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ON THE RELATION BETWEEN RECURRENCE AND ERGODICITY PROPERTIES IN DENUMERABLE MARKOV DECISION CHAINS

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This paper studies two properties of the set of Markov chains induced by the deterministic policies in a Markov decision chain. These properties are called μ-uniform geometric ergodicity and μ-uniform geometric recurrence. μ-uniform ergodicity generalises a quasi-compactness condition. It can be interpreted as a strong version of stability, as it implies that the Markov chains generated by the deterministic stationary policies are uniformly stable. μ-uniform geometric recurrence can be shown to be equivalent to the simultaneous Doeblin condition, if μ is bounded. Both properties imply the existence of deterministic average and sensitive optimal policies.

The second Key theorem in this paper shows the equivalence of μ-uniform geometric ergodicity and weak μ-uniform geometric recurrence under appropriate continuity conditions.

In the literature numerous recurrence conditions have been used. The first Key theorem derives the relation between several of these conditions, which interestingly turn out to be equivalent in most cases.

1. Introduction. In the literature on denumerable Markov decision chains (MDC's), two types of conditions play an important role. The first one is an ergodicity or quasi-compactness condition. In its most general form it has been introduced as μ-uniform geometric convergence in Dekker and Hordijk (1989), but in Hordijk and Spieksma (1992), Spieksma (1990) renamed as *μ-uniform geometric ergodicity* to have a better correspondence with the terminology for related concepts in Markov chain theory. *Uniform* is added in this paper to all concepts used in Hordijk and Spieksma (1992), to denote that the property holds for a set of Markov chains (MC's), for instance the set of MC's induced by the deterministic policies in an MDC.

μ-uniform geometric ergodicity requires geometrically fast convergence of the n step transition probabilities to the stationary probabilities in μ-norm, uniformly in the deterministic, stationary policies. This μ-norm is a weighted supremum norm with weighting or *bounding vector* μ.

Under additional compactness and continuity assumptions, the use of this property provides a direct method to establish the existence of maximising policies for sensitive optimality criteria (cf. Dekker and Hordijk (1989)), and all reward functions that are uniformly μ-bounded in the deterministic policies. Although an elegant analysis resutls when using ergodicity conditions, they are hard to verify in practice. Therefore most papers on denumerable MDC's use recurrence conditions.

Section 2 introduces various recurrence conditions, which describe different types of tail behaviour of the first hitting times of a fixed fintie set of states. These have been studied by Dekker and Hordijk (1992) to derive existence results for sensitive optimality criteria similar to their 1989 paper (Dekker and Hordijk 1989). The strongest version of these is called *μ-uniform geometric recurrence*. It requires the

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existence of a finite set M, such that the taboo transition matrix ${}_MP(f)$ with taboo set M is a contraction in μ-norm, uniformly in the deterministic policies $f \in \mathscr{F}$.

The first main result of the paper relates these recurrence conditions in Key Theorem I (see also Dekker (1985)); some of these relations have been derived in Dekker and Hordijk (1992) but are stated here to have a complete overview. Besides being of interest in itself, this theorem plays a crucial role in proving the second main result in Key Theorem II. The second Key theorem asserts the equivalence of *μ-uniform geometric ergodicity* and *weak μ-uniform geometric recurrence* for aperiodic MDC's, for which the number of closed classes depends continuously on the policy, and thus generalises a similar result for one MC in Hordijk and Spieksma (1992). Hence, the classes of models satisfying either the ergodicity condition or one of the recurrence conditions are essentially identical. In particular the desirable strong stability property μ-uniform geometric ergodicity can be checked through recurrence conditions and so this property is satisfied by controlled versions of the queueing models studied in Spieksma (1990, 1991a, b), Spieksma and Tweedie (1992), provided a uniform ergodicity condition holds. These models include the Jackson network, a coupled processors model, an ALOHA system, the K-competing queues model, a polling system and s-server retrial queues with a finite buffer and an infinite capacity orbit. In these models the arrival and/or service rates may be controlled to optimise the number of jobs at the various queues with respect to some (polynomial or exponential) cost function. In the case of the K-competing queues model, control is exercised by assigning the server to one of the queues.

We will briefly discuss some precursors to our results. The first results on the relation between μ-uniform geometric ergodicity and recurrence have been derived by Federgruen, Hordijk and Tijms (1978a, b) in the context of average optimality in MDC's. Actually they use the supremum norm, which is the same as the e-norm if e is the vector consisting of 1's only. For *unichain* MDC's they show the equivalence of e-uniform geometric ergodicity and the simultaneous Doeblin condition, thereby using a representation derived in Hordijk (1974). For one MC this result has already been proved in the fifties by Doob (cf. Doob 1953) and an elegant proof can be found in Neveu's book (Neveu 1965). The equivalence of the simultaneous Doeblin condition and μ-uniform geometric recurrence for μ a bounded vector and the relation between various recurrence conditions is studied in Hordijk (1974) as well as in Federgruen, Hordijk and Tijms (1978a, b). Thomas (1980) gives an extensive overview of various recurrence and ergodicity conditions used mainly for unichained MDC's.

The first extension of the results in Federgruen, Hordijk and Tijms (1978a, b) is due to Zijm (1985), who allows a *multichain* MC structure. Our results are for an arbitrary μ-vector and a multichain structure, and hence, extend all previously mentioned results. The extension to a general bounding vector is important. Indeed, when using the e-norm, the implications are valid only for uniformly bounded rewards. This is a severe restriction for many queueing models, as interesting reward functions, such as waiting costs in an open network, tend to be unbounded.

Additionally, allowing μ to be unbounded essentially extends the class of models that can be analysed using uniform recurrence or ergodicity properties. Consider for example a discrete time version of the $M/M/1$-queue (cf. Spieksma 1990). This has a left skipfree property and so for any n and any finite set, there does not exist a positive lower bound on the probability of reaching this finite set in n steps, uniformly in the initial states. Consequently, the $M/M/1$-queue does not satisfy the simultaneous Doeblin condition, and hence cannot be μ geometrically ergodic for bounded μ-vectors. However, it is easy to show, that the $M/M/1$-queue is μ-geometrically ergodic, if we take the ith component μ_i to equal $(1 + x)^i$, for some sufficiently small $x > 0$.

We would like to remark, that our analysis is more complicated than the analysis in Federgruen, Hordijk and Tijms (1978a, b) and Zijm (1985) in the following sense. Their proofs are facilitated by the knowledge that $\sum_j P_{ij}(f)\mu_j = 1$ for all initial states $i \in E$ and all deterministic policies, if $\mu = e$. For general μ, $\sum_j P_{ij}(f)\mu_j$ may be different for different starting states i and decision rules f and the only information available is the existence of some $c > 0$, such that $\sum_j P_{ij}(f)\mu_j \leqslant c\mu_i$, $\forall i, f$.

The paper by Lasserre (1988) poses the existence of the Laurent expansion of the probability generating functions, as a μ-bounded and μ-continuous operator. For μ-bounded rewards he applies the analysis in Dekker and Hordijk (1989) to obtain optimality results. Within the framework of MDC's with general but μ-bounded immediate rewards his conditions are quite weak. Spieksma (1990) shows that basically Lasserre's conditions and μ-uniform weak recurrence are equivalent. For the sake of completeness we state the precise result in this paper.

Finally, as mentioned before, the two Key theorems generalise the results in Hordijk and Spieksma (1992) for one MC, although the first Key theorem on the relation between the various recurrence conditions has been proved originally in Dekker (1985). In an earlier version of the present paper we based our proof of Key Theorem II on the proofs given in Hordijk and Spieksma (1992), but in this version we give a slightly different and much more compact derivation (when restricting to the case of MC only). The extension of this paper however is far from being straightforward, as we need continuity results and proper recurrence to a finite set, uniformly in the deterministic policies.

The outline of the paper is as follows. In §2 we introduce the model and we state our conditions and results. Sections 3 and 4 contain the proofs of the Key theorems.

2. The model, recurrence and ergodicity conditions and results. Consider the standard description of an MDC. A dynamical system is observed at discrete time points to be in one of a denumerable number of states. The state space is denoted as E. At each time point the controller of the system chooses an action a from the set of available actions $A(i)$, if the system is in state i. When action a is chosen, a reward r_{ia} is earned and the system moves to state j with probability P_{iaj}.

A decision rule π^n at time n is a function that assigns the probability of taking action a at time n; we allow this function to depend on all realised actions up to time n and all realised states up to and including time n. A policy R is then a sequence of decision rules $(\pi^0, \pi^1, \dots)$ and $\mathscr{C}$ is the class of all policies. A policy is called *stationary* if all decision rules are equal; it is *deterministic* if exactly one action is prescribed in each state. Denote with $\mathscr{F}$ the set of deterministic decision rules f, and with f^∞ the deterministic, stationary policy $(f, f, \dots)$.

Throughout this paper the following standard condition for denumerable MDC's will be assumed.

ASSUMPTION 2.1. (i) *$A(i)$ is a compact, metric set for all $i \in$ E.*
(ii) *P_{iaj}, r_{ia} are continuous functions of $a \in A(i)$.*

Combination of this assumption and a theorem of Tychonov's yields that $\mathscr{F}$ is a compact set in the weak topology of componentwise convergence. Moreover, $\mathscr{F}$ is metrisable by virtue of Theorem 4.14 in Kelley (1955).

For ease of notation we denote the transition matrix of the MC induced by decision rule f as $P(f)$, with ijth element $P_{ij}(f) = P_{if(i)j}$. $P^n(f)$ is the nth iterate of $P(f)$ and $P^0(f) = I$ with I the identity matrix. Similarly, $r(f)$ is the vector of immediate rewards with ith component $r_i(f) = r_{if(i)}$.

As the system operating under a stationary policy is a homogeneous MC, some concepts from MC theory need introduction. $\nu(f)$ denotes the number of closed classes in the MC with transition matrix $P(f)$, $B(f) \subset \mathsf{E}$ is called a *set of reference states* if it contains precisely one state from each closed class and no other states.

The taboo transition matrix ${}_MP(f)$ with taboo set $M \subset \mathsf{E}$ is defined as follows:

$$ {}_MP_{ij}(f) = \begin{cases} P_{ij}(f), & j \notin M, \\ 0, & j \in M, \end{cases} $$

with the convention that ${}_MP^n(f) = ({}_MP(f))^n$ and ${}_MP^0(f) = I$. Note that this definition slightly differs from Chung (1967), where the n-step taboo transition probabilities include transitions to the taboo set at time n. Then $F_{iM}^{(n)}(f)$ denotes the probability that the first hitting time of set M is n, $n \geq 1$, and $F_{iM}(f)$ the probability that M is eventually hit, when the system starts in state i and policy f^∞ is used. In formula

$$ F_{iM}^{(n)}(f) = \sum_{m \in M} \left({}_MP^{n-1}(f)P(f)\right)_{im}, \qquad n = 1,2,\ldots $$

$$ F_{iM}(f) = \sum_{n=1}^{\infty} F_{iM}^{(n)}(f). $$

The stationary matrix $\Pi(f)$ of the MC induced by f is

$$ \Pi_{ij}(f) = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} P_{ij}^n(f). $$

Finally, before stating the basic conditions in Dekker and Hordijk (1989, 1992), we need the notion of *weighted supremum norm*. Let μ be a positive vector on the state space E. The weighted supremum norm $|||\, x\, |||_\mu$ of a vector x on E is $\sup_{i \in \mathsf{E}} \mu_i^{-1}|x_i|$. The associated operator norm of a matrix A on $\mathsf{E} \times \mathsf{E}$ is $\sup_{i \in \mathsf{E}} \mu_i^{-1}\Sigma_j|A_{ij}|\mu_j$. μ is called the *weighting* or *bounding vector*.

The continuity of $P(f)r(f)$ on $\mathscr{F}$ is generally used to obtain the existence of maximising policies. For reward vectors bounded by μ, it is sufficient to require the following condition, which will be assumed throughout this section.

Assumption 2.2. *$\Sigma_{j \in \mathsf{E}} P_{iaj}\mu_j$ is continuous on $A(i)$.*

So, although $P(f)$ need not be a continuous operator in the space of μ-bounded linear operators, the assumptions imply that $P(f)$ satisfies a weaker concept for continuity of operators, called *μ-continuity*, which was introduced in Dekker and Hordijk (1989). Let $A(f)$ be a matrix function on $\mathsf{E} \times \mathsf{E}$, which is μ-bounded for any $f \in \mathscr{F}$.

Definition 2.1. $A(f)$ is μ-continuous on $\mathscr{F}$, if for any $i \in \mathsf{E}$, and any converging sequence $\{f_n\}_{n \in \mathbb{N}}$, with limit f^* say,

$$ \lim_{n \to \infty} \sum_{j \in \mathsf{E}} |A_{ij}(f_n) - A_{ij}(f^*)|\mu_j = 0. $$

By virtue of the characterisation Lemma 3.7 in Dekker and Hordijk (1992), μ-continuity is equivalent to componentwise continuity of $A(f)$ and $|A(f)|\mu$.

Let μ be a bounding vector with $\mu_i \geq 1$, $\forall i \in \mathsf{E}$. The following conditions play an important role in the analysis in Dekker and Hordijk (1989, 1992).

DEFINITION 2.2. The set of MC's with transition matrices $P(f)$, $f \in \mathscr{F}$, has property

$\cdot$ *μ-uniform geometric ergodicity* (μ-UGE), *if* $\exists c > 0$, $\beta < 1$, *such that for any* $f \in \mathscr{F}$,

$$\begin{cases} \| P^n(f) - \Pi(f) \|_\mu \leqslant c\beta^n, & n \in \mathbb{N}_0, \\ \| P(f) \|_\mu \leqslant c. \end{cases}$$

$\cdot$ *μ-uniform geometric recurrence* (*μ-UGR*), if a finite set M and a $\beta < 1$ exist, such that for any $f \in \mathscr{F}$,

$$\| {}_M P(f) \|_\mu \leqslant \beta.$$

Next we give a list of other useful recurrence properties, describing different types of uniform tail behaviour of the first hitting times of finite sets. Again let μ be a vector with $\mu_i \geqslant 1$ $\forall i \in \mathsf{E}$, $M \subset \mathsf{E}$ a finite set. $B(f)$ is said to be a set of reference states, if it contains precisely one state from each closed class in the MC generated by f.

DEFINITION 2.3. The set of MC's with transition matrices $P(f)$, $f \in \mathscr{F}$, satisfies condition *μ-UWGR(M)*, if $\exists c_i > 0$, $\beta < 1$, such that $\forall f \in \mathscr{F}$,

$$\| {}_M P^n(f) \|_\mu \leqslant c_1 \beta^n, \qquad n \in \mathbb{N}_0.$$

$\cdot$ *μ-UR(M)*, if $\exists n_0 \in \mathbb{N}_0$, $\beta < 1$, $c_2 > 0$, *such that* $\forall f \in \mathscr{F}$,

$$\| {}_M P^{n_0}(f) \|_\mu \leqslant \beta, \qquad \| P(f) \|_\mu \leqslant c_2.$$

$\cdot$ *μ-UBS(M)*, if $\exists c_3 > 0$, such that $\forall f \in \mathscr{F}$,

$$\left\| \sum_{n \in \mathbb{N}_0} {}_M P^n(f) \right\|_\mu \leqslant c_3.$$

$\cdot$ *μ-UWGRRS(M)*, if $\exists c_1 > 0$, $\beta < 1$, such that M contains a set of reference states $B(f)$ $\forall f \in \mathscr{F}$ with

$$\| {}_{B(f)} P^n(f) \|_\mu \leqslant c_1 \beta^n, \qquad n \in \mathbb{N}_0.$$

$\cdot$ *μ-URRS(M), if* $\exists n_0 \in \mathbb{N}_0$, $\beta < 1$, $c_2 > 0$, such that M contains a set of reference states $B(f) \forall f \in \mathscr{F}$ with

$$\| {}_{B(f)} P^{n_0}(f) \|_\mu \leqslant \beta, \qquad \| P(f) \|_\mu \leqslant c_2.$$

$\cdot$ *μ-UBSRS(M)*, if $\exists c_3 > 0$, such that M contains a set of reference states $B(f)$ $\forall f \in \mathscr{F}$ with

$$\left\| \sum_{n \in \mathbb{N}_0} {}_{B(f)} P^n(f) \right\|_\mu \leqslant c_3.$$

The letter combinations *U*, (*W*)*GR*, *BS*, *RS* stand for Uniform, (Weak) Geometric Recurrence, Bounded Sum, Reference States. If we want to specify the set M in the *μ-UGR*-property, we denote it as *μ-UGR(M)*. Notice that the nomenclature is slightly different from Dekker and Hordijk (1989, 1992). We adjusted it to have a better correspondence with the terminology for one MC.

For bounded μ, μ-*UBS*(M) is the same as requiring the first hitting times of a finite set M to be uniformly bounded in the starting states and the deterministic policies. μ-*UR*(M) then requires a positive lower bound on the probability of being in set M at time n_0 uniformly in the initial states and the deterministic policies. This is the simultaneous Doeblin condition (cf. Hordijk (1974)).

Federgruen, Hordijk and Tijms (1978a, b) prove the equivalence of e-*UR*(M), e-*UBS*(M), e-*URRS*(M) and e-*UBSRS*(M) for *unichain* MDC's. Moreover, these conditions are shown to be equivalent to e-*UGE*, if aperiodicity is assumed as well. In Zijm (1985) the same relations are established for *multichain* MDC's if in addition to conditions e-*UR*(M) or e-*UBS*(M), $\nu(f)$ (the number of closed classes under policy f^∞) is finite and continuous on $\mathscr{F}$. To verify continuity of $\nu(f)$ directly, sufficient conditions can be found in Schäll (1992). These are weaker than μ-*UBSRS*(M).

Combination of results in Dekker and Hordijk (1992) and the present paper yield the same assertions as in Zijm (1985) when we work in the space of μ-*bounded vectors* instead of *e-bounded* or uniformly bounded *vectors*. Indeed, the precise relations are formulated through the two following theorems. In the remainder of this section we suppose that Assumptions 2.1, 2.2 hold.

Key Theorem I. (i) μ-*UWGR*(M), μ-*UR*(M) *and* μ-*UBS*(M) *are equivalent.*

(ii) μ-*UWGRRS*(M), μ-*URRS*(M) *and* μ-*UBSRS*(M) *are equivalent.*

(iii) μ-*UWGR*(M), μ-*UR*(M) *and* μ-*UBS*(M) *together with continuity of* $\nu(f)$ *on* $\mathscr{F}$ *are equivalent to* μ-*UWGRRS*(M), μ-*URRS*(M) *and* μ-*UBSRS*(M).

(iv) μ-*UGR*$(M) \Rightarrow \mu$-*UWGR*(M).

(v) μ-*UWGR*$(M) \Rightarrow \tilde{\mu}$-*URG*$(M)$ *with* $\tilde{\mu} = \sup_{f\in\mathscr{F}} \sum_{n=0}^{\infty} {}_M P^n(f)\mu$.

Key Theorem II. *The two following sets of conditions are equivalent*

(i) $\begin{cases} \mu\text{-}UGE, \\ \nu(f) < \infty, \forall f \in \mathscr{F}. \end{cases}$

(ii) $\begin{cases} \mu\text{-}UWGR(M), \\ \nu(f) \text{ continuous on } \mathscr{F}, \\ P(f) \text{ aperiodic for any } f \in \mathscr{F}. \end{cases}$

The proof techniques in Dekker and Hordijk (189, 1992) are based on showing the existence and continuity of the Laurent expansion of the α-discounted rewards $\sum_n \alpha^n P^n(f) r(f)$ in $\alpha = 1$ as a function of the deterministic decision rules f. As

$$\sum_n \alpha^n P^n(f) = \frac{1}{1-\alpha}\Pi(f) + \sum_n \alpha^n (P^n(f) - \Pi(f))$$

for $\alpha \in \mathbb{C}$, $|\alpha| < 1$, μ-*UGE* can be easily shown to be equivalent to the following condition in the complex plane (cf. Spieksma 1990). We write $P(f,z) = \sum_n z^n P^n(f)$.

Condition 2.1. (i) There is an $R > 1$ such that $(1-z)P(f,z)$ can be elementwise continued as an analytic function in the disk $\mathscr{D}_{0,R} := \{z \in \mathbb{C} \mid |z - 0| < R\}$, for all $f \in \mathscr{F}$.

(ii) $\sup\{(1-z)|||P(f,z)|||_\mu : |z| = x, f \in \mathscr{F}\} < \infty, \forall x \in (0, R)$.

Using Cauchy's integral theorem, we can straightforwardly derive the required properties of the Laurent expansion of the discounted rewards for any μ-bounded immediate reward vector, if Assumptions 2.1 and 2.2 hold. However, Condition 2.1 looks fairly strong and the following condition seems the weakest condition to guarantee the necessary properties for the analysis by Dekker and Hordijk. Basically this is the condition used in Lasserre (1988), although we have stated it in a slightly different format.

CONDITION 2.2. (i) There is an $R > 0$, such that $(1 - z)P(f, z)$ can be elementwise continued as an analytic function in the disk $\mathscr{D}_{1,R}$, $\forall f \in \mathscr{F}$.

(ii) $\sup\{(1 - z)|||P(f, z)|||_\mu : |z - 1| = x, f \in \mathscr{F}\} < \infty$, $\forall x \in (0, R)$.

The following relation is shown in Spieksma (1990).

PROPOSITION 2.1. *Condition 2.2 together with $\nu(f) < \infty$ for $f \in \mathscr{F}$ is equivalent to μ-UWGRRS(M).*

Thus we conclude that in fact μ-*UWGRRS(M)*, hence all conditions used in this paper, are the weakest possible conditions to be used for the analysis in Dekker and Hordijk (1989, 1992), which allows general but μ-bounded immediate rewards. Notice that we require $\mu_i \geqslant 1$, $\forall i \in \mathrm{E}$, i.e. μ has to be bounded away from 0 in this paper. Allowing $\inf_i \mu_i = 0$ seems to be related to transience or null-recurrence of the corresponding Markov chains.

3. Proof of Key Theorem I: Equivalence of recurrence conditions. In this section we will prove our Key Theorem I concerning the relation between the recurrence conditions. Each part will be shown in separate lemmas. Furthermore, throughout this section *both* Assumptions 2.1 and 2.2 are supposed to hold. The section starts with a summary of some well-known results from MC theory that will be frequently used in the sequel (cf. Chung (1967)).

LEMMA 3.1. *Consider the MC with transition matrix $P(f)$.*

(i) *For i an essential state*

$$\sum_j \Pi_{ij}(f) = 1 \Leftrightarrow i \text{ is positive recurrent.}$$

(ii) *If j is transient or null recurrent then $\Pi_{ij}(f) = 0$, $\forall i \in \mathrm{E}$.*

(iii) *For j positive recurrent $\Pi_{ij}(f) = F_{ij}(f)\Pi_{jj}(f)$, $\forall i \in \mathrm{E}$.*

(iv) *For C a positive recurrent class*

$$F_{ij}(f) = F_{ik}(f), \quad \forall i \in \mathrm{E}, j, k \in C,$$

$$\Pi_{ik}(f) = \Pi_{jk}(f), \quad \forall i, j, k \in C.$$

(v) *If $\Pi(f)$ is stochastic, $F_{iB(f)}(f) = 1$ for all sets $B(f)$ of reference states, and all classes are positive recurrent.*

First we recall a result from Dekker and Hordijk (cf. Dekker (1985), Chapter 1) that has partially been published in Dekker and Hordijk (1992).

LEMMA 3.2. (i) *μ-UWGR(M), μ-UR(M) and μ-UBS(M) are equivalent.*

(ii) *μ-UWGRRS(M), μ-URRS(M) and μ-UBSRS(M) are equivalent.*

PROOF. Part (ii) is Theorem 5.2 from Dekker and Hordijk (1989) with a slight change of terminology. Part (i) is proved in a similar way. Q.E.D.

The next lemma shows the simplest relations between the various recurrence conditions.

LEMMA 3.3. (i) *μ-UGR(M) $\Rightarrow$ μ-UWGR(M)*

(ii) *μ-UWGR(M) $\Rightarrow$ $\tilde{\mu}$-UGR(M) with $\tilde{\mu} = \sup_{f \in \mathscr{F}} \sum_{n=0}^{\infty} {}_M P^n(f)\mu$.*

PROOF. Assume that the μ-*UGR*(M) property holds for $\beta < 1$. From a well-known property of norms we have

$$|||_M P^n(f)|||_\mu \leqslant |||_M P(f)|||_\mu^n \leqslant \beta^n,$$

so that μ-*UWGR*(M) holds for the same $\beta < 1$ and constant $c_1 = 1$. This proves (i). For the proof of (ii) we use results for positive dynamic programming from Hordijk (1974). Let $R = (\pi^0, \pi^1, \dots) \in \mathscr{C}$, and $P(\pi)$ the transition matrix of the MC induced by the stationary decision rule π. We write $_M P^n(R)$ for the matrix of marginal taboo probabilities of the system at time n, when policy R is used and transitions to set M are excluded. By the existence of nearly optimal policies we have that

$$\sup_{R \in \mathscr{C}} \sum_{n \in \mathbb{N}_0} {}_M P^n(R)\mu = \sup_{f \in \mathscr{F}} \sum_{n \in \mathbb{N}_0} {}_M P^n(f)\mu =: \tilde{\mu},$$

so that $\tilde{\mu}$ is μ-*excessive* with respect to $\{_M P(f) | f \in \mathscr{F}\}$, i.e.,

$$\mu + {}_M P(f)\tilde{\mu} \leqslant \tilde{\mu}, \quad \forall f \in \mathscr{F}.$$

If condition μ-*UWGR*(M) holds for the constants $c > 0$, $\beta < 1$, then $\mu \leqslant \tilde{\mu} \leqslant (1/(1-\beta))c\mu$. Hence,

$${}_M P(f)\tilde{\mu} \leqslant \tilde{\mu} - \mu \leqslant \left\{1 - \frac{1-\beta}{c}\right\}\tilde{\mu},$$

so that $\tilde{\mu}$-*UGR*(M) holds for the constant $\tilde{\beta} = 1 - (1-\beta)/c$. Q.E.D.

Notice that μ is bounded iff $\tilde{\mu}$ is. Before proving the only remaining equivalence, i.e. Key Theorem I (iii), we need the following lemma.

LEMMA 3.4. *If* $|||_M P^n(f)|||_\mu \leqslant \beta < 1$ *for some set* M *and integer* n, *and* $||| P(f) |||_\mu < \infty$, *then* $F_{iM}(f) = 1$ *for all* $i \in \mathsf{E}$.

PROOF. Suppose that $F_{iM}(f) < 1$ for some state $i \in \mathsf{E}$. Since

$$\sum_j {}_M P_{ij}^n(f) = 1 - \sum_{k=1}^{n} F_{iM}^{(k)}(f),$$

we have $\liminf_{n \to \infty} \sum_{jM} P_{ij}^n(f) > 0$ and accordingly, $\liminf_{n \to \infty} ({}_M P^n(f)\mu)_i > 0$. This would however imply that $\sum_{n=0}^{\infty} ({}_M P^n(f)\mu)_i = \infty$, which contradicts condition μ-*UBS*(M) and by Lemma 3.2 the assumptions of this lemma as well. Q.E.D.

The final proof of Key theorem I(iii) is split up into two parts.

LEMMA 3.5. μ-*URRS*(M) *implies* μ-*UR*(M) *and continuity of* $\nu(f)$ *on* $\mathscr{F}$.

PROOF. As the first part is easy to see, we only need to show continuity of $\nu(f)$ in f.

Suppose that condition μ-*URRS*(M) holds for constants $n_0, \beta, c,$and consider any converging sequence $\{f_n\}_n \subset \mathscr{F}$, with limit f^* say. Let $B(f_n)$, $n \in \mathbb{N}$, $B(f^*)$ be the corresponding sets of reference states, as specified by μ-*URRS*(M). Since these sets are all contained in M, there is at least one set B that occurs infinitely often in the sequence $\{B(f_n)\}_n$, say $B = B(f_{n_k})$, $k = 1, \dots$. Condition μ-*URRS*(M) states that

$$\left({}_B P^{n_0}(f_{n_k})\mu\right)_i \leqslant \beta\mu_i, \qquad i \in \mathsf{E}.$$

The μ-continuity of $P^{n_0}(f)$ (cf. Dekker and Hordijk (1992), Lemma 3.8) also implies

$$({}_BP^{n_0}(f^*)\mu)_i \leqslant \beta\mu_i, \qquad i \in \mathsf{E}.$$

By Lemma 3.4 we get $F_{iB}(f^*) = 1$ for all $i \in \mathsf{E}$, so that $\nu(f^*) \leqslant |B|$, where $|B|$ denotes the number of states contained in B. For any $i, j \in B$ we also have

$$P^m_{ij}(f_{n_k}) = 0, \qquad k, m = 1, 2, \ldots,$$

so that

$$P^m_{ij}(f^*) = 0, \qquad m = 1, 2, \ldots,$$

which implies that states in B are inaccessible from each other in the MC with transition matrix $P(f^*)$. Hence, we conclude that B is a set of reference states for policy $f^{*\infty}$ and that $\nu(f^*) = |B|$. We can apply this construction for any set B that occurs infinitely often in the sequence $\{B(f_n)\}_n$. Since there is an index N, such that for $n \geqslant N$ each $B(f_n)$ occurs infinitely often, we have $\nu(f_n) = |B(f_n)| = \nu(f^*)$ for $n \geqslant N$. This means that $\nu(f)$ is continuous in f. Q.E.D.

The last relation to prove is in fact the most difficult one, and it will be shown in multiple steps. Suppose that μ-$UR(M)$ holds and consider the embedded MC on M, a technique also applied by Çinlar (1975). Its transition probabilities $\hat{P}_{ij}(f)$, $i, j \in M$, are given by

$$\hat{P}_{ij}(f) = \sum_{n=0}^{\infty} \sum_{l \in \mathsf{E}} {}_MP^n_{il}(f) P_{lj}(f), \qquad i, j \in M. \tag{3.1}$$

$\hat{P}(f)$ is stochastic, as $F_{iM}(f) = 1$, $i \in M$, by Lemma 3.4. Notice further that a pair of states $i, j \in M$ communicates with respect to the embedded process if and only if they communicate with respect to the original one. Hence $\hat{\nu}(f) = \nu(f)$, for $\hat{\nu}(f)$ the number of closed classes in the embedded process.

The next step is to show that we can establish uniform recurrence to a set of reference states for the embedded chain. We would like to use a similar result by Zijm (1985), but he also requires aperiodicity, which we do not.

Lemma 3.6. *Suppose that condition μ-$UR(M)$ holds and that $\nu(f)$ is continuous in f. Then there exist $\beta < 1$, $c > 1$ and an index n_0, such that for all $f \in \mathscr{F}$ there is a set of reference states $B(f) \subset M$ with*

(i)
$$\sum_{j \in M \setminus B(f)} {}_{B(f)}\hat{P}^{n_0}_{ij}(f) < \beta, \qquad i \in M,$$

and

(ii)
$$\sum_{n=0}^{\infty} \sum_{j \in M \setminus B(f)} {}_{B(f)}\hat{P}^{n}_{ij}(f) \leqslant c, \qquad i \in M.$$

Proof. Recall that the relation with the original MC is given by (3.1). It follows from condition μ-$UR(M)$ and Lemma 3.2(i) that $|||\sum_{n=k}^{\infty} {}_MP^n(f)|||_\mu$ decreases monotonically to zero, uniformly in f. Note that ${}_MP(f)$ is μ-continuous by Assumptions 2.1 and 2.2, hence ${}_MP^n(f)$ is μ-continuous, $n = 2, 3, \ldots$. The μ-continuity of $\sum_{n=0}^{\infty} {}_MP^n(f)$ then follows in a standard way. We conclude that $\hat{P}(f)$ is μ-continuous, hence pointwise continuous. Since the embedded chain has finitely many states, the pointwise continuity of $\hat{P}^n(f)$ follows directly. Choose any positive $\beta < 1$. For $B \subset M$ and $n \in \mathbb{N}$ we define

$$F(B,n) = \left\{ f \in \mathscr{F} \,\middle|\, \begin{array}{l} B \text{ is a set of reference states for policy } f \\ \text{in the embedded MC on } M, \text{ and } \sum_{j \notin B} {}_B\hat{P}^n_{ij}(f) < \beta, i \in M \end{array} \right\}.$$

Consider any sequence $\{f_k\}_k$ with limit $f^* \in F(B,n)$. Then

$$\sum_{j \notin B} {}_B\hat{P}^n_{ij}(f^*) < \beta, \qquad i \in M.$$

By the pointwise continuity of $\hat{P}(f)$ there is an index K_1, such that

$$\sum_{j \notin B} {}_B\hat{P}^n_{ij}(f_k) < \beta, \qquad i \in M, k \geqslant K_1. \tag{3.2}$$

Hence, by similar arguments as in Lemma 3.4,

$$\hat{F}_{iB}(f_k) = 1, \qquad i \in M, k \geqslant K_1. \tag{3.3}$$

Since $\nu(f)$ is an integer-valued function in f, there is an index K_2, such that $\nu(f_k) = \nu(f^*)$ for $k \geqslant K_2$. Combining this with (3.3) we see that for $k \geqslant \max(K_1, K_2)$, B is a set of reference states for f_k. Together with (3.2) this implies that f_k is contained in $F(B,n)$ for $k \geqslant \max(K_1, K_2)$. Thus we have shown that $F(B,n)$ is an open set.

Consider the collection $\{F(B,n) | B \subset M, n \in \mathbb{N}\}$. It is well known that in a finite MC for every set of reference states there exists an index $n(f)$, such that

$$\sum_{j \notin B(f)} {}_{B(f)}\hat{P}^{n(f)}_{ij} < \beta, \qquad i \in M.$$

Hence every deterministic decision rule f is contained in at least one $F(B,n)$ and the collection is therefore an open covering of $\mathscr{F}$. Since $\mathscr{F}$ is compact, there is a finite sub-covering. Let n_0 be the maximum over the indices from this sub-covering. As for any set B, $\sum_{j \notin B} {}_B\hat{P}^n_{ij}(f)$ is nonincreasing in n, it is obvious that (i) holds for this n_0. Then (ii) follows immediately from Lemma 3.2 applied to $\hat{P}(f)$, $f \in \mathscr{F}$, with $\mu = e$. Q.E.D.

We are now able to prove the final relation, thus completing the proof of Key Theorem I.

LEMMA 3.7. *Condition μ-UR(M) together with continuity of $\nu(f)$ implies condition μ-UBSRS(M), hence μ-URRS(M).*

PROOF. Suppose that condition μ-$UR(M)$ holds for the constants n_0, β and $c > 1$. Let $B(f)$ be the reference set and c_1 the constant of Lemma 3.6(ii). Consider the expression $\sum_{n=0}^{\infty} ({}_{B(f)}P^n(f)\mu)_i$ for $i \in E$. We apply last exit decomposition to states of $M \setminus B(f)$, i.e.,

$${}_{B(f)}P^n_{ij}(f) = {}_MP^n_{ij}(f) + \sum_{k=0}^{n-1} \sum_{m \in M \setminus B(f)} {}_{B(f)}P^{n-k}_{im}(f)\, {}_MP^k_{mj}(f), \qquad n \geqslant 1.$$

Consequently,

(3.4)

$$\sum_{n=0}^{\infty} \left({}_{B(f)}P^n(f)\mu\right)_i = \sum_{n=0}^{\infty} \left({}_M P^n(f)\mu\right)_i + \sum_{m\in M\setminus B(f)} \sum_{n=1}^{\infty} {}_{B(f)}P^n_{im}(f) \sum_{k=0}^{\infty} \left({}_M P^k(f)\mu\right)_m.$$

The first term in (3.4) is smaller than or equal to $(1/(1-\beta))n_0 c^{n_0}\mu_i$. For the second term, note that $\sum_{n=0}^{\infty} {}_{B(f)}P^n_{im}(f)$ is the expected number of visits to state m (in the MC generated by f) before absorption into set $B(f)$, so that

$$\sum_{n=0}^{\infty} {}_{B(f)}P^n_{im}(f) \leqslant 1 + F_{iM}(f) \cdot \sum_{m'\in M\setminus B(f)} \sum_{k=0}^{\infty} {}_{B(f)}P^k_{m'm}(f).$$

Observe that $\sum_k {}_{B(f)}P^k_{m'm}(f) = \sum_k {}_{B(f)}\hat{P}^k_{m'm}(f)$, m', $m \in M$, since both represent the expected number of visits from m to m', before the corresponding processes are absorped into $B(f)$. Combination with Lemma 3.6(ii) yields that the second term in (3.4) is bounded by

$$(1+|M|c_1)\frac{1}{1-\beta} n_0 c^{n_0} \sum_{m\in M} \mu_m.$$

Combination of the two bounds completes the proof, as $\mu_i \geqslant 1$ for $i \in \mathsf{E}$. Q.E.D.

PROOF OF KEY THEOREM I. By combination of Lemmas 3.2, 3.3, 3.5, and 3.7. Q.E.D.

4. Proof of Key Theorem II: Equivalence of ergodicity and recurrence conditions. This section proves Key Theorem II. As in the previous section the derivation uses a sequence of lemmas. As stated before, Assumption 2.1 is supposed to hold. For most lemmas Assumption 2.2 is not required and we will assume it explicitly whenever necessary. The next lemma shows a simple consequence of the μ-*UGE* property.

LEMMA 4.1. *μ-UGE together with Assumption 2.2 implies, that $\Pi(f)$ is stochastic for all $f \in \mathcal{F}$ and μ-continuous.*

PROOF. Fix $i \in \mathsf{E}$ and suppose that μ-*UGE* holds for constants $c > 0$ and $\beta < 1$. Then

$$\left|1 - \sum_{j\in\mathsf{E}} \Pi_{ij}(f)\right| \leqslant \sum_{j\in\mathsf{E}} |P^n_{ij}(f) - \Pi_{ij}(f)|$$

$$\leqslant \sum_{j\in\mathsf{E}} |P^n_{ij}(f) - \Pi_{ij}(f)|\mu_j \leqslant c\beta^n\mu_i \to 0,$$

as $n \to \infty$, so that stochasticity follows.

From the componentwise continuity of $P(f)$ and $P(f)\mu$ on $\mathcal{F}$ componentwise continuity of $P^n(f)$ and $P^n(f)\mu$, hence μ-continuity of $P^n(f)$, are easily established by induction and Proposition 11.18 in Royden (1988). μ-continuity of $\Pi(f)$ then follows from Royden (1988, Problem 8.50), since the sequences $\{P^n_{ij}(f)\}_n$ and $\{(P^n(f)\mu)_i\}_n$ converge uniformly on $\mathcal{F}$ to their limits $\Pi_{ij}(f)$ and $(\Pi(f)\mu)_i$. Q.E.D.

To show that the recurrence conditions follow from μ-*UGE* under appropriate conditions, we will first prove that stochasticity and μ-continuity of $\Pi(f)$ implies recurrence to some finite set uniformly in f for the corresponding stationary MC's. We recall the following lemma from Zijm (1985, Lemma 3.2), albeit in a slightly modified version. The proof is essentially due to Deppe (1985).

LEMMA 4.2. *Suppose that $\nu(f) < \infty$ and that $\Pi(f)$ is stochastic for all $f \in \mathscr{F}$. Then there is a finite set $K \subset E$ containing a set $B(f)$ of reference states for any $f \in \mathscr{F}$.*

Under an additional continuity condition this can be strengthened.

LEMMA 4.3. *Suppose that $\nu(f) < \infty$ and that $\Pi(f)$ is stochastic for all $f \in \mathscr{F}$ and componentwise continuous on $\mathscr{F}$. Then there are a finite set $K \subset E$ and an $\epsilon > 0$, such that K contains for any $f \in \mathscr{F}$ a set $B(f)$ of reference states with*

$$\Pi_{bb}(f) \geqslant \epsilon, \quad \forall b \in B(f). \tag{4.1}$$

The proof of this lemma is implicit in the proof of Lemma 3.3 in Zijm (1985), but we prefer to give it explicitly here.

PROOF. Lemmas 3.1 and 4.2 imply the existence of a finite set $D \subset E$ containing a set $B(f)$ of reference states and with $F_{iD}(f) = 1$, $\forall i \in \mathsf{E}$, $f \in \mathscr{F}$. Choose $\delta \in (0,1)$. For $i \in D$ there is a finite set $D(i) \subset \mathsf{E}$, for which $\sum_{j \in D(i)} \Pi_{ij}(f) > \delta$, $\forall f \in \mathscr{F}$. To see this, we use similar arguments as in Theorem 3 of Federgruen, Hordijk and Tijms (1978b).

Let $\{S_n\}_{n \in \mathbb{N}}$ be a sequence of finite sets such that $S_n \subset \mathsf{E}$, $S_{n+1} \supseteq S_n$ and $\lim_{n \to \infty} S_n = \mathsf{E}$. Let $a_n(f) := \sum_{j \in S_n} \Pi_{ij}(f)$, $a(f) \equiv 1$, for all $f \in \mathscr{F}$. Then a, a_n, $n \in \mathbb{N}$, are continuous functions on $\mathscr{F}$, $a_n(f) \leqslant a_{n+1}(f)$, $n = 1, 2, \dots$, and $\lim_{n \to \infty} a_n(f) = 1 = a(f)$, for $f \in \mathscr{F}$. Since $\mathscr{F}$ is compact, we can apply Dini's theorem of uniform convergence (cf. Royden 1988) to obtain the uniform convergence of a_n to a.

Let now $N := \max_{i \in D} |D(i)|$. For all $i \in D$ and $f \in \mathscr{F}$, there is a state $s_{i,f} \in D(i)$ such that $\epsilon := \delta / N \leqslant \Pi_{is_{i,f}}(f)$. Then $\epsilon \leqslant \Pi_{s_{i,f}s_{i,f}}(f)$, by Lemma 3.1. Let further $K := \bigcup_{i \in D} D(i) \cup D$. For $f \in \mathscr{F}$, D contains a set of reference states $B(f)$. Then $\{s_{b,f} | b \in B(f)\} \subset K$ is a set of reference states satisfying the assertion of the lemma. Q.E.D.

Combination of Lemmas 3.1 and 4.3 yields the existence of a positive lower bound of the stationary probability on set K, uniformly in the initial states and the deterministic policies.

Componentwise continuity of $\Pi(f)$ on $\mathscr{F}$ is closely related to tightness of the collection $\{\Pi_{i\cdot}(f) | f \in \mathscr{F}\}$, for any $i \in \mathsf{E}$. Indeed, we can prove the following generalisation of Hordijk (1974, pp. 82–83).

LEMMA 4.4. *Let $\nu(f) < \infty$, and μ be a vector on E with $\mu_i \geqslant 1$, $\forall i \in \mathsf{E}$. Then the two following sets of conditions are equivalent:*

(i) $\begin{cases} \nu(f) \textit{ continuous on } \mathscr{F}, \\ \{\Pi_{i\cdot}(f) | f \in \mathscr{F}\} \textit{ is uniformly integrable with respect to } \mu \\ \qquad \textit{and a tight collection of probability measures}, \forall i \in \mathsf{E}. \end{cases}$

(ii)

$\begin{cases} \Pi(f) \textit{ stochastic}, \forall f \in \mathscr{F}, \\ \Pi(f)\ \mu\textit{-continuous on } \mathscr{F}, \textit{i.e.}, \Pi(f), \Pi(f)\mu \textit{ componentwise continuous on } \mathscr{F}. \end{cases}$

PROOF. (i) ⇒ (ii). The stochasticity follows directly from tightness. Next we prove the componentwise continuity of $\Pi(f)$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a convergent sequence in $\mathcal{F}$ with limit $f^* \in \mathcal{F}$. The corresponding collection $\{\Pi_{i\cdot}(f_n)|n \in \mathbb{N}\}$ is tight, for any $i \in \mathsf{E}$, since a subset of a tight set is tight. Choose $i \in \mathsf{E}$. By virtue of a theorem of Prohorov it contains a weakly convergent subsequence $\{\Pi_{i\cdot}(f_{n_k})\}_{k\in\mathbb{N}}$ with weak limit say Π_i. As the state space is discrete, all states are closed sets, hence

$$\lim_{k\to\infty} \Pi_{ij}(f_{n_k}) = \Pi_{ij}, \tag{4.2}$$

For $j \in \mathsf{E}$. By a diagonalisation procedure we obtain a subsequence, call it again $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$, and a stochastic matrix Π, for which (4.2) holds for all $i, j \in \mathsf{E}$. Clearly, $\Pi(f_{n_k})P(f_{n_k}) = \Pi(f_{n_k}) = P(f_{n_k})\Pi(f_{n_k})$. So, together with Royden (1988, Proposition 11.18) this yields $\Pi P(f^*) = \Pi = P(f^*)\Pi$, if we let k tend to infinity. Iterating, summing and averaging we obtain with Fubini's theorem

$$\Pi\frac{1}{N+1}\sum_{n=0}^{N} P^n(f^*) = \Pi = \frac{1}{N+1}\sum_{n=0}^{N} P^n(f^*)\Pi.$$

Again apply Royden's Proposition 11.18 to establish

$$\Pi \cdot \Pi(f^*) = \Pi, \tag{4.3}$$

$$\Pi(f^*) \cdot \Pi = \Pi. \tag{4.4}$$

The stochasticity of $\Pi(f)$ and Lemma 4.2 yield the existence of a finite set $D \subset E$ containing a set of reference states for any $f \in \mathcal{F}$. We claim that D contains a set B^*, such that B^* is a set of reference states for f^* and for a subsequence of $\{f_{n_k}\}_{k\in\mathbb{N}}$.

Obviously, since D is finite and $\nu(f)$ is integer valued and continuous on $\mathcal{F}$, there is a subsequence of $\{n_k\}_{k\in\mathbb{N}}$, call it again $\{n_k\}_{k\in\mathbb{N}}$ for simplicity, such that $B(f_{n_k}) = B$ and $\nu(f) = \nu(f^*) = \nu$, for some finite set $B \subset D$ and $\nu < \infty$.

If all $b \in B$ are positive recurrent states in the MC induced by f^*, we can choose $B^* = B$. Indeed, in this case $P^n_{bb'}(f^*) = \lim_{k\to\infty}P^n_{bb'}(f_{n_k}) = 0$, $n = 1, 2, \dots,$ for any $b, b' \in B$. On the other hand, suppose that $b \in B$ is transient under f^*. Since D contains a set of reference states for f^*, there is a state $b' \in D$, such that $F_{bb'}(f^*) > 0$, hence $P^n_{bb'}(f^*) > 0$ for some n. Consequently, the componentwise continuity of $P^n(f)$ implies the existence of a constant K_b, such that $P^n_{bb'}(f_{n_k}) > 0$ if $k \geqslant K_b$. This means that b' is in the same positive recurrent class as b in the MC induced by f_{n_k}, for $k \geqslant K_b$.

Adjust the subsequence $\{n_k\}_{k\in\mathbb{N}}$ and replace b by b'. Iterating the procedure for all states $b \in B$ that are transient under f^*, we end after finitely many steps with a set B^* that satisfies our claim.

Let $C_l(f^*)$, $l = 1, \dots, \nu$ be the positive recurrent classes and $T(f^*)$ be the set of transient states in the MC induced by f^*. We write $B^* = \{b_1, \dots, b_\nu\}$ with $b_l \in C_l(f^*)$, and show that $\Pi = \Pi(f^*)$. Then we consider the following cases:

(1) $j \in T(f^*)$: by (4.3) $\Pi_{ij} = \sum_{k\in E}\Pi_{ik}\Pi_{kj}(f^*) = 0$.

(2) $i \in C_l(f^*)$: for $b_l, b_m \in B^*$, $m \neq l$, clearly $\Pi_{b_l b_m}(f_{n_k}) = 0$, so that $\Pi_{b_l b_m} = 0$. Furthermore, combination with (1) and (4.3) yields for $m \neq l$,

$$0 = \Pi_{b_l b_m} = \sum_{k\notin T(f^*)} \Pi_{b_l k}\Pi_{k b_m}(f^*) = \sum_{k\in C_m(f^*)} \Pi_{b_l k}\Pi_{b_m b_m}(f^*).$$

Since $\Pi_{b_m b_m}(f^*) > 0$, $\Pi_{b_l k} = 0$ if $k \in C_m(f^*)$ for $m \neq l$. Together with (1) this gives that $\Pi_{b_l k} = 0$ if $k \notin C_l(f^*)$. Hence, $\sum_{k \in C_l(f^*)} \Pi_{b_l k} = 1$. Then, by (4.3) we have for $j \in C_l(f^*)$,

$$\Pi_{b_l j} = \sum_{k \in C_l(f^*)} \Pi_{b_l k} \Pi_{kj}(f^*) = 1 \cdot \Pi_{b_l j}(f^*).$$

Using (4.4) and

$$\Pi_{ij} = \sum_{k \in E} \Pi_{ik}(f^*) \Pi_{kj} = \sum_{k \in E} \Pi_{b_l k}(f^*) \Pi_{kj} = \Pi_{b_l j},$$

we obtain that $\Pi_{ij} = \Pi_{ij}(f^*)$ for any $i \notin T(f^*)$ and all $j \in E$.

(3) $i \in T(f^*)$. Use (4.4) and (2) to achieve for any $j \in E$,

$$\Pi_{ij} = \sum_{k \in E} \Pi_{ik}(f^*) \Pi_{kj} = \sum_{l=1}^{\nu} F_{ib_l}(f^*) \sum_{k \in C_l(f^*)} \Pi_{b_l k}(f^*) \Pi_{kj}$$

$$= \sum_{l=1}^{\nu} F_{ib_l}(f^*) \Pi_{b_l j} = \sum_{l=1}^{\nu} F_{ib_l}(f^*) \Pi_{b_l j}(f^*) = \Pi_{ij}(f^*).$$

(2) and (3) together prove that $\Pi_{ij} = \Pi_{ij}(f^*)$ for any $i, j \in E$. Combination with relation (4.3) yields the componentwise continuity of $\Pi(f)$ on $\mathscr{F}$.

Componentwise continuity of $\Pi(f)$ together with the uniform integrability condition establishes continuity of $(\Pi(f)\mu)_i$ on $\mathscr{F}$, for any $i \in E$.

(ii) $\Rightarrow$ (i). We first prove the continuity of $\nu(f)$ on $\mathscr{F}$. Let $\{f_n\}_{n \in \mathbb{N}}$ be a converging sequence in $\mathscr{F}$ with limit f^*. Let $D \subset E$ be a finite set as in the assertion of Lemma 4.3, i.e., D contains a set of reference states $B(f)$ with $\Pi_{bb}(f) \geqslant \epsilon$ for all $b \in B(f)$, for any $f \in \mathscr{F}$, and some $\epsilon > 0$.

Consider any set $B \subset D$ that occurs infinitely often in the sequence $\{B(f_n)\}_{n \in \mathbb{N}}$, and let $\{f_{n_k}\}_{k \in \mathbb{N}}$ be the subsequence for which $B(f_{n_k}) = B$. Since $\Pi_{bb'}(f_{n_k}) = 0$ for $b, b' \in B$ with $b \neq b'$, continuity of $\Pi_{bb'}(f)$ implies that $\Pi_{bb'}(f^*) = 0$. Similarly, $\Pi_{bb}(f) \geqslant \epsilon$, for $b \in B$. Hence, the states in B are positive recurrent and do not communicate in the MC induced by f^*. Thus $\nu(f^*) \geqslant |B|$.

Suppose that $\nu(f^*) > |B|$. Then there is a positive recurrent state $b^* \in D$ that does not communicate with the states in B in the MC induced by f^*. However,

$$\Pi_{b^*B}(f^*) = \lim_{k \to \infty} \Pi_{b^*B}(f_{n_k}) = \lim_{k \to \infty} \sum_{b \in B} F_{b^*b}(f_{n_k}) \Pi_{bb}(f_{n_k}) \geqslant \epsilon.$$

Consequently,

$$\nu(f^*) = |B| = \nu(f_{n_k}), \qquad k \in \mathbb{N}. \tag{4.5}$$

As B was an arbitrary limiting set, (4.5) holds for all limiting sets. Since there are only finitely many limiting sets, there is an $N_0 \in \mathbb{N}$, such that $\nu(f_n) = \nu(f^*)$ for $n \geqslant N_0$.

The componentwise continuity of $\Pi(f), \Pi(f)\mu$ together with Dini's theorem on uniform convergence establish the uniform integrability of the set $\{\Pi_i(f) | f \in \mathscr{F}\}$, similarly to the proof of Lemma 4.3. Q.E.D.

The next observation is trivial now, but convenient to state.

LEMMA 4.5. *Let $\nu(f) < \infty$ and μ be a vector with $\mu_i \geqslant 1$ for all $i \in$ E. Furthermore suppose that $\Pi(f)$ is stochastic for any $f \in \mathscr{F}$ and that $\{\Pi_i(f) | f \in \mathscr{F}\}$ is* *uniformly integrable with respect to* μ. *Then for any* $\epsilon > 0$ *there is a finite set* $K(\epsilon) \subset \mathsf{E}$ *containing a set of reference states* $B(f)$, *such that* $\sum_{j \notin K(\epsilon)} \Pi_{ij}(f)\mu_j \leqslant \epsilon$, $\forall i \in \mathsf{E}$, $f \in \mathscr{F}$. *Consequently the collection* $\{\Pi_i(f) | i \in \mathsf{E}, f \in \mathscr{F}\}$ *is uniformly integrable with respect to* μ *and hence tight.*

PROOF. By virtue of Lemma 4.2 we can find a finite set $D \subset \mathsf{E}$ containing a set of reference states $B(f)$ for any $f \in \mathscr{F}$. Choose $\epsilon > 0$.

The uniform integrability condition yields for each $i \in D$ the existence of a finite set $D(i, \epsilon)$ with $\sum_{j \notin D(i,\epsilon)} \Pi_{ij}(f)\mu_j < \epsilon$ for any $f \in \mathscr{F}$. Let $K(\epsilon) = \bigcup_{i \in D} D(i, \epsilon) \cup D$; then also $\sum_{j \notin K(\epsilon)} \Pi_{ij}(f)\mu_j < \epsilon$, for any $i \in D$ and $f \in \mathscr{F}$. For $i \notin D$,

$$\sum_{j \notin K(\epsilon)} \Pi_{ij}(f)\mu_j = \sum_{\substack{b \in B(f) \\ j \notin K(\epsilon)}} F_{ib}(f)\Pi_{bj}(f)\mu_j$$

$$\leqslant \sum_{b \in B(f)} F_{ib}(f) \cdot \epsilon = \epsilon. \quad \text{Q.E.D.}$$

We can now prove the first part of Key Theorem II.

LEMMA 4.6. *Suppose that Assumption* 2.2 *holds as well as condition* μ-*UGE together with* $\nu(f) < \infty$ *for all* $f \in \mathscr{F}$. *The condition* μ-*UR holds*, $P(f)$ *is aperiodic for* $f \in \mathscr{F}$ *and* $\nu(f)$ *is continuous on* $\mathscr{F}$.

PROOF. Aperiodicity follows directly, since by assumption, $P_{ij}^n(f) \to \Pi_{ij}(f)$, $n \to \infty$, for $i, j \in \mathsf{E}$ and $f \in \mathscr{F}$. Continuity of $\nu(f)$ follows from Lemmas 4.1 and 4.4.

Next choose positive $\epsilon < 1$. Since by virtue of Lemmas 4.1 and 4.4 the assumptions of Lemma 4.5 are satisfied, there thus is a finite set, say K, containing a set of reference states $B(f)$ for each $f \in \mathscr{F}$ with

$$\sum_{j \notin K} \Pi_{ij}(f)\mu_j \leqslant \epsilon/2.$$

Suppose that μ-*UGE* holds for constants $c > 0$, $\beta < 1$. Let further $n_0 \geqslant 1$ be such that $c\beta^{n_0} \leqslant \epsilon/2$. Then for $i \in \mathsf{E}$ and $f \in \mathscr{F}$,

$$\sum_{j \notin K} P_{ij}^{n_0}(f)\mu_j \leqslant \sum_{j \in \mathsf{E}} \left| P_{ij}^{n_0}(f) - \Pi_{ij}(f) \right| \mu_j + \sum_{j \notin K} \Pi_{ij}(f)\mu_j$$

$$\leqslant c \cdot \beta^{n_0}\mu_i + \epsilon/2 \leqslant \epsilon \cdot \mu_i,$$

as $\mu_i \geqslant 1$. Note, that by assumption, $c \geqslant \sup_{f \in \mathscr{F}} ||| P(f) |||_\mu$. Since

$$\sum_{j \in \mathsf{E}} {}_K P_{ij}^{n_0}(f)\mu_j \leqslant \sum_{j \notin K} P_{ij}^{n_0}(f)\mu_j \leqslant \epsilon\mu_i,$$

we thus have shown condition μ-*UR* for the finite set K and constants $n_0 \geqslant 1$, $\epsilon < 1$ and $c > 0$. Q.E.D.

In conjunction with Key Theorem I(i) this proves the first part of Key Theorem II. For the second part we need the following useful properties of condition μ-*UWGR*.

LEMMA 4.7. *Assume condition* μ-*UWGR and suppose that Assumption* 2.2 *holds. Then*

(i) *the set*

$$\{(P(f_1) \cdots P(f_N))_i | f_1, \ldots, f_N \in \mathscr{F}, N \in \mathbb{N}\}$$

is uniformly integrable with respect to μ *for any* $i \in E$;

(ii) *there is a constant* $c < \infty$, *such that*

$$|||P(f_1)\cdots P(f_N)|||_\mu \leqslant c,$$

for all $f_1, \ldots, f_N \in \mathscr{F}$ *and* $N \in \mathbb{N}$;

(iii) *for any* $\epsilon > 0$, *there are a finite set* $K(\epsilon) \subset \mathsf{E}$ *and a constant* $N(\epsilon)$, *such that* ${}_{K(\epsilon)}(P^n(f))$ *is an* ϵ-*contraction in* μ-*norm uniformly in* $f \in \mathscr{F}$ *for* $n \geqslant N(\epsilon)$, *i.e.*,

$$\sum_{j \notin K(\epsilon)} P_{ij}^n(f)\mu_j \leqslant \epsilon \cdot \mu_i, \qquad i \in \mathsf{E},\ n \geqslant N(\epsilon).$$

PROOF. Fix $i \in \mathsf{E}$. Choose any $\epsilon > 0$ and suppose that condition μ-*UWGR* holds for the finite set $M \subset \mathsf{E}$ and constants $c > 0$, $\beta < 1$. Then by virtue of Lemma 3.3(ii) there exists a vector $\tilde{\mu}$, with $\mu \leqslant \tilde{\mu} \leqslant c(1-\beta)^{-1}$, such that $\tilde{\mu} - UGR$ holds for the finite set M. Hence there exists $\tilde{\beta} < 1$ for which $|||\,{}_MP(f)\,|||_{\tilde{\mu}} \leqslant \tilde{\beta}$.

For any sequence $\{f_n\}_{n \in \mathbb{N}} \in \mathscr{F}$ we apply last exit decomposition to set M. Then, with the convention that the empty product equals I,

$$(4.6)\quad (P(f_1)\cdots P(f_n))_{ij} = ({}_MP(f_1)\cdots {}_MP(f_n))_{ij} + \sum_{k=0}^{n-1}\sum_{m \in M}(P(f_1)\cdots P(f_{n-k}))_{im}({}_MP(f_{n-k+1})\cdots {}_MP(f_n))_{mj}.$$

Let M' be any set containing M. Multiply both sides in (4.6) with $\tilde{\mu}_j$ and sum over all states outside M'. This gives

$$(4.7)\quad \sum_{j \notin M'}(P(f_1)\cdots P(f_n))_{ij}\tilde{\mu}_j = \sum_{j \notin M'}({}_MP(f_1)\cdots {}_MP(f_n))_{ij}\tilde{\mu}_j + \sum_{j \notin M'}\sum_{k=0}^{n-1}\sum_{m \in M}(P(f_1)\cdots P(f_{n-k}))_{im}({}_MP(f_{n-k+1})\cdots {}_MP(f_n))_{mj}\tilde{\mu}_j$$
$$\leqslant \tilde{\beta}^n\tilde{\mu}_i + \sum_{m \in M}\sum_{k=0}^{n-1}\sum_{j \notin M'}\left({}_MP(f_{n-k+1})\cdots {}_MP(f_n)\right)_{mj}\tilde{\mu}_j,$$

as $|||\,{}_MP(f_1)\cdots {}_MP(f_n)\,|||_{\tilde{\mu}} \leqslant \tilde{\beta}^n$ for any sequence $f_1, \ldots, f_n \in \mathscr{F}$, and any $n \in \mathbb{N}$. Choose $n_0 > 1$, such that $\tilde{\beta}^{n_0}(1-\tilde{\beta})^{-1}\max(\Sigma_{m \in M}\tilde{\mu}_m, \tilde{\mu}_i) < \epsilon/2$. Let $n > n_0$. Then

$$(4.8)\quad \sum_{m \in M}\sum_{k=n_0}^{n-1}\sum_{j \notin M'}({}_MP(f_{n-k+1})\cdots {}_MP(f_n))_{mj}\tilde{\mu}_j \leqslant \left(\tilde{\beta}^{n_0} + \cdots + \tilde{\beta}^{n-1}\right)\sum_{m \in M}\tilde{\mu}_m,$$

so that by combination with (4.7),

$$(4.9)\quad \sum_{j \notin M'}(P(f_1)\cdots P(f_n))_{ij}\tilde{\mu}_j < \frac{\epsilon}{2} + \sum_{m \in M}\sum_{k=0}^{n_0-1}\sum_{j \notin M'}({}_MP(f_{n-k+1}\cdots {}_MP(f_n))_{mj}\tilde{\mu}_j,$$

for all choices of $\{f_n\}_{n\in\mathbb{N}}$, any $n > n_0$ and any set $M' \supset M$. $(P(f)\tilde{\mu})_l$ is continuous on $\mathscr{F}$, $\forall l \in E$ by the dominated convergence theorem, since $(P(f)\mu)_l$ is continuous and $\tilde{\mu} \leqslant c(1-\beta)^{-1}\mu$. Furthermore, clearly $\tilde{\mu} - UGR$ implies that $\sup_{f\in\mathscr{F}} |||P(f)|||_{\tilde{\mu}} \leqslant c'$ for some $c' < \infty$. We invoke again Proposition 11.18 from Royden (1988) to obtain inductively the continuity of $[(P(f_1)\cdots P(f_n))\tilde{\mu}]_l$ on the compact set $\mathscr{F}^n$, $n \in \mathbb{N}$, $l \in \mathsf{E}$. Since n_0 is finite, we conclude the existence of a finite set $M^* \supset M$, such that for any $k \leqslant n_0 - 1$,

(4.10)

$$\sum_{j\notin M^*}(P(f_1)\cdots P(f_k))_{lj}\tilde{\mu}_j < \frac{\epsilon}{2|M|\cdot n_0}, \qquad l \in M, \text{or } l = i, \forall f_1,\ldots,f_k \in \mathscr{F}.$$

Combine this with (4.9) to establish that $\sum_{j\notin M^*}(P(f_1)\cdots P(f_n))_{ij}\tilde{\mu}_j < \epsilon$, for any $f_1,\ldots,f_n \in \mathscr{F}$ and $n > n_0$. By (4.10) this inequality holds for $n \leqslant n_0$ as well. Finally we use $\mu \leqslant \tilde{\mu}$ to complete the proof of (i). Setting $M' = M$, (ii) then follows directly from (4.7) and (4.8), the construction of $\tilde{\mu}$ and the fact that

$$\sum_{j\in M}(P(f_1)\cdots P(f_n))_{ij}\mu_j \leqslant M\cdot \max_{m\in M}\mu_m$$

for $f_1,\ldots,f_n \in \mathscr{F}$ and $n \in \mathbb{N}$. This leaves us to prove (iii). From (i) we have the existence of a finite set $M^* \supset M$, such that for all $m \in M$, $f \in \mathscr{F}$ and $n \in \mathbb{N}_0$,

$$\sum_{j\notin M^*} P^n_{mj}(f)\mu_j \leqslant \epsilon/2. \tag{4.11}$$

Choose $N(\epsilon)$ with $c\cdot\beta^{N(\epsilon)} \leqslant \epsilon/2$. Then applying first entrance decomposition to set M we find

$$\sum_{j\notin M^*} P^n_{ij}(f)\mu_j = \sum_{j\notin M^*} {}_MP^n_{ij}(f)\mu_j + \sum_{m\in M}\sum_{l=1}^{n}\left({}_MP^{l-1}(f)P(f)\right)_{im}\sum_{j\notin M^*}P^{n-l}_{mj}(f)\mu_j$$

$$\leqslant c\cdot\beta^n\mu_i + \sum_{l=1}^{n}F^{(l)}_{iM}(f)\cdot\epsilon/2 \leqslant \epsilon\cdot\mu_i,$$

for $i \in \mathsf{E}$, $n \geqslant N(\epsilon)$ and $f \in \mathscr{F}$. This proves (iii) by setting $K(\epsilon) \triangleq M^*$. Q.E.D.

The next two lemmas derive some consequences of this uniform integrability property that we need.

LEMMA 4.8. *Let μ be a vector on E with $\mu_i \geqslant 1$, $\forall i \in E$. Suppose, that the collection $\{(P(f_1)\cdots P(f_N))_{i\cdot}|f_1,\ldots,f_N \in \mathscr{F},\ N\in\mathbb{N}\}$ is uniformly integrable with respect to μ. Then $\Pi(f)$ is stochastic for any $f \in \mathscr{F}$ and the set $\{\Pi_i(f)|f\in\mathscr{F}\}$ is uniformly integrable with respect to μ.*

PROOF. Fatou's lemma yields that $\Pi(f)$ is substochastic. Moreover, the collection $\{P^n_{i\cdot}(f)|f\in\mathscr{F}, n\in\mathbb{N}\}$ is uniformly integrable with respect to μ, hence the set of all convex combinations of these measures. In particular,

$$\left\{\frac{1}{N+1}\sum_{n=0}^{N}P^n_{i\cdot}(f)|f\in\mathscr{F}, n\in\mathbb{N}\right\}$$

is uniformly integrable with respect to μ, consequently tight, for any $i \in \mathsf{E}$. Using the definition of the stationary matrix, this ensures the rowsums of $\Pi(f)$ to be equal to 1 and the uniform integrability of $\{\Pi_i(f) | f \in \mathscr{F}\}$. Q.E.D.

The assertion of Lemma 4.8 is used to reduce the proof of μ-*UGE* to the derivation of suitable bounds for the difference between $P_{ij}^n(f)$ and $\Pi_{ij}(f)$, for finitely many states i, j, uniformly in $f \in \mathscr{F}$. The existence of such bounds is guaranteed by Lemma 4.9. It is obvious that such a lemma is necessary as well if e-norms are used. However, Federgruen, Hordijk and Tijms (1978a) do not prove this explicitly, since the stronger MC structure required there, allows more direct arguments to establish the e-*UGE* property. The main idea of the proof of the following lemma, however, is contained in the proof of Theorem 2.5 in Federgruen, Hordijk and Tijms (1978a).

LEMMA 4.9. *Let* $\{(P(f_1) \cdots P(f_N))_i | f_1, \dots, f_N \in \mathscr{F},\ N \in \mathbb{N}\}$ *be tight for any* $i \in E$. *Furthermore we suppose that* $P(f)$ *is aperiodic for any* $f \in \mathscr{F}$, *and that* $\Pi(f)$ *is elementwise continuous in* f. *Then* $P_{ij}^n(f)$ *converges to* $\Pi_{ij}(f)$ *uniformly in* $f \in \mathscr{F}$ *for* $n \to \infty$, *for any* $i, j \in \mathrm{E}$.

PROOF. Choose $\epsilon > 0$, $i, j \in \mathrm{E}$. Let $K \subset E$ be a finite set, such that $\sum_{k \notin K} P_{ik}^n(f) < \epsilon/3$ and $\sum_{k \notin K} \Pi_{ik}(f) < \epsilon/3$, $\forall n \in \mathbb{N}$, $f \in \mathscr{F}$. We define

$$n(f) := \min\left\{n \in \mathbb{N} \,\middle|\, \left|P_{kj}^n(f) - \Pi_{kj}(f)\right| < \frac{\epsilon}{3}, \forall k \in K\right\}.$$

Note, that $n(f) < \infty$ since $P(f)$ is aperiodic for any $f \in \mathscr{F}$. Then the set $S_m := \{f \in \mathscr{F} | n(f) \geqslant m\} \subset \mathscr{F}$ is closed $\forall m \in \mathbb{N}$. Indeed, let $\{f_n\}_{n \in \mathbb{N}} \subset S_m$ be a converging sequence with limit $f^* \in \mathscr{F}$, for some $m \in \mathbb{N}$. Suppose that $f^* \notin S_m$. This means that $n(f^*) < m$, so that

$$\left|P_{kj}^{n(f^*)}(f^*) - \Pi_{kj}(f^*)\right| < \frac{\epsilon}{3}, \quad \forall k \in K.$$

Since $P_{kj}^n(f)$, $\Pi_{kj}(f)$ are continuous functions on $\mathscr{F}$: this contradicts the assumption that $f_n \in S_m$, $\forall n \in \mathbb{N}$.

We conclude that the function $n(f)$ is u.s.c. (upper semi-continuous) on the compact set $\mathscr{F}$, so that $M := \sup_{f \in \mathscr{F}} n(f) < \infty$ (cf. Royden (1988)). For $n > M$ and $f \in \mathscr{F}$ we obtain

$$\begin{aligned}\left|P_{ij}^n(f) - \Pi_{ij}(f)\right| &\leqslant \sum_{k \in E} \left|P_{ik}^{n-n(f)}(f)\left(P_{kj}^{n(f)}(f) - \Pi_{kj}(f)\right)\right| \\ &\leqslant \sum_{k \in K} P_{ik}^{n-n(f)}(f)\left|P_{kj}^{N(f)}(f) - \Pi_{kj}(f)\right| + \sum_{k \notin K} P_{ik}^{n-n(f)}(f) \cdot 2 \\ &\leqslant \frac{\epsilon}{3} + 2\frac{\epsilon}{3} = \epsilon, \quad \forall f \in \mathscr{F}. \quad \text{Q.E.D.}\end{aligned}$$

Finally we can complete the proof of Key theorem II.

LEMMA 4.10. *Suppose, that Assumption 2.2 hold, as well as* μ-*UWGR. Moreover,* $P(f)$ *is assumed to be aperiodic for* $f \in \mathscr{F}$ *and* $\nu(f)$ *is continuous on* $\mathscr{F}$. *Then condition* μ-*UGE holds and* $\nu(f) < \infty$ *for any* $f \in \mathscr{F}$.

PROOF. $\nu(f) < \infty$ follows from Key Theorem I(iii). Lemmas 4.7, 4.8 and 4.4(i) $\Rightarrow$ (ii) imply stochasticity and μ-continuity of $\Pi(f)$ as well as uniform integrability with respect to μ. Lemma 4.7(ii) implies, that it is sufficient for μ-*UGE* to show the existence of $n_0 \geqslant 1$ and $\delta < 1$, such that

$$\||P^{n_0}(f) - \Pi(f)\||_\mu \leqslant \delta. \tag{4.12}$$

Indeed, since $\sup_{f \in \mathscr{F}} \||P^n(f)\||_\mu \leqslant c'$, $n \in \mathbb{N}_0$, for some constant $c' < \infty$, Fatou's lemma implies that $\||\Pi(f)\||_\mu \leqslant c'$. Consequently, for $\beta \triangleq \delta^{1/n_0}$ and $n \in \{kn_0, \ldots, (k+1)n_0 - 1\}$, $k \in \mathbb{N}_0$,

$$\begin{aligned}\||P^n(f) - \Pi(f)\||_\mu &= \||(P^n(f) - \Pi(f)\||_u \\ &\leqslant \||P^{n_0}(f) - \Pi(f)\||_\mu^k \cdot \||P^{n-kn_0}(f) - \Pi(f)\||_\mu \\ &\leqslant \delta^k \cdot 2c' = \beta^{kn_0} \cdot 2c' \leqslant \beta^n \cdot \frac{2c'}{\delta}.\end{aligned}$$

We will hence prove (4.12). Choose $\epsilon > 0$. By virtue of Lemmas 4.5 and 4.7(iii) there are a finite set K and an integer N, such that

$$\begin{aligned}&\sum_{j \notin K} P_{ij}^n(f)\mu_j \leqslant \epsilon\mu_i, \qquad n \geqslant N,\\ &\sum_{j \notin K} \Pi_{ij}(f)\mu_j \leqslant \epsilon\mu_i.\end{aligned} \tag{4.13}$$

Clearly (4.13) implies

$$\sum_{j \notin K} \left|P_{ij}^n(f) - \Pi_{ij}(f)\right|\mu_j \leqslant 2\epsilon \cdot \mu_i. \tag{4.14}$$

Write $\tilde{P}(f) \triangleq P^N(f)$ and

$$p_{ik}^l(f) \triangleq \left({}_K\tilde{P}^{l-1}(f)\tilde{P}(f)\right)_{ik}$$

which is the probability that state k is hit at time l in the N-skeleton chain induced by f, without passing through set K at times $1, \ldots, l-1$. For $n > mN$,

$$\begin{aligned}P_{ij}^n(f) - \Pi_{ij}(f) = &\sum_{k \in K} p_{ik}^1(f)\left[P_{kj}^{n-N}(f) - \Pi_{kj}(f)\right]\\ &+ \sum_{k \notin K} \tilde{P}_{ik}(f)\left[P_{ij}^{n-N}(f) - \Pi_{kj}(f)\right].\end{aligned}$$

Decomposing $[P_{kj}^{n-N}(f) - \Pi_{kj}(f)]$ similarly and repeating this procedure, we find

$$\begin{aligned}P_{ij}^n(f) - \Pi_{ij}(f) = &\sum_{k \in K} \sum_{l=1}^{m} p_{ik}^l(f)\left[P_{kj}^{n-lN}(f) - \Pi_{kj}(f)\right]\\ &+ \sum_{k \notin K} {}_K\tilde{P}_{ik}^m(f)\left[P_{kj}^{n-mN}(f) - \Pi_{kj}(f)\right].\end{aligned} \tag{4.15}$$

Next let $m_0 = \max_{i \in K}\mu_i$ and choose $m \in \mathbb{N}_0$, such that $\delta \triangleq 3\epsilon + 2m_0\epsilon^m < 1$.

Lemma 4.9 implies the existence of $n^* \in \mathbb{N}_0$, for which

$$\sum_{j \in K} \left| P_{kj}^{l}(f) - \Pi_{kj}(f) \right| \mu_j < \epsilon, \quad \forall k \in K,$$

for $f \in \mathscr{F}$ and $l \geqslant n^*$, and so combination with (4.14) and (4.15) yields for $n - mN > n^*$,

$$\begin{aligned} \sum_{j \in E} \left| P_{ij}^{n}(f) - \Pi_{ij}(f) \right| \mu_j &\leqslant 2\epsilon \cdot \mu_i + \sum_{k \in K} \sum_{l=1}^{m} p_{ik}^{l}(f) \sum_{j \in K} \left| P_{kj}^{n-lN}(f) - \Pi_{kj}(f) \right| \mu_j \\ &\quad + \sum_{k \notin K} {}_K\tilde{P}_{ik}^{m}(f) \sum_{j \in K} \left| P_{kj}^{n-mN}(f) - \Pi_{kj}(f) \right| \mu_j \\ &\leqslant 2\epsilon \cdot \mu_i + \epsilon + \epsilon^m \cdot 2m_0 \cdot \mu_i \\ &\leqslant \delta \cdot \mu_i, \end{aligned}$$

where in the second inequality we have used

$$\sum_{j \in K} \left| P_{kj}^{n-mN}(f) - \Pi_{kj}(f) \right| \mu_j \leqslant 2m_0,$$

to bound the third term. This completes the proof of the lemma. Q.E.D.

The section ends by proving Key Theorem II.

PROOF OF KEY THEOREM II. By combination of Lemmas 4.6, 4.10 and Key theorem I(i) Q.E.D.

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