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ON THE $(S-1, S)$ STOCK MODEL FOR RENEWAL DEMAND PROCESSES

POISSON'S POISON

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In the standard $(S-1, S)$ stock model, demand follows a Poisson process. It has appeared to many stock analysts that this model causes an abundance of stock in reality. In case demand is caused by failure or is derived from another process, demand typically does *not* follow a Poisson process. In this paper, we discuss the $(S-1, S)$ stock model where demand follows a renewal process and the lead time is deterministic. Moreover, we will extend this to compound renewal demand and multi-echelon inventory systems. Our goal is to show the severe influence of taking the Poisson process for granted.

1. INTRODUCTION

For slow-moving stock, the lot-for-lot, or one-for-one, replenishment policy is a very popular policy in practice. To our surprise, literature has paid very little attention to $(S-1, S)$ stock inventory control with renewal demand. One paper worth mentioning is that by Kalpakam and Sapna [9]. In their paper, they consider perishable inventory and exponential lead times. They use semi-Markov modeling to compute the probability of a shortage. In numerous articles in which the $(S-1, S)$ stock system is analyzed, one usually makes the comfort-

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able assumption that demand follows a (compound) Poisson process (cf. Graves [4], Lee [11], Muckstadt [12], Nahmias [13], Schaeffer [15], Sherbrooke [16,18], and Sherbrooke and Feeney [19]). This assumption is usually justified by Palm's theorem (cf. Khintchine [10], stating that, under mild assumptions, the distribution of the total number of events in a small time interval coming from different independent renewal processes tends to a Poisson distribution as the number of independent processes grow. So, if demand originates from a large number of independent clients, a Poisson process on a small time scale may be justifiable.

The $(S-1, S)$ policy is often used in spare parts inventory control. Practical experience of stock managers of critical equipment is that the $(S-1, S)$ model with Poisson demand often advises more stock than they consider really necessary, especially if spares are installed in only a few machines. For example, Rooij [14] considers spare parts inventory control of a large petrochemical plant in Rotterdam. One of the spare parts, an oil baffle, installed in one unit, had an average demand of once every 20 months. The lead time was half a year, and a service level of 99.5% was required. The company had two spare oil baffles, which seemed to them a very reasonable amount of spares. However, the $(S-1, S)$ stock model with Poisson demand advised twice as many.

The observation is that although the failure process of equipment often follows a Poisson process, this is not the case at component level. Take, for example, a seal in a pump. Once replaced it is unlikely that it will fail shortly again. Instead, the failure process may be described by a renewal process where the interarrival distribution has an increasing failure rate. This lowers the probability of another demand during the lead time.

In case demand follows a renewal process, the $(S-1, S)$ policy is, in general, not optimal. When there is less variation in the demand, it can be more advantageous to delay an order for some period of time. Such a policy, however, is too complex for today's spare parts management systems. Therefore, we restrict ourselves in this paper to $(S-1, S)$ policies with immediate reordering.

We will assume that the lead time is deterministic, which is not an unrealistic assumption. There can be strict agreement between the client and the supplier about the lead time. Smaller as well as larger lead times are not desired (cf. Rooij [14]).

2. THE $(S-1, S)$ MODEL WITH BACKLOGGING

2.1 Single Echelon

Consider the $(S-1, S)$ model where demand follows a renewal process with a nonarithmetic arbitrary interarrival distribution, $G(.)$. We assume that a replenishment order takes a constant time D and that orders that cannot be delivered directly are backlogged. Furthermore, we assume that the demand process is not affected by stock-outs. This is an approximative assumption that is also made in most of the aforementioned literature. Because the $(S-1, S)$ stock model aims to choose S such that the stock-out probability is small, the influence of the assumption will be rather trifling. Moreover, in practice, there are different alternative actions possible when a stock-out occurs. A rush order, for example, can be placed or sometimes a different, better, but more expensive spare can be used or some provisional solution is found, such as replacing a v-snare by a panty. The steady-state probabilities of the number of outstanding orders follow directly from renewal theory.

LEMMA 1: *The steady-state probability, say, $p_n(D)$, that there are n outstanding replenishment orders is given by*

$$\begin{aligned} p_n(D) &= G_e * G^{(n-1)}(D) - G_e * G^{(n)}(D), \qquad n > 0, \\ p_0(D) &= 1 - G_e(D), \end{aligned} \tag{2.1}$$

where $*$ denotes a convolution, the superscript $^{(n)}$ the n-fold iterated convolution of a distribution with itself, and $G_e(t) = (1/\mu)\int_0^t (1 - G(x))\, dx$ the equilibrium distribution.

PROOF: When there are n outstanding replenishment orders, it follows that exactly n customers have arrived during the past D time. Because the interarrival distribution is nonarithmetic, it follows that the length of time since the last customer arrived is distributed according to the equilibrium excess distribution (cf. Tijms [20]). Therefore, the probability of n outstanding orders equals the probability that n events have happened in D time units in an equilibrium renewal process (cf. Cox and Miller [3]).

LEMMA 2: *The steady-state probability that there are n outstanding orders just after a customer has arrived is given by*

$$p_n^*(D) = G^{(n-1)}(D) - G^{(n)}(D), \qquad n > 0, \tag{2.2}$$

with $G^{(0)}(D) = 1$.

PROOF: When an order just has been made, the probability that there are n outstanding orders equals the probability of $n - 1$ events in an ordinary renewal process at time D (cf. Cox and Miller [3]). ∎

The main problem in evaluating p_n and p_n^* lies in the computation of the convolutions. A popular way to manage convolutions is to approximate the distribution by a phase-type distribution by fitting the first two or three moments and subsequently calculate the convolutions of these distributions analytically (cf. Harrison [5]). If we apply this approach directly to p_n, the problem occurs that we need to approximate two distributions, namely, $G_e(.)$ and $G(.)$. In the next theorem, we show how p_n can be written as a function of p_n^*. This way we only need to fit to the distribution $G(.)$ and *not* to the distribution $G_e(.)$, which simplifies the computation of p_n considerably. In the Appendix, we will elaborate on this topic.

Another method is to approximate the distribution by cubic splines, in which case a spline approximation of the n-fold convolution can be obtained by matrix multiplication (cf. Iseger, Smith, and Dekker [7]).

THEOREM 1: *The steady-state probabilities p_n can be expressed as*

$$p_n(D) = \frac{1}{\mu}\int_0^D \{p_n^*(t) - p_{n+1}^*(t)\}\,dt$$
$$= \frac{1}{\mu}\int_0^D \{G^{(n-1)}(t) - 2G^{(n)}(t) + G^{(n+1)}(t)\}\,dt, \qquad n > 0, \qquad (2.3)$$

and

$$p_0(D) = 1 - \frac{1}{\mu}\int_0^D p_1^*(t)\,dt = 1 - \frac{1}{\mu}\int_0^D \{1 - G(t)\}\,dt. \qquad (2.4)$$

PROOF: First notice that $p_n^*(t) - p_{n+1}^*(t) = G^{(n-1)}(t) - 2G^{(n)}(t) + G^{(n+1)}(t)$. Next, consider the probabilities $p_n(D)$ as a function of the lead time D. The Laplace transform is given by

$$\hat{p}_n(s) = \int_0^\infty e^{-sD} p_n(D)\,dD = \hat{g}_e(s)(\hat{g}^{n-1}(s) - \hat{g}^n(s))/s, \qquad (2.5)$$

with $\hat{g}(s)$ and $\hat{g}_e(s)$ the Laplace transform of G and G_e. Because $\hat{g}_e(s) = (1 - \hat{g}(s))/(\mu s)$, we obtain

$$\hat{p}_n(s) = (\hat{g}^{n-1}(s) - 2\hat{g}^n(s) + \hat{g}^{n+1}(s))/(\mu s^2). \qquad (2.6)$$

Formal inversion yields the desired result. ■

From these results, it easily follows that the long-term average proportion of time that there is a stock-out is given by

$$p_{loss}(D) = \sum_{i=S}^{\infty} p_i(D) = G_e * G^{(S-1)}(D) \qquad (2.7)$$

and that the probability of an arbitrary customer finding the stock empty is given by

$$p_{loss}^*(D) = \sum_{i=S+1}^{\infty} p_i^* = G^{(S)}(D). \qquad (2.8)$$

Another important aspect of the $(S-1, S)$ model is the waiting time distribution (cf. Higa, Feyerherm, and Machado [6] and Sherbrooke [17]). Let us denote the distribution of the waiting time of an arbitrary customer by $P(W \leq t)$. In the $(S-1, S)$ stock policy, we might say that a customer gets the part that was ordered as a replenishment for the Sth customer before him or her. Hence,

$$P(W \leq t) = 1 - p_{loss}^*(D - t) = 1 - G^{(S)}(D - t). \qquad (2.9)$$

Slightly different from the waiting time is the average uninterrupted time that there is no stock on hand. By using the level crossing method (cf. Tijms [20]), we find that this is given by $\mu p_{loss}(D)/p^*_{S+1}(D)$. In words, the average number of times per unit time that a stock-out appears, p^*_{S+1}/μ, should equal the average number of times per unit time that a stock-out disappears, $p_{loss}(D)/$(average stock-out length).

2.2. Extension: Multi-Echelon

In this section, we show that the analysis can easily be extended to the case that the demand stems from a superposition of renewal processes. Consider a two-level system where retailers order at a central warehouse. At each local retailer, there is a local inventory. Upon demand at the retailer, the retailer sets an order at the central warehouse to replenish its stock. In this case, the demand at the warehouse stems from a superposed renewal process. Here the usual Poisson assumption may be justified as an approximation when the lead time is relatively small. We will not pursue this approximation.

The lead time for replenishment orders of the warehouse will be denoted by D_w. The distribution of the interarrival times of customers at retailer i will be denoted by $G_i(t)$ with mean μ_i. The lead time for replenishment orders of retailer i from the warehouse is given by D_i. This constant lead time D_i is made under the assumption that there is no stock-out at the warehouse and can be interpreted as the transportation time between the warehouse and retailer i. There is no coordination of replenishment orders.

Let us denote by $p_{j_i}(D_w)$ and $p^*_{j_i}(D_w)$ the probability that there are j_i outstanding orders from retailer i at the warehouse, respectively, at an arbitrary moment and when retailer i has just ordered. These probabilities have been given in the previous section. The total number of outstanding orders now follows directly.

LEMMA 3: *Consider a warehouse and k retailers. The probabilities, say, $r_n(D_w)$, that at an arbitrary point in time there are n outstanding orders at the warehouse is given by*

$$r_n(D_w) = \sum_{\{j_1,\ldots,j_k \mid \sum_{i=1}^k j_i = n\}} \pi_{i=1}^k p_{j_i}(D_w). \tag{2.10}$$

PROOF: To have n outstanding orders at the warehouse at an arbitrary point in time, there have to be j_i outstanding orders of retailer i and the sum of j_i, $i = 1, \ldots, k$, should equal n. ∎

LEMMA 4: *The probabilities, say, $r^*_n(D_w)$, that there are n outstanding orders at the warehouse just after an order has been made is given by*

$$r^*_n(D_w) = \sum_{i=1}^{k} \sum_{j_i=1}^{n} \frac{1/\mu_i}{\sum_{l=1}^{k} 1/\mu_l}\, p^*_{j_i}(D_w) q_{n-j_i}(D_w), \tag{2.11}$$

with $q_{n-j_i}(D_w)$ the probability $r_{n-i_i}(D_w)$ as given in Theorem 3 without *retailer i.*

PROOF: When a customer arrives, then the steady-state probability that the customer is of type i is given by the ratio of the individual arrival frequency and the total arrival frequency. When the customer is of type i, then there will be j_i customers of type i in the system with probability $p^*_{j_i}(D_w)$. Because the distributions $G_i(.)$ are mutually independent and nonarithmetic, the arrival of a customer of type i will have happened at an arbitrary point in time as far as the other customers are concerned. Hence, the probability that the other customers total in the system equals $n - j_i$ is given by $q_{n-j_i}(D_w)$. ■

The stock-out probabilities are given by

$$r_{loss}(D_w) = \sum_{i=S}^{\infty} r_i(D_w) \quad \text{and} \quad r^*_{loss}(D_w) = \sum_{i=S+1}^{\infty} r^*_i(D_w) \tag{2.12}$$

and the waiting time distribution yields

$$P(W < t) = 1 - r^*_{loss}(D_w - t). \tag{2.13}$$

The average uninterrupted length of time that there is no stock on hand is given by

$$r_{loss}(D_w) \Big/ \left(r^*_{S+1}(D_w) \sum_{i=1}^{k} \frac{1}{\mu_i} \right). \tag{2.14}$$

The average lead time for the replenishment of retailer i is given by

$$\bar{D}_i = D_i + \int_0^D r^*_{loss}(t)\, dt. \tag{2.15}$$

Following the METRIC approximation (cf. Axsäter [1] and Sherbrooke [16]), we use $\bar{D}_i$ for the replenishment lead time of retailer i. The extension to more than two levels is straightforward.

3. RESULTS

To illustrate the influence of the arrival process, we consider again the example we mentioned in the introduction. One of the spare parts at the petrochemical plant is an oil baffle. It is installed in one vital machine for which breakdown would mean an immediate and serious interruption causing unsafe operations. Therefore, the management requires a high service level of 99.95%.

The oil baffle has an average demand interval of 20 months and a lead time of half a year. The price of the baffle is around $5250, which is reasonably expensive.

In Table 1, we compare the distributions of the number of outstanding orders in case the arrival process is a Poisson process and a renewal process with interarrival distribution Erlang-4. The number of spares on stock is determined by the tail of the distribution. We see that the Poisson process and the renewal process cause a completely different tail. The tail caused by the renewal process is much thinner.

TABLE 1. Distribution of the Number of Outstanding Orders at a Retailer Just after a Customer Has Arrived[a]

p_n^*	Poisson	Erlang_4
1	0.7408	0.9962
2	0.2222	0.0337
3	$3.33E{-2}$	$3.70E{-5}$
4	$3.33E{-3}$	$6.17E{-9}$
5	$2.50E{-4}$	$2.8E{-13}$
6	$1.50E{-5}$	$5.0E{-18}$
7	$7.50E{-7}$	$4.1E{-23}$
8	$3.21E{-8}$	$1.7E{-28}$
9	$1.21E{-9}$	$4.1E{-34}$
10	$4.0E{-11}$	$5.9E{-40}$

[a]The replenishment lead time is half a year, and the mean interarrival time is 20 months.

Furthermore, let us assume that the plant orders the oil baffles at some warehouse and that there are in total three more (identical) petrochemical plants that do the same. The distribution of the number of outstanding orders at the warehouse, with $D_w = 0.5$ year, is depicted in Table 2. Here we can draw the

TABLE 2. Distribution of the Number of Outstanding Orders at the Warehouse Just after a Customer Has Arrived[a]

p_n^*	Poisson	Erlang_4
1	0.3012	0.3348
2	0.3614	0.4338
3	0.2168	0.1956
4	$8.67E{-2}$	$3.41E{-2}$
5	$2.60E{-2}$	$1.59E{-3}$
6	$6.24E{-3}$	$2.77E{-5}$
7	$1.25E{-3}$	$2.25E{-7}$
8	$2.14E{-4}$	$8.8E{-10}$
9	$3.21E{-5}$	$1.6E{-12}$
10	$4.28E{-6}$	$1.6E{-15}$

[a]The demand is coming from four retailers. The replenishment lead time is half a year, and the mean interarrival time is 20 months

same conclusion; in case the interarrival distribution has a smaller variance, the tail of the distribution of the number of outstanding orders at the warehouse is thinner.

As a result, if management requires a service level of 99.95%, the $(S-1,S)$ stock model with Poisson demand would suggest four spare parts at the retailer ($p^*_{loss} = 3E-4$) and seven at the warehouse ($r^*_{loss} = 3E-4$). The $(S-1,S)$ model with demand following a renewal process with interarrival distribution Erlang$_4$ would suggest only two spares at the retailer ($p^*_{loss} = 4E-5$) and five at the warehouse ($r^*_{loss} = 1E-4$).

Further computational experiments confirm the hypothesis that substantially less stocks are needed for interarrival distributions with squared coefficient of variation less than 0.25.

As there are thousands of very slow-moving spare parts items at a petrochemical plant, each with a low stock level, the effect of a non-Poisson arrival distribution is considerable.

In case the replenishment lead time is relatively large, we can make the difference also insightful by looking at the mean, $m^*(D)$, and *especially* the variance, $V^*(D)$, of the number of outstanding orders at a retailer when a customer has just arrived. The variance is especially important because it can be used as a measure of the tail—consider, for example, Chebychev's inequality. From Theorem 2, it follows (cf. Cox and Miller [3]) that

$$m^*(D) = 1 + \frac{D}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + \mathcal{O}(1/D) \tag{3.1}$$

and

$$V^*(D) = \frac{\sigma^2 D}{\mu^3} + \left(\frac{1}{12} + \frac{5\sigma^4}{4\mu^4} - \frac{2\mu_3}{3\mu^3}\right) + \mathcal{O}(1/D) \tag{3.2}$$

with μ, μ_3, and σ^2 the mean, third central moment, and variance of the interarrival distribution. Note that in the leading term of the variance, $(\sigma^2 D)/\mu^3$, the variance of the interarrival distribution is presented. This shows again that a smaller variance of the interarrival distribution causes a smaller tail of the distribution of the number of orders in the lead time that determines the number of stock necessary. Furthermore, it can be derived (cf. Barbour, Holst, and Janson [2]) that in case $V^*(D) < m^*(D)$ we have

$$\sum_{n=0}^{\infty} \left| p_n^*(D) - \frac{e^{-m^*(D)} m^*(D)^n}{n!} \right| > c\epsilon \frac{\min\left\{1, \max(m^*(D),1)\left(1 + \ln\left(\frac{1}{\epsilon}\right)\right)^{-1}\right\}}{1 + \ln\left(\frac{1}{\epsilon}\right)}, \tag{3.3}$$

with

$$\epsilon = \min(1, m^*(D))(1 - V^*(D)/m^*(D)) \tag{3.4}$$

and c some universal constant. This formula shows us explicitly the order of the lower bound of the error as a function of the variance.

For Tables 1 and 2, the replenishment lead time is not large enough to use the preceding formulas, though. The mean and variance of the number of outstanding orders at a retailer are 1.30 versus 1.003 and 0.30 versus 0.09 for the Poisson and the Renewal processes, respectively. Formulas (3.1) and (3.2) give 1.30 versus 0.925 and 0.30 versus −1.39.

4. CONCLUSIONS

Practical experience shows that the standard models where the demand is assumed to follow a Poisson process may cause an abundance of stock. In case demand is caused by wear-out failures, demand typically does *not* follow a Poisson process. In this paper, we have discussed the $(S-1, S)$ stock model where demand follows a renewal process and the lead time is deterministic. We have extended the results to compound renewal demand and multi-echelon inventory systems. We have seen that the distribution of the number of orders depends heavily on the variance of the interarrival times. The influence of the variance of the demand process is such that it can easily lead a stock manager to make a wrong decision about the base stock level in case the Poisson process is mistakenly taken for granted.

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