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Int. J. Production Economics 46-47 (1996) 339-350

international journal of
production economics

An (s, Q) inventory model with remanufacturing and disposal

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Abstract

In this paper we analyse an (s, Q) inventory model in which used products can be remanufactured to new ones. We develop two approximations for the average costs and compare their performance with that of an approximation suggested by Muckstadt and Isaac. Next we extend the model with the option to dispose returned products and present a heuristic optimisation procedure which is checked with full enumeration.

Keywords: Inventory; Manufacturing; Optimisation

1. Introduction

Environmental pressures have more and more impact on the way manufacturers operate. An important issue in this respect is that manufacturers have to take care of their products after use. This may lead to recycling, in which the materials in the products are reused, or to remanufacturing, in which old products undergo some kind of manufacturing to make them as good as new (see [1]) for an overview of the area of product recovery management). The last variant is considered in this paper.

The result of remanufacturing is that producers have to take returns into account in their planning, next to outside procurement and internal production. This complicates their inventory control as the returns increase the fluctuations in inventory levels. At some point one may even go over to disposal of items.

Although there are many papers on inventory control of repairable items, they almost all concern systems in which the number of items remains constant (see [2, 4] for reviews). Few papers consider returns and outside procurement in a manufacturing environment, where the total number of items changes in course of time. The most relevant papers can be categorised in three groups: cash balance models, periodic and continuous review models (see [5] for a detailed review). In cash balance models demands and returns of money are explicitly modelled and various inventory control policies are considered, yet in all variants there is no leadtime which makes the inventory control much easier (see [6] for an overview). The same holds for the few papers [7, 8] for the periodic review case. In the continuous-review case the leadtimes and repair times can be modelled explicitly which makes them interesting from a manufacturing point of view. Yet also Heyman [9] and Hoadley and Heyman [10] who consider this case, assume zero repair times and no procurement leadtimes and do not take

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SSDI 0925-5273(95)00020-8

order costs into account. The only paper which does consider leadtimes is from Muckstadt and Isaac [11]. They develop an approximative method for the single-item (s, Q) model with fixed leadtimes, but without disposal and apply this method in a two-echelon warehouse-retailer problem.

In this paper we take Muckstadt and Isaac [11] as starting point in investigating the effects of remanufacturing on the inventory control of a single item. We develop alternative methods in Section 2 and extend them with the option to dispose items in Section 3. The justification for disposal is that accepting all returns leads to very high inventory levels in case of a high return rate. Apart from a performance evaluation and an approximative optimisation procedure, we give ample numerical results, showing the effect of disposal.

2. An (s, Q) inventory system with returns

We consider a single-item single-location inventory system (Fig. 1) with unit demands and returns according to independent Poisson processes with rates λ and γ, respectively. Every returned item has to undergo a repair in a repair facility before it is available to satisfy demands (in this paper we neglect testing and disassembly problems (see e.g. [12]; it is not allowed to hold up repairs). Initially we only assume independence of repair times with respect to the repair shop. To calculate performance measures, however, one needs to specify the repair capacity and for some cases we will give results. Apart from using the returned items, it is also possible to order items from outside against fixed order costs A. Orders arrive after a fixed leadtime τ. Demands not directly satisfied are backordered. The other costs considered consist of backorder costs π per item per unit of time, and holding costs h per item in serviceable inventory per unit of time. We assume that the inventory is continuously reviewed and that an (s, Q) inventory control policy is applied to the inventory position. Our objective is to determine those parameters s, Q that minimize the total long-term average costs.

For the analysis define the *net inventory* at time t as the number of on-hand serviceable units in the storage facility, $O(t)$, minus the number of outstanding backorders, $B(t)$. The *inventory position*, $I(t)$, is the sum of the net inventory, $N(t)$, the number of items in the repair system, $R(t)$, and the number of units on order, $P(t)$. Notice now that at time t all the outstanding orders from moment $t - \tau$ have arrived. Hence the net inventory at time t equals the inventory position at time $t - \tau$ minus the number in the repair shop at that moment minus the demand plus the output of the repair shop during the interval $[t - \tau, t]$. In formula, $N(t) = I(t - \tau) - R(t - \tau) + Z(t - \tau, t) - D(t - \tau, t)$, where the latter two indicate the output of the repair shop and the demand over the interval $[t - \tau, t]$ respectively. For the analysis we are interested in the average number of orders, the average on-hand inventory and the average number of backorders. Since both demand and return interarrival times are exponentially distributed, we can formulate a continuous-time Markov chain for the inventory position. Demands now decrease the inventory position with one item unless the level s is reached in which case a replenishment directly increases the inventory position to $s + Q$. A return increases the inventory position by one item. If

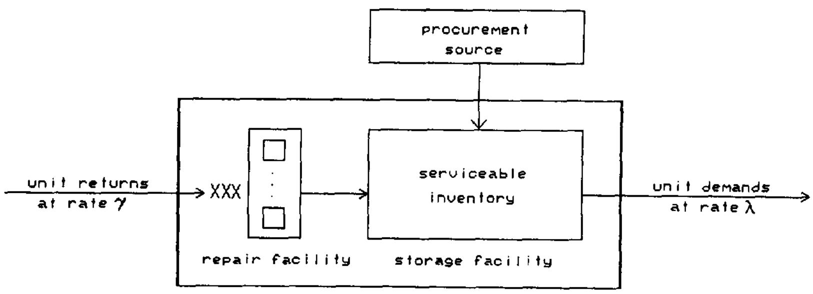

Fig. 1. A schematic representation of the inventory system.

$\gamma < \lambda$, then the Markov chain is ergodic and the limiting distribution of the inventory position equals the stationary distribution. The limiting values are denoted with the ∞ symbol replacing t, e.g. $I(\infty) = \lim_{t\to\infty} I(t)$. Let $p_i = \lim_{t\to\infty} P(I(t) = i)$. Using a generating function approach it is easy to obtain the following explicit expressions for the p_i's:

$$p_{s+i+1} = \begin{cases} 0, & i < 0, \\ \dfrac{[1-(\gamma/\lambda)^{i+1}]}{Q}, & 0 \leqslant i < Q, \\ \dfrac{(\gamma/\lambda)^{i-Q+1}[1-(\gamma/\lambda)^{Q}]}{Q}, & Q \leqslant i. \end{cases} \quad (1)$$

The resulting first two moments are

$$E(I(\infty)) = s + 1 + \frac{Q-1}{2} + \frac{\gamma}{\lambda-\gamma},$$

$$var(I(\infty)) = \frac{Q^2-1}{12} + \frac{\lambda\gamma}{(\lambda-\gamma)^2}, \quad (2)$$

Taking limits yields $E(N(\infty)) = E(I(\infty)) - E(R(\infty)) + (\gamma - \lambda)\tau$ and a similar formula can be derived for the variance. The long-run number of units in the repair shop, $E(R(\infty))$, abbreviated to R, can be calculated for various repair shops; for an M/M/∞ queue it equals γ/μ, where μ denotes the expected repair time.

In order to calculate the average number of backorders we need to get hold of the distribution of the net inventory at an arbitrary point in time. In case the repair times are exponentially distributed, it is possible to obtain the exact distribution. To this end a continuous-time Markov chain has to be formulated for both the inventory position and the number of items in the repair shop. Solving this Markov chain yields the joint stationary distribution of the inventory position and the number in the repair shop (note that they are not independent and that we need to truncate the state space to allow numerical computations). Next the distribution of the output of the repair shop over τ time units given an initial number in the repair shop has to be determined. Finally the distribution of the net inventory follows from the twofold convolution of the joint stationary inventory position and number in the repair shop with the output of the repair shop and the demand over τ time units (see [5] for details). It will be clear that this procedure requires considerable numerical effort and does not yield explicit formulas. Therefore we will develop two approximation procedures ([5] only provides exact results) and compare them with the exact solutions and with a procedure developed by Muckstadt and Isaac [11].

For both approximations we assume that at any point in time there is at most one order outstanding. The output of the repair shop is now approximated by an independent Poisson process with mean γ, which is exact for M/M/c and M/G/∞ queues [11]. In the first approximation we assume that the net demand (i.e. demand minus output of the repair shop) during the leadtime follows a normal distribution. Hence the expected number of backorders just before a replenishment, $F(s,\tau)$, is given by the expected surplus demand over a level $s - R$ of a normal distribution with mean $\mu_\tau = (\lambda - \gamma)\tau$ and variance $\sigma_\tau^2 = (\lambda + \gamma)\tau$. In formula

$$F(s,\tau) = (\mu_\tau + R - s)\Phi\left(-\frac{s-R-\mu_\tau}{\sigma_\tau}\right) + \sigma_\tau\phi\left(\frac{s-R-\mu_\tau}{\sigma_\tau}\right). \quad (3)$$

Assuming a linear increase of the number of backorders per time unit, it follows that the average number of backorders during the time that net inventory is negative equals $F(s,\tau)/2$. From the same assumption it also follows that the average time that net inventory is negative equals $F(s,\tau)/(\lambda-\gamma)$ divided by the average cycle length $Q/(\lambda-\gamma)$. Hence the expected number of backorders at a random point in time, $E(B(\infty))$ can be approximated by $F(s,\tau)^2/2Q$, which for reference we denote by $B_{L1}(s,\tau)$.

Notice next that the average number of orders per time unit is given by $(\lambda-\gamma)/Q$. Hence the average ordering costs equal $A(\lambda-\gamma)/Q$. In this way we obtain as total cost function $K(s,Q)$:

$$K(s,Q) = A\left(\frac{\lambda-\gamma}{Q}\right) + \pi E(B(\infty)) + h[E(N(\infty)) + E(B(\infty))]$$

$$\approx A\left(\frac{\lambda-\gamma}{Q}\right)+(\pi+h)\frac{F(s,\tau)^2}{2Q}$$
$$+h\left(s+1+\frac{Q-1}{2}+\frac{\gamma}{\lambda-\gamma}-R\right.$$
$$\left.-(\lambda-\gamma)\tau\right). \tag{4}$$

The function $K(s, Q)$ can be shown to be a strict convex function in s and Q (see Appendix A). The optimal value of Q given s^* is easily found by taking the first derivative of (4) and equals

$$Q^* = \sqrt{\frac{\pi+h}{h}F(s^*,\tau)^2+\frac{2(\lambda-\gamma)A}{h}}. \tag{5}$$

Notice that the second term within the square root resembles the well-known EOQ formula, adjusted for returns. Taking the derivative with respect to s and using (3) yields the following equation for the optimal value of s given Q^*:

$$\frac{F(s^*,\tau)}{Q^*}\Phi\left(-\frac{s^*-R-\mu_\tau}{\sigma_\tau}\right)=\frac{h}{\pi+h} \tag{6}$$

which can easily be solved with numerical techniques. Since both s and Q are integer valued, the final optimal combination is that neighbour which has lowest average costs.

In the second procedure we approximate the difference between the demand and the output process from the repair facility by a Brownian motion with drift equal to $(\lambda - \gamma)$ and variance parameter $(\lambda + \gamma)$. Moreover, we assume that at the moment of ordering there are R items in the repair shop. Consequentially the net inventory at t time units since the ordering of a replenishment follows a normal distribution with mean $s - R - (\lambda - \gamma)t$ and with variance $(\lambda + \gamma)t$. Hence the time-average amount backordered, $B_{L2}(s, \tau)$, equals

$$B_{L2}(s,\tau)=\frac{\lambda-\gamma}{Q}\int_0^\tau F(s,t)\,dt. \tag{7}$$

This leads to another total cost function which we denote by $K_2(s, Q)$. Again it is possible to show that this function is convex (see Appendix A). Using approximation (7), the optimal value of Q, Q^*, is computed as

$$Q^*=\sqrt{\left((\pi+h)\int_0^\tau F(s^*,t)\,dt+A\right)\frac{2(\lambda-\gamma)}{h}} \tag{8}$$

and the optimal value of s, s^*, must satisfy

$$-\int_0^\tau\frac{dF(s,t)}{ds}=\frac{h}{\pi+h}\cdot\frac{Q^*}{\lambda-\gamma}. \tag{9}$$

Muckstadt and Isaac [11] approximate the net inventory by a normal distribution, where they determine the moments from those of the stationary inventory position and those of the demand and output of the repair shop during the leadtime, while assuming independence of the random variables. As a result they obtain different formulas for the expected number of backorders. A disadvantage of their analysis is that the asymptotic properties of the approximation of the net inventory do not correspond to actual behaviour in some cases. We have in this connection Table 1 (the indices MI and L denote the approximations of the respective authors).

Moreover, it is more difficult to incorporate disposal in their method. Extensive simulation

Table 1
Summary of asymptotic results

	Procedure of Muckstadt and Isaac			Van der Laan et al. Procedure 1			Van der Laan et al. Procedure 2			Actual behaviour		
	s^*	Q^*	B_{M1}	s^*	Q^*	B_{L1}	s^*	Q^*	B_{L2}	s^*	Q^*	$E(B(\infty))$
$Q \to \infty$	—	—	∞	—	—	0	—	—	0	—	—	0
$\gamma \to \lambda$	$0/\infty$	const.	∞	const.	const.	const.	0	1	0	0	1	0
$s \to \infty$	—	—	0	—	—	0	—	—	0	—	—	0

experiments using the optimal values obtained by our two methods and method of Muckstadt and Isaac, and computing the accompanying *exact* costs, show that the three methods differ only slightly for moderate values of π (see Figs. 2(a) and 3(a)). For higher values of π, our first procedure does not perform very well for $\gamma/\lambda < 0.75$, whereas for values of γ close to λ the procedure of Muckstadt and Isaac performs considerably worse (see Figs. 2(b),(c), 3(a) to (c)). In all cases we considered, our second method generated results that are very close to optimal, and we conclude that this procedure is very accurate, and superior to the other methods in almost all cases. The computation time needed for the second procedure is slightly more than that of the first procedure and that of procedure of Muckstadt and Isaac but still factors less than the exact cost evaluation.

In Fig. 4, we compared the situation with a demand flow, with expected value λ, and a return flow, with expected value γ, to the situation where we have only a demand flow that is corrected for the expected return flow, thus with expected value $\lambda - \gamma$. It can be seen that the uncertainty of return flows gives rise to higher costs and higher variability of the processes involved. Moreover, as γ tends to λ the average costs tend to infinity. One way of reducing this explosive behaviour is applying a disposal strategy, in order to reduce the return flow.

3. Optimal disposal policies

An increase in the return rate of items, γ, does not result in lower costs, as can be seen in Figs 2(a) and (c). The increase in average costs is due to increasing inventory costs. Hence it makes sense to extend the model with the possibility to dispose items. Here we assume that disposal occurs in the first stage, i.e. in the repair shop, and is only based on local information, i.e. the number of items in the repair shop. We further assume that returned items are either repaired directly or disposed. Salomon et al. [5] also consider disposal on the inventory position. Disposal on the number in the repair shop may be easier implemented, especially if there is a limitation in storage space. Besides, the analysis is easier for this case than for disposal on the number in the inventory position. Yet the latter policy may sometimes be economically more attractive, see [13].

We will model the repair shop as an $M/M/c/c+N$ queue with a Poisson input (i.e. return) rate γ, with c parallel servers each having a negative exponentially distributed service time with mean $1/\mu$ and with a waiting room (for repair) of size N. Hence a returned item is disposed if there are N other items waiting for inspection and repair. Standard queueing theory yields the following expressions for the steady-state probabilities of the $M/M/c/c+N$ queue:

$$p_i = \begin{cases} 0, & i < 0, \\ p_0 \dfrac{(\gamma/\mu)^i}{i!}, & 0 \leqslant i \leqslant c-1, \\ p_0 \dfrac{(\gamma/\mu)^c}{c!}\left(\dfrac{\gamma}{c\mu}\right)^{i-c}, & c \leqslant i \leqslant c+N, \\ 0, & i > c+N, \end{cases} \tag{10}$$

with

$$p_0 = \left[\sum_{i=0}^{c-1} \frac{(\gamma/\mu)^i}{i!} + \frac{(\gamma/\mu)^c}{c!}\left(\frac{1-\left(\dfrac{\gamma}{c\mu}\right)^{N+1}}{1-\dfrac{\gamma}{c\mu}}\right)\right]^{-1}. \tag{11}$$

It follows that the long-run average number of items in the repair shop is given by

$$E(R(\infty)|c,N) = p_0\left(\sum_{i=0}^{c-1} i\cdot\frac{(\gamma/\mu)^i}{i!} + \frac{(\gamma/\mu)^c}{c!}\sum_{i=c}^{c+N} i \cdot\left(\frac{\gamma}{c\mu}\right)^{i-c}\right) \tag{12}$$

which can be rewritten as

$$E(R(\infty)|c,N) = p_0\left(\sum_{i=1}^{c-1}\frac{(\gamma/\mu)^i}{(i-1)!} + \frac{(\gamma/\mu)^c}{c!} \cdot \frac{\left(c+\dfrac{\rho}{1-\rho}\right)(1-\rho^{N+1}) - (N+1)\rho^{N+1}}{1-\rho}\right) \tag{13}$$

where $\rho = \gamma/c\mu$. For the expected number of items disposed per time unit, $D_{c,N}$, we obtain

$$D_{c,N} = \gamma p_{c+N} = \gamma\rho^N\left(V + \frac{1-\rho^{N+1}}{1-\rho}\right)^{-1} \tag{14}$$

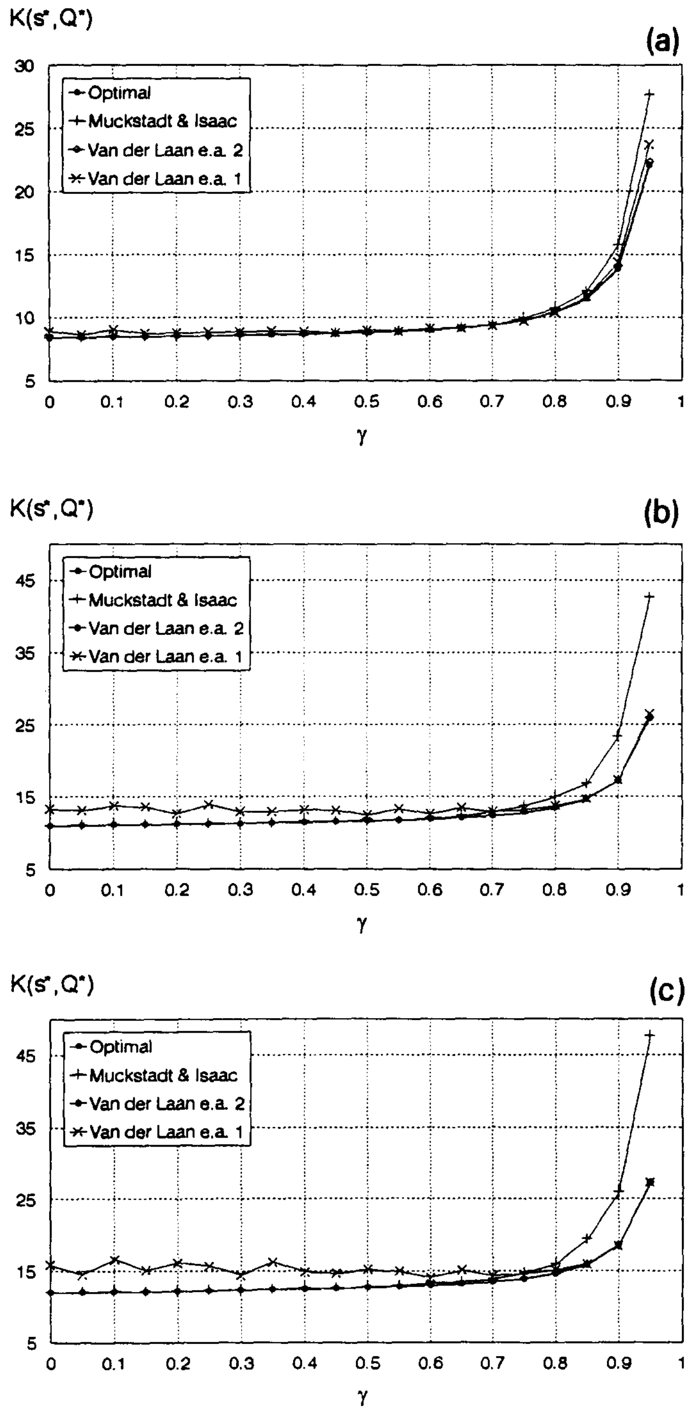

Fig. 2. (a) $h = 1$, $\lambda = 1$, $A = 10$, $\pi = 10$, $\tau = 10$. (b) $h = 1$, $\lambda = 1$, $A = 10$, $\pi = 50$, $\tau = 10$. (c) $h = 1$, $\lambda = 1$, $A = 10$, $\pi = 100$, $\tau = 10$

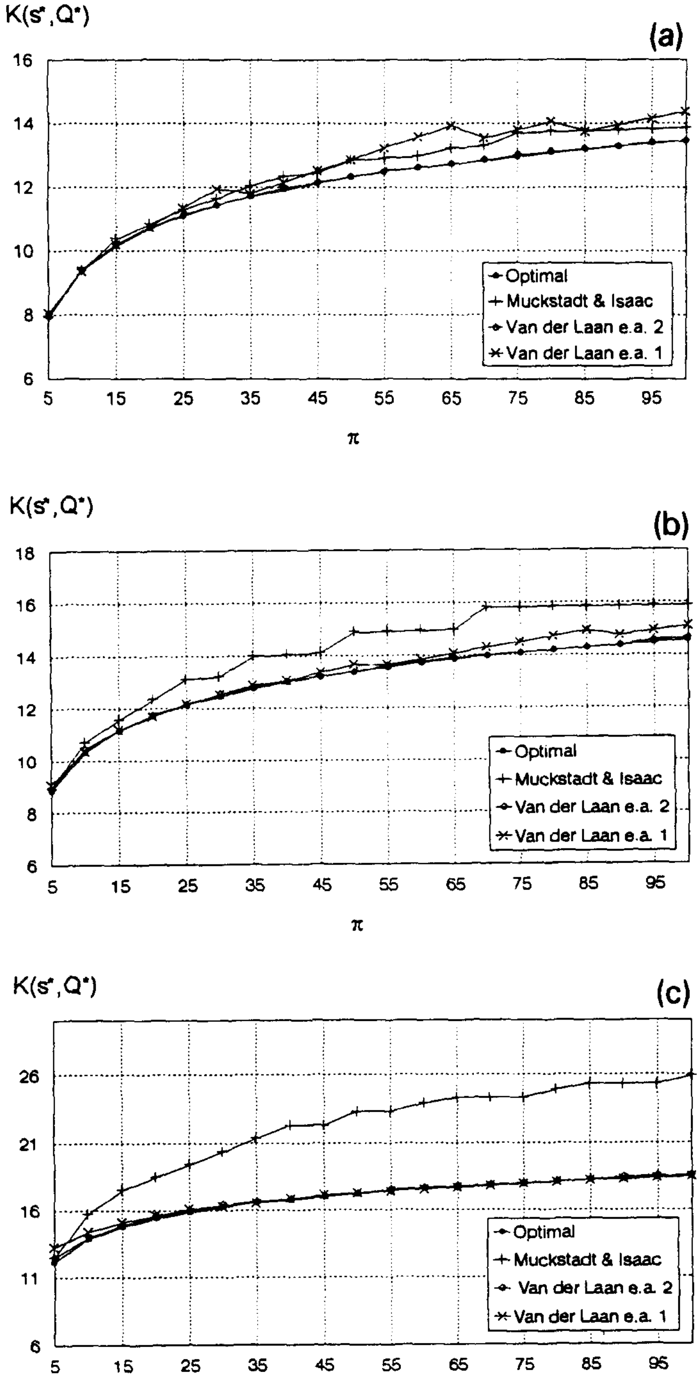

Fig. 3. (a) $h = 1$, $\lambda = 1$, $A = 10$, $\gamma = 0.7$, $\tau = 10$. (b) $h = 1$, $\lambda = 1$, $A = 10$, $\gamma = 0.8$, $\tau = 10$. (c) $h = 1$, $\lambda = 1$, $A = 10$, $\gamma = 0.9$, $\tau = 10$.

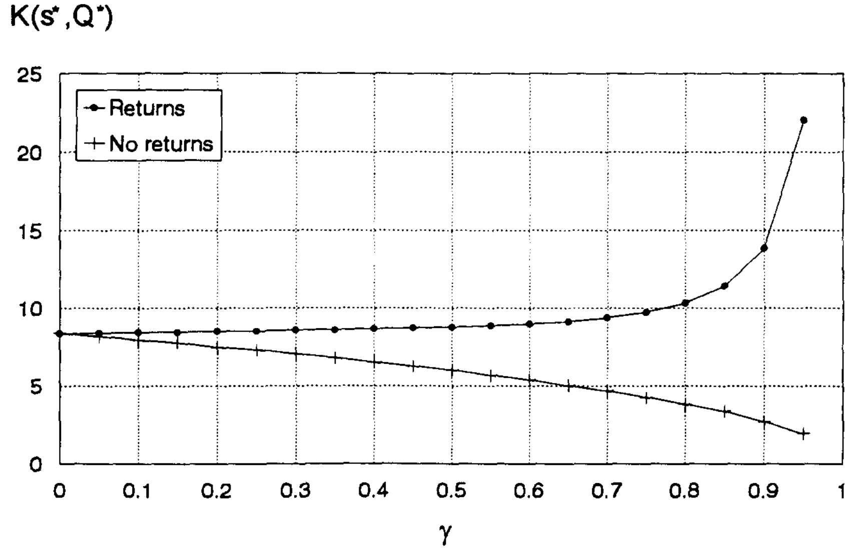

Fig. 4. $h = 1$, $\lambda = 1$, $A = 10$, $\tau = 10$, $\pi = 10$.

where

$$V = \sum_{i=0}^{c-1} \frac{c!}{i!}\left(\frac{\gamma}{\mu}\right)^{i-c}. \tag{15}$$

The exact procedure from Section 2 to calculate the costs can easily be applied in this case, since disposing only limits the state space. Yet the procedure will remain time consuming, especially if the return rate is high and we have to consider large state spaces. So we will also develop approximate cost function. Notice that the output process of the $M/M/c/c + N$ queueing system is no longer Poisson. Yet we will assume it is, and with a rate equal to $\gamma - D_{c,N}$, so that we can apply the analysis of the previous section. We now take into account costs δ_1, δ_2 and δ_3 for production, repair and disposal of an item, respectively. In the objective function we therefore have to replace γ by $\gamma - D_{c,N}$ and add a term $(\delta_1 - \delta_2 + \delta_3)D_{c,N}$. For the amount backordered we take the second approximation, since that yielded the best results. For a given disposal level N in the repair shop, the optimisation with respect to s and Q is similar to that in the previous section. The optimisation with respect to N, however, is far more difficult, since we cannot prove that the objective function is convex in N. To simplify the optimisation, we assume that the backorder costs are hardly influenced by N and can therefore be left out of consideration. Replacing γ by $\gamma - D_{c,N}$ into the objective function, neglecting the backordering costs and setting $k = s - R_{c,N}$, where $R_{c,N}$ indicates the expected number of items in the repair shop given c and N, yields the following minimisation problem:

$$\begin{aligned}\min_{N \geq 0} K(k^*, Q^*, N) = {} & A\left(\frac{\lambda - \gamma + D_{c,N}}{Q^*}\right) + h\Bigg(k^* + 1 \\ & + \frac{Q^* - 1}{2} + \frac{\gamma - D_{c,N}}{\lambda - \gamma + D_{c,N}}\Bigg) \\ & - h(\lambda - \gamma + D_{c,N})\tau \\ & + (\delta_1 - \delta_2 + \delta_3)D_{c,N}. \end{aligned} \tag{16}$$

From the derivative of this function it follows that a minimum exists if and only if

$$D_{c,N} = \sqrt{\frac{\lambda h}{H}} - (\lambda - \gamma), \tag{17}$$

where

$$H = \frac{A}{Q^*} - h\tau + \delta_1 - \delta_2 + \delta_3. \tag{18}$$

Since $0 \leqslant N \leqslant \infty$ it follows from (14) that $0 \leqslant D_{c,N} \leqslant \gamma/(1+V)$, and consequently it follows from (18) that a non-boundary minimum to (16) exists if and only if

$$\lambda h\left(\lambda - \gamma + \frac{\gamma}{1+V}\right)^{-2} \leqslant H \leqslant \lambda h(\lambda-\gamma)^{-2}. \qquad (19)$$

Now, if H does not satisfy the left inequality, or in other words, if $D_{c,N}$ should be larger than $\gamma/1+V$, then we want to dispose more items than is possible, since N is restricted to be non-negative. In this case the best we can do is to set $N = 0$. If H does not satisfy the right inequality, we do not want to dispose any item and in fact we want more items returned. Since in our model we cannot increase the rate of return, the best we can do is to set $N = \infty$. Finally, if H satisfies both inequalities it follows directly from (14) and (17) that we can compute the unique optimal value of N from the expression

$$\left(\frac{\gamma}{c\mu}\right)^N = \frac{U \cdot V}{\gamma} \cdot \left(\frac{1 - \dfrac{\gamma}{c\mu} + \dfrac{1}{V}}{1 - \dfrac{\gamma}{c\mu} + \dfrac{U}{c\mu}}\right) \qquad (20)$$

where

$$U = \sqrt{\frac{\lambda h}{H}} - (\lambda - \gamma). \qquad (21)$$

We now have obtained the following iterative optimisation scheme:

Step 0. Set $N = \infty$
Step 1. Compute s^* and Q^* given N from the adapted formulas (5) and (6)
Step 2. Compute $H = A/Q^* - h\tau + \delta_1 - \delta_2 + \delta_3$. If $H < \lambda h[\lambda - \gamma + \gamma/(1+V)]^{-2}$ then $N^* = 0$, else if H $> \lambda h(\lambda-\gamma)^{-2}$, then $N^* = \infty$, else solve N from (20)
Step 3. Repeat steps 1 and 2 until N converges.

We compared this iterative optimisation procedure with optimisation by a limited enumeration scheme of the approximate cost function *including* backorder costs and with a limited enumeration of the exact cost function. The results are given in Appendix B. For all resulting policies the associated average costs $K(s, Q, N)$ were evaluated using an exact procedure. It appeared that in 19 out of the 36 examples we considered, the iterative procedure yielded the optimum with respect to the full approximate cost function, but the costs difference can be substantial (up to 21%). This is due to the fact that in the first stage of the iterative procedure, backorders are not taken into account, causing an underestimate of N^* (in cases where the procedure estimated

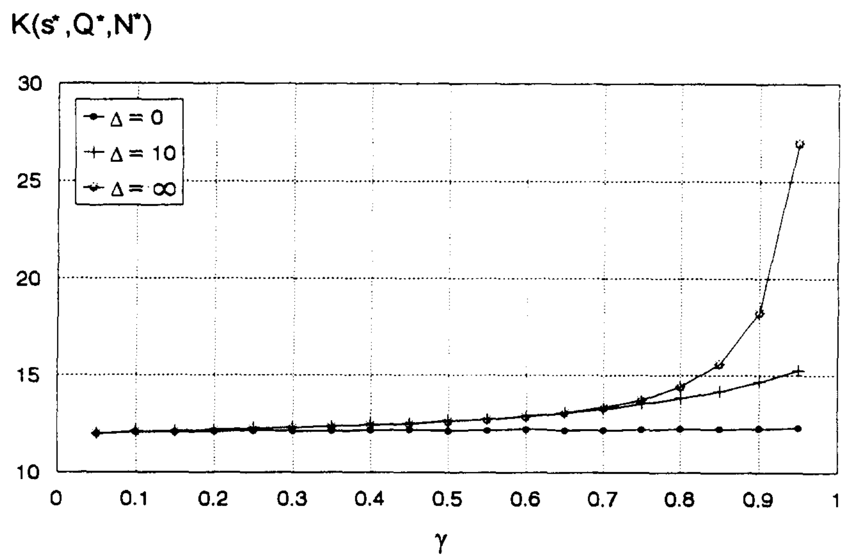

Fig. 5. $h = 1$, $\lambda = 1$, $A = 10$, $\tau = 10$, $\pi = 100$.

N^* to be ∞, the cost differential was small). To compensate for the consequential under-estimation of backorder costs, the second stage computes values of s and Q that are higher than the optimal values. However, still a substantial reduction in costs can be obtained using the iterative scheme with respect to the non-disposal case (last column Table 2). The enumerative scheme of the full approximate cost function performed very well with respect to the enumeration of the exact cost function. In 11 out of 36 cases, the optimal solution was found, whereas the maximum deviation in costs was 2% at the most. Concluding, we can say that the enumerative scheme is superior to the iterative scheme, but it is somewhat more time consuming. Yet the exact cost optimisation required more than 40 times more time, especially if a large state space had to be considered. Although we cannot guarantee convergence of the iterative scheme, less than four iterations were needed in all cases. We could improve our iterative procedure somewhat, if we would store the best solution encountered during the iterations and do a limited enumeration at the end.

From Table 2 we can also observe the following behaviour of s^*, Q^* and N^*. If γ/λ increases then both s^* and Q^* decrease, although not always monotonically as more experiments showed. The optimal value N^* decreases with γ/λ.

The effect of the value of $\Delta = \delta_1 - \delta_2 + \delta_3$, which we could denote as the net cost of disposal, is shown in Fig. 5, where we computed exact minimum costs for various values of Δ. As expected, average costs increase as Δ increases, but costs are always less than those of the non-disposal case, except for $\Delta = \infty$ when the minimum costs are equal to the minimum costs in the non-disposal case. This means that it is always meaningful to incorporate a disposal strategy, since it reduces costs in all instances. This can also be seen in Table 2.

4. Conclusions

In this note we presented an (s, Q) model for inventory control under remanufacturing and disposal. Although an exact analysis for this model is possible, we developed approximations for both the cost evaluation and the optimisation which perform reasonably well against much less numerical effort. Moreover, they provide more insight into the behaviour of the optimal parameters. We have shown that disposal is a necessary option because inventory levels may otherwise rise to very high values because of the variability in the return streams. Yet incorporating disposal does complicate the model and especially the optimisation. More research is needed to get insight into the value of this model in more complex production situations. Other possible extensions are with respect to a non-stationarity of the demand and return rates following life cycles of products, to consider other disposal strategies (with batch disposal) and to relax the Poisson assumptions of demand and return processes.

Appendix A: Convexity of the approximative cost functions

We start with proving that the first approximative cost function, given by (4) is convex. It is quite easy to show that the second derivative with respect to Q is positive. Next consider the second derivative with respect to s. The main complication lies in the term $\{F(s,t)\}^2$. Notice now that

$$\begin{aligned}\frac{\mathrm{d}^2F(s,t)^2}{\mathrm{d}s^2} &= 2F(s,t)\frac{\mathrm{d}^2F(s,t)}{\mathrm{d}s^2} + 2\left(\frac{\mathrm{d}F(s,t)}{\mathrm{d}s}\right)^2\\ &= \frac{2}{\sigma_L}F(s,t)\phi\left(\frac{s-R-\mu_L}{\sigma_L}\right)\\ &\quad + 2\left(\frac{\mathrm{d}F(s,t)}{\mathrm{d}s}\right)^2 > 0,\end{aligned}$$

which shows that the function is convex in s. To finish the proof we apply the following standard lemma on convexity, which we state without proof.

Lemma A.1. *Let $f(x)$ be a positive-valued, decreasing and convex function in x, and let $g(y)$ be a linear positive valued function in y, then $h(x,y) := f(x)/g(y)$ is convex in (x, y).*

Taking $f(x) = F(s, t)^2$ and $g(y) = Q/(\lambda - \gamma)$ yields the desired result.

For the second approximation given by (7) we have to consider the first and second derivative of

$$\int_0^\tau F(s,t)\,\mathrm{d}t$$

with respect to s.

$$\frac{\partial}{\partial s}\int_0^\tau F(s,t)\,\mathrm{d}t = \int_0^\tau \frac{\partial}{\partial s} F(s,t)\,\mathrm{d}t = \int_0^\tau -\Phi(z(t))\,\mathrm{d}t < 0$$

$$\frac{\partial^2}{\partial s^2}\int_0^\tau F(s,t)\,\mathrm{d}t = \int_0^\tau \frac{\partial^2}{\partial s^2} F(s,t)\,\mathrm{d}t = \int_0^\tau \frac{1}{\sigma_\tau}\Phi(z(t))\,\mathrm{d}t > 0$$

where $z(t) = (s - \mu_t - R)/\sigma_t$. Thus, $\int_0^\tau F(s,t)\,\mathrm{d}t$ is strictly decreasing and strictly convex in s. Applying the lemma with

$$f(s) = \int_0^\tau F(s,t)\,\mathrm{d}t \text{ and } g(y) = Q/(\lambda - \gamma)$$

yields the proof.

Appendix B. Comparison of numerical performance

Table 2
Performance of the (iterative) solution procedure for the M/M/1/N repair queue: $\lambda = 1.00$, $A = 10.0$, $h = 1.00$, $\tau = 10.0$, $\mu = 2.00$

γ/λ	π	Δ[a]	Exact cost function with enumeration				Approximative costs with enumeration				Approximative costs iterative procedure				Non-disposal case exact optimisation		
			s	Q	N	$K(s,Q,N)$	s	Q	N	$K(s,Q,N)$	s	Q	N	$K(s,Q,N)$	s	Q	$K(s,Q)$
0.30	10	0	9	6	0	8.4253	9	6	0	8.4253	9	6	0	8.4253	9	6	8.5735
0.50			8	5	0	8.4208	8	6	0	8.4518	8	6	0	8.4518	7	5	8.7410
0.70			7	5	0	8.4188	7	5	0	8.4188	7	5	0	8.4188	5	4	9.3044
0.80			6	5	0	8.3939	6	6	0	8.4615	6	6	0	8.4615	4	4	10.2307
0.90			5	5	0	8.4600	6	5	0	8.4924	6	5	0	8.4924	2	3	13.7291
0.95			5	5	0	8.4524	5	6	0	8.5250	5	6	0	8.5250	−1	3	21.9295
0.30		10	9	6	∞	8.5735	9	6	∞	8.5735	9	6	0	8.8166			
0.50			7	5	∞	8.7410	7	6	∞	8.7764	8	6	0	9.4518			
0.70			5	5	2	9.2493	5	5	∞	9.3093	7	5	0	10.2337			
0.80			5	4	1	9.8308	4	5	2	9.8937	6	6	0	10.7472			
0.90			4	4	1	10.5696	4	5	1	10.6870	6	5	0	11.2854			
0.95			3	4	1	11.1074	3	5	1	11.1893	5	6	0	11.5841			
0.30		20	9	6	∞	8.5735	9	6	∞	8.5735	9	6	∞	8.5735			
0.50			7	5	∞	8.7410	7	6	∞	8.7764	7	6	∞	8.7764			
0.70			5	4	7	9.3041	5	5	∞	9.3088	5	5	∞	9.3088			
0.80			4	4	3	10.1298	4	5	∞	10.3438	5	5	1	10.7273			
0.90			4	4	1	11.6724	4	5	1	11.7899	6	5	0	14.0791			
0.95			3	4	1	12.3677	3	5	1	12.4498	5	6	0	14.6426			
0.30	100	0	13	6	0	12.1248	13	6	0	12.1248	13	6	0	12.1248	13	5	12.2870
0.50			12	5	0	12.1365	12	5	0	12.1365	12	5	0	12.1365	11	5	12.6465
0.70			11	5	0	12.1742	11	5	0	12.1742	11	5	0	12.1742	10	4	13.3511
0.80			10	5	0	12.2386	11	5	0	12.3366	11	5	0	12.3366	9	3	14.4366
0.90			10	4	0	12.2537	10	5	0	12.2877	10	5	0	12.2877	7	3	18.2580
0.95			10	4	0	12.2961	10	5	0	12.3853	10	5	0	12.3853	6	2	27.0088
0.30		10	13	5	∞	12.2870	13	5	∞	12.2870	13	5	1	12.3111			
0.50			11	5	∞	12.6465	11	5	∞	12.6465	12	5	0	13.1365			
0.70			10	4	2	13.2528	10	4	∞	13.3511	11	5	0	13.9890			
0.80			9	4	1	13.8178	10	4	1	13.9535	11	5	0	14.6223			
0.90			8	4	1	14.6530	9	4	1	14.7705	10	5	0	15.0808			
0.95			8	3	1	15.2627	10	5	0	15.4445	10	5	0	15.4445			
0.30		20	13	5	∞	12.2870	13	5	∞	12.2870	13	5	∞	12.2870			
0.50			11	5	∞	12.6465	11	5	∞	12.6465	11	5	∞	12.6465			
0.70			10	4	5	13.3495	10	4	∞	13.3511	10	4	∞	13.3511			
0.80			9	4	2	14.2956	9	4	∞	14.4935	10	4	1	14.7740			
0.90			8	4	1	15.7558	9	4	1	15.8734	10	5	0	17.8739			
0.95			8	3	1	16.5231	8	4	1	16.5505	10	5	0	18.5039			

[a] $\Delta = \delta_1 - \delta_2 + \delta_3$.

References

[1] Thierry, M.C., Salomon, M., Van Nunen, J.A.E.E. and Van Wassenhove, L.N., 1995. Strategic production and operations management issues in product recovery management. Calif. Mgmt. Rev., 37: 114-135.

[2] Nahmias, S., 1981. Managing repairable item inventory systems: a review, in: TIMS Studies in Management Sciences, Vol. 16, pp. 253-277. North Holland, Amsterdam.

[3] Mabini, M.C. and Gelders, L.F., 1991. Repairable item inventory systems: a literature review. Belgian J. Oper. Res, Stat. and Comp. Sci., 30: 57-69.

[4] Cho, D.I. and Parlar, M. 1991. A survey of maintenance models for multi-unit systems. Eur. J. Oper. Res., 51: 1-23.

[5] Salomon, M., Laan, E.A. van der, Dekker, R., Thierry, M. and Ridder, A.A.N., 1994. Product remanufacturing and its effects on production and inventory control. ERASM Management Report Series no. 172, Erasmus University Rotterdam, Netherlands.

[6] Inderfurth, K., 1982. Zum Stand der betriebswirtschaflichen Kassenhaltungstheorie. Zeitschrift für Betriebswirtschaft, 3: 295-320.

[7] Simpson, V.P., 1978. Optimum solution procedure for a repairable inventory problem, Oper. Res., 26: 270-281.

[8] Kelle, P. and Silver, E.A., 1989. Purchasing policy of new containers considering the random returns of previously issued containers. IIE Trans., 21: 349-354.

[9] Heyman, D.P., 1977. Optimal disposal policies for a single-item inventory system with returns. Naval. Res. Logist. Quart., 24: 385-405.

[10] Haodley, B. and Heyman, D.P., 1977. A two-echelon inventory system with purchases, disposition, shipments, returns and transhipments. Naval. Res. Logist. Quart., 24: 1-19.

[11] Muckstadt, J.A. and Isaac, M.H., 1981. An analysis of single item inventory systems with returns. Naval. Res. Logist. Quart., 28: 237-254.

[12] Flapper, S.D.P., 1994. Matching material requirements and availabilities in the context of recycling: an MRP-I based heuristic. Proc. 8th Working seminar on Production Economics, IGLS, Austria.

[13] Laan, E.A. van der, 1993. On inventory control models where items are remanufactured or disposed, unpublished Master's Thesis, Econometric Institute, Erasmus University Rotterdam, Netherlands.