exponentially distributed r.v.'s with means $1/\lambda_1$ and $1/\lambda_2$, respectively, and $0 \le p \le 1$, $\lambda_1 > 0$ and $\lambda_2 > 0$. The K_2 distribution corresponds to the sojourn time (in states 1 or 2) in the following continuous-time Markov chain.

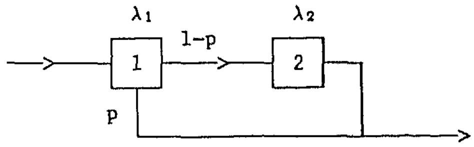

Figure 1. Markov Diagram of a K_2 Distribution

The pdf of a K_2 distribution is:

$$f(t) = \begin{cases} p\lambda \exp\{-\lambda t\} + (1-p)\lambda^2 t \exp\{-\lambda t\}, \ \lambda_1 = \lambda_2 = \lambda \\ \dfrac{p\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} \lambda_1 \exp\{-\lambda_1 t\} \\ \quad + \left(1 - \dfrac{p\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2}\right) \lambda_2 \exp\{-\lambda_2 t\}, \ \lambda_1 \ne \lambda_2 \end{cases}$$

Fitting a K_2 distribution on the first two moments leaves some degree of freedom in the choice of the parameters λ_1, λ_2, p. In some cases the third moment can also be fitted, but this is not always possible [12]. Therefore we used gamma normalization to obtain a unique fit. This means that λ_1, λ_2, p are chosen in such a way that the third moment of the K_2 distribution equals the third moment of a gamma distribution with the same first and second moment as X. This is always possible [11, pp 399–400].

The corresponding values of λ_1, λ_2, p are:

$$\lambda_1 = \frac{2}{\mathrm{E}\{X\}} \left(1 + \sqrt{\frac{c_X^2 - ½}{c_X^2 + 1}}\right), \quad \lambda_2 = \frac{4}{\mathrm{E}\{X\}} - \lambda_1,$$

$$p = (1 - \lambda_2 \mathrm{E}\{X\}) + \lambda_2/\lambda_1$$

The explicit expression (6) for the renewal function $M(t)$ of the K_2 distribution is obtained by first taking the Laplace transform of both sides of the renewal equation, $M(t) = F(t) + \int_0^t M(t-u)dF(u)$ and then applying the complex inversion formula.

The pdf of an $E_{k-1,k}$ distribution with parameters p and λ has the following form:

$$f(t) = p\lambda^{k-1} \frac{t^{k-2}}{(k-2)!} \exp\{-\lambda t\} + (1-p)\,\lambda^k \frac{t^{k-1}}{(k-1)!} \exp\{-\lambda t\}, \ t \ge 0$$

where $0 \le p \le 1$, $\lambda > 0$ and $k \ge 2$. In words, a random variable having this pdf is with probability p (respectively $1-p$) distributed as the sum of $k-1$ (k) independent exponentials with common mean $1/\lambda$. The $E_{k-1,k}$ distribution corresponds to the sojourn time (in states 1, 2,..., $k-1$ or k) in the following continuous time Markov chain:

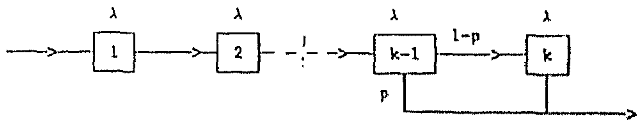

Figure 2. Markov Diagram of an $E_{k-1,k}$ Distribution

A 2-moment fit of the $E_{k-1,k}$ distribution is obtained by choosing the parameters k, p, λ in the following way:

$$\frac{1}{k} \le c_X^2 \le \frac{1}{k-1},$$

$$p = \frac{1}{1+c_X^2}\{kc_X^2 - [k(1+c_X^2) - k^2 c_X^2]^{½}\}, \ \lambda = \frac{k-p}{\mathrm{E}\{X\}}.$$

APPENDIX 2

Approximations to the Weibull Renewal Function
Comparison of exact values (Baxter et al [2]) with approximations from Kao [7], this paper and the asymptotic expansion $M_{stat}(t)$ based on (3).
scale $\sigma = 1.0$; shape $\beta = 2.0$; $E\{X\} = 0.88623$; $c_X^2 = 0.27323$
parameters of the $E_{k-1,k}$ distribution (fit on first and second moment):
$k=4$; $p=0.19137$; $\lambda= 4.29756$

t	$M(t)$ [2]	$M(t)$ [7]	$M_1(t)$	$M_2(t)$	$M_3(t)$	$M_{stat}(t)$
0.10	0.0100	0.01	0.0100	0.0100	0.0100	<0
0.20	0.0395	0.03	0.0392	0.0395	0.0395	<0
0.30	0.0874	0.08	0.0863	0.0874	0.0874	<0
0.40	0.1520	0.14	0.1491	0.1519	0.1519	0.0880
0.50	0.2308	0.22	0.2252	0.2307	0.2308	0.2008
0.60	0.3216	0.32	0.3126	0.3211	0.3215	0.3136
0.70	0.4216	0.42	0.4092	0.4206	0.4215	0.4265
0.80	0.5283	0.53	0.5131	0.5266	0.5282	0.5393
0.90	0.6397	0.64	0.6229	0.6368	0.6395	0.6522
1.00	0.7537	0.76	0.7358	0.7494	0.7534	0.7650
1.25	1.0427	1.05	1.0308	1.0336	1.0416	1.0471
1.50	1.3295	1.33	1.3261	1.3163	1.3265	1.3292
1.75	1.6126	1.61	1.6156	1.5985	1.6065	1.6113
2.00	1.8941	1.89	1.8995	1.8830	1.8844	1.8934

APPENDIX 3. Supplementary Tables

Comparison Between the Tables in Baxter et al [2] and the Approximation in Section 2.

Table I: Approximation to the Weibull renewal function with a shape parameter larger than 1
Table II: Approximation to the Weibull renewal function with a shape parameter less than 1
Table III: Approximation to the gamma renewal function
Table IV: Approximation to the truncated *s*-normal renewal function

Pdf's of the distributions

1. Weibull distribution with shape parameter β and scale parameter σ

$$f(t) = \frac{\beta}{\sigma}\left(\frac{t}{\sigma}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\sigma}\right)^{\beta}\right] \quad, t>0,\ \beta,\ \sigma>0$$

2. Gamma distribution with shape parameter β and scale parameter σ

$$f(t) = \frac{1}{\sigma^{\beta}\Gamma(\beta)}\, t^{\beta-1}\exp\left[-\frac{t}{\sigma}\right] \quad, t>0,\ \beta,\ \sigma>0$$

3. Truncated *s*-normal distribution with parameters μ and σ

$$f(t) = \frac{1}{a\sigma\sqrt{2\pi}}\exp[-(t-\mu)^2/2\sigma^2] \quad, t>0,\ \sigma>0$$

where $a \equiv \text{gaufc}(-\mu/\sigma)$, and gauf(z) & gaufc(z) are Cdf & Sf of the standard *s*-normal distribution.

TABLE I
Approximations to the Weibull Renewal Function
$\sigma = 1.0$; $\beta = 1.5$; $E\{X\} = 0.90330$; $c_X^2 = 0.45986$
parameters of the $E_{k-1,k}$ distribution
(fit on first and second moment):
$k = 3$; $p = 0.60884$; $\lambda = 2.64714$

t	$M(t)$ [2]	$M_1(t)$	$M_2(t)$	$M_3(t)$	$M_{stat}(t)$
0.10	0.0315	0.0312	0.0314	0.0314	<0
0.20	0.0879	0.0864	0.0878	0.0878	<0
0.30	0.1591	0.1554	0.1588	0.1590	0.0620
0.40	0.2408	0.2344	0.2403	0.2407	0.1728
0.50	0.3303	0.3210	0.3292	0.3302	0.2835
0.60	0.4257	0.4139	0.4239	0.4256	0.3942
0.70	0.5256	0.5118	0.5226	0.5253	0.5049
0.80	0.6287	0.6138	0.6243	0.6282	0.6156
0.90	0.7343	0.7189	0.7283	0.7334	0.7263
1.00	0.8416	0.8265	0.8340	0.8403	0.8370
1.25	1.1147	1.1024	1.1035	1.1115	1.1137
1.50	1.3910	1.3828	1.3777	1.3852	1.3905
1.75	1.6682	1.6639	1.6549	1.6597	1.6673
2.00	1.9455	1.9438	1.9339	1.9348	1.9440
2.25	2.2228	2.2224	2.2138	2.2107	2.2208
2.50	2.4998	2.4999	2.4937	2.4877	2.4976
2.75	2.7768	2.7767	2.7732	2.7658	2.7743
3.00	3.0538	3.0531	3.0520	3.0445	3.0511

TABLE II
Approximations to the Weibull Renewal Function with a Shape Parameter less than 1
$\sigma = 1.0$; $\beta = 0.75$; $E\{X\} = 1.1906$; $c_X^2 = 1.8304$
parameters of the K_2 distribution
fit on first and second moment and gamma normalisation:
$\lambda_1 = 2.831$; $\lambda_2 = 0.5281$; $p = 0.5577$
fit on first, second and third moment:
$\lambda_1 = 1.72010$; $\lambda_2 = 0.46367$; $p = 0.71751$

t	$M(t)$ [2]	$M_1(t)$ gamma normalization	$M_1(t)$ 3rd moment fit	$M_{stat}(t)$
0.10	0.1820	0.1741	0.1701	0.4992
0.20	0.3110	0.2991	0.2855	0.5832
0.30	0.4271	0.4162	0.3907	0.6672
0.40	0.5362	0.5299	0.4919	0.7512
0.50	0.6408	0.6412	0.5913	0.8352
0.60	0.7427	0.7502	0.6898	0.9191
0.70	0.8411	0.8571	0.7879	1.0031
0.80	0.9381	0.9617	0.8855	1.0871
0.90	1.0335	1.0643	0.9828	1.1711
1.00	1.1277	1.1648	1.0796	1.2551
1.25	1.3588	1.4079	1.3198	1.4651
1.50	1.5853	1.6412	1.5567	1.6751
1.75	1.8085	1.8672	1.7899	1.8850
2.00	2.0292	2.0879	2.0195	2.0950
2.25	2.2481	2.3046	2.2457	2.3050
2.50	2.4655	2.5188	2.4690	2.5150
2.75	2.6818	2.7311	2.6898	2.7250
3.00	2.8971	2.9422	2.9083	2.9349

TABLE III
Approximations to the Gamma Renewal Function
$\sigma = 3.5$; $\beta = 3.5$; $E\{X\} = 1$; $c_X^2 = 0.2857$
parameters of the $E_{k-1,k}$ distribution
(fit on first and second moment):
$k = 4$; $p = 0.3010$; $\lambda = 3.699$

t	$M(t)$ [2]	$M_1(t)$	$M_2(t)$	$M_3(t)$	$M_{stat}(t)$
0.10	0.0017	0.0017	0.0017	0.0017	<0
0.20	0.0145	0.0144	0.0144	0.0144	<0
0.30	0.0461	0.0459	0.0460	0.0460	<0
0.40	0.0978	0.0979	0.0978	0.0978	0.0429
0.50	0.1670	0.1675	0.1670	0.1670	0.1429
0.60	0.2495	0.2502	0.2494	0.2494	0.2429
0.70	0.3408	0.3420	0.3408	0.3408	0.3429
0.80	0.4376	0.4391	0.4377	0.4376	0.4429
0.90	0.5374	0.5393	0.5375	0.5373	0.5429
1.00	0.6383	0.6402	0.6386	0.6383	0.6429
1.25	0.8914	0.8929	0.8920	0.8913	0.8929
1.50	1.1428	1.1435	1.1438	1.1428	1.1429
1.75	1.3930	1.3930	1.3945	1.3933	1.3929
2.00	1.6430	1.6426	1.6444	1.6436	1.6429
2.25	1.8929	1.8923	1.8939	1.8939	1.8929
2.50	2.1429	2.1423	2.1433	2.1441	2.1429
2.75	2.3929	2.3924	2.3927	2.3941	2.3929
3.00	2.6429	2.6425	2.6424	2.6438	2.6429

TABLE IV
Approximations to the Truncated *s*-Normal Renewal Function
$\sigma = 1.0$; $\mu = 1.5$; $E\{X\} = 1.6389$; $c_X^2 = 0.2875$
parameters of the $E_{k-1,k}$ distribution
(fit on first and second moment):
$k = 4$; $p = 0.3171$; $\lambda = 2.2472$

t	$M(t)$ [2]	$M_1(t)$	$M_2(t)$	$M_3(t)$	$M_{stat}(t)$
0.10	0.0151	0.0149	0.0151	0.0151	<0
0.30	0.0529	0.0517	0.0529	0.0529	<0
0.50	0.1024	0.0984	0.1023	0.1024	<0
0.70	0.1647	0.1566	0.1644	0.1647	0.0709
0.90	0.2404	0.2267	0.2395	0.2403	0.1929
1.10	0.3292	0.3095	0.3275	0.3291	0.3149
1.30	0.4300	0.4050	0.4268	0.4298	0.4370
1.50	0.5410	0.5125	0.5357	0.5406	0.5590
1.75	0.6904	0.6610	0.6817	0.6896	0.7115
2.00	0.8472	0.8208	0.8343	0.8455	0.8641
2.25	1.0067	0.9866	0.9896	1.0040	1.0166
2.50	1.1662	1.1536	1.1451	1.1615	1.1692
2.75	1.3234	1.3186	1.2992	1.3164	1.3217
3.00	1.4781	1.4800	1.4535	1.4682	1.4742
3.50	1.7822	1.7915	1.7612	1.7661	1.7793
4.00	2.0848	2.0944	2.0728	2.0637	2.0844
4.50	2.3891	2.3954	2.3873	2.3667	2.3895

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