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Theory and Methodology

Marginal cost criteria for preventive replacement of a group of components

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Abstract

In this paper we introduce marginal cost criteria to determine the optimum time for preventively replacing a fixed group of components. The criteria have a clear interpretation and are very flexible: actual ages can be taken into account as well as discretely or continuously distributed lifetimes and any number of components. Since an overall optimal policy for the group replacement problem is difficult to establish, we compare the criteria with block replacement policies which replace the group at fixed intervals. Such policies, however, do not take failure renewals into account and may replace relatively new components. The performance of the criteria has been analysed both with discrete-time Markov decision chains and with simulation. In all cases considered the replacement criteria yield an improvement in average costs over the optimum block replacement policy varying between 0% and 10%, while the loss with respect to an optimal group age replacement policy (if it can be determined at all) is marginal.

Keywords: Maintenance; Planning; Markov decision chains

1. Introduction

Planned preventive maintenance is widely used to reduce unscheduled downtime. In many cases preventive maintenance is more economic when applied to an entire group of components than to individual ones. One of the problems encountered, however, is to decide when to maintain the entire group; this is often called the Group Replacement Problem (GRP). There are several variants of the GRP; in the one considered in this paper, failed components are replaced immediately and the objective criterion is the long-term average costs.

The GRP was actually encountered in setting up a long-term maintenance plan for large offshore-based gas turbines, which was to be incorporated in an operational decision support system (DSS). Preventive

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maintenance packages were devised, each consisting of a number of maintenance activities. Each activity dealt with one component and execution of it brought the component back to an as-good-as-new state (which makes such an activity identical to a replacement). Activities were grouped into a package if joint execution would be profitable (e.g. because of reduced setup costs). The packages would be executed at those moments when the unit in question was not required for production and hence downtime costs were almost zero. In case of failure, however, only the activity needed to restore the failure was carried out; no other activities of the same package could be done, because the costs of prolonged *unscheduled* downtime would outweigh the savings of a joint execution. Upon request, the DSS would advise on whether to execute the package, using information on the actual ages of components.

One standard policy which can be applied to our version of the GRP is block replacement (BR), i.e. replacing the group preventively after fixed time intervals, regardless of the number of failures in between. Determination of the long-run average costs and of an average optimal policy is basically the same as in the single component case. A disadvantage of block replacement, however, is that the replacement decision is not influenced by the occurrence of failures. Although there is an administrative advantage in not having to register failures, this advantage reduces if computerised information systems are used. Moreover, the advice of a decision support system to replace a group of components, while one component has just been replaced upon failure, will not be convincing. So, two questions arise: firstly, how large is the difference between an optimal BR policy and an overall optimal policy, and secondly, how can one arrive at an (almost) optimal policy?

Only few papers deal with multiple component replacement and in fact none solely with the problem described above. Multi-component models which allow a joint replacement, also allow individual replacement or opportunistic replacement upon failure of another component. The survey of Cho and Parlar [6] lists as examples Bäckert and Rippin [2] and Haurie and L'Ecuyer [10]. In these papers Markov decision chains were used with the state consisting of the vector of the components' ages. Hence these models are numerically tractable for a few (two, three) components only (the effort increases exponentially with the number of components). Optimal policies tend to have complex structures, e.g. a list indicating the action for each state, because of the multiple options. This method of analysis does not become much simpler when applied to our version of the GRP only. Other models of group replacement were given in Okumoto and Elsayed [11], Assaf and Shantikumar [1] and Sivazlian and Mahoney [14]. The first four authors allow that failed components are repaired later so that the repairs can be combined with preventive replacement. They consider identical components with exponentially distributed times to failure and arrive at single parameter control limit policies. Sivazlian and Mahoney [14] consider components with increasing operating costs which are not subject to failure. They apply variational calculus to obtain optimal policies. Solving the integro-differential equations is possible by analytical methods in some special cases or by discretisation, which suffers from the same computational problems as the Markov chain approach in general.

In this paper we present group age replacement criteria based on marginal cost considerations. Here the marginal costs are interpreted as the extra costs caused by deferring preventive replacement for an additional time unit. The criteria order a replacement if the marginal costs exceed minimum average costs. For the marginal costs simple formulas in the component ages can be derived, whereas the minimum average costs are approximated by the BR model. Marginal cost considerations were first introduced by Berg [4], later applied by Dekker and Smeitink [7,8] and by Savits [13], all for one component models. The major advantages of the criteria are that they have a clear interpretation, which is a very important aspect in practical applications, and the required numerical effort consists of the computation of the optimum BR policy, which effort increases linearly with the number of components. The criteria can be formulated both for continuous and for discrete lifetime distributions as well as for any number of non-identical components. Moreover, they can easily be extended to other models, eg.

opportunity models (Dekker and Smeitink [7]), they can be used as priority criteria (see Dekker and Smeitink [8]), and they can serve as penalty function in models aiming at short term combination of execution (see Dekker et al. [9]). This paper investigates the performance of the criteria by means of discrete time Markov decision chains for two components as well as by simulation for multiple components.

The structure of this paper is as follows. The group replacement problem is formulated in Section 2.1, while the first approach, block replacement, is presented in Section 2.2. Section 2.3 recapitulates the single component age replacement model, to give the background behind the group age replacement criteria, which we present in Section 2.5. The performance of the criteria is analysed in Section 3, 4 and 5. In Section 3 we consider the single component continuous time case, as then also an optimal age replacement policy can be determined. This case shows the influence of various model parameters. In Section 4 a discrete time Markov decision chain is formulated to evaluate the criteria for a two component case. Section 4.2 compares the replacement criteria with the policy improvement criterion. Finally, Section 5 presents simulation results for the general multi-component case.

2. The mathematical problem and the approaches

2.1. Formulation of the group replacement problem

Consider a preventive maintenance package addressing a group of n components. Let the r.v. X_i with mean μ_i and variance σ_i^2 denote the lifetime of component i ($i = 1,\dots,n$). Lifetimes are assumed to be independent and can follow either a discrete or continuous distribution. We furthermore assume that the failure rates of the components, $r_i(t)$, $i = 1,\dots,n$, are increasing. Failure and successive replacement of component i (by an identical one) induces cost c_i^{f}. Execution of the maintenance package involves cost c^{p} and leads to replacement of all components in the group. Individual components cannot be replaced preventively, neither can the execution of the package be combined with a failure replacement. The main problem is to determine the optimal moment to execute the maintenance package with the long term average costs as objective function. The two approaches to solve this problem, viz. block and group age replacement are treated in Sections 2.2 and 2.4.

2.2. Block replacement

The first and simplest approach is to execute the package at fixed intervals, hence the term block replacement (BR) is used. For this approach one does not need to keep track of ages of individual components. Since executions of the package can be considered to be renewals of the total process, we have the following expression for the long-run average costs $g_{\mathrm{b}}(t)$ for a maintenance interval of length t:

$$g_{\mathrm{b}}(t) = \frac{c^{\mathrm{p}} + \sum_{i=1}^{n} c_i^{\mathrm{f}} M_i(t)}{t}, \quad t > 0, \tag{2.1}$$

where $M_i(t)$ is the renewal function belonging to component i, indicating the expected number of failures in the interval $[0, t]$. The analysis of the BR model is standard (see Barlow and Proschan [3]) and will be reviewed for the continuous case only; for the discrete case it is similar. We assume that $M_i(t)$, $i = 1,\dots,n$, is differentiable in t and denote its derivative by $m_i(t)$. If there exists a minimum t_{b}^* to

$g_b(t)$, it has to satisfy the following equation, which follows directly from (2.1) by setting the derivative equal to zero and rearranging terms;

$$\sum_{i=1}^{n} c_i^f m_i(t) - g_b(t) = 0. \tag{2.2}$$

Similar to the single component case one can show that if $\Sigma_{i=1}^{n} c_i^f m_i(t)$ increases in t and $c^p < \Sigma_{i=1}^{n} c_i^f(1 - \sigma_i^2/\mu_i^2)$ then there exists a unique solution t_b^* to (2.2). Denote the corresponding minimum value of $g_b(t)$ by g_b^*. Assuming the aforementioned conditions, it can also be shown using (2.2) that we have the following equivalences:

$$\sum_{i=1}^{n} c_i^f m_i(t) - g_b^* \gtreqless 0, \quad t \gtreqless t_b^*. \tag{2.3}$$

Following Berg [4] we interpret (2.3) in the following way. Suppose at time t we consider the following two options:

(a) *Replace all components preventively.*

(b) *Defer the replacements for an infinitesimally small time* $\mathrm{d}t$.

For the second option the expected costs over the interval $[t, t + \mathrm{d}t]$ amount to $c^p + \Sigma_{i=1}^{n} c_i^f m_i(t)\,\mathrm{d}t$, where $m_i(t)\,\mathrm{d}t$ indicates the expected number of failures in $[t, t + \mathrm{d}t]$ of component i, given i was new at time 0 and no age was known at time t. For the first case we have costs c^p at time t and we associate costs $g_b^*\mathrm{d}t$ to the remaining time interval, representing the minimum average costs we would have over an arbitrary interval of length $\mathrm{d}t$. Eq. (2.3) now implies that comparing the options in this way yields that option (a) is better than option (b) if $t \geq t_b^*$. Hence $\Sigma_{i=1}^{n} c_i^f m_i(t) - g_b^*$ can be used as a replacement criterion. Replacing if it is nonnegative results in an average optimal BR policy.

An algorithm to compute t_b^* and g_b^* consists of two parts, one in which $g_b(t)$ is evaluated and one in which it is optimised. For calculation of the renewal function, needed to determine $g_b(t)$, we used a recursive scheme in the discrete-time (DT) case and an approximation from Smeitink and Dekker [15] in the continuous-time (CT) case.

2.3. Single component age replacement in continuous time

The group age replacement criteria, which will be introduced in Section 2.5, are based on a marginal cost formulation of the single component age replacement (AR) model, which we briefly recapitulate here. For ease of notation, we skip in this section the component index i.

Consider an AR policy which replaces the component at age t (where the age is reset to zero after both a failure and a preventive replacement) and denote the associated long term average costs by $g_a(t)$. Berg [4] showed that there exists a unique finite optimal replacement age t_a^* provided that $c^f > c^p$ and $\lim_{t\to\infty} r(t) > c^f/((c^f - c^p)\mu)$. This age t_a^* is the unique solution to the following optimality equation:

$$(c^f - c^p)r(t) - g_a(t) = 0. \tag{2.4}$$

Moreover, we also have the following equivalences:

$$(c^f - c^p)r(t) - g_a^* \lesseqgtr 0, \quad t \lesseqgtr t_a^*. \tag{2.5}$$

We interpret (2.5) in the following way. Suppose at age t we consider the following two options:

(a) *Replace the component preventively.*

(b) *Replace the component upon failure or preventively after an infinitesimally small time* $\mathrm{d}t$, *whichever comes first.*

For the second option the expected costs over the interval $[t, t + \mathrm{d}t]$ amount to $c^p + (c^f - c^p)r(t)\,\mathrm{d}t$, where $r(t)\,\mathrm{d}t$ is the probability of failure in $[t, t + \mathrm{d}t]$. For the first option we have costs c^p at age t and we associate costs $g_a^* \mathrm{d}t$ to the remaining time interval, representing the average costs we would have over an interval of length $\mathrm{d}t$. Eq. (2.5) now implies that comparing the options in this way yields that option (a) is better than (b) if $t \geq t_a^*$.

2.4. Group age replacement – General considerations

We now return to the group age replacement (GAR) problem and consider a replacement decision which is based on sufficient information about the history of the process, being the vector containing all component ages. Policies of this type will be called group age replacement (GAR) policies. Intuitively, it will be clear that the class of BR policies does not contain the overall optimum policy. For example, consider the case in which just before the moment of execution of the package all components fail and are replaced. The BR policy will still recommend execution, which may be deferred by GAR policies. As such an event will occur with a positive probability it will be clear that a BR policy can always be improved by a GAR policy. Berg [5] proved this fact for the single component case.

In the DT case the GAR problem can be formulated as a Markov decision problem (MDP), as we will do in Section 4. In Roelvink and Dekker [12] it is shown that within the class of control limit policies (which, if they order a replacement at age $x_1, \ldots, x_n$, also do so for all ages $y_1, \ldots, y_n$, with $y_i \geq x_i$, $i = 1, \ldots, n$) there exists a policy which is average optimal. Except for the identical component case, the dimension of the state space of the MDP equals the number of components, hence solving the MDP is computationally troublesome. For the CT case no results are yet available with respect to optimality. Furthermore, whereas in the single component case AR policies are easily described by a single threshold value, this is much harder in the multi-component case (see, e.g. Fig. 2). Even control limit policies are not easily described as they require the specification of an n-dimensional hull, unless they are simplified considerably, e.g. to rectangles. Using individual component threshold values leaves one with the problem to determine an optimal set of threshold values, which is an n-dimensional optimisation problem. On the other hand, using a fixed threshold value for all components (i.e. replace the group if one component is older than x) yields a policy which for many components is almost like a BR policy. So, in fact one requires an easily implementable replacement criterion. Such criteria are presented in the next section.

2.5. The group age replacement criteria

The basic idea of the replacement criteria is to use marginal costs and consider the problem in a myopic way: either to replace the group now or to defer replacement to the next decision moment. In the CT case the next decision moment is at time $t + \mathrm{d}t$, whereas in the DT case it is one epoch Δt ahead. We will now introduce the criteria for each case separately.

The replacement criteria for the continuous time case

The replacement criteria are derived from criterion (2.3) by taking the actual component ages into account. The option to defer replacement for an infinitesimally small time $\mathrm{d}t$ involves expected costs $c^p + \sum_{i=1}^n c_i^f r_i(x_i(t))\, \mathrm{d}t$ over the interval $[t, t + \mathrm{d}t]$, where $x_i(t)$, $r_i(x_i(t))$ indicate component's i age and failure rate at that age respectively. To the option which replaces directly we associate costs $c^p + g\, \mathrm{d}t$, where g is a decision parameter yet to be specified. Considering the same comparison for the single component age replacement we would like to use for g the minimum average costs under group age replacement; however, as this value is not available, we approximate it by g_b^*. Comparing the options yields, after deleting c^p on both sides and dividing by $\mathrm{d}t$, the following preventive replacement criterion:

$$\mathrm{RC}^{\mathrm{I}}(x_1(t), \ldots, x_n(t)) \equiv \sum_{i=1}^{n} c_i^f r_i(x_i(t)) - g_b^* \tag{2.6}$$

For shorthand we write RC^{I} instead of RC^{I} $(x_1(t),\dots,x_n(t))$. The probability that all components fail (in the multi-component case) in an infinitesimally samll time interval of length dt is negligible with respect to the probability of one failure. In the single component case, a failure of a component does imply a total renewal of the system, and in that case we do not need to replace the component preventively at the end of the interval. Therefore we also consider an adapted replacement criterion RC^{II} defined as:

$$RC^{II} \equiv (c^{f} - c^{p}) r(x(t)) - g_b^*. \tag{2.7}$$

In the sequel we associate to each replacement criterion RC the policy: 'replace if RC ≥ 0'. Notice that both RC^{I} and RC^{II} are increasing functions in the component ages and hence the associated policies are of the control limit type. An underlying assumption made in both criteria is that preventive replacements are done *at all*. As for many distributions the failure rate increases to infinity, RC^{I} will eventually be positive, even if $c^{p} > \sum_{i=1}^{n} c_i^{f}$; this is not optimal. So RC^{I} and RC^{II} should only be used in cases where preventive replacement is cost effective (which follows from the BR optimisation).

The replacement criterion for the discrete time case

Suppose the components are observed only every Δt time units, where Δt is small enough to preclude multiple failures of one component. Let $q_i(x_i(t))$ denote the probability that component i of age $x_i(t)$ fails just before the next time point $t + \Delta t$. In this case the expected costs associated with deferring replacement for a time Δt are $c^{p} + \sum_i c_i^{f} q_i(x_i(t))$. Similarly to the CT case we arrive at the following preventive replacement criterion:

$$RC^{I} \equiv \sum_{i=1}^{n} c_i^{f} q_i(x_i(t)) - g_b^* \Delta t. \tag{2.8}$$

Similar to the single-component case we also consider an alternative criterion RC^{II} in which we do not replace all components preventively if they all fail before the next decision moment:

$$RC^{II} \equiv \sum_{i=1}^{n} c_i^{f} q_i(x_i(t)) - c^{p} \cdot \prod_{i=1}^{n} q_i(x_i(t)) - g_b^* \Delta t. \tag{2.9}$$

Notice that for high age values RC^{II} may no longer be increasing in the ages and that it only generates a control limit policy if $\sum_{i=1}^{n} c_i^{f} - c^{p} - g_b^* \Delta t \geq 0$.

For ease of reference in other papers (e.g. Dekker et al. [9]) we here introduce the term *Modified Block Replacement policy* to indicate the policy which replaces as soon as $RC(t) \geq 0$ (where RC is either RC^{I} or RC^{II}).

3. Results for the continuous-time single-component case

In this case we only consider criterion RC^{II} as that is most appropriate for the single component case. Fig. 1 shows typical shapes of curves of the average costs $g_a(t)$ and marginal costs $(c^{f} - c^{p}) r(t)$.

Since by assumption $r(t)$ increases, there exists for any $g \in [g_a^*, c^{f}/\mu]$ an age $t_g > t_a^*$ such that $(c^{f} - c^{p}) r(t_g) - g = 0$. As $(c^{f} - c^{p}) r(t) > g_a(t)$ for $t > t_a^*$ we have $g_a(t_g) \leq g$, with equality only if $g = g_a^*$. If $g > c^{f}/\mu$, then either $t_g > t_a^*$ or $t_g = \infty$, and again $g_a(t_g) < g$. This proves that the average costs under RC^{II} are lower than the minimum BR costs. Use of $g = g_a^*$ would even lead to an average optimal policy.

We also did some numerical experiments to evaluate the performance of the RC^{II} policy and to assess the differences between AR and BR policies. Table 1 provides numerical results for Weibull lifetime

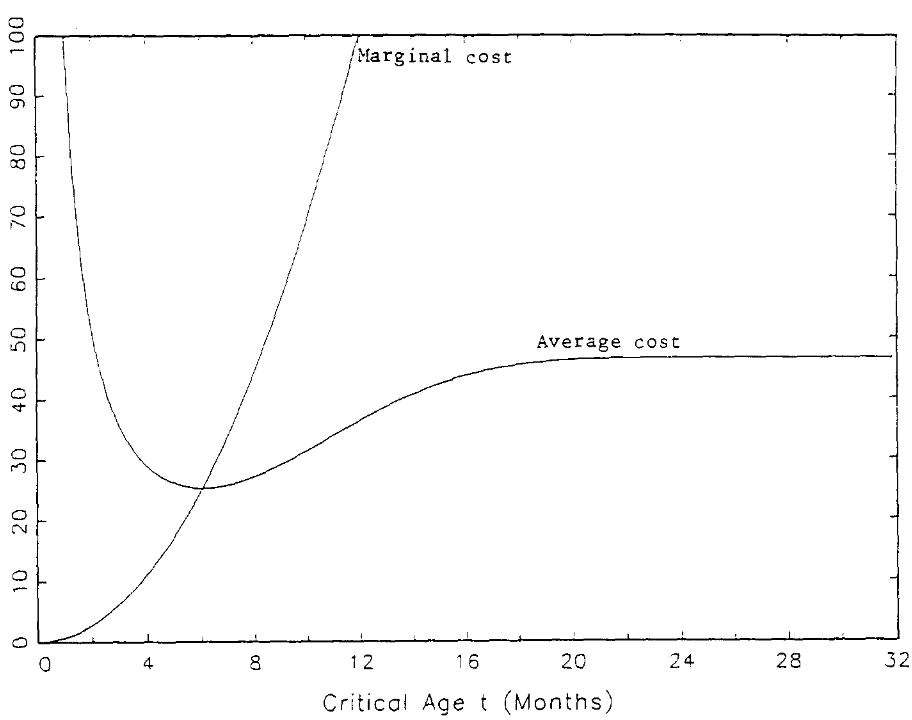

Fig. 1. Typical average cost and marginal cost curves.

distributions with constant mean and various shape and cost parameters. Herein t_{RC}^{II} indicates the control limit resulting from RC^{II} and g_{RC}^{II} the corresponding average costs.

From Table 1 we conclude that the RC^{II} policy yields a substantial improvement over the optimum BR policy and that it is only slightly inferior to the optimum AR policy. The differences in average costs between the optimum AR and optimum BR policies, however, are relatively small. As the policies mainly differ in case a failure occurs, we expect the difference to be related to the probability of failure before the optimum replacement time or age. If the ratio c^f/c^p is high, one is likely to replace at a relatively early time or age and thus allow only a small failure probability. Indeed we observe that the relative

Table 1
A comparison of optimum BR, AR policies with the RC^{II} policy

Case	β	$\frac{c^f}{c^p}$	$\frac{t_b^* - t_a^*}{t_a^*}$ (%)	$\frac{g_b^* - g_a^*}{g_a^*}$ (%)	$\frac{t_{RC}^{II} - t_a^*}{t_a^*}$ (%)	$\frac{g_{RC}^{II} - g_a^*}{g_a^*}$ (%)
1	2	2.5	+15.7	11.2	11.2	0.3
2	2	5	−0.6	5.4	4.7	0.1
3	2	10	−0.6	2.6	2.6	0.03
4	2	20	−0.5	1.3	1.4	0.01
5	6	2.5	−7.0	7.5	1.1	0.03
6	6	5	−3.2	3.4	0.4	0.01
7	12	2.5	−4.0	4.1	0.1	< 0.01
8	12	5	−1.8	1.9	< 0.1	< 0.01
9	12	10	−0.9	1.0	< 0.1	< 0.01
10	12	20	−0.5	0.5	< 0.1	< 0.01

difference between age and block policy decreases with the ratio c^{f}/c^{p}. The Weibull shape parameter β, or in fact the coefficient of variation c_x^2, as c_x^2 is monotonically related to β, has a similar effect. The larger c_x^2 (and the lower β), the more spread the distribution exhibits and the more difficult it is to predict a failure. The probability of failure before the optimum replacement time or age decreases with β and hence the difference between the average costs under the optimum AR and BR policy does so as well. If β, however, equals 1 (and the failure rate is constant) then both policies replace at failure only, and the difference in average costs is zero.

4. A Markov decision chain for the two-component replacement problem

4.1. Model formulation

In this section we consider discrete lifetime distributions and formulate the group age replacement problem as a Markov decision problem. The advantage of this approach is that we are able to determine an optimal policy and compare the replacement criteria with this policy. From other studies, although with slightly different models, we do not expect the optimal policy to exhibit a nice structure. Hence the only way to determine the optimal policy is to solve the Markov decision chain numerically, which, to limit computational effort, is done for two components only.

As it is our primary aim to compare various replacement policies, we have to take care of the discretisation effects, as changing the action in only one state may already have a large effect in case of a coarse discretisation. Therefore we take a continuous time example as starting point and consider various discretisation steps and two types of modelling.

Let $F(t)$ be a continuous distribution of the time to failure for a component and suppose we observe the component every Δt time units. Let $p(j)$, $j=0, 1, \ldots$, be a discrete time distribution, with $p(j)=F((j+1)\Delta t)-F(j\Delta t)$, $j=0, 1, \ldots$, denoting the probability of failure just before the next time point, i.e. at time $(j+1)\Delta t^{-}$. The probability that a component of age $j\ \Delta t$ fails just before the next time point, $q(j)$, is given by $q(j) \equiv p(j)/\sum_{k=j}^{\infty} p(k)$. $q(j)$ is the discrete analogon of the failure rate. We assume that $q(j)>0$, $j=0, 1, \ldots$. It is worth noting that the so-constructed discrete distribution is stochastically smaller than the continuous one and the average costs under any policy will therefore be lower than in the CT case. Notice further that if we replace a component immediately upon observation, we tend to disfavour preventive replacements against failure replacements, as the latter occur Δt^{-} units later. This may be disadvantageous for AR, which has a higher ratio of failure replacements to preventive replacements than BR. Hence we also consider another modelling, in which a preventive replacement takes Δt time units.

We now formulate the Markov chain for a two-component system, which is observed every Δt time units. Let the state (i, j) indicate that component 1 and 2 have age $i\ \Delta t$ and $j\ \Delta t$ respectively. The lifetimes are truncated to $N\ \Delta t$, where N is chosen large enough to have no effect on the optimal replacement decisions (we only consider cases in which replacement makes sense). In each state we have two actions, viz. action 1 implying a preventive replacement of both components, and action 0 implying a continuation of the operation. The transition probabilities, $P(s' \mid s, 0)$, are in case of action 0 and state $s=(i, j)$

$$P(s' \mid s, 0) = \begin{cases} [1-q_1(i)][1-q_2(j)], & s'=(i+1, j+1), \\ q_1(i)[1-q_2(j)], & s'=(0, j+1), \\ [1-q_1(i)]q_2(j), & s'=(i+1, 0), \\ q_1(i)q_2(j), & s'=(0, 0), \end{cases}$$

where the subindex k in $q_k(\cdot)$, $k = 1, 2$, refers to component k. Preventive replacement is modelled in two ways. In the first it takes Δt time units (denoted by $t_{pm} = 1$) and the components get age zero at the next inspection point, i.e. $P^A((0, 0) | (i, j), 1) = 1$, where the superscript A denotes the first modelling. In the second way (indicated by the superscript B) the components are replaced instantaneously ($t_{pm} = 0$) and have the same transition probabilities as two new components. That is, $P^B(s' | s, 1) = P(s' | (0, 0), 0)$. Notice that as we assumed that $q_k(j) > 0$ for all states j and components k, we have $P((0, 0) | s, a) > 0$ for all actions a in both modellings. Hence the Markov chain induced by any policy is unichain. Immediate costs are given by

$$c((i, j), 0) = c_1^f q_1(i) + c_2^f q_2(j),$$
$$c^A((i, j), 1) = c^p,$$
$$c^B((i, j), 1) = c^p + c_1^f q_1(0) + c_2^f q_2(0).$$

The replacement criteria are given by (2.8) and (2.9).

4.2. Relation between the replacement criteria and policy improvement

The replacement criteria are strongly related to the policy improvement procedure. The main difference however, is that the replacement criteria consider in each state only the actions: replace now or at the next time point, whereas the policy improvement procedure takes also other options into account. On the other hand, the policy improvement procedure requires the so-called value vector v, which, for any stationary policy R, is defined by the set of equations over the state space S:

$$g_R \Delta t + v_R(s) = c_R(s) + \sum_{s'} P_R(s' | s) v_R(s'), \quad s \in S, \tag{4.1}$$

where $c_R(s)$ stands for $c(s, R(s))$, $P_R(s' | s)$ for $P(s' | s, R(s))$ and $R(s)$ for the action prescribed by policy R in state s. The larger the number of components, the larger the set of equations to be solved, which we would like to avoid. Given a policy R with value vector $v_R(s)$ and average costs g_R, the policy improvement step can be formulated as

$$\max_{a \in A(s)} \left\{ c(s, a) + \sum_{s'} P(s' | s, a) v_R(s') - v_R(s) - g_R \Delta t \right\}. \tag{4.2}$$

A policy R for which (4.2) is zero, can no longer be improved and is average optimal. Inserting the two actions yields in case of modelling B

$$\max\left\{ c^p + w_R(0, 0), c_1^f q_1(i) + c_2^f q_2(j) + \sum_{s'} P(s' | s, 0) v_R(s') \right\} \tag{4.3}$$

where $w_R(0, 0)$ is short for $c_1^f q_1(0) + c_2^f q_2(0) + \sum_{s'} P(s' | s = (0, 0)) v_R(s')$. *If* the optimum policy R *would* replace in each of the next states s' we would have $v_R(s') = c^p + w_R(0, 0) - g_R \Delta t$, by (4.1). Inserting this into (4.2) yields the following maximization:

$$\max\left\{ c^p + w_R(0, 0), c_1^f q_1(i) + c_2^f q_2(j) + c^p + w_R(0, 0) - g_R \Delta t \right\},$$

which yields the same actions as RC^I if we replace g_R by g_b^*. RC^{II} now follows by remarking that we do not replace preventively if $s' = (0, 0)$. Hence, we obtain (after subtraction of c^p and remarking that $v_R(0, 0) = w_R(0, 0) - g_R \Delta t$)

$$\max\left\{ 0, c_1^f q_1(i) + c_2^f q_2(j) - g_R \Delta t - c^p q_1(i) q_2(j) \right\}.$$

For modelling A, a similar analysis can be done. The results, however, are only consistent with RC^I,

Table 2
Comparison of average costs for two-component problems
(a) Example 1

Δt	g_b^*	g_a^*	g_{RC}^{I}	g_{RC}^{II}	$\frac{g_b^* - g_a^*}{g_b^*}$ (%)	$\frac{g_b^* - g_{RC}^{I}}{g_b^*}$ (%)	$\frac{g_b^* - g_{RC}^{II}}{g_b^*}$ (%)
$t_{pm} = 1$:							
1	2.6686	2.6480	2.6480	2.6480	0.77	0.77	0.77
$\frac{1}{2}$	3.4515	3.3662	3.3662	3.3658	2.47	2.47	2.48
$\frac{1}{3}$	3.7711	3.6556	3.6704	3.6704	3.07	2.67	2.67
$t_{pm} = 0$:							
1	4.4133	4.0380	4.0383	4.0582	8.50	8.50	8.05
$\frac{1}{2}$	4.4467	4.1563	4.1600	4.2609	6.53	6.45	4.18
$\frac{1}{3}$	4.4462	4.1993	4.2305	4.2323	5.55	4.85	4.81

Component 1: $X_1 \sim$ Weibull, shape $\beta_1 = 2$, mean $\mu_1 = 3$ ($\lambda_1 = 3.385$), $c_1^f = 8$, $c^p = 4$.
Component 2: $X_2 \sim$ Weibull, shape $\beta_2 = 2$, mean $\mu_2 = 3$ ($\lambda_2 = 3.385$), $c_2^f = 8$.
Δt = the discretisation step; CT BRP: $g_b^* = 4.5024$; g_{RC}^{I} = average costs for RC^{I}.

(b) Example 2

Δt	g_b^*	g_a^*	g_{RC}^{I}	g_{RC}^{II}	$\frac{g_b^* - g_a^*}{g_b^*}$ (%)	$\frac{g_b^* - g_{RC}^{I}}{g_b^*}$ (%)	$\frac{g_b^* - g_{RC}^{II}}{g_b^*}$ (%)
$t_{pm} = 1$:							
1	1.3716	1.3709	1.3709	1.3709	0.05	0.05	0.05
$\frac{1}{2}$	1.8296	1.8162	1.8162	1.8163	0.73	0.73	0.73
$\frac{1}{3}$	2.0586	2.0370	2.0370	2.0372	1.05	1.05	1.04
$\frac{1}{4}$	2.1880	2.1666	2.1687	2.1689	0.98	0.88	0.87
$\frac{1}{5}$	2.2586	2.2323	2.2331	2.2331	1.17	1.13	1.13
$t_{pm} = 0$:							
1	2.7433	2.6602	2.6602	2.6603	3.03	3.03	3.03
$\frac{1}{2}$	2.6514	2.5987	2.6017	2.6018	1.99	1.87	1.87
$\frac{1}{3}$	2.6221	2.5694	2.5693	2.5695	2.01	2.01	2.00
$\frac{1}{4}$	2.6256	2.5720	2.5732	2.5732	2.04	2.00	2.00

Component 1: $X_1 \sim$ Weibull, shape $\beta_1 = 2.5$, mean $\mu_1 = 3$ ($\lambda_1 = 3.385$), $c_1^f = 8$, $c^p = 2$.
Component 2: $X_2 \sim$ Weibull, shape $\beta_2 = 2.5$, mean $\mu_2 = 3$ ($\lambda_2 = 3.385$), $c_2^f = 8$.
Δt = the discretisation step; CT BRP: $g_b^* = 2.6257$.

since in RC^{II} a preventive replacement is delayed if all components are replaced upon failure. In modelling A a preventive replacement takes Δt time units while a failure replacement is done instantaneously, so another term with g slips in. Anyhow, it will now be clear where the replacement criteria deviate from the optimal replacement policy. First of all, they use a time- and not a state-dependent value vector (because of the assumption that in the next state a preventive replacement will be done, regardless of which state it is) and secondly, they approximate g^* by g_b^*.

4.3. Numerical results

In this section we discuss numerical results obtained with the Markov chains. More results can be found in Roelvink and Dekker [11]. We first determined the optimal BR policy both for the CT case and

RC^I

age of comp. 2 \ age of comp. 1	0	1	2	3	4	5	6	7	8	9	10
10	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1
	0	0	1	1	1	1	1	1	1	1	1
6	0	0	0	1	1	1	1	1	1	1	1
	0	0	0	0	0	1	1	1	1	1	1
4	0	0	0	0	0	0	1	1	1	1	1
	0	0	0	0	0	0	1	1	1	1	1
2	0	0	0	0	0	0	0	1	1	1	1
	0	0	0	0	0	0	0	0	1	1	1
0	0	0	0	0	0	0	0	0	1	1	1

Optimal GAR ($t_{pm}=0$)

age of comp. 2 \ age of comp. 1	0	1	2	3	4	5	6	7	8	9	10
10	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1
8	0	0	1	1	1	1	1	1	1	1	1
	0	0	0	1	1	1	1	1	1	1	1
6	0	0	0	0	1	1	1	1	1	1	1
	0	0	0	0	0	1	1	1	1	1	1
4	0	0	0	0	0	0	1	1	1	1	1
	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	0	0	0	0	0	1	1	1
	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	1	1

Fig. 2. A comparison of the RC^I and the optimal GAR policy.

the DT case, thereby taking a possible time for preventive maintenance (t_{pm}) into account. We used a policy iteration scheme to obtain an average optimal AR policy, while starting from the policy obtained from the replacement criteria. In most cases only one or two policy iteration steps had to be made to obtain an optimal policy. The average costs for a given policy were obtained by solving (4.1) with the Householder method. Table 2 contains two typical examples for which, according to Section 3, a large difference in average costs between AR and BR policies was expected. Fig. 2 shows an example of the RC^I policy compared to an optimal GAR policy.

From the numerical experiments the following conclusions can be drawn:

(a) In all cases considered the replacement criteria yield an improvement over BR, varying between 0% and 8%. Roelvink and Dekker [11] report one case in which the improvement is negative, but in that case BR is not cost-effective, while GAR still is.

(b) The discretisation has a large effect on the difference in average costs between the optimal AR and BR policies and the RC^I and RC^{II} policies. Discretisation effects are in most cases monotonic. Low values for this difference were obtained in case $t_{pm} = 1$ and high values for $t_{pm} = 0$.

(c) The maximum relative difference in average costs between the RC^I and RC^{II} policies and the optimal AR policy as observed is 0.7%, which is about 10% of the difference between the optimum AR and BR policies. The RC^I and RC^{II} policies do not differ much from the optimal AR policy.

(d) There is not much difference in the average costs between RC^I and RC^{II}.

(e) The optimal AR policy does not allow a simple description.

5. Simulation results for the multi-component case

The performance of the RC^I policy was also tested by means of simulation. The simulations were carried out for cases with 2, 3, 5 and 10 identical components. For lifetime we used Weibull distributions with shape parameter $\beta = 2$ and mean 3. To investigate the effect of the number of components in the group, we varied the individual failure and preventive replacement costs in such a way that $\Sigma_{i=1}^{n} c_i^f$ and c^p remain constant for $n = 2, 3, 5, 10$. Hence, regardless of the number of components considered, BR yields the same optimum interval and the same average costs. We examined five different pairs of values for c^p and $\Sigma_{i=1}^{n} c_i^f$. The combinations are shown in Table 3a, together with the optimal replacement

Table 3a
Parameter values of the cases considered

	Case				
	1	2	3	4	5
$\Sigma_{i=1}^{n} c_i^{\mathrm{f}}$	8	8	8	6	20
c^{p}	1	2	2.5	2	1
t_{b}^{*}	1.285	2.011	2.425	2.588	0.777

Table 3b
Relative differences (%) in average costs, $(g_{\mathrm{RC}}^{\mathrm{I}} - g_{\mathrm{b}}^{*})/g_{\mathrm{b}}^{*}$, for the optimal BR and RC[I] policies

Case	g_{b}^{*}	Nr. of components				
		1	2	3	5	10
1	1.634	3.1	1.8	1.5	0.8	0.4
2	2.252	4.8	3.3	2.9	2.1	1.2
3	2.472	4.9	4.0	3.5	2.7	1.7
4	1.904	4.9	4.0	3.6	2.8	2.0
5	2.600	1.2	0.8	0.7	0.4	0.2

intervals t_{b}^{*}. Case 2 in fact corresponds to the first example of Section 4 (with all cost figures multiplied by 2).

In the simulation we used the regenerative approach, where a group replacement started and ended a renewal cycle. In every cycle we generated the same failure epochs for both the RC[I] and the optimal BR policy, until one of the strategies replaced the group (and the cycle of this policy ended). We then generated failure epochs for the other policy only, until this policy also replaced the group. In this way we got the best approximation for the difference between the two policies. The results are shown in Table 3b. In all cases considered the RC[I] policies yields an improvement over BR varying from almost 0.2% to 5%. The improvement is largest if the ratio $\Sigma_{i=1}^{n} c_i^{\mathrm{f}}/c^{\mathrm{p}}$ is smallest, which agrees with the conclusions from Section 3. The improvement over BR decreases with the number of components. In fact we conjecture that the difference in average costs between an optimum GAR policy and an optimum BR policy decreases to zero. To demonstrate this we give the following heuristic arguments. An age policy makes use of the actual behaviour of the components. The more components there are, the more this is close to the average behaviour on which BR is based. If there would be an infinite number of components, their behaviour would be fully described by the renewal function. That is, in a renewal cycle of length t, the ratio of failures to the total number of components would amount to the renewal function $M(t)$ and the cycle costs would amount to $c^{\mathrm{p}} + \Sigma_i^{\infty} c_i^{\mathrm{f}} M_i(t)$ with probability 1. Deviating from the optimum BR interval would therefore not be optimal.

6. Conclusions

From marginal cost considerations simple replacement criteria for group age replacement can be derived which perform better than the optimal block replacement policies and are only slightly worse than the optimal group age replacement policies. The differences in average costs between these policies are moderate. It depends upon the number of components and the probability of failure of a component before the replacement time. From a practical point of view, it may be more important that the replacement criteria give advice that better matches the intuition of users than block replacement does. The marginal cost approach to derive these age replacement polices has in fact been applied in a decision support system for opportunity maintenance. The replacement criteria could easily be adapted for this case, by taking the time between opportunities as time between consecutive decision moments. The stochasticness of the former did imply taking an integral, but posed no real problem.

Appendix A. Definition of the Weibull distribution

A r.v. X follows a two-parameter Weibull distribution with scale parameter λ and shape parameter β if its cdf $F(\cdot)$ is defined through

$$F(t) = 1 - \exp\left(-(t/\lambda)^{\beta}\right), \quad t > 0.$$

With respect to the mean $\mu = EX$ we have

$$\mu = \lambda \Gamma(1 + 1/\beta).$$

The squared coefficient of variation $c_x^2 (\equiv \mu^2/\sigma^2)$ equals

$$c_x^2 = \frac{\Gamma(1 + 2/\beta)}{[\Gamma(1 + 1/\beta)]^2} - 1,$$

where $\Gamma(\cdot)$ indicates the gamma function. If β goes from 1 to infinity, c_x^2 decreases from 1 to 0.

Appendix B. List of notations and abbreviations

AR(P)	– Age Replacement (Problem).
BR(P)	– Block Replacement (Problem).
GRP	– The Group Replacement Problem.
GAR	– Group Age Replacement.
DSS	– Decision Support System.
MDP	– Markov Decision Problem.
X_i	– Random variable indicating lifetime of component i.
μ_i	– Mean life of component i.
σ_i^2	– Variance of lifetime of component i.
$r_i(t)$	– Failure rate of component i.
c_x^2	– Coefficient of variation for r.v. X.
n	– Number of components in the maintenance package.
c_i^{f}	– Costs of failure replacement of component i.
c^{p}	– Costs of executing the maintenance package.
$M_i(t)$	– The renewal function of component i, indicating the expected number of failures in $[0, t]$.
$m_i(t)$	– Derivative of $M_i(t)$.
$g_{\mathrm{b}}(t))$	– Long term average costs under a BR policy with interval t.
$g_{\mathrm{a}}(t)$	– Long term average costs under an AR policy with replacement age t.
t_{b}^*, t_{a}^*	– Optimal interval for BR, optimal replacement age for AR resp.
g_{b}^*, g_{a}^*	– Minimal long term average costs under BR, AR resp.
$\mathrm{RC}^1(t)$	– The group age replacement criterion which considers the options replace now or at the next decision moment.

$RC^{II}(t)$ – The group age replacement criterion which in its considerations does not replace if all components fail at the next decision moment.
$q_i(x)$ – Probability that component i at age x fails before the next decision moment.
$x_i(t)$ – Age of component i at t time units since the last execution of the package.
β – Shape factor of Weibull distribution.
R – Stationary policy in the Markov decision chain.
v – Value vector in the Markov decision chain.
$c(s, a)$ – Immediate expected costs associated with action a in state s.
$P(s' \mid s, a)$ – Transition probability from state s to state s' under action a.

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