Opportunity-Based Age Replacement: Exponentially Distributed Times Between Opportunities

Rommert Dekker* and Matthijs C. Dijkstra†
Koninklijke/Shell-Laboratorium, Amsterdam (Shell Research B.V.)
P.O. Box 3003, 1003 AA Amsterdam, The Netherlands

This article gives a full analysis of a component-replacement model in which preventive replacements are only possible at maintenance opportunities. These opportunities arise according to a Poisson process, independently of failures of the component. Conditions for the existence of a unique average optimal control limit policy are established and an equation characterizing the optimal policy and minimal average costs is derived. An important result is that the optimal policy can be described as a so-called one-opportunity-look-ahead policy. Such policies play an important role as heuristics in more general models. It is shown that there is a correspondence with the well-known age-replacement model, which can be considered as an extreme case of the model. Finally, some numerical results are given.

1. INTRODUCTION

Many replacement models assume that a preventive replacement can be carried out at any moment in time. If the units or components to be replaced are frequently idle, this assumption is not too crucial. However, many units in industry, e.g., power generators, are used continuously. For preventive replacements of components of these units to be cost effective, execution has to be delayed to some moment in time at which the unit is not required for service. Such idle moments can be created by many mechanisms, e.g., by breakdowns of other units in a series configuration with the unit in question, and in such cases we speak of maintenance opportunities. Unfortunately, in most cases opportunities cannot be predicted in advance, and because of this random occurrence, traditional maintenance planning fails to make effective use of them.

Within the Koninklijke/Shell-Laboratorium, Amsterdam, a decision support system for opportunity-based maintenance has recently been developed. In this article we deal with one of its underlying models, viz., the opportunity-based age replacement model. In this model a component can only be replaced preventively at an opportunity, contrary to a failure, at which the component is

*Corresponding author; present address: Shell Internationale Petroleum Maatschappij B.V., P.O. Box 162, 2501 AN The Hague, The Netherlands.
†Present address: Rotterdam School of Management, Erasmus University, Rotterdam.

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directly replaced. The occurrence of opportunities is described by a renewal process, independently of the state of the component.

Although the simple age-replacement model has been studied widely (see e.g., Barlow and Proschan [1] or Berg [4]) models for opportunity (or opportunistic) age replacement are scarce. Here we give a brief overview.

In one type of opportunity replacement models (introduced by Jorgenson, McCall, and Radner [10]) there are two classes of components. Failure of components in one class creates opportunities for the preventive replacement of components in the other. There are many variants of this model; e.g., the opportunity-creating component may or may not itself be preventively replaced, it may have an exponentially or a generally distributed lifetime, other components may be replaced at higher costs outside opportunities, etc. Jorgenson et al. [10] considered this type of model and provided formulas for the operating characteristics; however, they did not establish optimality results. Sethi [11] considered the case of two independent components with general discrete IFR distributions and showed that there exists an optimal policy of the control-limit type (a control-limit policy orders a component to be replaced at an opportunity if its age has passed a certain critical value). However, he did not show how the optimal policy can be determined. Other examples of this type of opportunity model are those given by Vergin and Ṡcriabin [14], Van der Duyn Schouten and Vanneste [12], and Bäckert and Rippin [2], who used discrete-time Markov decision chains. A disadvantage of their analysis is that it does not provide an interpretation; furthermore, their method becomes computationally intractable, if there are many discrete states or if there are more than three components. Finally, Berg [3] also considered a two-unit opportunity model with continuous time and derived partial differential equations for the joint p.d.f. of the ages of the components. In the case of Erlang distributed lifetime distributions he was able to express the joint p.d.f. as the solution to a set of linear equations. Although in some special cases (e.g., one- or two-stage Erlang) his approach yields explicit results, it is in general not suited for optimization.

Our model belongs to another type, in which opportunities are generated independently of the components considered. It corresponds to a special case of the first type, in which there is one opportunity-generating component and one independent component which can only be replaced preventively at opportunities. Woodman [15], Zaino [16], and Duncan and Scholnick [7] provide numerical results in the case of, respectively, exponentially and geometrically distributed times between opportunities. None of them, however, established optimality results. Vanneste [13] considered a component which deteriorates according to a continuous-time Markov chain (with observable states) and opportunities which arise according to a Poisson process. Apart from establishing average optimality of control-limit policies, he also presents an algorithm for evaluating the average costs.

This article can be regarded as a sequel to the one by Dekker and Smeitink [5] which dealt with opportunity block replacement. In this article we consider age replacement, principally being better than block replacement. However, an elegant analysis for age replacement can only be given for the case of exponentially distributed times between opportunities, whereas for block replacement a more general analysis is possible. We further focus on establishing optimality results and on giving an interpretation of the optimality equation. To this end we use the class of so-called one-opportunity-look-ahead policies, as introduced by Dekker and Smeitink [5], and show that the average optimal policy belongs to this class. These policies play an important role in setting priorities for execution and in developing heuristics for more general models, as used in the aforementioned decision support system for the opportunity maintenance. We further make a comparison between our model and the age replacement model, which can be regarded as an extreme case of our model. We finish this article by giving some numerical results.

2. THE AGE REPLACEMENT MODELS

Consider a component with a stochastic lifetime X with c.d.f. $F_X(t)$, p.d.f. $f_X(t)$ and failure rate $r_X(t)$. We assume that $f_X(t)$ is continuous in t. Upon failure, the component is replaced at costs c_f. A preventive replacement of the component against costs c_p is possible only at opportunities, which are supposed to be generated according to a Poisson process, independently of failures of the component. Let the random variable Y denote the time between successive opportunities, and let EY denote its finite mean and $F_Y(\cdot)$ its c.d.f. The main problem is to determine a strategy for preventive replacements which minimizes the total long-term average costs. In this article we will restrict ourselves to the class of so-called age-based control limit policies. Under these policies a component is preventively replaced at an opportunity if its age has passed the control limit. Besides their practical advantage, justifications for this restriction have been given by Sethi [10] and Vanneste [13]. In our analysis we will frequently make comparisons with an extreme case of the model in which the time between opportunities is infinitesimally small. It will be called the planned case and the model reduces to the well-known age replacement model. Our analysis can be considered as an extension of the marginal cost analysis (see Berg [4]) for that model. We will first start with a brief review of the analysis for the age replacement model.

2.1. The Standard Age Replacement Model

In the standard age replacement model (ARP) a component can be replaced preventively at any time. Therefore, under a strategy with control limit t a component is replaced preventively if its age equals t, and according to the renewal reward theorem the corresponding average costs $g_p(t)$ are equal to the average cycle cost divided by the average cycle length $L_p(t)$. Hence we have

$$g_p(t) = \frac{c_p + (c_f - c_p)P(X < t)}{L_p(t)}, \tag{1}$$

where the expected cycle length $L_p(t)$ is given by the following formula:

$$L_p(t) = E(\min(X, t)) = \int_0^t (1 - F_X(x))\, dx = tP(X > t) + \int_0^t x f_X(x)\, dx. \tag{2}$$

As $f_X(t)$ is continuous we can easily differentiate $L_p(t)$ and $g_p(t)$ and obtain after some algebra

$$L_p'(t) = P(X > t) \tag{3}$$

and

$$g_p'(t) = \frac{P(X > t)}{L_p(t)} [(c_f - c_p)r_X(t) - g_p(t)]. \tag{4}$$

Let $\eta_p(t) \equiv (c_f - c_p)r_X(t)$. We are now able to formulate the main theorem for the ARP model the proof of which can be found in, e.g., Berg [4]. To avoid confusion, in this article the word increasing has the same meaning as strictly increasing.

THEOREM 1:

(a) If $r_X(t)$ increases in t and if $\lim_{t\to\infty} r_X(t) > c_f/(c_f - c_p)EX$, then there exists a unique minimum g_p^* of $g_p(t)$ in t_p^*.
(b) t_p^* is the unique solution to the optimality equation

$$\eta_p(t) - g_p(t) = 0. \tag{5}$$

(c) $\eta_p(t) - g_p^* < 0 \Leftrightarrow t < t_p^*$.

2.2. The Opportunity Age Replacement Model

In the opportunity age replacement model (OARP) preventive replacements are allowed at opportunities only, which occur according to a Poisson process with rate $1/EY$. Notice that both after a failure and after a preventive replacement the (residual) time to the next opportunity is again exponentially distributed with mean EY and that both events therefore can be considered as the end of a renewal cycle. According to the renewal reward theorem we have the following formula for the long-term average costs $g_{\text{op}}(t)$ under a policy with control limit t ("op" indicates the opportunity model):

$$g_{\text{op}}(t) = \frac{C_{\text{op}}(t)}{L_{\text{op}}(t)} = \frac{c_p + (c_f - c_p)P(X < t + Y)}{L_{\text{op}}(t)}, \tag{6}$$

where $C_{\text{op}}(t)$, $L_{\text{op}}(t)$ denote the expected cycle cost, length, respectively. The latter is given by the following formula:

$$L_{\text{op}}(t) = E(\min(X, t + Y)) = \int_0^\infty \int_0^{t+y} (1 - F_X(x))\,dx\,dF_Y(y). \tag{7}$$

In the sequel it will be useful to have other expressions for $L_{\text{op}}(t)$ as well. Note therefore that by change of integration order we obtain

$$L_{\text{op}}(t) = \int_0^t (1 - F_X(x))\,dx + \int_t^\infty (1 - F_X(x))\,(1 - F_Y(x - t))\,dx. \tag{8}$$

Remark further that as Y is exponentially distributed with mean EY we have $(1 - F_Y(t)) = EY\, dF_Y(t)/dt$. Hence we can rewrite (8) into

$$L_{op}(t) = \int_0^t (1 - F_X(x))\, dx + \int_0^\infty (1 - F_X(x + t))\, EY\, dF_Y(x)$$

$$= E(\min(X, t)) + EY\, P(X > t + Y). \quad (9)$$

The analysis of the OARP model is similar to that of the ordinary ARP model and is directed at establishing a formula for the derivative of the average costs. Optimality results then follow easily. To differentiate $P(X < t + Y)$ first notice that by uniform convergence of its constituting integral we have

$$\frac{d}{dt} P(X < t + Y) = \int_0^\infty f_X(t + y) f_Y(y)\, dy,$$

which, by partial integration [with $f_X(\cdot)$ as derivative function] is equal to

$$= -\frac{P(X < t)}{EY} - \int_0^\infty P(X < t + y) \frac{-1}{EY} f_Y(y)\, dy$$

$$= \frac{1}{EY} P(t < X < t + Y). \quad (10)$$

With respect to the derivative of the expected cycle length $L_{op}(t)$ we notice that by combining Eqs. (3), (9), and (10) we obtain

$$L'_{op}(t) = P(X > t) - P(t < X < t + Y) = P(X > t + Y). \quad (11)$$

Hence, we arrive at the following expression for $g'_{op}(t)$

$$\begin{aligned} g'_{op}(t) &= \frac{1}{[L_{op}(t)]^2} [C'_{op}(t) L_{op}(t) - C_{op}(t) L'_{op}(t)] \\ &= \frac{L'_{op}(t)}{L_{op}(t)} \left[\frac{C'_{op}(t)}{L'_{op}(t)} - g_{op}(t) \right] \\ &= \frac{P(X > t + Y)}{L_{op}(t)} \left[(c_f - c_p) \frac{P(t < X < t + Y)}{EY\, P(X > t + Y)} - g_{op}(t) \right]. \end{aligned} \quad (12)$$

Notice that both $L_{op}(t)$ and $L'_{op}(t)$ are positive for all $t > 0$; hence, we have established the following relation:

$$g'_{op}(t) = 0 \Leftrightarrow (c_f - c_p) \frac{P(t < X < t + Y)}{EY\, P(X > t + Y)} - g_{op}(t) = 0. \quad (13)$$

Let us now define

$$\eta_{\mathrm{op}}(t) = (c_f - c_p)\frac{P(t < X < t + Y)}{EY\,P(X > t + Y)}.$$

To prove our main theorem, on the existence of optimal policies, it is essential to establish monotonicity of $\eta_{\mathrm{op}}(t)$. This rather intricate problem is solved in the following lemma.

LEMMA 2: If $r_X(t)$ is increasing in t for $t > 0$, then (a) both $P(X < t + Y|X > t)$ and $\eta_{\mathrm{op}}(t)$ are increasing in t, for $t > 0$, (b) $\eta_{\mathrm{op}}(t) > \eta_p(t)$, $t > 0$.

PROOF: Part (a). First notice that we can rewrite $\eta_{\mathrm{op}}(t)$ into

$$\eta_{\mathrm{op}}(t) = (c_f - c_p)\frac{1}{EY}\left[\frac{P(X > t)}{P(X > t + Y)} - 1\right].$$

Next remark that

$$\frac{P(X > t + Y)}{P(X > t)} = \int_0^\infty \frac{P(X > t + y)}{P(X > t)}\,dF_Y(y) \tag{14}$$

and that from the well-known relationship between the failure rate $r_X(t)$ and the cumulative distribution function $P(X < t)$ we have

$$\frac{P(X > t + y)}{P(X > t)} = \exp\left[-\int_t^{t+y} r_X(x)\,dx\right]. \tag{15}$$

As $\int_t^{t+y} r_X(x)\,dx$ increases in t, $P(X > t + y)/P(X > t)$ is decreasing in t by (15). By (14) $P(X > t + Y)/P(X > t)$ is also decreasing in t, from which part (a) of the lemma readily follows.

Part (b). First notice that

$$P(t < X < t + Y) = \int_0^\infty \int_t^{t+y} f_X(x)\,dx\,dF_Y(y)$$

$$= \int_0^\infty \int_t^{t+y} r_X(x)[1 - F_X(x)]\,dx\,dF_Y(y) > r_X(t)\int_0^\infty \int_t^{t+y} (1 - F_X(x))\,dx\,dF_Y(y),$$

which, by (7)–(9) equals $r_X(t)\;EY\;P(X > t + Y)$. This directly implies part (b). □

Now we are able to state the main theorem of this section.

THEOREM 3: If $r_X(t)$ increases in t and $\lim_{t\to\infty} r_X(t) > c_f/(c_f - c_p)EX$, then there exists a unique t^*_{op} which minimizes $g_{\text{op}}(t)$ and which is the unique solution to the optimality equation

$$\eta_{\text{op}}(t) - g_{\text{op}}(t) = 0. \tag{16}$$

Moreover,

$$\eta_{\text{op}}(t) - g_{\text{op}}(t) > 0 \Leftrightarrow t > t^*_{\text{op}}$$

and

$$\eta_{\text{op}}(t) - g^*_{\text{op}} > 0 \Leftrightarrow t > t^*_{\text{op}},$$

where $g^*_{\text{op}} = g_{\text{op}}(t^*_{\text{op}})$.

PROOF: First notice that for $t = 0$ we have

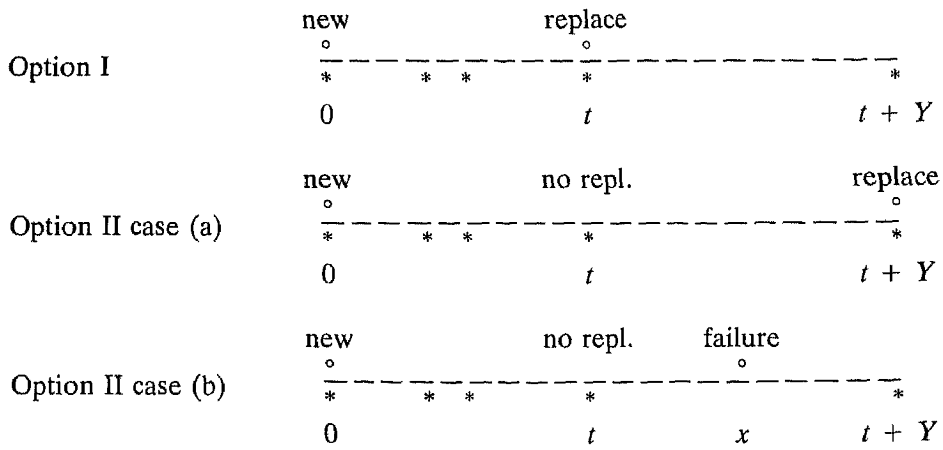
$$g_{\text{op}}(0) = \frac{c_p + (c_f - c_p)P(X < Y)}{E(\min(X, Y))} < \infty$$

and that $\eta_{\text{op}}(0) = (c_f - c_p)P(X < Y)/EY\ P(X > Y)$, which by (9) is equal to $(c_f - c_p)P(X < Y)/E(\min(X, Y))$. Hence $\eta_{\text{op}}(0) < g_{\text{op}}(0)$, and therefore, $g'_{\text{op}}(0) < 0$. Next consider the behavior of $\eta_{\text{op}}(t)$ and $g_{\text{op}}(t)$ for large t. It is easy to show that $\lim_{t\to\infty} g_{\text{op}}(t) = c_f/EX$. Furthermore, by Lemma 2 we have $\lim_{t\to\infty} \eta_{\text{op}}(t) > c_f/EX$. Hence there is at least one point in which $\eta_{\text{op}}(t)$ and $g_{\text{op}}(t)$ intersect and by (13) it is an extremum of $g_{\text{op}}(t)$. Let t_0 be the smallest intersection point and t_1 the next one (if it exists). Hence t_0 is a minimum and t_1 a maximum. As $g'_{\text{op}}(t_1) = 0$ it follows that $g_{\text{op}}(t)$ is nonincreasing for $t_1 < t < t_1 + \delta$, for some $\delta > 0$, which is in contradiction with the fact that as $\eta_{\text{op}}(t)$ is increasing in t, $g_{\text{op}}(t)$ should be increasing on $(t_1, t_1 + \delta)$ by (13). Hence $g_{\text{op}}(t)$ has one extremum only and the rest of the theorem easily follows. □

3. THE ONE-OPPORTUNITY-LOOK-AHEAD POLICIES

In this section we will give an interpretation of the optimality equation derived in Section 2.2. To this end we use the concept of one-opportunity-look-ahead policies as introduced by Dekker and Smeitink [5]. These policies can be considered as generalizations of marginal cost policies (see Berg [3]).

Consider an opportunity and suppose that t is the age of the component. We consider two options, one in which we replace the component preventively at the present opportunity, and one in which we do not replace at the present opportunity but at the next opportunity, Y time units ahead, if it has not been replaced upon failure in between. Both options are sketched in the following figure.



where asterisks denote an opportunity and open circles a renewal of the component, either preventive or upon failure. We assume that after both options a policy is followed with long-term average costs $g > 0$. Comparing Options I and II is only possible if we compensate for the different moments at which the component is renewed. Therefore we associate costs $c_p + gEY$ over the interval $[t, t + Y]$ with Option I. Costs associated with Option II over the interval $[t, t + Y]$ depend on which case occurs. Let the r.v. X_t denote the conditional residual lifetime of the component with age t; i.e., $P(X_t \le y) = P(X - t \le y | X > t)$. Case (a) occurs with probability $P(X_t > Y)$ and the associated costs amount to c_p. Case (b) occurs with probability $P(X_t < Y)$; the conditional expected time of the failure is $E(X_t|X_t < Y)$; hence we associate costs $c_f + gE(Y - X_t|X_t < Y)$ with case (b). Taking the expectation over both cases yields as total expected costs for Option II $c_pP(X_t > Y) + P(X_t < Y)[c_f + gE(Y - X_t|X_t < Y)]$. A cost comparison of the two options now reveals that Option I is to be preferred to Option II if $(c_f - c_p)P(X_t < Y) - g\{EY - P(X_t < Y)E(Y - X_t|X_t < Y)\} \ge 0$. Notice now that $EY = P(X_t < Y)\, E(Y|X_t < Y) + P(X_t > Y)\, E(Y|X_t > Y)$; hence

$$EY - P(X_t < Y)\, E(Y - X_t|X_t < Y) = P(X_t < Y)\, E(X_t|X_t < Y) + P(X_t > Y)\, E(Y|X_t > Y) = E(\min(X_t, Y)).$$

This implies that Option I is to be preferred if

$$(c_f - c_p)P(X_t < Y) - g\, E(\min(X_t, Y)) \ge 0. \tag{17}$$

We now define the one-opportunity-look-ahead (OOLA) policy with threshold value g (>0) by "replace the component preventively at an opportunity if (17) is fulfilled, where t is the age of the component."

Next we will show under which conditions the OOLA policies belong to the class of control-limit strategies.

THEOREM 4: (a) If $r_X(t)$ is increasing in t for all $t > 0$ then every OOLA policy is a control-limit policy, and (b) if also $\lim_{t\to\infty} r_X(t) > c_f/(c_f - c_p)EX$, then the OOLA policy with threshold value g^*_{op} is equivalent to the control-limit strategy with control limit t^*_{op} and is average optimal.

PROOF: First notice that by Lemma 2, $P(X_t < Y) = P(X < t + Y|X > t)$ is increasing in t. Next consider $E(\min(X_t, Y)$. Remark that analogous to Eqs. (7)–(9) we can derive that

$$E(\min(X_t, Y)) = EY\, P(X_t > Y). \tag{18}$$

Hence $E(\min(X_t, Y))$ is decreasing in t. This implies that if inequality (17) for a certain g is fulfilled for some $t_0 > 0$, it is also fulfilled for every $t > t_0$. In other words, the OOLA policy is of the control-limit type. To prove part (b) consider the OOLA policy with threshold value g^*_{op} and let t_c be the control limit of the corresponding control-limit policy, implying that $(c_f - c_p)P(X_{t_c} < Y) - g^*_{\text{op}}$ $E(\min(X_{t_c}, Y) = 0$. By eq. (18) we now have

$$[(c_f - c_p)P(t_c < X < t_c + Y)/P(X > t_c)] - g^*_{\text{op}}EY\, P(X > t_c + Y)/P(X > t_c) = 0,$$

and hence

$$[(c_f - c_p)P(t_c < X < t_c + Y)/EY\, P(X > t_c + Y)] - g^*_{\text{op}} = 0.$$

From Theorem 3 it then follows that $t_c = t^*_{\text{op}}$ and the OOLA policy with threshold value g^*_{op} is average optimal. □

The importance of OOLA policies over control-limit policies is that they yield a criterion stating not only *whether* to replace, but also *how important* it is to replace. A control-limit policy in fact only states the first part, and the amount of time passed since the control limit is not a good measure of the importance of preventive replacement. In practice, one frequently has to set priorities for execution, in particular for opportunity maintenance, and OOLA policies provide a good priority criterion. OOLA policies have been applied in the aforementioned decision support system for opportunity maintenance. Their use in setting priorities (in case of block replacement) has been investigated in Dekker and Smeitink [6].

4. RELATIONS BETWEEN THE OARP AND THE ARP MODEL

In this section we consider both the OARP and the ARP model and establish some relationships. The first question is about the relation between the minimum t^*_{op} in the OARP and the minimum t^*_p in the ARP model. As t^*_{op} is a control limit implying that preventive replacements are executed at the first opportunity occurring after it, one would expect t^*_{op} to be smaller than t^*_p. Next to that, one wonders where the graph of $g_p(t)$ would intersect the graph of $g_{op}(t)$. Both questions are tackled in the next theorem.

THEOREM 5: (a) For all values of EY we have $g_p(t) = g_{op}(t) \Leftrightarrow g'_{op}(t) = 0$. (b) If the assumptions of Theorem 3 are fulfilled then $t^*_{op} < t^*_p$ and $g_p(t) < g_{op}(t)$ for $t > t^*_p$.

PROOF: For part (a) notice first the following equivalences (provided that no numerator equals 0):

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{a}{b} = \frac{c - a}{d - b} \Leftrightarrow \frac{c}{d} = \frac{c - a}{d - b}.$$

Accordingly, from Eqs. (1), (2), (6), and (9), it follows that $g_p(t) = g_{op}(t)$ is equivalent to

$$g_{op}(t) = \frac{(c_f - c_p)[P(X < t + Y) - P(X < t)]}{EY\, P(X > t + Y)}.$$

As the right-hand side equals $\eta_{op}(t)$, it follows directly from Eq. (13) that $g_p(t) = g_{op}(t)$ is equivalent to $g'_{op}(t) = 0$. For part (b) recall that under the assumptions both the respective minima t^*_p and t^*_{op} exist, that $g_p(0) = \infty$, while $g_{op}(0) < \infty$, and that both $g_p(t)$ and $g_{op}(t)$ are decreasing up to their absolute minimum and are increasing afterwards. Let us now assume that $t^*_{op} > t^*_p$. This implies that $g_p(t)$ is increasing at t^*_{op} and intersects $g_{op}(t)$ therefore from below. However, from $g_{op}(0) < g_p(0)$, it follows that $g_p(t)$ and $g_{op}(t)$ should intersect also before t^*_{op}, which contradicts part (a). Therefore we have $t^*_{op} \leq t^*_p$. To prove the remaining part of (b) notice that by Eqs. (6) and (7) we can write $g_{op}(t)$ as

$$g_{op}(t) = \frac{\int_0^\infty [c_p + (c_f - c_p)P(X < t + y)]\, dF_Y(y)}{\int_0^\infty E(\min(X, t + y)\, dF_Y(y)}. \tag{19}$$

As $g_p(t)$ is strictly increasing in t from t^*_p onward, we have

$$c_p + (c_f - c_p)P(X < t^*_p + y) > g^*_p\, E(\min(X, t^*_p + y)), \qquad y > 0. \tag{20}$$

Since Y is nondegenerate, it directly follows that $g_{op}(t^*_p) > g_p(t^*_p)$ by inserting inequality (20) in the integral in the numerator of Eq. (19). Together with part (a) it also implies that $t^*_{op} \neq t^*_p$, which finishes the proof. □

5. NUMERICAL RESULTS

5.1. Methods

Calculation of $g_{op}(t)$ using Eqs. (6) and (9) basically requires calculation of $P(X < t + Y)$ and $E(\min(X, t))$. The latter can be done with any numerical integration routine, since for most distributions, e.g., Weibull, there is no explicit algebraic expression for the truncated expectation. Truncating the constituting integral for the first one is not advisable, since its tail can have a large contribution. It is better to use a quadrature method or to split the integral up into two parts and convert the indefinite part into a finite integral by using a $1/t$ transformation.

Determining the optimum t_{op}^* can be done by evaluating next to $g_{\mathrm{op}}(t)$ $\eta_{\mathrm{op}}(t)$ as well and making use of Theorem 3. By means of a search and bisection procedure the minimum is then easily determined. On the other hand, just evaluating $g_{\mathrm{op}}(t)$ for many values of t to determine the minimum is also quite easy, especially since by Theorem 5 $t_{\mathrm{op}}^* < t_p^*$.

5.2. Numerical Observations

We have evaluated the OARP model for many different values of the input parameters. Table 1 shows some results. The conclusions which can be drawn from these experiments on the sensitivity of the minimal average costs g_{op}^* and the optimal critical age t_{op}^* are basically the same as for the ARP model (see, e.g., Glasser [9] and Geurts [8]). They are

(i) The value of preventive maintenance, expressed as $[g(\infty) - g_{\mathrm{op}}^*]/g(\infty)$, where $g(\infty) = \lim_{t\to\infty} g_{\mathrm{op}}(t) = \lim_{t\to\infty} g_p(t)$, increases with an increasing cost ratio $NR \equiv c_f/c_p$ and with a decreasing coefficient of variation of the lifetime distribution. It decreases with an increasing mean time between opportunities, EY.

(ii) The position of the minimum t_{op}^* decreases with an increasing cost ratio c_f/c_p and with an increasing time between opportunities. The coefficient of variation $c_X(\equiv\sigma(X)/EX)$ has a hybrid effect. For c_X decreasing from 1, the minimum t_p^* decreases from infinity to values below EX. However, from a certain value of c_X it starts increasing again to EX for c_X decreasing to zero.

Figures 1–3 typically show these effects.

Table 1. The minimal average costs g_{op}^* and optimal control limit t_{op}^* as functions of the mean time between opportunities EY.

	$R = 20, \beta = 1.5$		$R = 20, \beta = 2.0$		$R = 20, \beta = 3.0$		$R = 10, \beta = 2.0$	
EY	g_{op}^*	t_{op}^*	g_{op}^*	t_{op}^*	g_{op}^*	t_{op}^*	g_{op}^*	t_{op}^*
0.00	0.612	2.50	0.388	2.60	0.226	3.35	0.537	3.80
0.50	0.617	2.10	0.395	2.15	0.231	2.85	0.541	3.35
0.75	0.622	1.90	0.402	2.00	0.238	2.60	0.546	3.15
1.00	0.629	1.80	0.412	1.80	0.247	2.40	0.552	2.95
1.50	0.646	1.60	0.438	1.55	0.274	2.05	0.568	2.70
2.00	0.664	1.40	0.466	1.40	0.307	1.75	0.586	2.45
3.00	0.699	1.20	0.525	1.15	0.380	1.35	0.626	2.10
5.00	0.757	1.00	0.622	0.90	0.510	1.00	0.695	1.75
7.00	0.799	0.90	0.691	0.80	0.602	0.85	0.747	1.60

Where $R = c_f/c_p$, $c_f = 10$.
Lifetime distribution: Weibull with mean 10 and shape β.

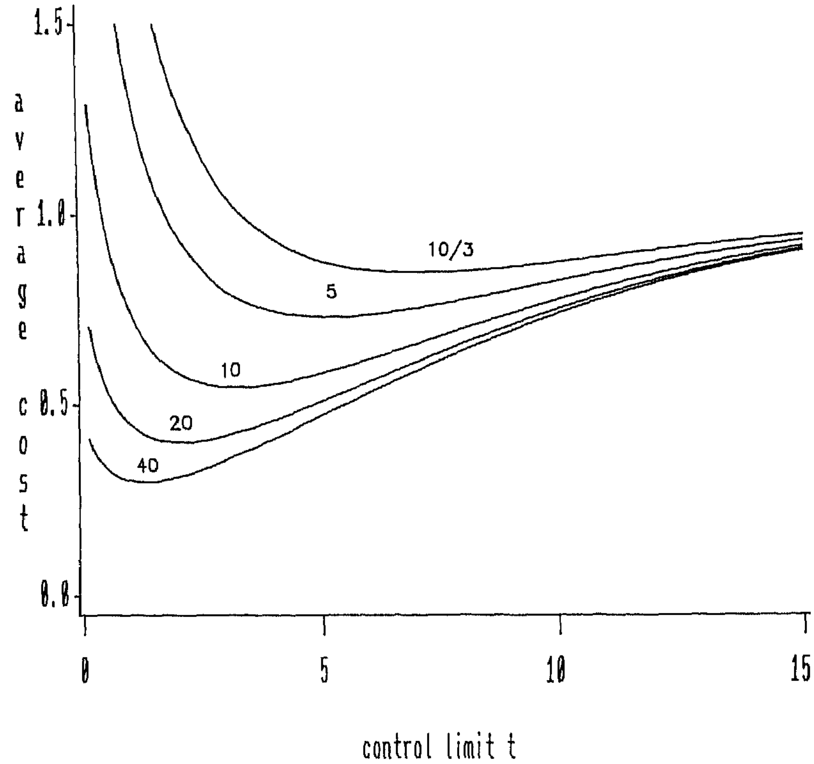


Figure 1. Sensitivity to the cost ratio c_f/c_p. Cost ratio c_f/c_p = 40, 20, 10, 5, 10/3 Lifetime distr. Weibull, mean 10, $\beta = 2$, $c_f = 10$, $EY = 0.75$

We have also tried to determine some yardstick for the position of t^*_{op}, using the value of t^*_p. As it is most cost effective to do preventive replacements when the component has about age t^*_p and as the expected time to the first opportunity after time t is equal to $t + EY$, we have investigated the performance of $t^*_p - EY$ as approximation for t^*_{op}. However, as this appoximation can be negative for large values of EY, and as it is easily shown that accepting opportunities before $t = c_pEX/c_f$ leads to cycles with average costs larger than $g(\infty)$, we altered the approximation into $t_a \equiv \max(c_pEX/c_f,\ t^*_p - EY)$. Table 2 gives an overview of its performance in case of Weibull lifetime distributions.

6. THE CASE OF NONEXPONENTIAL TIMES BETWEEN OPPORTUNITIES

In this section we will briefly investigate the case of nonexponentially distributed times between opportunities. In the foregoing analysis we have frequently used properties of the exponential distribution. Therefore, it will be clear that the analysis cannot easily be extended to the nonexponential case. The main bottleneck is that failure replacements can no longer be regarded as renewals of the opportunity process and one has to keep track of the residual time to the next opportunity. Using phase-type distributions for the opportunity process

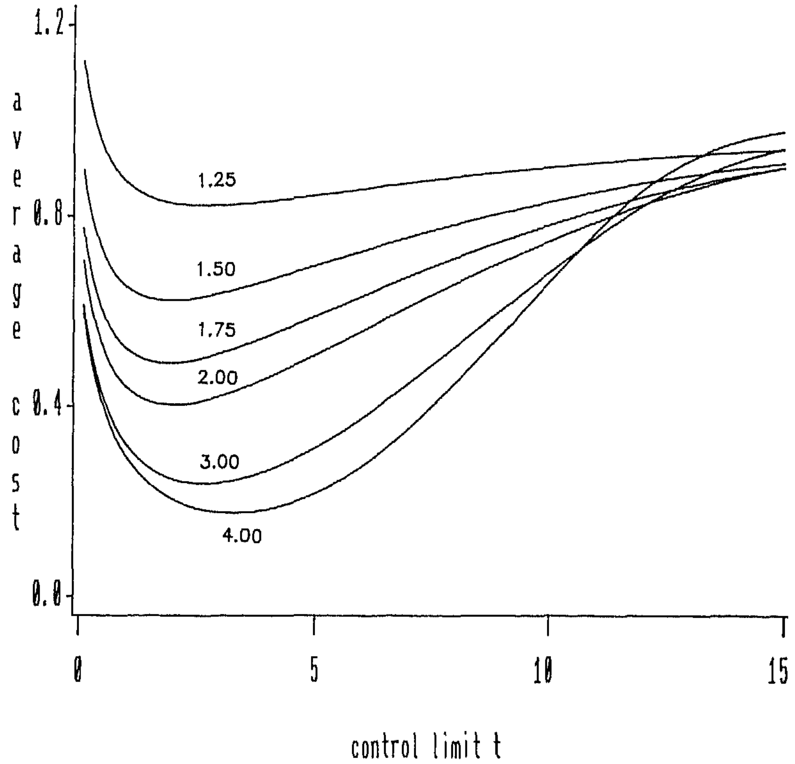


Figure 2. Sensitivity to the Weibull shape factor. Weibull shape factor β = 1.25, 1.50, 1.75, 2.00, 3.00, 4.00 Lifetime distr. Weibull, mean 10, c_f = 10, c_p = 0.5, EY = 0.75

does not help unless the lifetime distribution also has a phase-type distribution. In fact, the only general approach is to discretize both the lifetime and the time between opportunities. One can set up a Markov decision chain with a two-dimensional state space, with one state component for the lifetime and one for the remaining time to the next opportunity. As decisions only need to be taken at opportunities, one can reduce the dimension of the state space by reducing the Markov chain to a semi-Markov chain. We will not discuss this approach here in detail. However, we would like to make one further remark on the nonexponential case. The next counterexample shows that Theorem 4 is no longer valid and that the optimal policy no longer can be described as a one-opportunity-look-ahead policy.

EXAMPLE: Let the r.v.'s X and Y, denoting the lifetime and the time between opportunities, respectively, be almost degenerate, so that for the calculations we can consider them as being deterministic. This is no restriction, as for X we can, e.g., choose a Weibull distribution with a very large shape factor. Since the coefficient of variation goes to zero if the shape factor goes to infinity, we can approximate any deterministic number by an IFR distribution with small enough coefficient of variation. The same argument applies to Y, although for Y we do not need the IFR property. Let $Y = 2 + \delta$, $X = 3 + \varepsilon$, where δ, ε are infinitesimally small and $\delta \ll \varepsilon$. The only purpose of δ and ε is to indicate the order of occurrence of events. The cost parameters have the values c_f =

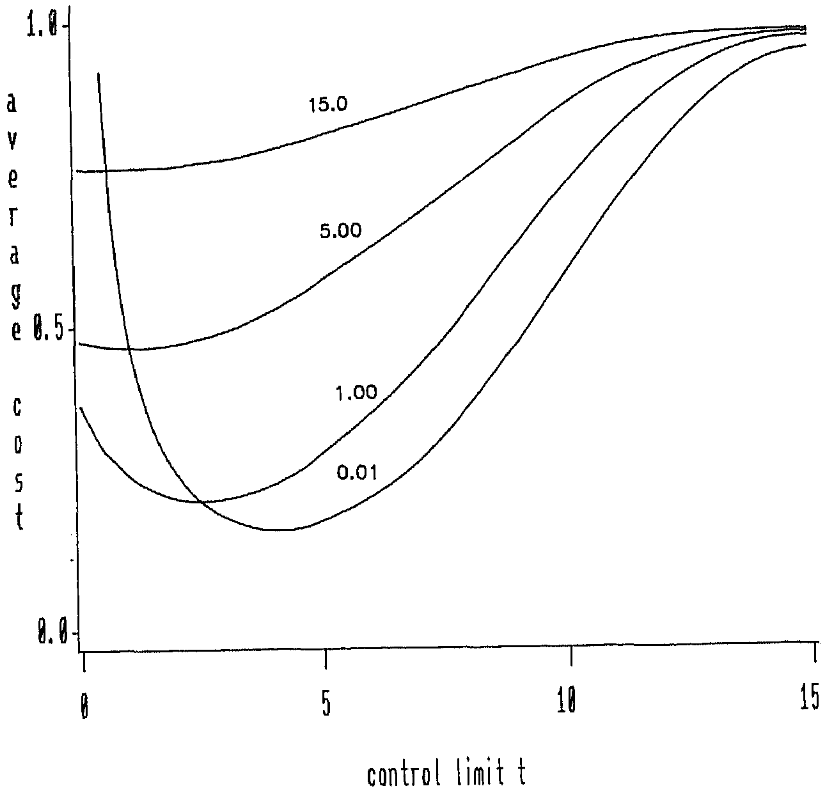


Figure 3. Sensitivity to the mean time between opportunities EY. EY = 0.01, 1.00, 5.00, 15.0 Lifetime distr. Weibull, mean 10, $\beta = 4$, $c_f = 10$, $c_p = 0.5$

5 and $c_p = 3$. We denote by t the control-limit policy with threshold t. For the average costs we obtain

t	$g_{op}(t)$
[0, 2]	1.50
(2, 3)	1.33
[3, ∞]	1.66

Table 2. The maximal relative error (in %) in average costs by the approximation $t_a = \max(c_p EX/c_f,\ t_p^* - EY)$ of t_{op}^*.

R	40		20		10		5		10/3	
β	Max	at EY	Max	at EY	Max	at EY	Max	at EY	Max	at EY
1.25	2.3	15	1.9	30	2.4	50	—			
1.50	2.3	15	3.1	20	2.4	30	2.3	50		
1.75	3.3	15	3.6	20	2.9	30	2.4	50	2.5	70
2.00	4.1	15	3.5	20	3.1	30	1.8	50	2.9	50
3.00	2.1	20	3.2	30	2.8	30	1.9	30	1.7	50
4.00	1.7	30	1.3	30	1.4	50	1.9	50	1.7	50

Max: the maximum error $100\% \cdot (g_{op}(t_a) - g_{op}^*)/g_{op}^*$, taken over the following range of EY: 5%, 7.5%, 10%, 15%, 20%, 30%, 50%, and 70% of the mean life; lifetime distribution: Weibull; $R = c_f/c_p$.

Notice that $t = 3 - \varepsilon$ is an optimal control limit and that the minimal average costs g^*_{op} amount to 1.33. Now consider the OOLA policy with threshold value g^*_{op}. For $t = 2$ we have $(c_f - c_p)P(X_2 < Y) = 2$, which is larger than $g^*_{\text{op}}E(\min(X_2, Y))$. Hence the OOLA policy with threshold g^*_{op} orders that the component be replaced at an opportunity if it has age 2, which is not average optimal. It is easily checked that for any $g \leq 2$ we will replace preventively at $t = 2$, while for $g > 2$ we will never replace preventively. Hence the optimal policy cannot be described as an OOLA policy.

REMARK: Essential aspects of the example are that doing two preventive replacements is more costly than one failure replacement, that there are few opportunities compared to the mean lifetime and, finally, that both the opportunity and lifetime process are deterministic. In fact, the optimal policy first allows the component to fail and then makes use of the good coordination between opportunity and failure time. As the OOLA policy only looks one opportunity ahead, it does not succeed in making this comparison. In most practical applications the lifetime process is not at all deterministic (e.g., from Weibull analyses hardly any shape $\beta > 5$ is observed) and preventive replacements are only realistic if made before the mean lifetime. Therefore, it is likely that OOLA policies will behave better in realistic cases with a nonexponential time between opportunities. We conjecture that the OOLA policy with threshold value $g^*_{b,\text{op}}$ (the minimal average costs in the opportunity block replacement problem) would be a good approximation for the optimal policy.

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